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FRACTIONAL BOUNDARY VALUE PROBLEM WITH NABLA DIFFERENCE EQUATION*

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Abstract In the paper, using Z-transform technique, we study eigenvalues and eigenfunctions for a fractional boundary value problem with linear nabla difference equation. Furthermore, by topological degree theory and the obtained results of eigenvalues, we get at least one nontrivial solution for relevant nonlinear fractional boundary value problem.

Keywords Fractional difference equation, existence, eigenvalue.

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1. Introduction

In recent years, the theory of fractional ordinary differential equations (FODE) has been received many attentions due to its wide applications. Many results have been obtained about this class of equations (see [1–8, 10, 12, 14–17]). For instance, in 2011, Abdeljawad [1] studied the Caputo fractional difference equation. And Abdeljawad and Baleanu [2] introduced the fractional differences and integration by parts. In [5], the authors considered a two-point BVP for fractional difference equation. In 2014, Wu and Baleanu [16] gave some applications for the Caputo fractional difference to chaotic maps. Cheng [7] introduced the theory of fractional nabla difference equations and gave the definition of fractional sum, fractional nabla difference and basic calculus theory.

Following the trend, we consider the following boundary value problem with linear fractional difference equation

$$\begin{cases} {}_{0}^{c}\nabla_{n}^{\alpha}u(n) + \lambda u(n) = 0, 0 \le n \le N, \\ \gamma_1 u(-1) = \delta_1 u(N), \\ \gamma_2 \nabla u(-1) = \delta_2 \nabla u(N), \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, ${}_{0}^{c}\nabla_{n}^{\alpha}$ is the Caputo fractional difference, $\nabla u(n) := u(n) - u(n-1)$. Moreover we consider the following boundary value problem with nonlinear

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fractional difference equation

$$\begin{cases} {}_0^c\nabla_n^\alpha u(n) + f(n,\, u(n)) = 0, 0 \le n \le N, \\ \gamma_1 u(-1) = \delta_1 u(N), \\ \gamma_2 \nabla u(-1) = \delta_2 \nabla u(N). \end{cases} \tag{1.2}$$

By using Z-Transformation, we get eigenvalues and eigenfunctions of (1.1). By applying the topological degree theory combining the eigenvalue theory, we obtain the existence of nontrival solution of (1.2).

The plan of the article is as follows: in section 2, we provide basic definitions and some useful Lemmas; in section 3, we prove the existence of solution by using topological degree theory and an example is also given.

2. Preliminaries

In this segment, we recall some essential definitions and fractional difference calculus. For more details, we refer to the literature [7].

Definition 2.1. For any $x \in R$, $n \in N^+$, one define

$$\begin{bmatrix} x \\ n \end{bmatrix} \triangleq \frac{x(x+1)\cdots(x+n-1)}{n!}.$$

Definition 2.2. The νth fractional sum of a function f is

$$\nabla^{-\nu} f(n) = \begin{bmatrix} \nu \\ n \end{bmatrix} * f(n) = \sum_{r=0}^{n} \begin{bmatrix} \nu \\ n-r \end{bmatrix} f(r),$$

where $\nu > 0$.

Definition 2.3. Assume that $0 \le m-1 \le \nu < m$, the νth Caputo fractional difference for $\nu > 0$ defined by

$${}_a^c\nabla_n^\nu f(n) \triangleq_a \nabla_n^{-m+\nu}[\nabla^m f(n)],$$

where ${}_a\nabla_n^{-\alpha} f(n) = \sum\limits_{r=a}^{n} \begin{bmatrix} \alpha \\ n-r \end{bmatrix} f(r)$ for $\alpha > 0$.

Lemma 2.1. *Assume $m-1 \le \alpha < m$, then the following relation holds*

$$Z[{}_0^c\nabla_n^\alpha f(n)] = (\frac{z-1}{z})^\alpha F(z) - \sum_{k=0}^{m-1} (\frac{z-1}{z})^{\alpha-k-1} \nabla^k f(-1),$$

where Z-Transform defined by $Z[f(n-k)] = \sum\limits_{n=0}^{\infty} f(n-k) z^{-n}$.

Definition 2.4. The discrete Mittag-Leffer type function is defined by

$$F_{\alpha,\,\beta}(\lambda,\,n)=\sum_{k=0}^{\infty}\lambda^{k}\begin{bmatrix}\alpha k+\beta\\ n\end{bmatrix},\quad (|\lambda|<1),$$

where $\alpha,\ \beta\in R^{+},\ \lambda\in C$.

Lemma 2.2. *The Z-Transform of function* $F_{\alpha,\,\beta}(\lambda,\,n)$ *is*

$$Z[F_{\alpha,\,\beta}(\lambda,\,n)]=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda},\ (|\lambda|<|s|^{\alpha}),$$

where $s=\frac{z-1}{z}$.

Lemma 2.3. *Assume* $\alpha>0$, *then*

$${}_{a}\nabla_{n}^{-\alpha}[{}_{a}^{c}\nabla_{n}^{\alpha}f(n)]=f(n)-\sum_{k=0}^{[\alpha]}\nabla^{k}f(a-1)\begin{bmatrix}k+1\\ n-\alpha\end{bmatrix}.$$

3. Main Results

In the paper, we assume that the following conditions hold.
(H1) $\delta_1>\gamma_1>0,\ \gamma_2>\delta_2>0$.
(H2) $f\in C([-1,N]\times R,\ R^{+})$.

Theorem 3.1. *The eigenfunction of* (1.1) *is*

$$u(n)=[F_{\alpha,\,1}(-\lambda,\,n)+\frac{F_{\alpha,\,2}(-\lambda,\,n)(\gamma_1-\delta_1F_{\alpha,1}(-\lambda,N))}{\delta_1F_{\alpha,2}(-\lambda,N)}]C,$$

(C is a constant) and the corresponding eigenvalue λ *is the solution of equation*

$$\Big(1-\frac{\delta_2}{\gamma_2}F_{\alpha,1}(-\lambda,N)\Big)\Big(\gamma_1-\delta_1F_{\alpha,1}(-\lambda,N)\Big)-\frac{\delta_1\delta_2}{\gamma_2}F_{\alpha,0}(-\lambda,N)F_{\alpha,2}(-\lambda,N)=0.$$

Proof. For the equation of (1.1), we take Z-Transformation and get

$$\begin{aligned}&Z({}_{0}^{c}\nabla_{n}^{\alpha}u(n))+\lambda Z(u(n))\\ =&(\frac{z-1}{z})^{\alpha}F(z)-(\frac{z-1}{z})^{\alpha-1}u(-1)-(\frac{z-1}{z})^{\alpha-2}\nabla u(-1)+\lambda F(z)=0,\end{aligned}$$

which together with Lemma 2.2 yields

$$\begin{aligned}F(z)=&\frac{(\frac{z-1}{z})^{\alpha-1}}{(\frac{z-1}{z})^{\alpha}+\lambda}u(-1)+\frac{(\frac{z-1}{z})^{\alpha-2}}{(\frac{z-1}{z})^{\alpha}+\lambda}\nabla u(-1)\\ =&Z[F_{\alpha,1}(-\lambda,n)]u(-1)+Z[F_{\alpha,2}(-\lambda,n)]\nabla u(-1).\end{aligned}$$

That is

$$u(n)=F_{\alpha,\,1}(-\lambda,\,n)u(-1)+F_{\alpha,\,2}(-\lambda,\,n)\nabla u(-1).\tag{3.1}$$

By (3.1) one has $u(N)=F_{\alpha,1}(-\lambda,\ N)u(-1)+F_{\alpha,2}(-\lambda,\ N)\nabla u(-1)$, which together with the boundary condition $\gamma_1 u(-1)=\delta_1 u(N)$, we obtain

$$\nabla u(-1)=\frac{\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,\ N)}{\delta_1 F_{\alpha,2}(-\lambda,N)}u(-1). \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$u(n)=\Big(F_{\alpha,1}(-\lambda,\ n)+\frac{F_{\alpha,2}(-\lambda,n)(\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,\ N))}{\delta_1 F_{\alpha,2}(-\lambda,N)}\Big)u(-1).$$

$$\nabla u(n)=\Big(\nabla F_{\alpha,1}(-\lambda,\ n)+\nabla F_{\alpha,2}(-\lambda,n)\frac{\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,\ N)}{\delta_1 F_{\alpha,2}(-\lambda,N)}\Big)u(-1).$$

Further by using $\gamma_2\nabla u(-1)=\delta_2\nabla u(N)$, we get

$$\begin{aligned}&\gamma_2\cdot\frac{\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,N)}{\delta_1 F_{\alpha,2}(-\lambda,N)}u(-1)\\ =&\delta_2\Big(F_{\alpha,0}(-\lambda,N)+F_{\alpha,1}(-\lambda,N)\frac{\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,N)}{\delta_1 F_{\alpha,2}(-\lambda,N)}\Big)u(-1).\end{aligned}$$

So

$$\Big(1-\frac{\delta_2}{\gamma_2}F_{\alpha,1}(-\lambda,N)\Big)\Big(\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,N)\Big)-\frac{\delta_1\delta_2}{\gamma_2}F_{\alpha,0}(-\lambda,N)F_{\alpha,2}(-\lambda,N)=0.$$

That is, the eigenvalues of (1.1) are the solution of the above equation. And the corresponding eigenfunction is

$$u(n)=[F_{\alpha,1}(-\lambda,\ n)+\frac{F_{\alpha,2}(-\lambda,\ n)(\gamma_1-\delta_1 F_{\alpha,1}(-\lambda,N))}{\delta_1 F_{\alpha,2}(-\lambda,N)}]C,$$

where C is a constant. □

Remark 3.1. In [13], the authors investigate the eigenvalues and eigenfunctions of fractional differential equation as follows

$$\begin{cases}{}_0^cD_t^\alpha u(t)+\lambda u(t)=0,\ 0<t<1,\\ u(0)=au(1),u'(0)=bu'(1),\end{cases}$$

where $1<\alpha<2, a\neq 0,\ b\neq 0,\ {}_0^cD_t^\alpha$ is the Caputo fractional derivative. The Theorem 3.1 is a discrete form of the corresponding results in reference [13].

Using Lemma 2.3, we can get the following theorem.

Theorem 3.2. *Let (H1) hold,* $1<\alpha<2,\ v\in C[-1,N]$. *The solution of*

$$\begin{cases}{}_0^c\nabla_n^\alpha u(n)+v(n)=0,\ 0\le n\le N,\\ \gamma_1 u(-1)=\delta_1 u(N),\\ \gamma_2\nabla u(-1)=\delta_2\nabla u(N),\end{cases} \tag{3.3}$$

is given by $u(n)=\sum_{r=0}^{N}G(n,r)v(r)$, *where*

$$G(n,r)=\begin{cases}\left(\frac{(\gamma_1+\delta_1N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)}+\frac{\delta_2 n}{\delta_2-\gamma_2}\right)\begin{bmatrix}\alpha-1\\ N-r\end{bmatrix}+\frac{\delta_1}{\delta_1-\gamma_1}\begin{bmatrix}\alpha\\ N-r\end{bmatrix}-\begin{bmatrix}\alpha\\ n-r\end{bmatrix}, & \\ \qquad\qquad 0\le r\le n, & \\ \left(\frac{(\gamma_1+\delta_1N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)}+\frac{\delta_2 n}{\delta_2-\gamma_2}\right)\begin{bmatrix}\alpha-1\\ N-r\end{bmatrix}+\frac{\delta_1}{\delta_1-\gamma_1}\begin{bmatrix}\alpha\\ N-r\end{bmatrix}, n\le r\le N, & \end{cases}$$

and $G(n,r)>0$.

Proof. Using Lemma 2.3 with $a=0$, we can get

$$u(n)=c_1+c_2n-\nabla_n^{-\alpha}v(n).$$

Since $u(-1)=c_1-c_2,\ u(N)=c_1+c_2N-\nabla^{-\alpha}f(N),\ \gamma_1u(-1)=\delta_1u(N)$, we know

$$(\gamma_1-\delta_1)c_1-(\gamma_1+\delta_1N)c_2=-\delta_1\nabla^{-\alpha}f(N).$$

By $\nabla u(n)=c_2-\nabla^{1-\alpha}f(n)$ and $\gamma_2\nabla u(-1)=\delta_2\nabla u(N)$, we obtain $c_2=\frac{\delta_2\nabla^{-(\alpha-1)}f(N)}{\delta_2-\gamma_2}$. So

$$c_1=\frac{1}{\delta_1-\gamma_1}\Big((\gamma_1+\delta_1N)\frac{\delta_2\nabla^{-(\alpha-1)}f(N)}{\gamma_2-\delta_2}+\delta_1\nabla^{-\alpha}f(N)\Big).$$

That is

$$\begin{aligned}y(n)=&(\frac{(\gamma_1+\delta_1N)\delta_2}{(\delta_1-\gamma_1)(\delta_2-\gamma_2)}+\frac{\delta_2 n}{\delta_2-\gamma_2})\nabla^{-(\alpha-1)}f(N)\\ &+\frac{\delta_1}{\delta_1-\gamma_1}\nabla^{-\alpha}f(N)-\nabla^{-\alpha}f(n).\end{aligned}\tag{3.4}$$

So we know the Green's function $G(n,r)$ can be written as above. From (H1), we obtain the following conclusions.

(i) $\frac{\gamma_1+\delta_1N}{\delta_1-\gamma_1}-n>0$;

(ii) $\frac{\delta_1}{\delta_1-\gamma_1}\begin{bmatrix}\alpha\\ N-r\end{bmatrix}>\begin{bmatrix}\alpha\\ N-r\end{bmatrix}>\begin{bmatrix}\alpha\\ n-r\end{bmatrix}$.

So $G(n,r)>0$. □

Let $E=\{y|y:[-1,\,N]\to R,\gamma_1y(-1)=\delta_1y(N),\ \gamma_2\nabla y(-1)=\delta_2\nabla y(N)\}$. It is clear that E is a Banach space with the norm $||y||=\max\limits_{n\in[-1,\,N]}|y(n)|$. Now we define the operator $(Ty)(n)=\sum_{r=0}^{N}G(n,r)y(r),\ (\Phi y)(n)=\sum_{r=0}^{N}G(n,r)f(r,y(r))$. It is easy to see that $u=u(n)$ is a solution of problem (1.2) if and only if $u=u(n)$ is a fixed point of Φ.

Lemma 3.1 ([9]). *Let Ω be a bounded open set in infinite dimensional real Banach space E, $\theta \notin \partial\Omega$ and $A : \bar{\Omega} \to E$ be completely continuous. Suppose that $||Ax|| \geq ||x||$, $Ax \neq x$, $\forall x \in \partial\Omega$. Then $\deg(I - A, \Omega, \theta) = 0$.*

Lemma 3.2 ([11]). *Let A be a completely continuous operator which is defined on a Banach space E. Assume that 1 is not an eigenvalue of the asymptotic derivative. The completely continuous vector field $I - A$ is then nonsingular on spheres $S_\rho = \{x|||x|| = \rho\}$ of sufficiently large radius ρ and $\deg(I - A, B(\theta, \rho), \theta) = (-1)^k$, where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\infty)$ in $(1, \infty)$.*

Theorem 3.3. *Assume (H1) and (H2) hold. Moreover we assume that*
(H3) there exists a constant c such that $|f(n, u(n))| > \frac{c}{A}$ for $|u| < c$, where

$$A = \sum_{r=0}^{N} \Big(\frac{(\gamma_1 + \delta_1 N)\delta_2}{(\delta_1 - \gamma_1)(\gamma_2 - \delta_2)} + \frac{\delta_2 N}{\delta_2 - \gamma_2} \Big) \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix};$$

(H4) $\beta_\infty := \lim\limits_{x\to\infty} |\frac{f(x)}{x}|$ is not the eigenvalues of (1.1).
Then (1.2) has at least one nontrivial solution.

Proof. We first prove $\Phi : E \to E$ is completely continuous.

By (H2), it is easy to know $\Phi : E \to E$ is continuous. Let $O \subset B(\theta, M) \subset E$, for $u \in O$, we have $||u|| \leq M$. Using the continuous of f, there exists M' such that $|f(n, u(n))| < M'$. So

$$\begin{aligned} |(\Phi u)(n)| =& \Big| \frac{1}{\delta_1 - \gamma_1} \Big(\sum_{r=0}^{N} \frac{(\gamma_1 + \delta_1 N)\delta_2}{\gamma_2 - \delta_2} \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix} f(r, u(r)) + \delta_1 \begin{bmatrix} \alpha \\ N - r \end{bmatrix} f(r, u(r)) \Big) \\ &+ \frac{\delta_2}{\delta_2 - \gamma_2} \sum_{r=0}^{N} \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix} f(r, u(r))n - \sum_{r=0}^{n} \begin{bmatrix} \alpha \\ N - r \end{bmatrix} f(r, u(r)) \Big| \\ \leq& \frac{1}{\delta_1 - \gamma_1} \sum_{r=0}^{N} \Big(\frac{(\gamma_1 + \delta_1 N)\delta_2}{\gamma_2 - \delta_2} \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix} |f(r, u(r))| + \delta_1 \begin{bmatrix} \alpha \\ N - r \end{bmatrix} |f(r, u(r))| \Big) \\ \leq& \frac{M'}{\delta_1 - \gamma_1} \sum_{r=0}^{N} \Big(\frac{(\gamma_1 + \delta_1 N)\delta_2}{\gamma_2 - \delta_2} \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix} + \delta_1 \begin{bmatrix} \alpha \\ N - r \end{bmatrix} \Big). \end{aligned}$$

This proves Φ is uniformly bounded on the space E. It is easy to know Φ is equicontinuous. So Φ is completely continuous.

Similar to the proof of Lemma 3.3 of [13], we can obtain Φ is Fréchet differentiable at ∞, and $\Phi'(\infty) = \beta_\infty T$.

Choose $B(\theta, c)$, from the above discussion, we know $\Phi : \overline{B(\theta, c)} \to E$ is completely continuous. For $||u|| = c$ and $n \in [0, N]$

$$|(\Phi u)(n)| = \Big| \sum_{r=0}^{n} \Big(\big(\frac{(\gamma_1 + \delta_1 N)\delta_2}{(\delta_1 - \gamma_1)(\gamma_2 - \delta_2)} + \frac{\delta_2 n}{\delta_2 - \gamma_2} \big) \begin{bmatrix} \alpha - 1 \\ N - r \end{bmatrix} + \frac{\delta_1}{\delta_1 - \gamma_1} \begin{bmatrix} \alpha \\ N - r \end{bmatrix}$$

$$- \begin{bmatrix} \alpha \\ N-r \end{bmatrix} \Big) f(r,u(r)) + \sum_{r=n}^{N} \Big(\Big(\frac{(\gamma_1+\delta_1 N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)} + \frac{\delta_2 n}{\delta_2-\gamma_2} \Big) \begin{bmatrix} \alpha-1 \\ N-r \end{bmatrix}$$

$$+ \frac{\delta_1}{\delta_1-\gamma_1} \begin{bmatrix} \alpha \\ N-r \end{bmatrix} \Big) f(r,u(r)) \Big|$$

$$> \Big| \sum_{r=0}^{N} \Big(\Big(\frac{(\gamma_1+\delta_1 N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)} + \frac{\delta_2 n}{\delta_2-\gamma_2} \Big) \begin{bmatrix} \alpha-1 \\ N-r \end{bmatrix} + \frac{\delta_1}{\delta_1-\gamma_1} \begin{bmatrix} \alpha \\ N-r \end{bmatrix}$$

$$- \begin{bmatrix} \alpha \\ N-r \end{bmatrix} \Big) f(r,u(r)) \Big|$$

$$> \sum_{r=0}^{N} \Big(\frac{(\gamma_1+\delta_1 N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)} + \frac{\delta_2 N}{\delta_2-\gamma_2} \Big) \begin{bmatrix} \alpha-1 \\ N-r \end{bmatrix} f(r,u(r)) > c = ||u||.$$

That is $||\Phi u|| > ||u||$ for $u \in \partial(B(\theta, c))$. By Lemma 3.1,

$$\deg(I - \Phi, B(\theta, c), \theta) = 0. \tag{3.5}$$

By Lemma 3.2, we get $\frac{\beta_\infty}{\lambda} \neq 1$. So

$$\deg(I - \Phi, B(\theta, \rho), \theta) = (-1)^k, \ k \geq 1. \tag{3.6}$$

Using (3.5) and (3.6), we have $\deg(I - \Phi, B(\theta, \rho) \backslash B(\theta, c), \theta) = (-1)^k$.

So there exists at least one $u \in B(\theta, \rho) \backslash B(\theta, c)$ such that it is a fixed point of Φ. The proof is complete. □

Example 3.1. Consider

$$\begin{cases} {}_0^c\nabla_n^{\frac{5}{3}} u(n) + f(n, u(n)) = 0, \ 0 \leq n \leq N, \\ 2u(-1) = 3u(N), \\ 5\nabla u(-1) = 4\nabla u(N), \end{cases} \tag{3.7}$$

where

$$f(n, u(n)) = \begin{cases} -2u(n), u(n) \leq -\frac{1}{A}, \\ \frac{2}{A}, -\frac{1}{A} \leq u(n) \leq \frac{1}{A}, \\ 2u(n), u(n) \geq \frac{1}{A}, \end{cases}$$

$$A = \sum_{r=0}^{N} \Big(\frac{(\gamma_1+\delta_1 N)\delta_2}{(\delta_1-\gamma_1)(\gamma_2-\delta_2)} + \frac{\delta_2 N}{\delta_2-\gamma_2} \Big) \begin{bmatrix} \alpha-1 \\ N-r \end{bmatrix}.$$

It is easy to know that f satisfy:

(i) $f(n, u(n)) \in C([-1, N] \times R, R^+)$;

(ii) $\beta_\infty = 2$, and

$$2 \notin \{\lambda | \Big(1 - \frac{\delta_2}{\gamma_2} F_{\alpha,1}(-\lambda, N)\Big)\Big(\gamma_1 - \delta_1 F_{\alpha,1}(-\lambda, N)\Big) - \frac{\delta_1 \delta_2}{\gamma_2} F_{\alpha,0}(-\lambda, N) F_{\alpha,2}(-\lambda, N) = 0\};$$

(iii) Choose $c = \frac{3}{2}$ such that $|f(n, u(n))| > \frac{c}{A}$. In fact, for $-\frac{1}{A} \leq u(n) \leq \frac{1}{A}$, we have $|f(n, u(n))| = \frac{2}{A} > \frac{c}{A}$ and for $|u(n)| \geq \frac{1}{A}$, $|f(n, u(n))| = |2u(n)| \geq \frac{2}{A} \geq \frac{c}{A}$.
By Theorem 3.3, we get the equation (3.7) has at least one nontrival solution.

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