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AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION MODELLING A PRODUCTION PLANNING PROBLEM

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Abstract Our purpose is to investigate the existence and uniqueness of positive solutions for an elliptic partial differential equation. The considered problem describes many real-world models and the obtained solutions can be useful in industry and manufacturing.

Keywords Elliptic equation, production problem, stochastic control.

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1. Introduction

The problem of establishing the existence, uniqueness, asymptotic behavior, determination or numerical approximation of positive solutions for the partial differential equation

$$\begin{cases} \Delta u = f\left(x,u\right) & \text{in} \quad \Omega, \quad f:\Omega\times\mathbb{R}\to\mathbb{R},\, \Omega\subseteq\mathbb{R}^N \ (N\geq 1), \\ u = g & \text{on} \quad \partial\Omega \quad g:\partial\Omega\to\mathbb{R}, \end{cases} \tag{1.1}$$

are in the attention of several of researchers.

In general, the theoretical methods for approaching the problem (1.1) under different classes of functions f and g are different from researcher to researcher and depends on the desired response to the applicative models; for more on this see Baalal-Berghout [2], Chang, Li, Yue, Lee, Chiang-Lin [7], He-Wu [14], Zhang [18], Zhang-Xu-Jiang-Wu-Cui [17] and references therein.

In our work we consider the problem

$$\Delta u\left(x\right) = \frac{1}{\sigma^4}a\left(x\right)u\left(x\right) + \frac{2\alpha}{\sigma^2}u\left(x\right)\ln u\left(x\right) \ \text{ for } x\in\Omega\subseteq\mathbb{R}^N(N\geq 1), \tag{1.2}$$

where σ and α are positive real numbers. We show that this problem is important from applications point of view and also interesting from the theoretical point of view.

For example, let us consider the production planning problem for the continuous-time case, with a factory producing N types of economic goods which stores them in an inventory designated place. In describing the model the following notations are used:

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- $\alpha > 0$ is the constant discount rate,
- $\sigma > 0$ the diffusion coefficient,
- $p(t) = (p_1(t), ..., p_N(t))$, with $p_i(t) \geq 0$ for all i and t, the production rate at time t (control variable) adjusted for the demand rate,
- $|\cdot|$ the Euclidean norm,
- $y(t) = (y_1(t), ..., y_N(t))$ the inventory level for production rate at time t (state variable) adjusted for demand,
- y_i^0 the initial inventory level of good i,
- $(\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq T \leq \infty}, \mathcal{F}, P)$ is a complete probability space on which lives a N-dimensional Brownian motion denoted by $w = (w_1, ..., w_N)$, with T the length of planning period generated by w.

Next, we define the cost functional

$$J(p_1, ..., p_N) := E \int_0^T (|p(t)|^2 + a(y(t)))e^{-\alpha t} dt, \tag{1.3}$$

and our theoretical problem is reduced to finding the value function

$$z(y_1^0, y_2^0, \ldots y_N^0) = \inf_{p \in \mathbb{R}^N} \{J(p_1, ..., p_N)\}, \tag{1.4}$$

subject to Itö stochastic differential equation

$$dy_i(t) = p_i dt + \sigma dw_i, \; y_i(0) = y_i^0, \; i = 1, ..., N. \tag{1.5}$$

The Hamilton-Jacobi-Bellman equation (HJB) associated with the problem (1.4)-(1.5) is

$$\alpha z - \frac{\sigma^2}{2} \Delta z - a(x) = \inf_{p \in \mathbb{R}^N} \{p \nabla z + |p|^2\}, \tag{1.6}$$

and it can be shown from an elementary computation originally due to [8, 9] that $z := z(x_1, ..., x_N)$ is a $C^2(\Omega)$ function satisfying

$$-2\sigma^2 \Delta z + |\nabla z|^2 + 4\alpha z = 4a(x) \text{ for } x \in \Omega, \tag{1.7}$$

where $x \in \mathbb{R}^N$ assumes values $(y_1(0), ..., y_N(0))$.

By using the change of variable $u(x) = e^{-\frac{z(x)}{2\sigma^2}}$, the problem (1.7) is reduced to the partial differential equation (1.2).

In the case $\alpha = 0$, a general and rigorous mathematical theory referring to the problem (1.2) can be found in many articles from the literature, since is the case when the problem become the well known stationary Schrödinger equation, see [6] for details.

When $\Omega = \mathbb{R}^N$ $(N = 1)$, $\alpha \neq 0$ and $a(x) = |x|^2$, with a long time ago Bensoussan, Sethi, Vickson and Derzko [4], observed that the equation (1.7) subject to the boundary condition

$$z(x) \to \infty \text{ as } |x| \to \infty, \tag{1.8}$$

has a unique classical convex solution.

Many works have been done after Bensoussan et all.'s results to the study of equation (1.7) and the corresponding system of equations, some of them are [5, 10–12]. Unfortunately, all these works was restricted to the case $N = 1$, $T = \infty$, $a(x) = x^2$ and the boundary condition (1.8) which simply means that the stochastic production planning problem is considered for one economical good in an infinite horizon case with a quadratic loss function.

Among all of the results, the extension to several economic goods was recently given in the paper [9], where the problem is treated both from theoretical and practical point of view in the entire Euclidean space $\mathbb{R}^N$.

Regarding our mathematical contribution about (1.2), an existence and uniqueness result is obtained and presented for the case when $\Omega = B_R$ is the ball of radius $R > 0$ centered at the origin (zero element) and a closed form solution for the case

$$\Omega = \mathbb{R}^N \text{ and } a(x) = |x|^2,$$

is given. Then, this work is a follow-up of the papers by [4], [8] and [9] where the authors have investigates the existence and uniqueness of solutions in $\mathbb{R}^N$. A relevant work in our direction is the paper of Barles and Murat [3].

We are ready to state our first result.

Theorem 1.1. *Suppose $\Omega = B_R$ and $a : \overline{B}_R \to [0, \infty)$ is a continuous function satisfying*

$$a(x) \le K\left(|x|^2 + 1\right) \text{ such that } K > 0. \tag{1.9}$$

Then, the problem (1.2) subject to the Dirichlet boundary condition

$$u(x) = 1, \text{ for } x \in \partial B_R, \tag{1.10}$$

has a unique solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ with $0 < u(x) \le 1$ for any $x \in \overline{B}_R$.

Theorem 1.1 can also be very useful in many economic models. Indeed, let $T < \infty$ be the stopping time representing the moment when the inventory level reaches some threshold R, i.e.,

$$T = \inf_{t>0}\{|y(t)| \ge R\}.$$

From the applications point of view, the objective is to minimize this cost functional (1.3) subject to the Itö stochastic differential equation (1.5) in order to obtain the problem (1.2) subject to the Dirichlet boundary condition (1.10).

Our next result refer to the entire Euclidean space $\mathbb{R}^N$ and relies on an elementary computation.

Theorem 1.2. *Assume $\Omega = \mathbb{R}^N$ and $a(x) = |x|^2$. The partial differential equation (1.2) with boundary condition*

$$u(x) \to 0 \text{ as } |x| \to \infty, \tag{1.11}$$

has a unique positive classical convex solution with quadratic growth.

To conclude, let me mention that a different abbordation to the proof of Theorem 1.2 has been developed in Cadenillas, Lakner and Pinedo [5], again for the case $N = 1$.

The rest of the paper is structured as follows. In Section 2 we will prove Theorem 1.1. The main tool we use here is the classical sub-supersolution method. The difficulty that arises is the construction of a sub-supersolution with order. Section 3 is dedicated to the proof of Theorem 1.2. Here, we exploit our intuition.

2. The Proof of Theorem 1.1

The proof is divided into two parts: uniqueness and existence. We begin with

Uniqueness. If one solution exists we show that it is unique. We proceed in a similar way to the paper [16]. Suppose that u_1 and u_2 are classical solutions of (1.2) subject to the Dirichlet boundary condition (1.10) with $0 < u_1(x) \leq 1$ and $0 < u_2(x) \leq 1$ for any $x \in \overline{B}_R$. Let us show that $u_1(x) \leq u_2(x)$ for any $x \in \overline{B}_R$. Assume on the contrary that there exists $x_0 \in \overline{B}_R$ such that $u_1(x_0) > u_2(x_0)$.

Now, we set

$$w(x) := \frac{u_1(x)}{u_2(x)} - 1.$$

Note that from (1.10), we have

$$w(x) = 0 \text{ for } x \in \partial B_R,$$

which implies in fact that

$$\max_{\overline{B}_R} w(x),$$

exists and is positive. At that point, say $x_1 \in B_R$, we have

$$\nabla w(x_1) = 0 \text{ and } \Delta w(x_1) \leq 0.$$

It is clear from the definition of $w(x)$ that

$$\begin{aligned} -\operatorname{div}\left(u_2^2(x_1)\nabla w(x_1)\right) &= -\operatorname{div}\left(u_2^2(x_1)\right)\nabla w(x_1) - u_2^2(x_1)\Delta w(x_1) \\ &= -u_2^2(x_1)\Delta w(x_1) \geq 0. \end{aligned}$$

A straightforward computation shows that

$$-\operatorname{div}\left(u_2^2(x_1)\nabla w(x_1)\right) = -u_2(x_1)\Delta u_1(x_1) + u_1(x_1)\Delta u_2(x_1).$$

As a consequence, we have

$$-u_2(x_1)\Delta u_1(x_1) + u_1(x_1)\Delta u_2(x_1) \geq 0,$$

or, equivalently

$$\frac{\Delta u_1(x_1)}{u_1(x_1)} - \frac{\Delta u_2(x_1)}{u_2(x_1)} \leq 0.$$

The above relation produces

$$0 \geq \frac{1}{\sigma^4}a(x) + \frac{2\alpha}{\sigma^2}\ln u_1(x_1) - \frac{1}{\sigma^4}a(x) - \frac{2\alpha}{\sigma^2}\ln u_2(x_1) = \frac{2\alpha}{\sigma^2}\ln\frac{u_1(x_1)}{u_2(x_1)},$$

which is a contradiction, since

$$\frac{u_1(x_1)}{u_2(x_1)} - 1 > 0 \Longrightarrow \frac{2\alpha}{|\sigma|^2}\ln\frac{u_1(x_1)}{u_2(x_1)} > 0.$$

A similar argument can be made to produce $u_2(x) \leq u_1(x)$ for any $x \in \overline{B}_R$. Therefore, (1.2) has a unique such solution if it exists. We next establish

Existence. We apply the sub- and supersolution method due to Sattinger [15]. In the following we construct the functions $\underline{u}$, $\overline{u} \in C^2(\overline{B}_R)$ such that

$$\begin{cases} \Delta \underline{u} \geq \frac{1}{\sigma^4} a(x) \underline{u}(x) + \frac{2\alpha}{\sigma^2} \underline{u}(x) \ln \underline{u}(x) \ \text{ for } x \in B_R, \\ 0 < \underline{u}(x) \leq 1 \text{ for } x \in B_R, \\ \underline{u}(x) = 1 \text{ for } x \in \partial B_R, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Delta \overline{u} \leq \frac{1}{\sigma^4} a(x) \overline{u}(x) + \frac{2\alpha}{\sigma^2} \overline{u}(x) \ln \overline{u}(x) \ \text{ for } x \in B_R, \\ 0 < \overline{u}(x) \leq 1 \text{ for } x \in B_R, \\ \overline{u}(x) = 1 \text{ for } x \in \partial B_R. \end{cases} \tag{2.2}$$

We point that the function $\underline{u}$ (resp. $\overline{u}$) is called a subsolution (resp. supersolution) for the problem (1.2) subject to the Dirichlet boundary condition (1.10). A simple calculation shows that $\overline{u}(x) = 1$ is a supersolution for the problem (1.2) and that

$$\underline{u}(x) = e^{-\frac{1}{4\sigma^2}\left(\alpha + \sqrt{\alpha^2 + 4K}\right)\left(R^2 - |x|^2\right)},$$

is a subsolution for (1.2). As a consequence of the above construction, one obtains

$$\underline{u}(x) \leq \overline{u}(x) \ \text{ for } x \in \overline{B}_R.$$

We are showing that, the problem (1.2) admits a unique solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ such that

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \ \text{ for } x \in \overline{B}_R.$$

Denote

$$M_1 = e^{-\frac{1}{4\sigma^2}\left(\alpha + \sqrt{\alpha^2 + 4K}\right)R^2} \text{ and } M_2 = 1,$$

and let $g : \overline{B}_R \times [M_1, M_2] \to \mathbb{R}$ defined by

$$g(x,t) = \frac{1}{\sigma^4} a(x) t + \frac{2\alpha}{\sigma^2} t \ln t.$$

Since g is a continuous function with respect to the first variable in $\overline{B}_R$ and continuously differentiable with respect to the second in $[M_1, M_2]$, it allows to choose $\Lambda < 0$ such that

$$-\Lambda \geq \frac{g(x,s) - g(x,t)}{s - t},$$

for every t, s with $\underline{u} \leq t < s \leq \overline{u}$ and $x \in B_R$. Starting with $u_0 = \underline{u}$ we inductively define a sequence $\{u_k\}_{k \in \mathbb{N}}$ such that

$$\begin{cases} \Delta u_k + \Lambda u_k = g(x, u_{k-1}) + \Lambda u_{k-1} & \text{for } x \in B_R, \\ u_k(x) = \overline{u}(x) & \text{for } x \in \partial B_R. \end{cases}$$

Next, assuming that $u_{k-1} \leq u_k$ on $\overline{B}_R$ we prove that $u_k \leq u_{k+1}$ on $\overline{B}_R$. The constant Λ was chosen so that

$$(\Delta + \Lambda)(u_{k+1}(x) - u_k(x)) \leq 0 \text{ in } B_R, \tag{2.3}$$

if $u_{k-1} \leq u_k$ on B_R for $k = 1, 2, ...$, which is true for $k = 1$ and thus for every larger k by (2.3) and the maximum principle (see [13]).

Finally, by induction we get a monotone increasing sequence $\{u_k\}_{k\geq 1}$ of iterates

$$\underline{u} \leq u_1 \leq u_2 \leq ... \leq u_{k-1} \leq u_k \leq u_{k+1} \leq ... \leq \overline{u} \text{ on } \overline{B}_R.$$

Therefore the sequences $\{u_k\}_{k\in\mathbb{N}}$ converge (since they are monotone and bounded by some constants independent of k). It is perfectly clear that the limit

$$\lim_{k\to\infty} u_k(x) = u(x), \text{ for all } x \in \overline{B}_R,$$

exists as a continuous function. A standard bootstrap argument, implies that $u_k(x) \to u(x)$ in $C^2(B_R) \cap C(\overline{B}_R)$ and that u is a solution of problem (1.2) satisfying $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for $x \in \overline{B}_R$. Thanks to the uniqueness the **Theorem 1.1** is proved. We now present

3. The proof of Theorem 1.2

In the following we construct the function u which satisfies (1.2) with boundary condition (1.11). More exactly, we observe that there exist

$$u(x) = e^{B|x|^2+D} \text{ with } B, D \in (-\infty, 0),$$

that solve (1.2).

This is reduced to find B, $D \in (-\infty, 0)$ such that

$$2B\left(2|x|^2 B + 1\right) + 2B(N-1) = \frac{1}{\sigma^4}|x|^2 + \frac{2\alpha}{\sigma^2}\left(B|x|^2 + D\right),$$

or, after rearranging the terms

$$|x|^2\left[4B^2 - \frac{1}{\sigma^4} - \frac{2\alpha B}{\sigma^2}\right] + 2BN - \frac{2\alpha D}{\sigma^2} = 0.$$

Now, we observe that the system of equations

$$\begin{cases} 4B^2 - \frac{1}{\sigma^4} - \frac{2\alpha B}{\sigma^2} = 0, \\ 2BN - \frac{2\alpha D}{\sigma^2} = 0, \end{cases}$$

has the solution

$$B = \frac{1}{4\sigma^2}\left(\alpha - \sqrt{\alpha^2+4}\right), \ D = \frac{1}{4}\frac{N}{\alpha}\left(\alpha - \sqrt{\alpha^2+4}\right). \tag{3.1}$$

Then $u(x) = e^{B|x|^2+D}$, with B, D defined by (3.1), is a solution for (1.2) with boundary condition (1.11).

It remains now to establish the uniqueness of the solution to (1.2) with boundary condition (1.11). Our argument relies on a simple maximum principle used in [1, p. 118] (see [13] for details). Suppose that v is another solution of the problem (1.2) with boundary condition (1.11). Let us show that $u(x) \leq v(x)$ for any $x \in \mathbb{R}^N$. Assume the contrary, there exists $x_0 \in \mathbb{R}^N$ such that $u(x_0) > v(x_0)$. Since

$$\lim_{|x|\to\infty} (u(x) - v(x)) = 0,$$

we deduce that

$$\max_{\mathbb{R}^N} \left(u\left(x\right) - v\left(x\right)\right),$$

exists and is positive. At that point, say x_1, we have

$$\begin{aligned}
0 &\geq \Delta u\left(x_1\right) - \Delta v\left(x_1\right) \\
&= \left(4\left|x_1\right|^2 B^2 + 2BN\right) u\left(x_1\right) - \frac{1}{\sigma^4}\left|x_1\right|^2 v\left(x_1\right) - \frac{2\alpha}{\sigma^2} v\left(x_1\right) \ln v\left(x_1\right) \\
&> (4B^2\left|x_1\right|^2 + 2BN - \frac{\left|x_1\right|^2}{\sigma^4}) v\left(x_1\right) - \frac{2\alpha}{\sigma^2} v\left(x_1\right) \ln v\left(x_1\right) \\
&= (2BN + \frac{2\alpha B\left|x_1\right|^2}{\sigma^2}) v\left(x_1\right) - \frac{2\alpha}{\sigma^2} v\left(x_1\right) \ln v\left(x_1\right) \\
&= [(\frac{2\alpha D}{\sigma^2} + \frac{2\alpha B\left|x_1\right|^2}{\sigma^2}) - \frac{2\alpha}{\sigma^2} \ln v\left(x_1\right)] v\left(x_1\right) \\
&= \left[\left(D + B\left|x_1\right|^2\right) - \ln v\left(x_1\right)\right] \frac{2\alpha}{\sigma^2} v\left(x_1\right) = \left(\ln u\left(x_1\right) - \ln v\left(x_1\right)\right) \frac{2\alpha}{\sigma^2} v\left(x_1\right) > 0,
\end{aligned}$$

which is a contradiction. So, $u\left(x\right) \leq v\left(x\right)$ for any $x \in \mathbb{R}^N$. A similar argument can be made to produce $v\left(x\right) \leq u\left(x\right)$ for any $x \in \mathbb{R}^N$.

Setting

$$B = \frac{1}{4\sigma^2}\left(\alpha - \sqrt{\alpha^2 + 4}\right),\ D = \frac{1}{4}\frac{N}{\alpha}\left(\alpha - \sqrt{\alpha^2 + 4}\right),$$

clearly $z\left(x\right) = -2\sigma^2\left(B\left|x\right|^2 + D\right)$, is the unique solution of (1.7) subject to the boundary condition (1.2). This finishes the proof of **Theorem 1.2**.

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