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CONSERVATION LAWS, EXACT SOLUTIONS OF TIME-SPACE FRACTIONAL GENERALIZED GINZBURG-LANDAU EQUATION FOR SHALLOW WAKE FLOWS

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Abstract In this study, starting from the rigid-lid shallow water equations, an amplitude evolution equation is derived by using the multi-scale and perturbation analysis method. The resulting equation has complex coefficients and is called (2+1)-dimensional generalized Ginzburg-Landau(gGL) equation. Then, the (2+1)-dimensional time-space fractional gGL equation is obtained by using the semi-inverse method and fractional variational principle in the first time. Finally, the conservation laws and exact solutions of the fractional gGL equation are discussed on the basis of Lie symmetry analysis and $\exp(-\phi(\zeta))$-expansion method. By analyzing these solutions, we conclude that there are solitary waves and rogue waves in shallow wake flows.

Keywords Shallow wake flows, time-space fractional generalized Ginzburg-Landau equation, conservation laws.

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1. Introduction

On June 4, 1942, at 10 am, the US B-17 bomber fleet spotted the Japanese destroyer "arashi" in a clear zigzagged line on the sea, and tracked down the unsuspecting Japanese aircraft carriers. In the ensuing minutes, three Japanese aircraft carriers, the "akita", "kaga" and "soru", became fireballs and sank. This battle rewrote the end of midway and the whole of the Second World War. What looks like an ordinary shallow wake flow can be deadly.

Shallow wake flows are defined as the tail-like flows that are formed when the flows are split by obstacles(such as islands and headlands) and the transverse length scale of the flow, is much larger than the water depth. In daily life, the most common shallow wake flows are the wakes formed by ships pushing the sea water forward

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and under the action of wave-making resistance. Ship wake flows [7] are usually herringbone line of waves, consisting of diffuse and transverse waves. The existence of these wake flows will directly affect the ship's navigation safety and pose a potential threat to the ship's concealment. In the Marine environment, shallow wake flows also be formed, which are the island wake flow. With the development of science and technology, satellite images and aerial photographs show that the wake creates vortices in the leeward areas of islands and headlands. This indicates that the water flow patterns of island wake flows presents a complex vortex pattern. These vortices trap sediments and pollutants, worsening water quality around islands and affecting entire ecosystems. Therefore, the study of island wake flows plays a decisive role in the location of drainage outlets, sludge treatment outlets, Marine parks and nature reserves [5].

The interest in shallow wake flows has been growing continually during the last few years because of their practical significance. In 1997, Chen & Jirka deduced the modified Orr-Sommerfeld equation to describe the stability characteristics of wake flows. After that, Shumm, Berger & Monkewitz and Leweke & Provansal discovered that the Ginzburg-Landanu(GL) equation could better describe the wake flows motion, and they determined the coefficients of the equation based on experimental data. In 2003, Kolyshkin & Ghidaoui derived the (1+1)-dimensional GL equation from the two-dimensional shallow water equation under the rigid-lid assumption [14]. On such basis, starting from the shallow water equation, we use the multiple-scale and perturbation method to deduce a higher dimensional model, that is (2+1)-dimensional gGL equation. The one-dimensional equations in space [29] have limitations in the study of shallow wake flows, while the two-dimensional equations we consider are more in line with the actual environment.

A fractional partial differential equation(FPDE) is an equation that contains fractional derivatives or fractional integrals. The FPDE is an important branch of modern mathematics, and it can be defined in a variety of ways [26, 28]. In the real world, because many phenomena cannot be described by integral order differential equations [3], fractional order differential equations [16] have been highly valued by the mathematical circle in recent years, and have been proposed in many fields such as fluid mechanics, material mechanics, biology, plasma physics, finance and so on [27]. The conservation law of constructing differential equations is an important subject of mathematical physics research. The conservation law reflects the phenomenon that physical quantities do not change with time [13, 25]. Lie symmetry analysis is an effective tool for studying conservation laws of FPDEs [19–22]. In general, Lie symmetry analysis can obtain the symmetry and invariance of the system. A symmetry corresponds to a conservation law, and the invariant performance is used to check the accuracy of numerical results.

In order to explain more characteristics related to shallow wake flow through the gGL equation, the equation was converted into time-space fractional equation, and multiple conservation laws were obtained by applying the method of lie symmetry analysis. In addition to, the search for exact solutions of FPDEs [18] is becoming an emerging area of current research. Up to now, many effective and powerful methods to obtaining exact solutions have been proposed, such as finite difference method [6], sub-equation method [30], $\tan(\phi/2)$-expansion method [17], fractional variational iteration method [24], exp-function method [4], (G'/G)-expansion method [8] and so on [9, 31, 32]. In this paper, we use the $\exp(-\phi(\zeta))$-expansion method [1] to solve the fractional gGL equation, and finally obtain some more accurate results.

2. Derivation procedure of (2+1)-dimensional gGL equation

In the realm of hydraulic, ocean and atmospheric engineering, the shallow water equations can be used to explain various engineering applications. The shallow water equations can be derived from the Navier-Stokes equations by using vertically-averaged quantities. In this section, based on the two-dimensional shallow water equations under the rigid-lid [14], (2+1)-dimensional gGL equation is derived by using the multi-scale and perturbation analysis method. The two-dimensional shallow water equations in the following form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_f}{2h}u\sqrt{u^2+v^2} = 0, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + \frac{c_f}{2h}v\sqrt{u^2+v^2} = 0, \tag{2.3}$$

where x and y are the spatial coordinates, u and v are the velocity components, p is the dimensionless pressure, c_f is the friction coefficient and h is the water depth.

Introducing the stream function $\varphi(x, y, t)$ as well as $u = \frac{\partial \varphi}{\partial y}$, $v = \frac{\partial \varphi}{\partial x}$ to represent the perturbation field. Then, combining them with Eqs.(2.1)-(2.3) to eliminate pressure p, we can obtain

$$\begin{aligned}(\Delta\varphi)_t + \varphi_y(\Delta\varphi)_x - \varphi_x(\Delta\varphi)_y + \frac{s}{2}\Delta\varphi\sqrt{\varphi_x^2+\varphi_y^2} \\ + \frac{s}{2\sqrt{\varphi_x^2+\varphi_y^2}}(\varphi_y^2\varphi_{yy} + 2\varphi_x\varphi_y\varphi_{xy} + \varphi_x^2\varphi_{xx}) = 0,\end{aligned} \tag{2.4}$$

where Δ is two-dimensional Laplace operator and $s = \frac{c_f}{h}$, subscripts express the partial derivatives of the independent variables x,y and t. The slow variation of this function in time and space can denote the wave packet.

In order to balance the nonlinearity of space, we transform the scale of time and space with the small parameter ε and a group velocity c_g

$$T = \varepsilon^2 t, \qquad X = \varepsilon(x - c_g t), \qquad Y = \varepsilon y, \qquad S = s(1-\varepsilon^2), \tag{2.5}$$

where $\varepsilon \ll 1$ is used to measure the weakness of nonlinearity.

Using the chain rule, the differential operators $\partial/\partial t, \partial/\partial x$ and $\partial/\partial y$ in the form

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g\frac{\partial}{\partial X} + \varepsilon^2\frac{\partial}{\partial T}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} \to \frac{\partial}{\partial y} + \varepsilon\frac{\partial}{\partial Y}.$$

Supposing the stream function has the following expansion form

$$\varphi = \varphi_0(y) + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \cdots, \tag{2.6}$$

where $\varphi_0 = u_0(y)y$, $\varphi_i, (i = 1, \cdots, n)$ are functions of x, y, t, X, Y, T.

Substituting (2.5) and (2.6) into Eq.(2.4), then taking the square root and collecting the terms of ε gives

$$\begin{aligned}\varepsilon^1 :& \varphi_{1xxt} + \varphi_{1yyt} + \varphi_{0y}\varphi_{1yyx} + \varphi_{0y}\varphi_{1xxx} - \varphi_{1x}\varphi_{0yyy} \\ &+ \frac{1}{2}S[\varphi_{0y}\varphi_{1xx} + 2\varphi_{0y}\varphi_{1yy} + \varphi_{1y}\varphi_{oyy}] = 0,\end{aligned} \tag{2.7}$$

$$\varepsilon^2:\begin{cases}\varphi_{2xxt}+\varphi_{2yyt}+\varphi_{0y}\varphi_{2yyx}+\varphi_{0y}\varphi_{2xxx}-\varphi_{2x}\varphi_{0yyy}+\dfrac{1}{2}S[\varphi_{0y}\varphi_{2xx}\\ +2\varphi_{0y}\varphi_{2yy}+\varphi_{2y}\varphi_{oyy}]=c_g(\varphi_{1Xxx}+\varphi_{1Xyy})-2\varphi_{1Xxt}\\ -3\varphi_{0y}\varphi_{1Xxx}-\varphi_{1y}\varphi_{1xxx}-\varphi_{1y}\varphi_{1yyx}-\varphi_{0y}\varphi_{1Xyy}+\varphi_{1x}\varphi_{1yyy}\\ +\varphi_{1X}\varphi_{0yyy}-2\varphi_{1Yyt}-2\varphi_{0y}\varphi_{1Yyx}-\dfrac{1}{2}S[\varphi_{1xx}\varphi_{1y}+2\varphi_{1Xx}\varphi_{0y}\\ +2\varphi_{1yy}\varphi_{1y}+\varphi_{1x}\varphi_{1xy}-2\varphi_{0y}\varphi_{0yy}+4\varphi_{0y}\varphi_{1Yy}],\end{cases}\tag{2.8}$$

$$\varepsilon^3:\begin{cases}\varphi_{3xxt}+\varphi_{3yyt}+\varphi_{0y}\varphi_{3yyx}+\varphi_{0y}\varphi_{3xxx}-\varphi_{3x}\varphi_{0yyy}+\dfrac{1}{2}S[\varphi_{0y}\cdot\\ \varphi_{3xx}+2\varphi_{0y}\varphi_{3yy}+\varphi_{3y}\varphi_{oyy}]=c_g(\varphi_{2Xyy}+\varphi_{2Xxx}+2\varphi_{1XXx}+\\ 2\varphi_{1YYy})-\varphi_{1xxT}-\varphi_{XXt}-\varphi_{1yyT}-\varphi_{1YYt}-2\varphi_{2Xxt}-2\varphi_{2Yyt}-\\ 3\varphi_{0y}\varphi_{2Xxx}-3\varphi_{0y}\varphi_{1XXx}-\varphi_{0y}\varphi_{2Xyy}-\varphi_{1y}\varphi_{2xxx}-3\varphi_{1y}\varphi_{1Xxx}-\\ \varphi_{1y}\varphi_{2yyx}-\varphi_{1y}\varphi_{1yyX}-\varphi_{2y}\varphi_{1xxx}+\varphi_{1x}\varphi_{2xxy}+2\varphi_{1x}\varphi_{1Xxy}+\\ \varphi_{1x}\varphi_{2yyy}+\varphi_{2x}\varphi_{1xxy}+\varphi_{2x}\varphi_{1yyy}+\varphi_{1X}\varphi_{1xxy}+\varphi_{1X}\varphi_{1yyy}+\\ \varphi_{2X}\varphi_{0yyy}-2\varphi_{0y}\varphi_{2Yyx}-2\varphi_{0y}\varphi_{1YyX}-\varphi_{0y}\varphi_{1YYx}-2\varphi_{1y}\varphi_{1Yyx}\\ -\varphi_{1Y}\varphi_{1xxx}-\varphi_{1Y}\varphi_{1yyx}+\varphi_{1x}\varphi_{1xxY}+3\varphi_{1x}\varphi_{1Yyy}\\ +3\varphi_{1x}\varphi_{1Yyy}-\dfrac{1}{2}S[\varphi_{2x}\varphi_{1xx}+2\varphi_{0y}\varphi_{2Xx}+\varphi_{0y}\varphi_{1XX}+\varphi_{1y}\cdot\\ \varphi_{2xx}+2\varphi_{1y}\varphi_{2yy}+2\varphi_{2y}\varphi_{1yy}+\varphi_{1x}\varphi_{2xy}+\varphi_{2x}\varphi_{1xy}+\varphi_{1x}\varphi_{1Xy}\\ +\varphi_{1X}\varphi_{1xy}+\dfrac{\varphi_{1x}^2\varphi_{1xx}}{\varphi_{0y}}-\varphi_{0y}\varphi_{1xx}-2\varphi_{0y}\varphi_{1yy}-2\varphi_{1y}\\ +\varphi_{0yy}+4\varphi_{0y}\varphi_{2Yy}+2\varphi_{0y}\varphi_{1YY}4\varphi_{1y}\varphi_{1Yy}+2\varphi_{1Y}\varphi_{1yy}\\ +2\varphi_{2Y}\varphi_{0yy}+\varphi_{1x}\varphi_{1Yx}].\end{cases}\tag{2.9}$$

where on account of the small parameter ε is much less than one, so the high power of ε can be omitted.

We can assume that perturbation function φ_1 in Eq.(2.6) has the form

$$\varphi_1=A(X,Y,T)\phi_1(y)\exp[ik(x-\omega t)]+c.c,\tag{2.10}$$

where A is function of slowly varying amplitude, k is parameter, ω is the wave speed and $c.c$ is complex conjugate.

Substituting this function into Eq.(2.8), the function φ_2 can be obtained as

$$\begin{aligned}\varphi_2&=\varphi_{20}+\varphi_{21}+\varphi_{22}+\varphi_{23}\\&=AA^*\phi_{20}(y)+A_X\phi_{21}(y)\exp[ik(x-\omega t)]\\&\quad+A_Y\phi_{22}(y)\exp[ik(x-\omega t)]+A^2\phi_{23}(y)\exp[2ik(x-\omega t)]+c.c,\end{aligned}\tag{2.11}$$

here φ_{21} is to balance the nonlinear terms which are proportional to A_X and $\exp[ik(x-\omega t)]$ on the right-hand side of Eq.(2.8), and substituting φ_2 into the Eq.(2.8) to get the corresponding terms as follows

$$\begin{aligned}&\phi_{22y}(u_0-\omega+\frac{u_0S}{ik})+\phi_{22y}\frac{Su_{0y}}{2ik}+\phi_{21}(k^2\omega-k^2u_0-u_{0yy}-\frac{u_0kS}{2i})\\=&\phi_{1y}[2\omega-2u_0+\frac{S(2u_0-u_{0y})i}{2k}]+\phi_1\frac{u_{0y}S}{2k},\end{aligned}$$

similarly, $\varphi_{20}, \varphi_{23}$ and φ_{24} exist to balance corresponding nonlinear terms, then arranging the terms proportional to $AA^*\phi_{20}(y), A_Y\phi_{22}(y)\exp[ik(x-\omega t)]$ and $A^2\phi_{23}(y)\exp[2ik(x-\omega t)]$ respectively yields

$$\begin{aligned}
&2Su_0\phi_{20yy} + u_{0y}S\phi_{20y}\\
=&ik(\phi_{1y}\phi^*_{1yy} - \phi^*_{1y}\phi_{1yy} + \phi_1\phi^*_{1yyy} - \phi^*_1\phi_{1yyy}) - \frac{1}{2}S[k^2(\phi^*_1\phi_{1y} + \phi_1\phi^*_{1y})\\
&+ 2(\phi^*_{1y}\phi_{1yy} + \phi^*_{1yy}\phi_{1y})],\\
&\phi_{21yy}(u_0 - \omega + \frac{u_0S}{ik}) + \phi_{21y}\frac{u_{0y}S}{2ik} + \phi_{21}(k^2\omega - k^2u_0 - u_{0yy} - \frac{u_0kS}{2i})\\
=&\phi_1\frac{1}{ik}(-2k^2\omega + 3k^2u_0 + u_{0yy} - k^2c_g - iku_0S) + \phi_{1yy}\frac{1}{ik}(c_g - u_0),\\
&\phi_{23yy}[Su_0 + 2ik(u_0 - \omega)] + \phi_{23y}Su_0 - \phi_{23}[8ik^3(u_0 - \omega) + 2iku_{0yy} + 2Sk^2u_0]\\
=&ik(\phi_1\phi_{1yyy} - \phi_{1y}\phi_{1yy}) - \frac{1}{2}S(2\phi_{1y}\phi_{1yy} - 3k^2\phi_1\phi_{1y}).
\end{aligned}$$

Putting φ_1 and φ_2 as well as Eqs.(2.10) and (2.11) into Eq.(2.9), sorting out the linear and non-linear terms which proportional to $\exp[ik(x-\omega t)]$. A whole new model to describe the amplitude evolution of shallow wake flows can be given as

$$A_T + a_1A + a_2A_{XX} + a_3A_{YY} + a_4A_{XY} + a_5|A|^2A = 0, \tag{2.12}$$

where

$$\begin{cases}
a_1 =(Su_0\varphi_{1yy} + Su_{0y} - \frac{1}{2}Sk^2u_0\varphi_1)/\xi,\\
a_2 =[\varphi_1(2ic_gk + ik\omega - 3iku_0 - \frac{1}{2}Su_0) - \varphi_{21}(c_gk^2 + 2k^2\omega\\
\qquad - 3k^2u_0 - u_{0yy} + iSu_0k) + \varphi_{21yy}(c_g - u_0)]/\xi\\
a_3 =[\varphi_1(ik\omega - iku_0 + Su_0) + \varphi_{1y}2c_g + \varphi_{22}Su_{0y}\\
\qquad + \varphi_{22y}(2ik\omega - 2iku_0 + 2Su_0)]/\xi,\\
a_4 =[-\varphi_{1y}2u_o - \varphi_{21}Su_{0y} + \varphi_{21y}(2ik\omega - 2iku_0 - 2Su_0) - \varphi_{22}(c_gk^2\\
\qquad + 2k^2\omega - u_{0yy} - 3u_0k^2 + ikSu_0) + \varphi_{22yy}(c_g - u_0)]/\xi,\\
a_5 =[\varphi_1(k^3\varphi_{20yyy} + \frac{1}{2}Sk^2\varphi_{20y}) + \varphi^*_1(k^3\phi_{23yyy} + \frac{1}{2}Sk^2\varphi_{23y}) + \varphi^*_{1y}\\
\qquad - S\varphi_{23yy} + \varphi_{1y}S\varphi_{20yy} - \varphi^*_{1yy}\varphi_{23y} - \varphi_{1yy}\varphi_{20y}]/\xi,\\
\xi =k^2\varphi_1 - \varphi_{1yy}.
\end{cases}$$

The (2+1)-dimensional gGL equation is obtained by considering the spatial factor y on the basis of the (1+1)-dimensional GL equation. It does not simply increase derivative terms related to y, but generates a cross-coupling term. Similar to the form of the nonlinear Schrödinger equation, it is also a complex function. We hypothesized that the new gGL equation could exhibit more propagation characteristics of shallow wake flows and have many characteristics similar to Schrödinger equation.

3. Time-space fractional (2+1)-dimensional gGL equation

In the last section, we successfully derived the (2+1)-dimensional gGL equation describing shallow wake flows. However, in the actual atmospheric and ocean system, the propagation process of shallow wake flows is complex and diverse, and the partial differential equation of integer order is far from enough to describe. Therefore, in this section, we will introduce another brand-new model called the time-space fractional (2+1)-dimensional gGL equation to further explore more about the propagation properties of shallow wake flows.

In order to get the target function, we suppose that $X \to ix, Y \to iy, T \to it$ Eq.(2.12) can become the follow nonstationary gGL equation

$$iA_t + a_1 A - a_2 A_{xx} - a_3 A_{yy} - a_4 A_{xy} + a_5 |A|^2 A = 0. \tag{3.1}$$

Because $A = A(x, y, t)$ is a complex function and not a real function, we introduce a pair of potential functions $p(x, y, t)$ and $q(x, y, t)$, substitute $A(x, y, t) = p(x, y, t) + iq(x, y, t)$ into Eq.(3.1), we obtain

$$\begin{aligned} &- q_t + a_1 p - a_2 p_{xx} - a_3 p_{yy} - a_4 p_{xy} + a_5 p(p^2 + q^2) \\ &+ i[p_t + a_1 q - a_2 q_{xx} - a_3 q_{yy} - a_4 q_{xy} + a_5 q(q^2 + p^2)] = 0, \end{aligned} \tag{3.2}$$

which can be divided into two second-order equations as follows

$$- q_t + a_1 p - a_2 p_{xx} - a_3 p_{yy} - a_4 p_{xy} + a_5 p(p^2 + q^2) = 0, \tag{3.3}$$

$$p_t + a_1 q - a_2 q_{xx} - a_3 q_{yy} - a_4 q_{xy} + a_5 q(q^2 + p^2) = 0. \tag{3.4}$$

Furthermore, a trial-functional $H(p)$ is constructed to find the variational principle of the Eqs.(3.3) and (3.4)

$$\begin{aligned} J(p, q) = \int_R \int_R \int_T [&p_t q + \frac{a_1}{2} q^2 + \frac{a_2}{2} q_x^2 + \frac{a_3}{2} q_y^2 + \frac{a_4}{2} q_x q_y \\ &+ \frac{a_5}{4}(2p^2 q^2 + q^4) + H(p)] dxdydt, \end{aligned} \tag{3.5}$$

where $H(p)$ is an unknown function consisting of p and the derivatives of p which will be solved later. Then, taking the variation of p, we get the following Euler-Lagrange equation [2]

$$- q_t + a_5 p q^2 + \frac{\delta H}{\delta P} = 0, \tag{3.6}$$

there the meaning of $\frac{\delta H}{\delta P}$ is He's variational differential [10]

$$\begin{aligned} \frac{\delta H}{\delta p} =& \frac{\partial H}{\partial p} - \frac{\partial}{\partial t}(\frac{\partial H}{\partial p_t}) - \frac{\partial}{\partial x}(\frac{\partial H}{\partial p_x}) - \frac{\partial}{\partial y}(\frac{\partial H}{\partial p_y}) \\ &+ \frac{\partial^2}{\partial x^2}(\frac{\partial H}{\partial p_{xx}}) + \frac{\partial^2}{\partial y^2}(\frac{\partial H}{\partial p_{yy}}) + \frac{\partial^2}{\partial x \partial y}(\frac{\partial H}{\partial p_{xy}}) + \cdots . \end{aligned}$$

Since both Eq.(3.3) and Eq.(3.6) are satisfied, $\frac{\delta H}{\delta P}$ can be expressed as

$$\frac{\delta H}{\delta p} = a_1 p - a_2 p_{xx} - a_3 p_{yy} + a_5 p^3, \tag{3.7}$$

therefore, $H=\frac{a_1}{2}p+\frac{a_2}{2}p_x^2+\frac{a_3}{2}p_y^2+\frac{a_4}{2}p_xp_y+\frac{a_5}{4}p^4$, Eq.(3.5) can be rewritten as

$$\begin{aligned} J(p,q)=\int_R\int_R\int_T[p_tq+\frac{a_1}{2}(p^2+q^2)+\frac{a_2}{2}(p_x^2+q_x^2)+\frac{a_3}{2}(p_y^2+q_y^2)\\ +\frac{a_4}{2}(p_xp_y+q_xq_y)+\frac{a_5}{4}(p^2+q^2)^2]dxdydz. \end{aligned} \tag{3.8}$$

On substituting $p=\frac{A+A^*}{2}, q=i\frac{A^*-A}{2}$ into the functional, where A^* is the conjugate function of A and $A^*=p-iq$, the following needed variation principle is expressed as

$$\begin{aligned} J(A)=\int_R\int_R\int_T[\frac{i}{4}(A-A^*)(A_t+A_t^*)+\frac{a_1}{2}AA^*+\frac{a_2}{2}(A_xA_x^*)\\ +\frac{a_3}{2}(A_yA_y^*)+\frac{a_4}{2}(A_xA_y^*+A_x^*A_y)+\frac{a_5}{4}(AA^*)^2]dxdydz, \end{aligned} \tag{3.9}$$

form this we have the Lagrangian form of (2+1)-dimensional nonstationary gGL equation

$$\begin{aligned} La=&\frac{i}{4}(A^*-A)(A_t+A_t^*)+\frac{a_1}{2}AA^*+\frac{a_2}{2}(A_xA_x^*)\\ &+\frac{a_3}{2}(A_yA_y^*)+\frac{a_4}{2}(A_xA_y^*+A_x^*A_y)+\frac{a_5}{4}(AA^*)^2. \end{aligned} \tag{3.10}$$

Equivalently, the Lagrangian form of the time-space fractional order of (2+1)-dimensional nonstationary gGL equation can be written as

$$\begin{aligned} Fr=&\frac{i}{4}(A-A^*)(D_t^\alpha A+D_t^\alpha A^*)+\frac{a_1}{2}AA^*+\frac{a_2}{2}(D_x^\beta A\cdot D_x^\beta A^*)\\ &+\frac{a_3}{2}(D_y^\omega A\cdot D_y^\omega A_y^*)+\frac{a_4}{2}(D_x^\beta A\cdot D_y^\omega A^*+D_x^\beta A^*\cdot D_y^\omega A)+\frac{a_5}{4}(AA^*)^2, \end{aligned} \tag{3.11}$$

where something like $D_z^\gamma g(z)$ represents the mRL fractional definition [11,12] and is specifically defined as

$$D_z^\gamma g(z)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\left\{\int_a^z d\delta\frac{[(\delta)-f(a)]}{(z-\delta)^\gamma}\right\},\quad 0\le\gamma<1,$$

and, the integration by parts rule is

$$\begin{aligned} &\int_a^b(d\kappa)^\gamma f(z)D_z^\gamma g(z)=\Gamma(1+\gamma)[g(z)f(z)]_a^b-\int_a^b(dz)^\gamma g(z)D_z^\gamma f(z),\\ &f(z),g(z)\in[a,b]. \end{aligned}$$

Then, the functional of the time-space fractional order of (2+1)-dimensional nonstationary gGL equation has the form

$$J_{Fr}(A^*)=\int_x(dx)^\beta\int_y(dy)^\omega\int_t(dt)^\alpha Fr(A^*,D_t^\alpha A^*,D_x^\beta A^*,D_y^\omega A^*). \tag{3.12}$$

Applying the rule of integration by parts, optimizing the variation of the functional and making the $\delta J_{Fr}(A)=0$, as well as

$$\begin{aligned} J_{Fr}(A)=\int_x(dx)^\beta\int_y(dy)^\omega\int_t(dt)^\alpha[\frac{\partial Fr}{\partial A^*}-D_t^\alpha(\frac{\partial Fr}{\partial D_t^\alpha A^*})\\ -D_x^\beta(\frac{\partial Fr}{\partial D_x^\beta A^*})-D_y^\omega(\frac{\partial Fr}{\partial D_y^\omega A^*})]\delta JF_r=0. \end{aligned} \tag{3.13}$$

The Euler-Lagrange equation of (2+1)-dimensional time-space fractional nonstationary gGL equation can be written in the form

$$\frac{\partial Fr}{\partial A^*} - D_t^\alpha\left(\frac{\partial Fr}{\partial D_t^\alpha A^*}\right) - D_x^\beta\left(\frac{\partial Fr}{\partial D_x^\beta A^*}\right) - D_y^\omega\left(\frac{\partial Fr}{\partial D_y^\omega A^*}\right) = 0. \tag{3.14}$$

Finally, substituting Fr which is the Eq.(3.11) into the Euler-Lagrange equation, the (2+1)-dimensional time-space fractional nonstationary gGL equation takes the form

$$iD_t^\alpha A + a_1 A - a_2 D_x^{2\beta} A - a_3 D_y^{2\omega} A - a_4 D_x^\beta D_y^\omega A + a_5 |A|^2 A = 0. \tag{3.15}$$

Introducing a inverse transformation $x \to ix', y \to iy', t \to it'$, and the apostrophe of independent variable is ignored. The fractional nonstationary gGL equation is changed into the (2+1)-dimensional time-space fractional gGL equation as follows

$$D_t^\alpha A + a_1 A + a_2 D_x^{2\beta} A + a_3 D_y^{2\omega} A + a_4 D_x^\beta D_y^\omega A + a_5 |A|^2 A = 0. \tag{3.16}$$

The fractional derivative of the time-space fractional gGL equation are related to the wave propagation in shallow wake flows with fractal properties. The fractional derivative may due to superdiffusive wave propagation, other terms correspond to wave interaction due to the nonlinear properties of the media. Thus, Eq.(3.16) can describe fractal processes of the shallow wake flows.

4. Conservation laws of time-space fractional gGL equation

There are many methods to construct the conservation law of fractional partial differential equations. Considering the close relationship between symmetry and conservation law, in this section, we use the Lie symmetry method to obtain new conservation quantities of time-space fractional (2+1)-dimensional gGL equation by utilizing the invariance of the equation under infinitesimal transformation. For the convenience of understanding, we divide the content into two parts to elaborate.

4.1. Lie symmetry analysis

Introducing the following one-parameter lie group of point transformations, the time-space fractional (2+1)-dimensional gGL equation is invariant under the lie group

$$\begin{cases} \bar{x} \to x + \varepsilon\xi_1(x,y,t,A) + o(\varepsilon^2), \\ \bar{y} \to y + \varepsilon\xi_2(x,y,t,A) + o(\varepsilon^2), \\ \bar{t} \to t + \varepsilon\tau(x,y,t,A) + o(\varepsilon^2), \\ \bar{A} \to A + \varepsilon\eta(x,y,t,A) + o(\varepsilon^2), \\ D_t^\alpha \bar{A} \to D_t^\alpha A + \varepsilon\eta_t^\alpha(x,y,t,A) + o(\varepsilon^2), \\ D_x^{2\beta} \bar{A} \to D_x^{2\beta} A + \varepsilon\eta_x^{2\beta}(x,y,t,A) + o(\varepsilon^2), \\ D_y^{2\omega} \bar{A} \to D_y^{2\omega} A + \varepsilon\eta_y^{2\omega}(x,y,t,A) + o(\varepsilon^2), \\ D_x^\beta D_y^\omega \bar{A} \to D_x^\beta D_y^\omega A + \varepsilon\eta_{x,y}^{\beta\omega}(x,y,t,A) + o(\varepsilon^2), \end{cases} \tag{4.1}$$

where ε is the infinitesimal group parameter, ξ_1, ξ_2, τ, η are infinitesimal generator functions and $\eta_t^\alpha, \eta_x^{2\beta}, \eta_y^{2\omega}, \eta_{x,y}^{\beta\omega}$ are extension functions defined as

$$\begin{aligned}\eta_t^\alpha =& D_t^\alpha(\eta) + \xi_1 D_t^\alpha(A_x) - D_t^\alpha(\xi_1 A_x) + \xi_2 D_t^\alpha(A_y) - D_t^\alpha(\xi_2 A_y)\\ &+ D_t^\alpha(D_t(\tau)A) - D_t^{\alpha+1}(\tau A) + \tau D_t^{\alpha+1}(A),\end{aligned}\tag{4.2}$$

$$\begin{aligned}\eta_x^{2\beta} =& D_x^{2\beta}(\eta) + D_x^\beta(\tau D_x^\beta(A_t)) - D_x^{2\beta}(\tau A_t) + D_x^\beta(\xi_2 D_x^\beta(A_y))\\ &- D_x^{2\beta}(\xi_2 A_y) + D_x^{2\beta}(D_x(\xi_1)A) - D_x^{2\beta+1}(\xi_1 A) + D_x^\beta(\xi_1 D_x^{\beta+1}(A))\\ &+ \xi_1 D_x^{2\beta}(A_x) + \tau D_x^{2\beta}(A_t) + \xi_2 D_x^{2\beta}(A_y),\end{aligned}\tag{4.3}$$

$$\begin{aligned}\eta_y^{2\omega} =& D_y^{2\omega}(\eta) + D_y^\omega(\tau D_y^\omega(A_t)) - D_y^{2\omega}(\tau A_t) + D_y^\omega(\xi_1 D_y^\omega(A_x))\\ &- D_y^{2\omega}(\xi_1 A_x) + D_y^{2\omega}(D_y(\xi_2)A) - D_y^{2\omega+1}(\xi_2 A) + D_y^\omega(\xi_2 D_y^{\omega+1}(A))\\ &+ \xi_2 D_y^{2\omega}(A_y) + \tau D_y^{2\omega}(A_t) + \xi_1 D_y^{2\omega}(A_x),\end{aligned}\tag{4.4}$$

$$\begin{aligned}\eta_{x,y}^{\beta\omega} =& D_x^\beta D_y^\omega(\eta) + D_x^\beta(\tau D_y^\omega(A_t)) - D_x^\beta D_y^\omega(\tau A_t) + D_x^\beta(\xi_1 D_y^\omega(A_x))\\ &- D_x^\beta D_y^\omega(\xi_1 A_x) + D_x^\beta D_y^\omega(D_y(\xi_2)A) - D_x^\beta D_y^{\omega+1}(\xi_2 A) + D_x^\beta(\xi_2 D_y^{\omega+1}(A))\\ &+ \xi_2 D_x^\beta D_y^\omega(A_y) + \tau D_x^\beta D_y^\omega(A_t) + \xi_1 D_x^\beta D_y^\omega(A_x),\end{aligned}\tag{4.5}$$

here $D_z^{2\gamma} g = D_z^\gamma[D_z^\gamma g]$ and $D_t^\alpha, D_x^\beta, D_y^\omega$ are the total differential operators with respect to the independent variables x, y, t respectively, which have the forms

$$\begin{cases} D_t = \dfrac{\partial}{\partial t} + A\dfrac{\partial}{\partial A} + A_{tt}\dfrac{\partial}{\partial A_t} + A_{xt}\dfrac{\partial}{\partial A_x} + A_{yt}\dfrac{\partial}{\partial A_y} + \cdots,\\ D_x = \dfrac{\partial}{\partial x} + A_x\dfrac{\partial}{\partial A} + A_{xx}\dfrac{\partial}{\partial x} + A_{tx}\dfrac{\partial}{\partial A_t} + A_{yx}\dfrac{\partial}{\partial A_y} + \cdots,\\ D_y = \dfrac{\partial}{\partial y} + A_y\dfrac{\partial}{\partial A} + A_{yy}\dfrac{\partial}{\partial A_y} + A_{yt}\dfrac{\partial}{\partial A_t} + A_{xy}\dfrac{\partial}{\partial A_x} + \cdots. \end{cases}\tag{4.6}$$

Taking the infinitesimal transformations to generate the symmetry vector N

$$N = \xi_1(x,y,t,A)\frac{\partial}{\partial x} + \xi_2(x,y,t,A)\frac{\partial}{\partial y} + \tau(x,y,t,A)\frac{\partial}{\partial t} + \eta(x,y,t,A)\frac{\partial}{\partial A},\tag{4.7}$$

where the coefficient functions ξ_1, ξ_2, τ, η are to be determined. Due to the invariance of the equation Eq.(3.16), the vector field can be obtained

$$Pr^{(4)}N(\Lambda)|_{\Lambda=0} = 0,\tag{4.8}$$

here Λ denotes the time-space fractional (2+1)-dimensional gGL equation. The invariant conditions yields

$$\tau(x,y,t,A)|_{t=0} = 0, \quad \xi_1(x,y,t,A)|_{x=0} = 0, \quad \xi_2(x,y,t,A)|_{y=0} = 0.$$

In consideration of the generalized Leibnitz rule [23] and the generalized chain rule, the prolongation form of symmetry operator η_t^α after calculation is

$$\begin{aligned}\eta_t^\alpha =&D_t^\alpha\eta+(\eta_A-\alpha D_t(\tau))D_t^\alpha A-AD_t^\alpha\eta_A+\mu_\alpha\\&+\sum_{m=1}^{\infty}[\begin{pmatrix}\alpha\\m\end{pmatrix}D_t^\alpha\eta_A-\begin{pmatrix}\alpha\\m+1\end{pmatrix}D_t^{m+1}(\tau)]D_t^{\alpha-m}(A)\\&-\sum_{m=1}^{\infty}\begin{pmatrix}\alpha\\m\end{pmatrix}D_t^m(\xi_1)D_t^{\alpha-m}A_x-\sum_{m=1}^{\infty}\begin{pmatrix}\alpha\\m\end{pmatrix}D_t^m(\xi_2)D_t^{\alpha-m}A_y,\end{aligned}\tag{4.9}$$

where

$$\mu_\alpha=\sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{k=2}^{m}\sum_{n-\alpha}^{r}\begin{pmatrix}\alpha\\m\end{pmatrix}\frac{1}{r!}\frac{t^{m-l}}{\Gamma(m+1-\alpha)}(-A)^r\frac{\partial^n}{\partial t^n}(A^{k-r})\frac{\partial^{m-n+k}n}{\partial t^{m-n}\partial A^k}.$$

And similarly, the prolongation forms of $\eta_x^{2\beta},\eta_y^{2\omega},\eta_{x,y}^{\beta\omega}$ can be written as

$$\begin{aligned}\eta_x^{2\beta} =&D_x^{2\beta}\eta+(\eta_A-\beta D_x(\xi_1))D_x^{2\beta}A-AD_x^{2\beta}\eta_A+\mu_{2\beta}+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}2\beta\\m\end{pmatrix}\\&\cdot D_x^m\eta_A-\begin{pmatrix}2\beta\\m+1\end{pmatrix}D_x^{m+1}(\xi_1)+\begin{pmatrix}\beta\\m+1\end{pmatrix}D_x^{\beta}(\xi_1)\Big]D_x^{2\beta-m}A\\&+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}\beta\\m\end{pmatrix}-\begin{pmatrix}2\beta\\m\end{pmatrix}\Big]D_x^{\beta}(\tau)D_x^{2\beta-m}(A_t)+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}\beta\\m\end{pmatrix}\\&-\begin{pmatrix}2\beta\\m\end{pmatrix}\Big]D_x^m(\xi_2)\cdot D_x^{2\beta-m}(A_y)+\xi_1D_x^{2\beta}(A_x)+\tau D_x^{2\beta}(A_t)+\xi_2D_x^{2\beta}(A_y),\end{aligned}\tag{4.10}$$

$$\begin{aligned}\eta_y^{2\omega} =&D_y^{2\omega}\eta+(\eta_A-\omega D_y(\xi_2))D_y^{2\omega}A-AD_y^{2\omega}\eta_A+\mu_{2\omega}+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}2\omega\\m\end{pmatrix}\\&\cdot D_y^m\eta_A-\begin{pmatrix}2\omega\\m+1\end{pmatrix}D_y^{m+1}(\xi_2)+\begin{pmatrix}\omega\\m+1\end{pmatrix}D_y^{m}(\xi_2)\Big]D_y^{2\omega-m}A\\&+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}\omega\\m\end{pmatrix}-\begin{pmatrix}2\omega\\m\end{pmatrix}\Big]D_y^{\omega}(\tau)D_y^{2\omega-m}(A_t)+\sum_{m=1}^{\infty}\Big[\begin{pmatrix}\omega\\m\end{pmatrix}-\begin{pmatrix}2\omega\\m\end{pmatrix}\Big]\\&\cdot D_y^m(\xi_1)D_y^{2\omega-m}(A_x)+\xi_1D_y^{2\omega}(A_x)+\tau D_y^{2\omega}(A_t)+\xi_2D_y^{2\omega}(A_y),\end{aligned}\tag{4.11}$$

$$
\begin{aligned}
\eta_{x,y}^{\beta\omega} =& D_x^{\beta}D_y^{\omega}\eta + (\eta_A - \omega D_y(\xi_2))D_x^{\beta}D_y^{\omega}A - AD_x^{\beta}D_y^{\omega}\eta_A + \mu_{\beta+\omega} \\
&+ \sum_{m=1}^{\infty}\Big[\begin{pmatrix}\beta+\omega\\ m\end{pmatrix}D_y^m\eta_A - \begin{pmatrix}\omega+\omega\\ m+1\end{pmatrix}D_y^{m+1}(\xi_2) + \begin{pmatrix}\omega\\ m+1\end{pmatrix}D_y^m(\xi_2)\Big] \\
&\cdot D_x^{\beta}D_y^{\omega-m}A + \sum_{m=1}^{\infty}\Big[\begin{pmatrix}\omega\\ m\end{pmatrix} - \begin{pmatrix}\beta+\omega\\ m\end{pmatrix}\Big]D_y^{\omega}(\tau)D_x^{\beta}D_y^{\omega-m}(A_t) \\
&+ \sum_{m=1}^{\infty}\Big[\begin{pmatrix}\omega\\ m\end{pmatrix} - \begin{pmatrix}\omega+\beta\\ m\end{pmatrix}\Big]D_y^m(\xi_1)D_x^{\beta}D_y^{\omega-m}(A_x) + \xi_1 D_x^{\beta}D_y^{\omega}(A_x) \\
&+ \tau D_x^{\beta}D_y^{\omega}(A_t) + \xi_2 D_x^{\beta}D_y^{\omega}(A_y).
\end{aligned} \tag{4.12}
$$

Applying the second prolongation to Eq.(3.16),the symmetry determining equation is given as

$$\eta_t^{\alpha} + a_2\eta_x^{2\beta} + a_3\eta_y^{2\omega} + a_4\eta_{x,y}^{\beta\omega} + (a_1 + 2a_5AA^*)\eta = 0. \tag{4.13}$$

Substituting Eq.(4.9)-Eq.(4.12) into Eq.(4.13),and sorting out similar terms, the infinitesimal generator functions can be determined in the form as

$$\xi_1 = \frac{b_1x + b_2y}{\beta} + b_3, \quad \xi_2 = \frac{b_1y + b_2x}{\omega} + b_4, \quad \tau = \frac{2b_1t}{\alpha} + b_5, \quad \eta = b_1A, \tag{4.14}$$

where $b_i, i = 1\cdots 4$ are arbitrary constants. Hence, we obtain the explicit expression of the infinitesimal operators as follows

$$
\begin{aligned}
&N_1 = \frac{\partial}{\partial x},\ N_2 = \frac{\partial}{\partial y},\ N_3 = \frac{\partial}{\partial t},\ N_4 = \frac{y}{\beta}\frac{\partial}{\partial x} + \frac{x}{\omega}\frac{\partial}{\partial y}, \\
&N_5 = \frac{x}{\beta}\frac{\partial}{\partial x} + \frac{y}{\omega}\frac{\partial}{\partial y} + \frac{2t}{\alpha}\frac{\partial}{\partial t} + A\frac{\partial}{\partial A}.
\end{aligned} \tag{4.15}
$$

Since we have solved the symmetric operators and the conservation vector associated with the time-space fractional (2+1)-dimensional gGL equation, multiple conservation laws are obtained in the next section.

4.2. Conservation laws

It is well known that the conservation laws of the (2+1)-dimensional time-space fractional gGL equation satisfy the equation given by

$$D_t(C^t) + D_x(C^x) + D_y(C^y) = 0, \tag{4.16}$$

where C^t, C^x and C^y are called the conserved vectors.

A formal Lagrangian of Eq.(3.16) can be presented as

$$= \varsigma(x, y, t)(D_t^{\alpha}A + a_1A + a_2D_x^{2\beta}A + a_3D_y^{2\omega}A + a_4D_x^{\beta}D_y^{\omega}A + a_5|A|^2A), \tag{4.17}$$

here $\varsigma(x, y, t)$ denotes a new dependent variable. The adjoint equation is introduced as

$$F = \frac{\delta}{\delta A} = 0, \tag{4.18}$$

where $\frac{\delta}{\delta A}$ is Euler-Lagrange operator can be defined as follows

$$\begin{aligned}\frac{\delta}{\delta A} =&\frac{\partial}{\partial A} + (D_t^{\alpha})^* \frac{\partial}{\partial D_t^{\alpha} A} + (D_x^{2\beta})^* \frac{\partial}{\partial D_x^{2\beta} A} \\ &+ (D_y^{2\omega})^* \frac{\partial}{\partial D_y^{2\omega} A} + (D_x^{\beta} D_y^{\omega})^* \frac{\partial}{\partial D_y^{2\omega} A},\end{aligned} \tag{4.19}$$

where $(D_t^{\alpha})^*$, $(D_x^{2\beta})^*$, $(D_y^{2\omega})^*$ and $(D_x^{\beta} D_y^{\omega})^*$ are the adjoint operators of the (D_t^{α}), $(D_x^{2\beta})$, $(D_y^{2\omega})$ and $(D_x^{\beta} D_y^{\omega})$ respectively.

Then, the Lie characteristic function $\aleph$ is given by

$$\aleph = \eta - \tau A_t - \xi_1 A_x - \xi_2 A_y, \tag{4.20}$$

as well as

$$\begin{aligned}&\aleph_1 = -A_x,\ \aleph_2 = -A_y,\ \aleph_3 = -A_t,\ \aleph_4 = -\frac{y}{\beta} A_x - \frac{x}{\omega} A_y, \\ &\aleph_5 = -A - \frac{x}{\beta} A_x + \frac{y}{\omega} A_y - \frac{2t}{\alpha} A_t.\end{aligned}$$

The fractional Noether operator for the variable t has the following from

$$\begin{aligned}C^t = \tau &+ \sum_{k=0}^{m-1} (-1)^k D_t^{\alpha-1-k}(\aleph) D_t^k \Big(\frac{\partial}{\partial (D_t^{\alpha} A)} \Big) \\ &- (-1)^m J\Big(\aleph, D_t^m \Big(\frac{\partial}{\partial (D_t^{\alpha} A)} \Big) \Big),\end{aligned} \tag{4.21}$$

where $m = [\alpha] + 1$ and ,for any two functions $f(x, y, t), g(x, y, t)$,J is defined as

$$J(f, g) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^b \frac{f(x, y, r) g(x, y, s)}{(s-r)^{\alpha+1-m}} dsdr.$$

Equivalently, the fractional Noether operators for the x and y are writing as

$$\begin{aligned}C^x =&\xi_1 + \sum_{k=0}^{n-1} (-1)^k D_x^{\beta-1-k}(\aleph) D_x^k \Big(\frac{\partial}{\partial (D_x^{\beta} A)} \Big) \\ &- (-1)^n J_1\Big(\aleph, D_x^n \Big(\frac{\partial}{\partial (D_x^{\beta} A)} \Big) \Big),\end{aligned} \tag{4.22}$$

$$\begin{aligned}C^y =&\xi_2 + \sum_{k=0}^{e-1} (-1)^k D_y^{\omega-1-k}(\aleph) D_t^k \Big(\frac{\partial}{\partial (D_y^{\omega} A)} \Big) \\ &- (-1)^e J_2\Big(\aleph, D_y^e \Big(\frac{\partial}{\partial (D_y^{\omega} A)} \Big) \Big),\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}J_1(f, g) &= \frac{1}{\Gamma(n-\beta)} \int_0^x \int_x^b \frac{f(r, y, t) g(s, y, t)}{(s-r)^{\beta+1-n}} dsdr, \\ J_2(f, g) &= \frac{1}{\Gamma(e-\omega)} \int_0^y \int_y^b \frac{f(x, r, t) g(x, s, r)}{(s-r)^{\omega+1-n}} dsdr.\end{aligned}$$

By putting $\aleph_i, i = 1 \cdots 5$ into the Noether operators, the conservation laws of the (2+1)-dimensional time-space fractional gGL equation can be obtained. For the sake of brevity, let's take $\aleph_5$ as an example to calculate by using the preceding formula

$$\begin{aligned} C^{t_1} =& \varsigma D_t^{\alpha-1}(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y) \\ &+ J[(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y), \varsigma_t], \\ C^{x_1} =& a_2 \varsigma D_x^{2\beta-1}(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y) \\ &+ J_1[(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y), a_2 \varsigma_x], \\ C^{y_1} =& a_3 \varsigma D_y^{2\omega-1}(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y) \\ &+ J_2[(\eta - \tau A_t - \xi_1 A_x - \xi_2 A_y), a_3 \varsigma_y]. \end{aligned}$$

5. Exact solutions of time-space fractional gGL equation

In this section, the $\exp(-\phi(\zeta))$-expansion method [1] has been utilized for finding new exact solutions of (2+1)-dimensional time-space fractional gGL equation. New soliton solutions of the model have been constructed.

For our goal, we wish to obtain the following travelling wave solution of the form

$$\begin{aligned} &A(x, y, t) = U(\zeta) e^{i\tau}, \quad \zeta = \frac{x^\beta}{\Gamma(1+\beta)} + \frac{y^\omega}{\Gamma(1+\omega)} - \frac{vt^\alpha}{\Gamma(1+\alpha)}, \\ &\tau = \frac{-kx^\beta}{\Gamma(1+\beta)} + \frac{-ly^\omega}{\Gamma(1+\omega)} + \frac{wt^\alpha}{\Gamma(1+\alpha)} + \theta, \end{aligned} \tag{5.1}$$

where v is the soliton velocity, k and l are the soliton frequencies, w is the soliton wave number and θ is the phase constant.

Putting the above wave transformation into the fractional gGL equation Eq.(3.16) and decomposing into real and imaginary parts as follows

$$(a_1 + k^2 a_2 + l^2 a_3 + kla_4)U - vU' + (a_2 + a_3 + a_4)U'' + a_5 U^3 = 0, \tag{5.2}$$

and

$$wU - [2a_2 k + 2a_3 l + (k + l)a_4]U' = 0, \tag{5.3}$$

where $'$ denotes the derivative with respect to ζ. Substituting Eq.(5.3) into Eq.(5.2), an ordinary differential equation is obtained as

$$b_1 U + b_2 U'' + a_5 U^3 = 0, \tag{5.4}$$

where

$$b_1 = a_1 + k^2 a_2 + l^2 a_3 + kla_4 - \frac{wv}{2a_2 k + 2a_3 l + (k + l)a_4}, \quad b_2 = a_2 + a_3 + a_4.$$

Balancing the highest order derivative and nonlinear term in the above equation gives $N = 1$. Then we assume that Eq.(5.4) has a truncated series as follows

$$U(\zeta) = c_0 + c_1 \exp(-\phi(\zeta)), \tag{5.5}$$

where c_0, c_1 are constants and $\phi(\zeta)$ satisfies the auxiliary ordinary differential equation given by

$$\phi^{'}(\zeta) = \exp(-\phi(\zeta)) + \mu\exp(\phi(\zeta)) + \lambda. \tag{5.6}$$

Substituting Eq.(5.5) into Eq.(5.4) gets a polynomial in $\exp(-\phi(\zeta))$, collecting all terms with the same degree of $\exp(-\phi(\zeta))$ and equating each coefficient to zero yields, a set of equations is obtained as following

$$\begin{aligned}
&(\exp(-\phi(\zeta))^0 : b_1c_0 + b_2c_1\lambda\mu + a_5c_0^3 = 0,\\
&(\exp(-\phi(\zeta))^1 : b_1c_1 + 2b_2c_1\mu + b_2c_1\lambda^2 + 3a_5c_0^2 = 0,\\
&(\exp(-\phi(\zeta))^2 : 3b_2c_1\lambda + 3a_5c_0c_1 = 0,\\
&(\exp(-\phi(\zeta))^3 : 2b_1c_1 + a_5c_1^3 = 0.
\end{aligned}$$

Maple software was used to solve the above equations, the values of the constants c_0, c_1, μ can be calculated as **set 1**

$$\begin{aligned}
c_1 &= \pm i\frac{\sqrt{2}\sqrt{b_2}}{\sqrt{3}},\\
c_0 &= \pm i\frac{\sqrt{b_1b_2^2a_5c_1 + 2b_1b_2^2}}{\sqrt{3b_2^2a_5^2 + b_1a_5c_1 + 2b_2^3a_5}},\\
\mu &= -\frac{b_1}{a_5c_1} + \frac{b_2^2}{a_5^2c_1}\lambda^2.
\end{aligned}$$

and **set 2**

$$\begin{aligned}
c_1 &= \pm i\frac{\sqrt{2}\sqrt{b_2}}{\sqrt{3}},\\
c_0 &= \lambda = 0,\\
\mu &= -\frac{b_1}{2b_2},
\end{aligned}$$

For **set 1**, as when the following solutions can be obtained, when $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, then

$$\begin{aligned}
A_{11}(x,y,t) =& \pm ie^{i}(\frac{-kx^{\beta}}{\Gamma(1+\beta)} + \frac{-ly^{\omega}}{\Gamma(1+\omega)} + \frac{wt^{\alpha}}{\Gamma(1+\alpha)} + \theta)\\
&\times\Big\{\frac{\sqrt{b_1b_2^2a_5c_1 + 2b_1b_2^2}}{\sqrt{3b_2^2a_5^2 + b_1a_5c_1 + 2b_2^3a_5}} \pm \sqrt{\frac{2b_2c_1}{3}}\\
&\cdot\frac{(\frac{a_5b_1c_1 - b_2^2\lambda^2}{a_5^2c_1})}{\sqrt{\frac{4a_5c_1b_1 + (a_5^2c_1 - 4b_2^2)\lambda^2}{4a_5^2c_1}}\tanh\big[\sqrt{\frac{4a_5c_1b_1 + (a_5^2c_1 - 4b_2^2)\lambda^2}{4a_5^2c_1}}(\zeta + C)\big] - \lambda}\Big\}.
\end{aligned}$$

When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, then

$$\begin{aligned}
A_{12}(x,y,t) =& \pm ie^{i}(\frac{-kx^{\beta}}{\Gamma(1+\beta)} + \frac{-ly^{\omega}}{\Gamma(1+\omega)} + \frac{wt^{\alpha}}{\Gamma(1+\alpha)} + \theta)\\
&\times\Big\{\frac{\sqrt{b_1b_2^2a_5c_1 + 2b_1b_2^2}}{\sqrt{3b_2^2a_5^2 + b_1a_5c_1 + 2b_2^3a_5}} \mp \sqrt{\frac{2b_2c_1}{3}}
\end{aligned}$$

$$\cdot \frac{(\frac{a_5b_1c_1-b_2^2\lambda^2}{a_5^2c_1})}{\sqrt{\frac{4a_5c_1b_1+(a_5^2c_1-4b_2^2)\lambda^2}{4a_5^2c_1}}tan\big[\sqrt{\frac{4a_5c_1b_1+(a_5^2c_1-4b_2^2)\lambda^2}{4a_5^2c_1}}(\zeta+C)\big]-\lambda}\Big\}.$$

When $\lambda^2-4\mu>0$ and $\mu=0$ and $\lambda\neq 0$, then

$$A_{13}(x,y,t)=\pm\, ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)$$
$$\times\Big\{\frac{\sqrt{b_1b_2^2a_5c_1+2b_1b_2^2}}{\sqrt{3b_2^2a_5^2+b_1a_5c_1+2b_2^3a_5}}+\frac{\sqrt{\frac{2b_2c_1}{3}}\lambda}{\cosh(\lambda(\zeta+C))+\sinh(\lambda(\zeta+C))-1}\Big\}.$$

When $\lambda^2-4\mu=0$ and $\mu\neq 0,\lambda\neq 0$, then

$$A_{14}(x,y,t)=\pm\, ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)$$
$$\times\Big[\frac{\sqrt{b_1b_2^2a_5c_1+2b_1b_2^2}}{\sqrt{3b_2^2a_5^2+b_1a_5c_1+2b_2^3a_5}}+\frac{\sqrt{2b_2c_1}\lambda^2(\zeta+C)}{2\sqrt{3}\lambda(\zeta+C)+2}\Big].$$

When $\lambda^2-4\mu=0$ and $\mu=0,\lambda=0$, then

$$A_{15}(x,y,t)=\pm\, ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)$$
$$\times\Big[\frac{\sqrt{b_1b_2^2a_5c_1+2b_1b_2^2}}{\sqrt{3b_2^2a_5^2+b_1a_5c_1+2b_2^3a_5}}+\frac{\sqrt{2b_2c_1}}{2\sqrt{3}(\zeta+C)}\Big].$$

For **set 2** the following solutions can be obtained, when $\lambda^2-4\mu>0$ and $\mu\neq 0$, then

$$A_{21}(x,y,t)=\pm\, 2ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)$$
$$\cdot\Big\{\sqrt{\frac{b_1}{3}}\coth\Big[\sqrt{\frac{2b_1}{b_2}}(\zeta+C)\Big]\Big\}.$$

When $\lambda^2-4\mu<0$ and $\mu\neq 0$, then

$$A_{22}(x,y,t)=\pm\, 2ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)$$
$$\cdot\Big\{\sqrt{\frac{b_1}{3}}\cot\Big[\sqrt{\frac{2b_1}{b_2}}(\zeta+C)\Big]\Big\}.$$

When $\lambda^2-4\mu=0$ and $\mu=0,\lambda=0$, then

$$A_{23}(x,y,t)=\pm 2ie^i(\frac{-kx^\beta}{\Gamma(1+\beta)}+\frac{-ly^\omega}{\Gamma(1+\omega)}+\frac{wt^\alpha}{\Gamma(1+\alpha)}+\theta)\Big[\frac{\sqrt{b_1}}{\sqrt{3}(\zeta+C)}\Big],$$

where C is the constant of integration.

These solutions are travelling wave solutions that provide strong evidence for the existence of solitary waves in shallow wake flows. Obviously, $A_{11},A_{13},A_{21},A_{22}$ are hyperbolic function solutions, A_{12} are trigonometric function solutions and periodic solutions with the spatial variables x and y, which indicates that the propagation of shallow wake flows is a cyclical phenomena. In addition to, A_{14}, A_{15}, A_{23} are rational function solutions, in particular, as ζ approaches zero, the A_{15}, A_{23} get bigger. This reflects that the (2+1)-dimensional fractional gGL equation has rogue wave solution, and the shallow wake flows have rogue wave with a high crest at a certain moment.

6. Conclusion

In this paper, for the first time, a (2+1)-dimensional gGL equation describing shallow wake flows propagation is obtained by using the multi-scale and perturbation analysis method. Furthermore, in order to explore more propagation rules, we extended the integer (2+1)-dimensional gGL equation to the fractional order equation of time-space, and found for the first time that the (2+1)-dimensional time-space fractional gGL equation is more suitable to describe the actual situation in the atmosphere and ocean. In addition to,using the Lie symmetry, the conservation laws of the (2+1)-dimensional time-space fractional gGL equation are explored. Finally, $\exp(-\phi(\zeta))$-expansion method is used to obtain the exact solutions of the new model. We find that there are solitary waves and rogue wave in the shallow wake flows, and the propagation has periodicity.

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