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DARBOUX TRANSFORMATIONS, MULTISOLITONS, BREATHER AND ROGUE WAVE SOLUTIONS FOR A HIGHER-ORDER DISPERSIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract In this paper, dynamic of a higher-order dispersive nonlinear Schrödinger equation is investigated. Firstly, we obtain the determinant representation of the N-fold Darboux transformations of the Schrödinger equation. Then based on the above analysis, we get the one-soliton, two-soliton and the breather wave solution. Furthermore, the first-order rogue wave is derived by means of a Taylor expansion of the breather wave. Finally, by selecting some special parameters and drawing the 3-D and 2-D graphs to better describe the dynamic traits of those solutions.

Keywords Darboux transformation, higher-order dispersive nonlinear Schrödinger equation, solitons, breather waves, rogue waves.

MSC(2010) 35C08, 35G35.

1. Introduction

Nonlinear partial differential equations (NLPDEs) can be used as models for many complex physical phenomena, including mathematics, plasma physics, fluid mechanics, aerodynamics, atmospheric oceans, etc [3–5, 7, 12, 17, 22, 28, 31]. Therefore, searching exact solutions of NLPDEs plays a rather significant part in the fields of nonlinear science. Recently, the completely integrable nonlinear Schrödinger equations (NLSEs) attract an increasing attention in natural science and mathematics. The Hirota bilinear method [1, 6], Darboux transformation scheme [24, 30], Riemann-Hilbert approach [9] and the inverse scattering method [23] are used to solve NLSE. Moreover, the research of solution of NLSE promotes better analysis of NLPDEs [13, 20, 21]. Akhmediev and Ankiewicz have obtained rogue wave solutions and rational solutions of the standard self-focusing NLSE through traditional Darboux transformation scheme [2]. Li-Ping Xu has constructed exact solutions of two higher order nonlinear Schrödinger equations by applying homogeneous balance principle and F-expansion method [27]. The rogue wave solution is a type of special rational solution, rogue waves be called killer waves, are higher and steeper than all the other waves around them [8]. In addition, rogue waves have been considered by sailors as a threaten to shipping and are believed to have been responsible for the

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unexplained losses of vessels [8]. Due to this feature of rogue waves, the research of rogue waves has become more and more important.

In this paper, we mainly study the higher-order dispersive nonlinear Schrödinger equation, reads

$$iq_t + q_{xx} + 2q|q|^2 + \tau(q_{xxxx} + 6q_x^2 q^* + 4q|q_x|^2 + 8|q|^2 q_{xx} + 2q^2 q_{xx}^* + 6|q|^4 q) = 0, \quad (1.1)$$

where $q(x,t)$ is the complex envelope and τ denotes the strength of higher-order linear and nonlinear effects [25]. Hai-Qiang Zhang and Bo Tian have obtained conservation laws, soliton solutions and modulational instability of Eq. (1.1) based on linear eigenvalue problem and Darboux transformation method [29]. Porsezian and Daniel have studied the effect of perturbation on the nonintegrable GNLSE by using perturbation method [18]. As far as we know, the multisolitons, breather wave solutions, rogue wave solutions of Eq. (1.1) by applying the Darboux transformation have never been researched.

In this paper, the multisolitons, breather wave solution, rogue wave solution of Eq. (1.1) are obtained by applying the N-fold Darboux transformation. In Section 2, the N-fold Darboux transformations of Eq. (1.1) are researched in detail. In Section 3, based on obtained N-fold Darboux transformations, the one-soliton solution and two-soliton solution of Eq. (1.1) are derived. In Section 4, we obtain the breather wave solution through the eigenfunctions associated with a periodic seed solution. In Section 5, the rogue wave solution is derived by using the Taylor expansion of the breather wave solution. The last Section includes a conclusion and further discussion.

2. Darboux transformation

In this section, we would like to research the Darboux transformation of Eq. (1.1). Firstly, the Lax pair of Eq. (1.1) are derived by utilizing the Ablowitz-Kaup-Newell-Segur scheme [14,18]

$$\varpi_x = U\varpi, \quad \varpi_t = V\varpi, \quad (2.1)$$

where $\varpi = (\phi, \varphi)^T$ is the vector eigenfunction, and matrices U and V satisfy the following forms

$$U = i\lambda U_0 + U_1 = i\lambda \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} = \begin{pmatrix} -i\lambda & q \\ -q^* & i\lambda \end{pmatrix}, \quad (2.2)$$

$V = 8i\gamma_1 V_4 - 2iV_2$, here

$$V_2 = \begin{pmatrix} \lambda^2 - \frac{1}{2}qq^* & iq\lambda - \frac{1}{2}q_x \\ -iq^*\lambda - \frac{1}{2}q_x^* & -\lambda^2 + \frac{1}{2}qq^* \end{pmatrix}, V_4 = \begin{pmatrix} A_4 & B_4 \\ C_4 & -A_4 \end{pmatrix},$$

$$A_4 = \lambda^4 - \frac{1}{2}qq^*\lambda^2 + \frac{i}{4}(qq_x^* - q_xq^*)\lambda + \frac{1}{8}(3q^2q^{*2} + q^*q_{x,x} + qq_{x,x}^* - q_xq_x^*), \quad (2.3)$$

$$B_4 = iq\lambda^3 - \frac{1}{2}q_x\lambda^2 - \frac{i}{4}(q_{x,x} + 2q^2q^*)\lambda + \frac{1}{8}(q_{x,x,x} + 6qq^*q_x),$$

$$C_4 = -iq^*\lambda^3 - \frac{1}{2}q_x^*\lambda^2 + \frac{i}{4}(q_{x,x}^* + 2qq^{*2})\lambda + \frac{1}{8}(q_{x,x,x}^* + 6qq^*q_x^*).$$

Furthermore,

$$\varpi(\lambda) = \begin{pmatrix} \phi(\lambda) \\ \varphi(\lambda) \end{pmatrix} = \begin{pmatrix} \phi(\lambda; x, t) \\ \varphi(\lambda; x, t) \end{pmatrix}, \tag{2.4}$$

indicates the eigenfunction of Lax pair Eq. (1.1) related to λ, and λ is a constant spectral parameter. Then we introduce a simple gauge transformation

$$\varpi^{[1]} = T\varpi. \tag{2.5}$$

It can be clearly seen that the linear problem Eq. (2.5) is transformed into

$$\begin{aligned} \varpi_x^{[1]} &= U^{[1]}\varpi^{[1]}, \quad U^{[1]}T = T_x + TU, \\ \varpi_t^{[1]} &= V^{[1]}\varpi^{[1]}, \quad V^{[1]}T = T_t + TV. \end{aligned} \tag{2.6}$$

In view of compatibility conditions $\varpi_{xt}^{[1]} = \varpi_{tx}^{[1]}$, we can obtain the following relationship

$$U_t^{[1]} - V_x^{[1]} + \left[U^{[1]}, V^{[1]}\right] = T(U_t - V_x + [U, V])T^{-1}, \tag{2.7}$$

where $U^{[1]}, V^{[1]}$ have the same structure as U and V, the q, q^* in the matrices U,V are replaced with $q^{[1]}$ and $q^{[1]^*}$ in the matrices $U^{[1]}$, $V^{[1]}$. It can be seen that the solution of matrix T is rather vital to solve Eq. (1.1). Moreover, the seed solution (q_1, q_2) of Eq.(1.1) in U, V is transformed into new solution $(q_1^{[1]}, q_2^{[1]})$ in $U^{[1]}, V^{[1]}$. The Darboux matrix T of Eq. (2.1) is assumed as follows

$$T = T(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{2.8}$$

where $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ are functions of x, t. Substituting the specific form of matrix T (2.8) into $U^{[1]}T = T_x + TU$ of Eq. (2.6) to calculate the relationships among all functions of Eq. (2.8), that is

$$\begin{aligned} &\begin{pmatrix} a_{1x} & b_{1x} \\ c_{1x} & d_{1x} \end{pmatrix} \lambda + \begin{pmatrix} a_{0x} & b_{0x} \\ c_{0x} & d_{0x} \end{pmatrix} \\ &= \begin{pmatrix} -ia_1\lambda^2 - ia_0\lambda + c_1 q^{[1]}\lambda + c_0 q^{[1]} & -ib_1\lambda^2 - ib_0\lambda + d_1 q^{[1]}\lambda + d_0 q^{[1]} \\ ic_1\lambda^2 + ic_0\lambda - a_1 q^{[1]^*}\lambda - a_0 q^{[1]^*} & id_1\lambda^2 + id_0\lambda - b_1 q^{[1]^*}\lambda - b_0 q^{[1]^*} \end{pmatrix} \\ &\quad - \begin{pmatrix} -ia_1\lambda^2 - ia_0\lambda - b_1 q^*\lambda - b_0 q^* & ib_1\lambda^2 + ib_0\lambda + a_1 q\lambda + a_0 q \\ -ic_1\lambda^2 - ic_0\lambda - d_1 q^*\lambda - d_0 q^* & id_1\lambda^2 + id_0\lambda + c_1 q\lambda + c_0 q \end{pmatrix}. \end{aligned} \tag{2.9}$$

By comparing the coefficients of $\lambda^n (n = 0, 1, 2)$, we obtain

$n = 2,\ b_1 = c_1 = 0.$

$n = 1,\ a_{1x} = d_{1x} = 0,\ \ -2ib_0 + d_1 q^{[1]} - a_1 q = 0,\ 2ic_0 - a_1 q^{[1]^*} + d_1 q^* = 0.$

$n = 0, a_{0x} = c_0 q^{[1]} + b_0 q^*, b_{0x} = d_0 q^{[1]} - a_0 q, c_{0x} = -a_0 q^{[1]^*} + d_0 q^*,\ d_{0x} = -b_0 q^{[1]^*} - c_0 q.$

We can know that $b_1 = c_1 = 0$, $a_1,\ d_1$ are constants. Without loss of generality, we derive the DT of Eq. (1.1) in the following form

$$\varpi^{[1]} = T\varpi = (\lambda I - R)\varpi, \tag{2.10}$$

where λ is a complex spectral parameter, I is a 2×2 identity matrix, and R is a nonsingular matrix. Substituting expression of $U, U^{[1]}, T$ into $U^{[1]}T = T_x + TU$ of Eq. (2.6), then comparing the coefficient of λ, one can obtain

$$\begin{pmatrix} 0 & q^{[1]} \\ -q^{[1]^*} & 0 \end{pmatrix} \lambda = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} + i\left[R, \sigma\right], \tag{2.11}$$

where $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$. Therefore a new solution is obtained

$$q^{[1]} = q - 2ir_{12}, \quad q^{[1]^*} = q^* - 2ir_{21}, \tag{2.12}$$

under a constraint $r_{12}^* = -r_{21}$.

According to the examples of the NLSE [15, 16], in order to derive the explicit formula of R by applying the solution of the Lax pair, we can presume that

$$R = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} \times \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \times \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}^{-1}, \tag{2.13}$$

where $(f_1, f_2)^T$ is a solution of the eigenvalue equation of Eq. (2.1) with $\lambda = \lambda_1$. Moreover, $(g_1, g_2)^T = (-f_2^*, f_1^*)$ is a solution of Eq. (2.1) when $\lambda = \lambda_1^*$. To meet the constraint of R, letting $\lambda_2 = \lambda_1^*$, we have

$$R = \frac{1}{|f_1|^2 + |f_2|^2} \begin{pmatrix} \lambda_1|f_1|^2 + \lambda_1^*|f_2|^2 & (\lambda_1 - \lambda_1^*)f_1 f_2^* \\ (\lambda_1 - \lambda_1^*)f_1^* f_2 & \lambda_1|f_2|^2 + \lambda_1^*|f_1|^2 \end{pmatrix}. \tag{2.14}$$

Based on the above analysis, we can obtain a new solution of Eq. (1.1) as follows

$$q^{[1]} = q - \frac{2i}{|f_1|^2 + |f_2|^2}(\lambda_1 - \lambda_1^*)f_1 f_2^*. \tag{2.15}$$

In addition, the DT can be written determinant representation to obtain the higher order transformation. The one-fold DT is as follows

$$q^{[1]} = q - 2i\frac{R_2}{M_2} = q - 2i\frac{\begin{vmatrix} f_1 & \lambda_1 f_1 \\ g_1 & \lambda_2 g_1 \end{vmatrix}}{\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}}, \tag{2.16}$$

under the reductions $g_1 = -f_2^*, \quad g_2 = f_1^*, \quad \lambda_2 = \lambda_1^*$. As for the two-fold DT, we have

$$q^{[2]} = q - 2i\frac{R_4}{M_4}, \tag{2.17}$$

here

$$R_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1^2 f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2^2 g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3^2 f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4^2 g_3 \end{vmatrix}, \quad M_4 = \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 \end{vmatrix}, \tag{2.18}$$

under the reductions $g_1 = -f_2^*, \; g_2 = f_1^*, \; g_3 = -f_4^*, \; g_4 = f_3^*, \; \lambda_2 = \lambda_1^*, \; \lambda_4 = \lambda_3^*$. In a similar way, the n-fold DT can be written

$$q^{[n]} = q - 2i\frac{R_{2n}}{M_{2n}}, \tag{2.19}$$

where

$$\begin{aligned} R_{2n} &= \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \cdots & \lambda_1^{n-1} f_1 & \lambda_1^n f_1 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \cdots & \lambda_2^{n-1} g_1 & \lambda_2^n g_1 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \cdots & \lambda_3^{n-1} f_3 & \lambda_3^n f_3 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \cdots & \lambda_4^{n-1} g_3 & \lambda_4^n g_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \cdots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^n g_{2n-1} \end{vmatrix}, \\ M_{2n} &= \begin{vmatrix} f_1 & f_2 & \lambda_1 f_1 & \lambda_1 f_2 & \cdots & \lambda_1^{n-1} f_1 & \lambda_1^{n-1} f_2 \\ g_1 & g_2 & \lambda_2 g_1 & \lambda_2 g_2 & \cdots & \lambda_2^{n-1} g_1 & \lambda_2^{n-1} g_2 \\ f_3 & f_4 & \lambda_3 f_3 & \lambda_3 g_3 & \cdots & \lambda_3^{n-1} f_3 & \lambda_3^{n-1} f_4 \\ g_3 & g_4 & \lambda_4 g_3 & \lambda_4 g_4 & \cdots & \lambda_4^{n-1} g_3 & \lambda_4^{n-1} g_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{2n-1} & g_{2n} & \lambda_{2n} g_{2n-1} & \lambda_{2n} g_{2n} & \cdots & \lambda_{2n}^{n-1} g_{2n-1} & \lambda_{2n}^{n-1} g_{2n} \end{vmatrix}. \end{aligned} \tag{2.20}$$

As far as we know [10, 15, 19], the Darboux transformation is more convenient to obtain the multisolitons, multibreathers and higher order rogue waves of the nonlinear Schrödinger equation.

3. Solitons and Breather wave solutions

According to the analysis of the determinant representation of DT, we construct one-soliton solution and two-soliton solution by taking a zero seed solution. Then by taking a periodic seed solution, the breather wave solution of Eq. (1.1) is derived.

3.1. One-Soliton solution

It can be seen that the accurate form of the one-soliton solution has been given in Eq. (2.15). Setting the seed $q = 0$ and $\lambda_1 = \xi + i\eta$, and substituting them into the Lax pair Eq. (2.1). By calculation with the help of MAPLE, we obtain

$$\begin{aligned} f_1 &= e^{-i(\xi+i\eta)x+(-2i(\xi+i\eta)^2+8i\tau(\xi+i\eta)^4)t}, \\ f_2 &= e^{i(\xi+i\eta)x+(2i(\xi+i\eta)^2-8i\tau(\xi+i\eta)^4)t}, \end{aligned} \tag{3.1}$$

and $g_1 = -f_2^*, g_2 = f_1^*$, $\lambda_2 = \lambda_1^*$. Then substituting Eq. (3.1) into Eq. (2.15), the one-soliton solution of Eq. (1.1) is obtained

$$\begin{aligned} q^{[1]} =& 2\eta e^{(4it\eta^2+16it\tau\xi^4+16it\tau\eta^4-2ix\xi-4it\xi^2-96it\tau\xi^2\eta^2)} \\ & \times \sec h(64\eta^3 t\tau\xi - 64\eta t\tau\xi^3 + 8\eta t\xi + 2\eta x). \end{aligned} \tag{3.2}$$

3.2. Two-Soliton solution

In this subsection, taking the seed solution $q = 0$, and $\lambda_1 = \xi + i\eta, \lambda_3 = \theta + i\upsilon$, to derive two-soliton solution. Where

$$\begin{aligned} f_1 &= e^{-i(\xi+i\eta)x+i(-2(\xi+i\eta)^2+8\tau(\xi+i\eta)^4)t}, \\ f_2 &= e^{i(\xi+i\eta)x+i(2(\xi+i\eta)^2-8\tau(\xi+i\eta)^4)t}, \\ f_3 &= e^{-i(\theta+i\upsilon)x+i(-2(\theta+i\upsilon)^2+8\tau(\theta+i\upsilon)^4)t}, \\ f_4 &= e^{i(\theta+i\upsilon)x+i(2(\theta+i\upsilon)^2-8\tau(\theta+i\upsilon)^4)t}, \end{aligned} \tag{3.3}$$

under the reductions $g_1 = -f_2^*, \quad g_2 = f_1^*, \quad g_3 = -f_4^*, \quad g_4 = f_3^*, \quad \lambda_2 = \lambda_1^*$ and $\lambda_4 = \lambda_3^*$. Substituting Eq. (3.3) into the two-fold DT of Eq. (2.16), the two-soliton solution of Eq. (1.1) can be derived

$$q^{[2]} = \frac{[2A\cosh(B)+iC\sinh(B)]e^{-2i(\theta x+Dt)}+[2E\cosh(F)-iC\sinh(F)]e^{-2i(\xi x+Gt)}}{2(\psi+2\eta\upsilon)\cosh(H)+2(\psi-2\eta\upsilon)\cosh(I)-8\eta\upsilon\cos(J)}, \tag{3.4}$$

where

$$\begin{aligned} A &= 4\eta^2\upsilon - 4\theta^2\upsilon + 8\theta\xi\upsilon - 4\xi^2\upsilon - 4\upsilon^3, \\ B &= 64\eta^3 t\tau\xi - 64\eta t\tau\xi^3 + 8\eta t\xi + 2\eta x, \\ C &= -16\xi\eta\upsilon + 16\theta\eta\upsilon, \\ D &= (96t\upsilon^2 - 4)\theta^2 - 16\tau\theta^4 - 16\tau\upsilon^4 + 4\upsilon^2, \\ E &= -4\eta^3 - 4\eta\theta^2 + 8\eta\theta\xi - 4\eta\xi^2 + 4\eta\upsilon^2, \\ F &= 64t\tau\theta^3\upsilon - 64t\tau\theta\upsilon^3 - 8t\theta\upsilon - 2x\upsilon, \\ G &= (96\eta^2\tau + 4)\xi^2 - 16\xi^4\tau - 16\eta^4\tau, \\ \psi &= \eta^2 + \theta^2 - 2\theta\xi + \xi^2 + \theta^2, \\ H &= (-64\tau\theta\upsilon^3 + (64\tau\theta^3 - 8\theta)\upsilon - 64\xi(\xi^2\tau - \eta^2\tau - \frac{1}{8})\eta)t + 2x(\eta - \upsilon), \\ I &= (64t\theta\upsilon^3 + (-64\tau\theta^3 + 8\theta)\upsilon - 64\xi(\xi^2\tau - \eta^2\tau - \frac{1}{8})\eta)t + 2x(\eta + \upsilon), \\ J &= ((16\eta^4 - 96\eta^2\xi^2 - 16\theta^4 + 96\theta^2\upsilon^2 + 16\xi^4 - 16\upsilon^4)\tau - 4\xi^2 + 4\eta^2 + 4\theta^2 - 4\upsilon^2)t - 2x(\xi - \theta). \end{aligned} \tag{3.5}$$

Figs. 1-2 reveal the characters of the one-soliton solution and two-soliton solution, respectively. From Fig. 1, it can be seen that the high peak keeps constant over time. Fig. 2 displays the interaction phenomenon of two single solitons.

3.3. Breather wave solution

According to the determinant presentation of DT, taking a periodic seed solution $q^{[0]}$, the breather wave solution can be derived. The $q^{[0]}$ is defined as

$$q^{[0]} = ce^{i\rho}, \tag{3.6}$$

here $\rho = ax + bt$. Substituting $q^{[0]}$ into Eq.(1.1), we get $b = a^4\tau - 12a^2c^2\tau + 6c^4\tau - a^2 + 2c^2$. By using the MAPLE, the solution of eigenvalue equations of the Lax pair is as follows

$$\begin{aligned} f_1 &= ce^{i(\frac{1}{2}a+\upsilon_1)x+i(\frac{1}{2}b+\upsilon_1\upsilon_2)t}, \\ f_2 &= (\frac{1}{2}a + \lambda_1 + \upsilon_1)e^{i(-\frac{1}{2}a+\upsilon_1)x+i(-\frac{1}{2}b+\upsilon_1\upsilon_2)t}, \end{aligned} \tag{3.7}$$

where

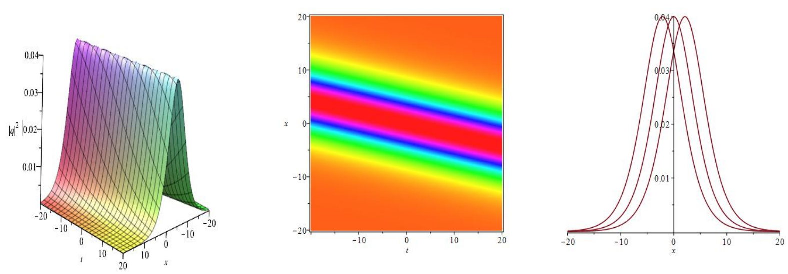

Figure 1. The one-soliton solution of the higher-order dispersive nonlinear Schrödinger equation with $\eta = 0.1, \xi = 0.05$ and $\tau = 1$: a three-dimensional plot, b density plot, c the two-dimensional plot at different $t = -10$(left), $t = 0$(middle), $t = 10$(right).

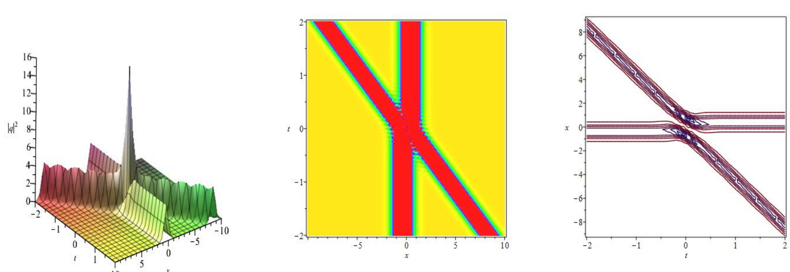

Figure 2. The two-soliton solution of the higher-order dispersive nonlinear Schrödinger equation with $\eta = 1, \xi = 1, \upsilon = 1, \theta = 0$, and $\tau = 1$: a three-dimensional plot, b density plot, c the contour plot.

$$\begin{aligned} \upsilon_1 &= \frac{1}{2}(a^2 + 4a\lambda + 4c^2 + 4\lambda^2)^{\frac{1}{2}}, \\ \upsilon_2 &= \tau(a^3 - 2a^2\lambda + (-6c^2 + 4\lambda^2)a + 4c^2\lambda - 8\lambda^3) - a + 2\lambda. \end{aligned} \tag{3.8}$$

Then by applying the one-fold DT and principle of the superposition of linear differential equation, the first-order breather wave solution of Eq. (1.1) can be derived

$$q^{[1]} = q - \frac{2i}{|F_1|^2 + |F_2|^2}(\lambda_1 - \lambda_1^*)F_1 F_2^*, \tag{3.9}$$

and $F_1 = f_1 - f_2^*$, $F_2 = f_1^* + f_2$. Without loss of generality, taking $a = -2\mathrm{Re}(\lambda_1) = -2\xi$, then substituting Eq. (3.7) into Eq. (3.9), the exact expression of breather wave solution is as follows

$$q^{[1]} = e^{i\rho}\left[c + \frac{2\eta\left\{[\kappa_1\cos(2S) - \kappa_2\cos(2W)] - i\left[\left(\kappa_1 - 2c^2\right)\sin(2S) - \kappa_3\sinh(2W)\right]\right\}}{\kappa_1\cosh(2W) - \kappa_2\cos(2S)}\right], \tag{3.10}$$

where

$$\begin{aligned} \kappa_1 &= c^2 + \eta^2 + \chi^2, \\ \kappa_2 &= 2c\eta, \\ \kappa_3 &= 2c\chi, \\ W &= (-48\xi^2\eta\tau + 8\eta^3\tau + 4c^2\eta\tau + 2\eta)\chi t, \\ S &= (\chi x + \chi(-32\tau\xi^3 + 16c^2\xi\tau + 32\eta^2\tau\xi + 4\xi)t, \end{aligned} \tag{3.11}$$

and

$$\chi = c^2 - \frac{1}{6}\frac{-24\xi^2\tau + \sqrt{480\tau^2\xi^4 - 24\tau\xi^2 + 6b\tau + 1} + 1}{\tau}. \tag{3.12}$$

The character of breather wave solution is illustrated by Fig. 3.

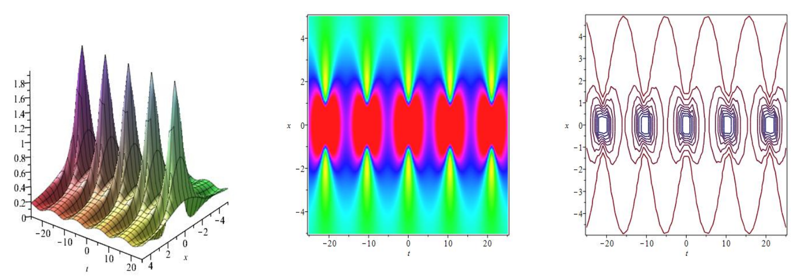

Figure 3. The first-order breather solution of higher-order dispersive nonlinear Schrödinger equation with $\eta = 0.2, \xi = 0, b = 1, c = 1$, and $\tau = 1$: a three-dimensional plot, b density plot, c the contour plot.

4. Rogue wave solution

In this section, the rogue wave solution of the higher-order dispersive nonlinear Schrödinger equation is researched. On the basis of [26], we know that when the period of breather wave Eq. (3.10) tends to infinity, the breather wave can translate to the rogue wave. According to [11], the first-order rogue wave of Eq. (1.1) is obtained through the coefficient of the Taylor expansion

$$q^{[1]} = \left(\frac{P_1 + iQ_1}{\gamma_1} - 1\right)ce^{i\rho}, \tag{4.1}$$

where

$$
\begin{aligned}
&P_1 = 4,\\
&Q_1 = 16(1+6c^2\tau - 6a^2\tau)c^2 t,\\
&\gamma_1 = 4c^2x^2 + (32a^3c^2\tau - 16ac^2(1+12\tau c^2))xt + (64c^2\tau^2a^6 - 64c^2\tau a^4(1+3\tau c^2)\\
&\qquad + 16c^2(1+12\tau c^2 + 72c^4\tau^2)a^2 + 16c^4(6\tau c^2+1)^2)t^2 + 1.
\end{aligned} \tag{4.2}
$$

The traits of rogue wave solution are shown by Fig. 4.

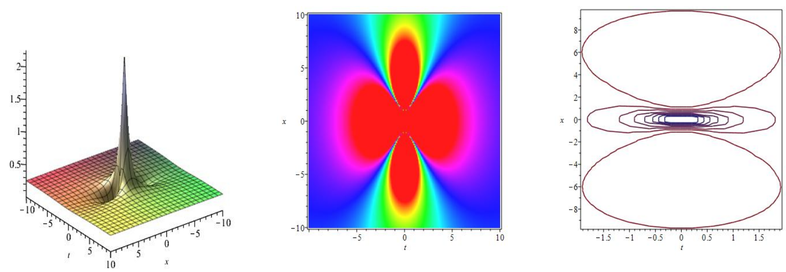

Figure 4. The first-order rogue wave solution of higher-order dispersive nonlinear Schrödinger equation with $\xi = 0, \eta = 0.5, a = 0, b = 1, c = 0.5$, and $\tau = 0.5$: a three-dimensional plot, b density plot, c the contour plot.

5. Conclusion and Discussion

In this paper, the high-order solitons, breather wave solution and rogue wave solution for the higher-order dispersive nonlinear Schrödinger equation by applying the Darboux transformation method, are obtained. Firstly, the soliton solutions by taking zero-seed solution are derived. Then the breather wave solution and rogue wave solution by taking the period seed solutions are obtained. These solutions can be used to explain some phenomena appear in fluid and plasma mechanics via 3-D and 2-D plots in detail. It can be known that the solutions of NLSE is rather important for the research and development of nonlinear phenomena. Nextly, we will dedicated to the other dynamic characters of NLSE.

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