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THE EXACT BLOW-UP RATE OF LARGE SOLUTIONS TO INFINITY-LAPLACIAN EQUATION*

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Abstract Under new conditions on weight functions $b(x)$, this paper mainly considers the exact boundary behavior of solutions to the following boundary blow-up elliptic problems $\triangle_\infty u = b(x)f(u),\ x \in \Omega,\ u|_{\partial\Omega} = +\infty$ for more general nonlinearities f, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, and $b \in C(\bar{\Omega})$ which is positive in Ω.

Keywords Infinity-Laplacian equation, blow-up solutions, asymptotic behavior.

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1. Introduction and the main results

In this paper, we analyze the exact asymptotic behavior of viscosity solutions to the following problem

$$\triangle_\infty u = b(x)f(u),\ u > 0,\ x \in \Omega,\quad u|_{\partial\Omega} = \infty, \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N (N \geq 2)$, the last condition means that $u(x) \to \infty$ as $d(x) = dist(x, \partial\Omega) \to 0$, and the solution is called "boundary blow-up solution," "large solution" or "explosive solution."

The operator $\triangle_\infty$ is the ∞-Laplacian, a highly degenerate elliptic operator given by

$$\triangle_\infty u := \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^{N} D_i u D_{ij} u D_j u,$$

b satisfies

$(\mathbf{b_1})$ $b \in C(\bar{\Omega})$ is positive in Ω,

and f satisfies

$(\mathbf{f_1})$ $f \in C^1(0,\infty),\ f(0) = 0,\ f$ is increasing on $(0,\infty)$;

$(\mathbf{f_2})$ $\int_1^\infty \frac{d\nu}{(f(\nu))^{\frac{1}{3}}} < \infty$.

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Aronsson [1] established that the ∞-Laplacian equation $\triangle_\infty u = 0$ is the Euler-Lagrange equation for smooth absolute minimizers. Because of the high degeneracy of the ∞-Laplacian, the associated Dirichlet problems may not have classical solutions. Therefore, Crandall, Lions [5], Crandall, Evans, Lions [6] and Crandall, Ishii, Lions [7] introduced the concept of viscosity solution. Later, Jensen [9] proved the equivalence of absolute minimizers and viscosity solutions of the Dirichlet problem to the infinity harmonic equation and the uniqueness of the viscosity solutions. Since then, the infinity Laplace equation has been considered widely, see [2–10] and the references therein.

Juutinen and Rossi [10] first studied the existence, uniqueness and boundary behavior of solutions to the following problem

$$\triangle_\infty^N u = u^q,\ q > 1,\ x \in \Omega,\quad u|_{\partial\Omega} = \infty,$$

with the normalized $\infty-$Laplacian

$$\triangle_\infty^N u := \frac{1}{|Du|^2}\langle D^2uDu, Du\rangle.$$

When b satisfies (b_1) and f satisfies (f_1), Mohammed and Mohammed [11, 12] first supplied a necessary and sufficient condition

$$\int_a^\infty \frac{ds}{\sqrt[4]{F(s)}} < \infty,\quad \forall a > 0,\quad F(s) = \int_0^s f(\nu)d\nu, \tag{1.2}$$

for the existence of solutions to problem (1.1). Moreover, they showed that

(i) if

$$b_2 diam(\Omega) < \Psi(0) = \int_0^\infty \frac{ds}{\sqrt[4]{F(s)}},$$

then there are constants $c_0 > 0$ and $\delta > 0$ sufficiently small such that for any solution $u \in C(\Omega)$ to problem (1.1), it holds

$$\psi_1(b_2 d(x)) \le u(x) \le \psi_1(b_1 d(x)) + c_0,\ x \in \Omega_\delta := \{x \in \Omega : d(x) < \delta\},$$

where $b_1 = \big(\min_{x\in\Omega} b(x)\big)^{1/4}, b_2 = \big(\max_{x\in\Omega} b(x)\big)^{1/4}$, and ψ_1 satisfies

$$\int_{\psi_1(t)}^\infty \frac{ds}{\sqrt[4]{F(s)}} = t,\quad \forall t > 0; \tag{1.3}$$

(ii) if $b \equiv b_0$ in Ω and f satisfies the condition that

$$f(s)/s^3 \text{ is nondecreasing on } (0, \infty),$$

then problem (1.1) has a unique solution.

By means of Karamata regularly varying theory, Wang et al. [18] and [19], the author [14] and Zhang [21] further showed the boundary behavior of solutions to problem (1.1).

Next, we introduce two classes of functions.

Firstly, we denote Λ the set of all positive non-decreasing functions $k \in C^1(0,\nu)$ which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = C_k, \quad where\ K(t) = \int_0^t k(s)ds. \tag{1.4}$$

It is easy to see that for each $k \in \Lambda$,

$$\lim_{t\to 0^+} \frac{K(t)}{k(t)} = 0 \text{ and } C_k \in [0,1].$$

Next, let Θ denote the set of all Karamata functions $\hat{L}$ which are **normalized** slowly varying at zero (the definition can be found in Section 2.) defined on $(0,\sigma]$ for some $\sigma > 0$ by

$$\hat{L}(s) = c_0 \exp\left(\int_s^\sigma \frac{y(\tau)}{\tau} d\tau\right), \qquad s \in (0,\sigma], \tag{1.5}$$

where $c_0 > 0$ and the function $y \in C[0,\sigma]$ with $y(0) = 0$.

To the best of the author's knowledge, in the previous papers that consider the exact asymptotic behavior of the solution u to problem (1.1) near $\partial\Omega$, the assumption on the weight function $b(x)$ mainly depends on the set Λ. In this paper, the assumption on the weight function $b(x)$ mainly depends on the set Θ.

Inspired by the above works, in this paper, we also consider the exact asymptotic behavior of the solution u to problem (1.1) near $\partial\Omega$ under the following structure conditions on b and f.

Suppose b also satisfies the following condition:

($\mathbf{b_2}$) there exist some $L \in \Theta$ and a positive constant b_0 such that

$$\lim_{d(x)\to 0} \frac{b(x)}{(d(x))^{-\lambda} L(d(x))} = b_0$$

where

$$\lambda \le 0 \quad and \quad \int_0^a s^{\frac{1-\lambda}{3}} \big(L(s)\big)^{\frac{1}{3}} ds < \infty \text{ for some } a > 0, \tag{1.6}$$

and f also satisfies

($\mathbf{f_3}$) there exists $C_f > 0$ such that

$$\lim_{s\to+\infty} H'(s) \int_s^\infty \frac{1}{H(\nu)} d\nu = C_f, \qquad H(s) := \big(f(s)\big)^{1/3}, \quad \forall s > 0.$$

In this paper, we mainly use the solution of the problem

$$\int_{\phi(t)}^\infty \frac{ds}{H(s)} = t, \quad t > 0, \tag{1.7}$$

to get our estimates.

Our main results are summarized as follows.

Theorem 1.1. *Let f satisfy (f_1)-(f_3) and b satisfy (b_1)-(b_2). If $C_f > 1$ and $4C_f + \lambda(1 - C_f) > 1$, for the unique solution u of problem* (1.1)*, it holds that*

$$\lim_{d(x)\to 0} \frac{u(x)}{\phi\big(h(d(x))\big)} = \xi_0, \tag{1.8}$$

where ϕ is uniquely determined by (1.7)*,*

$$h(t) = \int_0^t s^{\frac{1-\lambda}{3}} L^{\frac{1}{3}}(s)ds, \tag{1.9}$$

and

$$\xi_0 = \left(\frac{3b_0}{(4-\lambda)C_f + (\lambda - 1)}\right)^{\frac{1-C_f}{3}}. \tag{1.10}$$

Remark 1.1. For the existence of solutions for problem (1.1), see A. Mohammed and S. Mohammed [11, 12].

Remark 1.2. By the following Proposition 2.6, one can see that when $\lambda < 0$, h in (1.9) satisfies

$$h(t) \cong \frac{3}{4-\lambda} t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t).$$

Remark 1.3. Some basic examples of the functions which satisfy (f_3) are

($\mathbf{i_1}$) When $f(s) = s^p$, $p > 3$, $C_f = \frac{p}{p-3}$,

$$\phi(t) = \big(((p-3)t)/3\big)^{\frac{3}{3-p}},\ \forall\, t > 0.$$

($\mathbf{i_2}$) When $f(s) = s^p(\ln s)^\beta$, $p > 3$, $\beta \in \mathbb{R}$, $s \ge S_0$, $C_f = \frac{p}{p-3}$.

($\mathbf{i_3}$) When $f(s) = s^p e^{(\ln s)^\beta}$, $p > 3$, $0 < \beta < 1$, $s \ge S_0$, $C_f = \frac{p}{p-3}$.

($\mathbf{i_4}$) When $f(s) = e^{s^\beta}$, $\gamma > 0$, $s \ge S_0$, $C_f = 1$.

($\mathbf{i_5}$) When $f(s) = e^{s \ln s}$, $s \ge S_0$, $C_f = 1$.

The outline of this paper is as follows. In sections 2-3, we give some useful results that will be used in the next section. The proof of Theorem 1.1 will be given in section 4.

2. Preparation

In this section, we first give a brief account of the definition and properties of regularly varying functions that will be used in this paper (see [15–17]).

Definition 2.1. A positive measurable function f defined on $[a, \infty)$, for some $a > 0$, is called **regularly varying at infinity** with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called **slowly varying at infinity**.

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Definition 2.2. A positive measurable function f defined on $[a,\infty)$, for some $a>0$, is called **rapidly varying at infinity** if for each $\rho>1$

$$\lim_{s\to\infty}\frac{f(s)}{s^\rho}=\infty. \tag{2.2}$$

A positive measurable function g defined on $(0,a)$ for some $a>0$, is **regularly varying at zero** with index σ (written as $g\in RVZ_\sigma$) if $t\to g(1/t)$ belongs to $RV_{-\sigma}$. Similarly, g is called **rapidly varying at zero** if $t\to g(1/t)$ is rapidly varying at infinity.

Proposition 2.1 (Uniform convergence theorem)**.** *If $f\in RV_\rho$, then* (2.1) *holds uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$. Moreover, if $\rho<0$, then uniform convergence holds on intervals of the form (a_1,∞) with $a_1>0$; if $\rho>0$, then uniform convergence holds on intervals $(0,a_1]$ provided f is bounded on $(0,a_1]$ for all $a_1>0$.*

Proposition 2.2 (Representation theorem)**.** *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s)=\varphi(s)\exp\left(\int_{a_1}^{s}\frac{y(\tau)}{\tau}d\tau\right),\ s\ge a_1, \tag{2.3}$$

for some $a_1\ge a$, where the functions φ and y are measurable and for $s\to\infty$, $y(s)\to 0$ and $\varphi(s)\to c_0$, with $c_0>0$.

We call that

$$\hat{L}(s)=c_0\exp\left(\int_{a_1}^{s}\frac{y(\tau)}{\tau}d\tau\right),\ s\ge a_1, \tag{2.4}$$

is **normalized** slowly varying at infinity and

$$f(s)=s^\rho\hat{L}(s),\ s\ge a_1, \tag{2.5}$$

is **normalized** regularly varying at infinity with index ρ (and written as $f\in NRV_\rho$).

Similarly, g is called **normalized** regularly varying at zero with index σ, written as $g\in NRVZ_\sigma$ if $t\to g(1/t)$ belongs to $NRV_{-\sigma}$.

A function $f\in RV_\rho$ belongs to NRV_ρ if and only if

$$f\in C^1[a_1,\infty)\text{ for some }a_1>0\text{ and }\lim_{s\to\infty}\frac{sf'(s)}{f(s)}=\rho. \tag{2.6}$$

Proposition 2.3. *If functions L, L_1 are slowly varying at infinity, then*

(i) *L^σ for every $\sigma\in\mathbb{R}$, $c_1L+c_2L_1$ ($c_1\ge 0,\ c_2\ge 0$ with $c_1+c_2>0$), $L\circ L_1$ (if $L_1(t)\to+\infty$ as $t\to+\infty$), are also slowly varying at infinity.*

(ii) *For every $\theta>0$ and $t\to+\infty$, $t^\theta L(t)\to+\infty$, $t^{-\theta}L(t)\to 0$.*

(iii) *For $\rho\in\mathbb{R}$ and $t\to+\infty$, $\frac{\ln(L(t))}{\ln t}\to 0$ and $\frac{\ln(t^\rho L(t))}{\ln t}\to\rho$.*

Proposition 2.4. **(i)** *If $f_1\in RV_{\rho_1}, f_2\in RV_{\rho_2}$ with $\lim_{t\to\infty}f_2(t)=\infty$, then $f_1\circ f_2\in RV_{\rho_1\rho_2}$.*

(ii) *If $f\in RV_\rho$, then $f^\alpha\in RV_{\rho\alpha}$ for every $\alpha\in\mathbb{R}$.*

Proposition 2.5. *If a function L be defined on $(0,\eta]$, is slowly varying at zero. Then we have*

$$\lim_{t\to 0^+}\frac{L(t)}{\int_t^\eta \frac{L(s)}{s}ds}=0. \tag{2.1}$$

If further $\int_0^\eta \frac{L(s)}{s}ds$ converges, then we have

$$\lim_{t\to 0^+}\frac{L(t)}{\int_0^t \frac{L(s)}{s}ds}=0. \tag{2.2}$$

Proposition 2.6 (Asymptotic behavior)**.** *If a function L is slowly varying at infinity, then for $a\ge 0$ and $t\to\infty$,*

(i) $\int_a^t s^\beta L(s)ds \cong (\beta+1)^{-1}t^{1+\beta}L(t),\ for\ \ \beta>-1;$

(ii) $\int_t^\infty s^\beta L(s)ds \cong (-\beta-1)^{-1}t^{1+\beta}L(t),\ for\ \ \beta<-1.$

Next, we give the precise definition of viscosity solutions for the problem (1.1).

Definition 2.3. A function $u\in C(\Omega)$ is a viscosity subsolution of the PDE $\Delta_\infty u=b(x)f(u)$ in Ω if for every $\varphi\in C^2(\Omega)$, with the property that $u-\varphi$ has a local maximum at some $x_0\in\Omega$, then

$$\Delta_\infty\varphi(x_0)\ge b(x_0)f(u(x_0)).$$

Definition 2.4. We say a function $u\in C(\Omega)$ is a viscosity supsolution of the PDE $\Delta_\infty u=b(x)f(u)$ in Ω if for every $\varphi\in C^2(\Omega)$, with the property that $u-\varphi$ has a local minimum at some $x_0\in\Omega$, then

$$\Delta_\infty\varphi(x_0)\le b(x_0)f(u(x_0)).$$

Definition 2.5. A function $u\in C(\Omega)$ is a viscosity solution of the PDE $\Delta_\infty u=b(x)f(u)$ in Ω if it is both a subsolution and a supersolution. Finally, by a solution of (1.1), we mean a function u that is a solution of the PDE $\triangle_\infty u=b(x)f(u)$such that $u=\infty$ on $\partial\Omega$.

Remark 2.1. It is easy to prove that if $u\in C^2(\Omega)$ is a classical subsolution (supersolution) of the PDE $\Delta_\infty u=b(x)f(u)$, then u is a viscosity subsolution (supersolution) of the PDE $\Delta_\infty u=b(x)f(u)$.

3. Some auxiliary results

Some auxiliary results that will be used in the proof of the theorem are given in this section.

Lemma 3.1. *Let*

$$a(t)=t^{-\lambda}L(t)$$

and

$$h(t)=\int_0^t s^{\frac{1-\lambda}{3}}(L(s))^{\frac{1}{3}}ds,$$

where $t\in(0,\delta_0),\ \lambda\le 4,\ \int_0^\eta s^{\frac{1-\lambda}{3}}(L(s))^{\frac{1}{3}}ds<\infty\ for\ some\ \eta>0$ and $L\in\Theta$. Then

(i) $\lim\limits_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \frac{4-\lambda}{3}$ *and* $\lim\limits_{t\to 0^+} \frac{th'(t)}{h(t)} = \frac{4-\lambda}{3}$;

(ii) $\lim\limits_{t\to 0^+} \frac{th''(t)}{h'(t)} = \frac{1-\lambda}{3}$;

(iii) $\lim\limits_{t\to 0^+} \frac{(h'(t))^2 h''(t)}{a(t)} = \frac{1-\lambda}{3}$.

Proof. (i) Since $h'(t) = t^{\frac{1-\lambda}{3}}(L(t))^{\frac{1}{3}}$, then

$$\frac{(h'(t))^4}{h(t)a(t)} = \frac{t^{\frac{4-4\lambda}{3}} L^{\frac{4}{3}}(t)}{t^{-\lambda} L(t) \int_0^t s^{\frac{1-\lambda}{3}} (L(s))^{\frac{1}{3}} ds} = \frac{t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t)}{\int_0^t s^{\frac{1-\lambda}{3}} (L(s))^{\frac{1}{3}} ds}$$

and

$$\frac{th'(t)}{h(t)} = \frac{t^{\frac{4-\lambda}{3}} L^{\frac{1}{3}}(t)}{\int_0^t s^{\frac{1-\lambda}{3}} (L(s))^{\frac{1}{3}} ds}.$$

Hence, when $\lambda < 4$, it follows by Proposition 2.6 that $\lim\limits_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \lim\limits_{t\to 0^+} \frac{th'(t)}{h(t)} = \frac{4-\lambda}{3}$;
when $\lambda = 4$, it follows by Proposition 2.5 that $\lim\limits_{t\to 0^+} \frac{(h'(t))^4}{h(t)a(t)} = \lim\limits_{t\to 0^+} \frac{th'(t)}{h(t)} = 0$.
(ii) By a direct calculation, we obtain

$$h''(t) = \frac{1-\lambda}{3} t^{-\frac{2+\lambda}{3}} (L(t))^{\frac{1}{3}} + \frac{1}{3} t^{\frac{1-\lambda}{3}} (L(t))^{-\frac{2}{3}} L'(t)$$

and

$$\frac{th''(t)}{h'(t)} = \frac{1}{3} \frac{tL'(t)}{L(t)} + \frac{1-\lambda}{3}.$$

Since $L \in \Theta$, $\lim\limits_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$. Therefore,

$$\lim_{t\to 0^+} \frac{th''(t)}{h'(t)} = \frac{1-\lambda}{3}.$$

(iii) Since

$$\frac{(h'(t))^2 h''(t)}{a(t)} = \frac{th''(t)}{h'(t)} \frac{(h'(t))^3}{ta(t)} = \frac{th''(t)}{h'(t)},$$

by (ii), we have

$$\lim_{t\to 0^+} \frac{(h'(t))^{p-2} h''(t)}{a(t)} = \frac{1-\lambda}{3}.$$

□

Lemma 3.2. *Let f satisfy* (f_1)-(f_2), *we get*

(i) *if f satisfies* (f_3), *then $C_f \in [1, \infty)$;*

(ii) (f_3) *holds for $C_f > 1$ if and only if $f \in NRV_\rho$ with $\rho > 3$. In this case $\rho = 3C_f/(C_f - 1)$.*

(iii) *if f satisfies* (f_3) *with $C_f = 1$, f is rapidly varying at infinity.*

(iv) *if* $f \in C^2(S_0, \infty)$ *for some large* $S_0 > 0$ *and*

$$\lim_{s\to\infty} \frac{f(s)f''(s)}{(f'(s))^2} = 1, \tag{3.1}$$

then f *satisfies* (f_3) *with* $C_f = 1$.

Proof. (i) Let $C_f \in (0, \infty]$ and

$$I(s) = H'(s) \int_s^\infty \frac{d\tau}{H(\tau)}, \quad \forall s > 0.$$

Integrating $I(v)$ from a ($a > 0$) to s and integration by parts, we obtain

$$\int_a^s I(t)dt = H(s) \int_s^\infty \frac{1}{H(\nu)} d\nu - H(a) \int_a^\infty \frac{1}{H(\nu)} d\nu + s - a, \quad \forall s > a.$$

It follows by the l'Hospital's rule that

$$0 \le \lim_{s\to\infty} \frac{H(s)\int_s^\infty \frac{d\nu}{H(\nu)}}{s} = \lim_{s\to\infty} \frac{\int_a^s I(\nu)d\nu}{s} - 1 = \lim_{s\to\infty} I(s) - 1 = C_f - 1, \tag{3.2}$$

i.e., $C_f \ge 1$, so (i) holds.
(ii) When (f_3) holds with $C_f \in (1, \infty)$, by (3.10) and (f_3), we have that

$$\begin{aligned}\lim_{s\to\infty} \frac{f(s)}{sf'(s)} &= \lim_{s\to\infty} \frac{f^{1/3}(s)\int_s^\infty \frac{1}{f^{1/3}(\nu)} d\nu}{sf'(s)\int_s^\infty \frac{1}{f^{1/3}(\nu)} d\nu f^{\frac{1}{3}-1}(s)} \\ &= \lim_{s\to\infty} \frac{H(s)\int_s^\infty \frac{d\nu}{H(\nu)}}{3sH'(s)\int_s^\infty \frac{d\nu}{H(\nu)}} = \frac{C_f - 1}{3C_f},\end{aligned} \tag{3.3}$$

i.e., $f \in NRV_{3C_f/(C_f-1)}$.

Conversely, when $f \in NRV_\rho$ with $\rho > 3$, i.e., $\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \rho$ and there exist sufficiently large constant $S_0 > 0$ and $\hat{L} \in \Lambda_1$ such that $f(s) = s^\rho \hat{L}(s)$, $\forall s \ge S_0$.

By 2.2 and Proposition 2.6(i), we have

$$\begin{aligned}&\lim_{s\to\infty} H'(s) \int_s^\infty \frac{d\nu}{H(\nu)} = \frac{1}{3} \lim_{s\to\infty} \frac{sf'(s)}{f(s)} \lim_{s\to\infty} \frac{f^{1/3}(s)}{s} \int_s^\infty f^{-1/3}(\nu)d\nu \\ &= \frac{\rho}{3} \lim_{s\to 0^+} s^{\frac{\rho}{3}-1}(\hat{L}(s))^{\frac{1}{3}} \int_s^\infty \nu^{-\frac{\rho}{3}} \big(\hat{L}(\nu)\big)^{-\frac{1}{3}} d\nu \\ &= \frac{\rho}{\rho - 3} = C_f.\end{aligned}$$

(iii) By $C_f = 1$ and (3.8), one can see that

$$\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \infty.$$

Consequently, for an arbitrary $p > 1$, there exists $S_0 > 0$ such that

$$\frac{f'(s)}{f(s)} \ge (p+1)s^{-1} \qquad \forall s \ge S_0.$$

Integrating the above inequality from S_0 to s, we get

$$\ln\big(f(s)\big) - \ln\big(f(S_0)\big) \ge (p+1)\big(\ln s - \ln S_0\big) \qquad \forall s \ge S_0,$$

i.e.,

$$\frac{f(s)}{s^p} \ge \frac{f(S_0)s}{S_0^{p+1}} \qquad \forall s \ge S_0. \tag{3.4}$$

Letting $s \to \infty$, by Definition 2.2, we obtain that f is rapidly varying at infinity.

(v) It follows by (3.1) and the l'Hospital's rule that

$$\lim_{s\to\infty} \frac{f(s)}{sf'(s)} = \lim_{s\to\infty} \frac{\frac{f(s)}{f'(s)}}{s} = \lim_{s\to\infty} \frac{d}{ds}\left(\frac{f(s)}{f'(s)}\right) = 1 - \lim_{s\to\infty} \frac{f(s)f''(s)}{(f'(s))^2} = 0, \tag{3.5}$$

i.e.

$$\lim_{s\to\infty} \frac{sf'(s)}{f(s)} = \infty.$$

By the similar proof of (iv), we have that for an arbitrary $p > 3$, there exists $S_1 > 0$ such that

$$f(s) \ge \frac{f(S_0)}{S_0^{p+1}} s^{p+1} \qquad \forall s \ge S_1. \tag{3.6}$$

Hence,

$$\frac{\big(f(s)\big)^{\frac{1}{3}}}{s} \ge \left(\frac{f(S_0)}{S_0^{p+1}}\right)^{\frac{1}{3}} s^{\frac{p-2}{3}} \qquad \forall s \ge S_1. \tag{3.7}$$

Letting $s \to \infty$, it follows that

$$\lim_{s\to\infty} \frac{\big(f(s)\big)^{\frac{1}{3}}}{s} = \infty. \tag{3.8}$$

So, combining with (3.5), we get that

$$\lim_{s\to\infty} \frac{f^{\frac{2}{3}}(s)}{f'(s)} = \lim_{s\to\infty} \frac{f(s)}{sf'(s)} \frac{s}{f^{\frac{1}{3}}(s)} = \lim_{s\to\infty} \frac{f(s)}{sf'(s)} \lim_{s\to 0} \frac{s}{f^{\frac{1}{3}}(s)} = 0. \tag{3.9}$$

By the l'Hospital's rule and (3.9), we get that

$$\begin{aligned}
&\lim_{s\to\infty} H'(s) \int_s^\infty \frac{1}{H(\nu)} d\nu \\
&= \lim_{s\to\infty} \frac{1}{3f^{\frac{2}{3}}(s)} f'(s) \int_s^\infty f^{-1/3}(\nu) d\nu \\
&= \lim_{s\to\infty} \frac{1}{3} \frac{\int_s^\infty f^{-1/3}(\nu)d\nu}{\frac{f^{\frac{2}{3}}(s)}{f'(s)}} = \lim_{s\to\infty} \frac{1}{3} \frac{1}{\frac{2}{3} - \frac{f''(s)f(s)}{(f'(s))^2}} \\
&= 1,
\end{aligned}$$

i.e. $C_f = 1$.

$\square$

Lemma 3.3. *Let f satisfy* (f_1)-(f_3) *and ϕ be the solution to the problem*

$$\int_{\phi(t)}^{\infty} \frac{ds}{(f(s))^{\frac{1}{3}}} = t, \ \forall\ t > 0.$$

Then

(i) $-\phi'(t) = \big(f(\phi(t))\big)^{\frac{1}{3}}, \ \phi(t) > 0, \ t > 0, \ \phi(0) := \lim_{t\to 0^+} \phi(t) = +\infty$ *and* $\phi''(t) = \frac{1}{3}\big(f(\phi(t))\big)^{-\frac{1}{3}} f'(\phi(t)), \ t > 0;$

(ii) $\phi \in NRVZ_{-(C_f-1)}$;

(iii) $\phi' \in NRVZ_{-C_f}$;

(iv) $\lim_{t\to 0^+} \frac{\ln(\phi(t))}{-\ln t} = (C_f - 1)$ and $\lim_{t\to 0^+} \frac{\ln(f(\phi(t)))}{-\ln t} = 3C_f$.

Proof. By the definition of ϕ and a direct computation, we show that (i) holds.

(ii) It follows from the proof of Lemma 3.1 and Proposition 2.4(ii) that $f^{-\frac{1}{3}} \in RV_{-\frac{C_f}{C_f-1}}$. Define $L_1(t) := f^{-\frac{1}{3}}(t)/t^{-\frac{C_f}{C_f-1}}$. Then L_1 is slowly varying, and $-\frac{C_f}{C_f-1} < -1$ due to $C_f \geq 1$. So, It follows by Proposition 2.6, that

$$\lim_{t\to\infty} \frac{t f^{-\frac{1}{3}}(t)}{\int_t^{\infty} f^{-\frac{1}{3}}(s) ds} = \lim_{t\to\infty} \frac{t L_1(t) t^{-\frac{C_f}{C_f-1}}}{\int_t^{\infty} L_1(s) s^{-\frac{C_f}{C_f-1}} ds} = \frac{1}{C_f - 1}. \tag{3.10}$$

Therefore,

$$\begin{aligned}
\lim_{t\to 0^+} \frac{t\phi'(t)}{\phi(t)} &= -\lim_{t\to 0^+} \frac{t(f(\phi(t)))^{\frac{1}{3}}}{\phi(t)} \\
&= -\lim_{s\to+\infty} \frac{(f(s))^{\frac{1}{3}} \int_s^{\infty} \frac{d\nu}{(f(\nu))^{\frac{1}{3}}}}{s} = -(C_f - 1),
\end{aligned}$$

i.e., $\phi \in NRVZ_{-(C_f-1)}$.

(iii) It follows by Lemma 3.2, (i) and (3.10) that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{t\phi''(t)}{\phi'(t)} &= -\frac{1}{3} \lim_{t\to 0^+} \frac{f'(\phi(t)) \int_{\phi(t)}^{\infty} (f(s))^{-\frac{1}{3}} ds}{(f(\phi(t)))^{\frac{2}{3}}} \\
&= -\lim_{s\to+\infty} \frac{f'(s) \int_s^{\infty} (f(\nu))^{-\frac{1}{3}} d\nu}{3(f(s))^{\frac{2}{3}}} \\
&= -\frac{1}{3} \lim_{s\to+\infty} \frac{s f'(s)}{f(s)} \frac{\int_s^{\infty} (f(\nu))^{-\frac{1}{3}} d\nu}{s(f(s))^{-\frac{1}{3}}} = -C_f.
\end{aligned}$$

The last result (iv) follows from (ii)-(iii) and Proposition 2.3 (iii). □

4. Proof of the Theorem

Theorems 1.1 will be proved in this section.

First, we need the following result.

Lemma 4.1 (the comparison principle, [12, Lemma 2.5])**.** *Let b satisfy* (b_1)*, and f satisfy* (f_1)*. Suppose $u, v \in C(\bar{\Omega})$ such that*

$$\Delta_\infty u \geq b(x)f(u) \quad in \quad \Omega \quad and \ \Delta_\infty v \leq b(x)f(v) \quad in \quad \Omega$$

in the viscosity sense. If $u \leq v$ on $\partial\Omega$ and $0 \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

For any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

Because Ω is smooth, there exists $\delta_0 \in (0, \sigma)$ (σ is given as in the definition of Θ) such that $d \in C^2(\Omega_{\delta_0})$ and $|\nabla d(x)| = 1, \ \forall \ x \in \Omega_{\delta_0}$. Hence, $\Delta_\infty d = 0$ in Ω_{δ_0} in the viscosity sense.

Proof of Theorem 1.1. Fix a small $\varepsilon > 0$. Let $\delta_\varepsilon \in (0, \frac{\delta_0}{2})$, $\rho \in (0, \delta_\varepsilon)$ and define

$$\bar{u}_\varepsilon = (\xi_0 + \varepsilon)\phi\big(h(d(x)) - h(\rho)\big) \quad \text{ for } \ \text{any } x \in \Omega_{2\delta_\varepsilon} \backslash \bar{\Omega}_\rho =: \Omega_\rho^-$$

and

$$\underline{u}_\varepsilon = (\xi_0 - \varepsilon)\phi\big(h(d(x)) + h(\rho)\big) \quad \text{ for } \ \text{any } x \in \Omega_{2\delta_\varepsilon - \rho} =: \Omega_\rho^+,$$

where h is given as in (1.9) and ξ_0 is given as in (1.10).

Let

$$\eta(t) = (\xi_0 + \varepsilon)\phi\big(h(t) - h(\rho)\big), \quad t \in (\rho, 2\delta_\varepsilon).$$

Obviously, h and ϕ are increasing and decreasing in their respective definition domains. So, when δ_ε is small enough, η is decreasing in $(\rho, 2\delta_\varepsilon)$. Let ζ be the inverse of η. By a direct computation, we have

$$\zeta'(t) = \frac{1}{\eta'(\zeta(t))} = \Big((\xi_0 + \varepsilon)\phi'\big(h(\zeta(t)) - h(\rho)\big)h'(\zeta(t))\Big)^{-1} \tag{4.1}$$

and

$$\begin{aligned}\zeta''(t) = &-(\xi_0 + \varepsilon)^{-2}\left(\phi'\big(h(\zeta(t)) - h(\rho)\big)h'(\zeta(t))\right)^{-3} \\ &\times \Big(\phi''\big(h(\zeta(t)) - h(\rho)\big)\big(h'(\zeta(t))\big)^2 + \phi'\big(h(\zeta(t)) - h(\rho)\big)h''(\zeta(t))\Big). \end{aligned} \tag{4.2}$$

Let$(x_0, \psi) \in \Omega_\rho^- \times C^2(\Omega_\rho^-)$ be a pair such that $\bar{u}_\varepsilon \geq \psi$ in a neighborhood N of x_0 and $\bar{u}_\varepsilon(x_0) = \psi(x_0)$ Then $\varphi = \zeta(\psi) \in C^2(\Omega_\rho^-)$, and

$$d(x) \leq \varphi(x) \quad \text{in } N, \qquad d(x_0) = \varphi(x_0).$$

Because $\Delta_\infty d = 0$ in Ω_ρ^-, we get $\Delta_\infty \varphi(x_0) \geq 0$. A simple computation shows that

$$\Delta_\infty \varphi = \zeta''(\psi)(\zeta'(\psi))^2 |D\psi|^4 + (\zeta'(\psi))^3 \Delta_\infty \psi.$$

Since $\Delta_\infty \varphi(x_0) \geq 0$ and $\zeta' < 0$, we have

$$\Delta_\infty \psi(x_0) \leq -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-1} |D\psi(x_0)|^4.$$

Moreover, since $|Dd(x)| = 1$ for $x \in \Omega_\rho^-$ and $d - \varphi$ attains a local maximum at x_0, it follows that

$$|Dd(x_0)| = |\zeta'(\psi(x_0))D\psi(x_0)|.$$

So

$$\Delta_\infty \psi(x_0) \le -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-5}.$$

Combing with (4.1) and (4.2), we further get

$$\begin{aligned}\Delta_\infty \psi(x_0) \le &(\xi_0+\varepsilon)^3 \left(\phi'\big(h(\zeta(\psi(x_0)))-h(\rho)\big)\right)^3 \\ &\times\left[\frac{\phi''\big(h(\zeta(\psi(x_0)))-h(\rho)\big)\big(h'(\zeta(\psi(x_0)))\big)^4}{\phi'\big(h(\zeta(\psi(x_0)))-h(\rho)\big)}+h''(\zeta(\psi(x_0)))\big(h'(\zeta(\psi(x_0)))\big)^2\right].\end{aligned}$$

Since

$$\lim_{\rho\to 0}\frac{h(d(x))}{h(d(x))-h(\rho)}=1, \quad \text{for any } x\in\Omega_{2\delta_\varepsilon}\backslash\bar{\Omega}_\rho =: \Omega_\rho^-,$$

we can choose ρ small enough such that

$$1<\frac{h(d(x))}{h(d(x))-h(\rho)}<1+\alpha\varepsilon, \quad \text{for } \forall\varepsilon>0 \text{ and any } x\in\Omega_{2\delta_\varepsilon}\backslash\bar{\Omega}_\rho =: \Omega_\rho^-,$$

where α is a sufficiently small positive constant.

Combing with $-\frac{\phi''(t)t}{\phi'(t)}>0, \quad t>0$, we can obtain

$$\begin{aligned}&\Delta_\infty\psi(x_0)-b(x_0)f(\bar{u}_\varepsilon(x_0))\\ \le\ &(\xi_0+\varepsilon)^3\left(-\phi'\big(h(d(x_0))-h(\rho)\big)\right)^3 a(d(x_0))\left[-\frac{\phi''\big(h(d(x_0))-h(\rho)\big)\big(h(d(x_0))-h(\rho)\big)}{\phi'\big(h(d(x_0))-h(\rho)\big)}\right.\\ &\times\frac{\big(h'(d(x_0))\big)^4}{h(d(x_0))a(d(x_0))}\frac{h(d(x_0))}{h(d(x_0))-h(\rho)}-\frac{h''(d(x_0))\big(h'(d(x_0))\big)^2}{a(d(x_0))}\\ &\left.-(\xi_0+\varepsilon)^{-3}\frac{b(x_0)}{a(d(x_0))}\frac{f(\bar{u}_\varepsilon(x_0))}{\left(-\phi'\big(h(d(x_0))-h(\rho)\big)\right)^3}\right]\\ \le\ &(\xi_0+\varepsilon)^3\left(-\phi'\big(h(d(x_0))-h(\rho)\big)\right)^3 a(d(x_0))\left[-\frac{\phi''\big(h(d(x_0))-h(\rho)\big)\big(h(d(x_0))-h(\rho)\big)}{\phi'\big(h(d(x_0))-h(\rho)\big)}\right.\\ &\times\frac{\big(h'(d(x_0))\big)^4}{h(d(x_0))a(d(x_0))}(1+\alpha\varepsilon)-\frac{h''(d(x_0))\big(h'(d(x_0))\big)^2}{a(d(x_0))}\\ &\left.-(\xi_0+\varepsilon)^{-3}\frac{b(x_0)}{a(d(x_0))}\frac{f(\bar{u}_\varepsilon(x_0))}{\left(-\phi'\big(h(d(x_0))-h(\rho)\big)\right)^3}\right]\\ =:\ &((\xi_0+\varepsilon))^3\left(-\phi'\big(h(d(x_0))-h(\rho)\big)\right)^3 a(d(x_0)))I(x_0).\end{aligned}$$

Notice that $h(d(x_0))\to 0$ as $\delta_\varepsilon\to 0$ (and thereby x_0 tends to the boundary of Ω.) Then, by Lemmas 3.1 and 3.3, we get that

$$I(x_0)\to\frac{C_f(4-\lambda)-(1-\lambda)}{3}-b_0\,(\xi_0+\varepsilon)^{\frac{3}{C_f-1}}+\frac{C_f(4-\lambda)}{3}\alpha\varepsilon \qquad \text{as } \delta_\varepsilon\to 0.$$

By the choice of ξ_0, we have $I(x_0)<0$ provided $\alpha>0$ and $\delta_\varepsilon\in(0,\frac{\delta_0}{2})$ small enough. Thus

$$\Delta_\infty\psi(x_0)\le b(x_0)f(\bar{u}_\varepsilon(x_0)),$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of equation (1.1) in Ω_ρ^-.

Next, in a similar way, we will prove that $\underline{u}_\varepsilon$ is a subsolution of equation (1.1) in Ω_ρ^+.

Let

$$\eta(t) = (\xi_0 - \varepsilon)\phi\big(h(t) + h(\rho)\big), \quad t \in (0, 2\delta_\varepsilon - \rho).$$

Obviously, h and ϕ are increasing and decreasing in their respective definition domains. So, when δ_ε is small enough, η is decreasing in $(0, 2\delta_\varepsilon - \rho)$. Let ζ be the inverse of η. By a direct computation, we have

$$\zeta'(t) = \frac{1}{\eta'(\zeta(t))} = \big((\xi_0 - \varepsilon)\phi'\big(h(\zeta(t)) + h(\rho)\big)h'(\zeta(t))\big)^{-1} \tag{4.3}$$

and

$$\begin{aligned}\zeta''(t) = &-(\xi_0 - \varepsilon)^{-2}\left(\phi'\big(h(\zeta(t)) + h(\rho)\big)h'(\zeta(t))\right)^{-3} \\ &\times \Big(\phi''\big(h(\zeta(t)) + h(\rho)\big)\big(h'(\zeta(t))\big)^2 + \phi'\big(h(\zeta(t)) + h(\rho)\big)h''(\zeta(t))\Big).\end{aligned} \tag{4.4}$$

Let$(x_0, \psi) \in \Omega_\rho^+ \times C^2(\Omega_\rho^+)$ be a pair such that $\underline{u}_\varepsilon \leq \psi$ in a neighborhood N of x_0 and $\underline{u}_\varepsilon(x_0) = \psi(x_0)$ Then $\varphi = \zeta(\psi) \in C^2(\Omega_\rho^+)$, and

$$d(x) \geq \varphi(x) \quad \text{in } N, \qquad d(x_0) = \varphi(x_0).$$

Because $\Delta_\infty d = 0$ in Ω_ρ^+, we get $\Delta_\infty \varphi(x_0) \leq 0$. A simple computation shows that

$$\Delta_\infty \varphi = \zeta''(\psi)(\zeta'(\psi))^2|D\psi|^4 + (\zeta'(\psi))^3\Delta_\infty\psi.$$

Since $\Delta_\infty\varphi(x_0) \leq 0$ and $\zeta' < 0$, we have

$$\Delta_\infty\psi(x_0) \geq -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-1}|D\psi(x_0)|^4.$$

Moreover, since $|Dd(x)| = 1$ for $x \in \Omega_\rho^+$ and $d - \varphi$ attains a local minimum at x_0, it follows that

$$|Dd(x_0)| = |\zeta'(\psi(x_0))D\psi(x_0)|.$$

So

$$\Delta_\infty\psi(x_0) \geq -\zeta''(\psi(x_0))(\zeta'(\psi(x_0)))^{-5}.$$

Combing with (4.3) and (4.4), we further get

$$\begin{aligned}\Delta_\infty\psi(x_0) \geq\, &(\xi_0 + \varepsilon)^3\left(\phi'\big(h(\zeta(\psi(x_0))) + h(\rho)\big)\right)^3 \\ &\times \left[\frac{\phi''\big(h(\zeta(\psi(x_0))) + h(\rho)\big)\big(h'(\zeta(\psi(x_0)))\big)^4}{\phi'\big(h(\zeta(\psi(x_0))) + h(\rho)\big)} + h''(\zeta(\psi(x_0)))\big(h'(\zeta(\psi(x_0)))\big)^2\right].\end{aligned}$$

Since

$$\lim_{\rho\to 0}\frac{h(d(x))}{h(d(x)) + h(\rho)} = 1, \text{ for any } x \in \Omega_{2\delta_\varepsilon - \rho} =: \Omega_\rho^+,$$

we can choose ρ small enough such that

$$1 - \alpha\varepsilon < \frac{h(d(x))}{h(d(x)) + h(\rho)} < 1, \text{ for } \forall\varepsilon > 0 \text{ and any } x \in \Omega_{2\delta_\varepsilon - \rho} =: \Omega_\rho^+,$$

where α is a sufficiently small positive constant.

Combing with $-\frac{\phi''(t)t}{\phi'(t)} > 0, \quad t > 0$, we can obtain

$$
\begin{aligned}
&\Delta_\infty \psi(x_0) - b(x_0) f(\bar{u}_\varepsilon(x_0)) \\
\geq\ & (\xi_0+\varepsilon)^3 \left(-\phi'\big(h(d(x_0))+h(\rho)\big)\right)^3 a(d(x_0)) \Bigg[-\frac{\phi''\big(h(d(x_0))+h(\rho)\big)\big(h(d(x_0))+h(\rho)\big)}{\phi'\big(h(d(x_0))+h(\rho)\big)} \\
&\times \frac{\big(h'(d(x_0))\big)^4}{h(d(x_0))a(d(x_0))} \frac{h(d(x_0))}{h(d(x_0))+h(\rho)} - \frac{h''(d(x_0))\big(h'(d(x_0))\big)^2}{a(d(x_0))} \\
&- (\xi_0+\varepsilon)^{-3} \frac{b(x_0)}{a(d(x_0))} \frac{f(\bar{u}_\varepsilon(x_0))}{\big(-\phi'\big(h(d(x_0))+h(\rho)\big)\big)^3}\Bigg] \\
\geq\ & (\xi_0+\varepsilon)^3 \left(-\phi'\big(h(d(x_0))+h(\rho)\big)\right)^3 a(d(x_0)) \Bigg[-\frac{\phi''\big(h(d(x_0))+h(\rho)\big)\big(h(d(x_0))+h(\rho)\big)}{\phi'\big(h(d(x_0))+h(\rho)\big)} \\
&\times \frac{\big(h'(d(x_0))\big)^4}{h(d(x_0))a(d(x_0))}(1-\alpha\varepsilon) - \frac{h''(d(x_0))\big(h'(d(x_0))\big)^2}{a(d(x_0))} \\
&- (\xi_0+\varepsilon)^{-3} \frac{b(x_0)}{a(d(x_0))} \frac{f(\bar{u}_\varepsilon(x_0))}{\big(-\phi'\big(h(d(x_0))+h(\rho)\big)\big)^3}\Bigg] \\
=:\ & ((\xi_0+\varepsilon))^3 \left(-\phi'\big(h(d(x_0))+h(\rho)\big)\right)^3 a(d(x_0))) I(x_0).
\end{aligned}
$$

Notice that $h(d(x_0)) \to 0$ as $\delta_\varepsilon \to 0$ (and thereby x_0 tends to the boundary of Ω.) Then, by Lemmas 3.1 and 3.3, we get that

$$
I(x_0) \to \frac{C_f(4-\lambda)-(1-\lambda)}{3} - b_0 (\xi_0-\varepsilon)^{\frac{3}{C_f-1}} - \frac{C_f(4-\lambda)}{3}\alpha\varepsilon \qquad \text{as } \ \delta_\varepsilon \to 0.
$$

By the choice of ξ_0, we have $I(x_0) > 0$ provided $\alpha > 0$ and $\delta_\varepsilon \in (0, \frac{\delta_0}{2})$ small enough. Thus

$$
\Delta_\infty \psi(x_0) \geq b(x_0) f(\underline{u}_\varepsilon(x_0)),
$$

i.e., $\underline{u}_\varepsilon$ is a supersolution of equation (1.1) in Ω_ρ^+.

Now let u be an arbitrary solution of problem (1.1). We assert that there exists a positive constant M such that

$$
u \leq M + \bar{u}_\varepsilon, \ x \in \Omega_\rho^-, \tag{4.5}
$$

$$
\underline{u}_\varepsilon \leq u + M, \ x \in \Omega_\rho^+. \tag{4.6}
$$

In fact, we may choose a large M such that

$$
u \leq M + \bar{u}_\varepsilon \text{ on } \Gamma_{2\delta} : = \{x \in \Omega : d(x) = 2\delta_\varepsilon\},
$$

where $M := \max\{u(x) : d(x) \geq 2\delta_\varepsilon\}$.

By (f_1), we see that $\bar{u}_\varepsilon + M$ is also a supersolution of equation (1.1) in Ω_ρ^-. Since $u < \bar{u}_\varepsilon$ on $\Gamma_\beta := \{x \in \Omega : d(x) = \rho\}$, (3.6) follows by Lemma 3.1.

In a similar way, we can show (3.7).

Hence, $x \in \Omega_\rho^- \cap \Omega_\rho^+$, by letting $\rho \to 0$, we have

$$
\xi_0 - \varepsilon - \frac{M}{\phi\big(h(d(x))\big)} \leq \frac{u(x)}{\phi\big(h(d(x))\big)}
$$

and

$$\frac{u(x)}{\phi\big(h(d(x))\big)} \le \xi_0 + \varepsilon + \frac{M}{\phi\big(h(d(x))\big)}.$$

Moreover, it follows by Lemma 3.3 that $\phi(0) = \infty$, hence, we obtain

$$\xi_0 - \varepsilon \le \liminf_{d(x)\to 0} \frac{u(x)}{\phi\big(h(d(x))\big)} \quad \text{and} \quad \limsup_{d(x)\to 0} \frac{u(x)}{\phi\big(h(d(x))\big)} \le \xi_0 + \varepsilon.$$

Thus the proof is finished by letting $\varepsilon \to 0$.

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