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BIFURCATION ANALYSIS OF A DIFFUSIVE PREDATOR-PREY MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

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Abstract In this paper, we consider a diffusive predator-prey model with Beddington-DeAngelis functional response. The Turing instability and Hopf bifurcation of the coexisting equilibrium are investigated. We also use bifurcation parameters m, d_2 to study the Turing-Hopf bifurcation. In addition, we compute the normal form for the Turing-Hopf bifurcation. On the basis of the corresponding normal form, there exists complex spatiotemporal dynamics near Turing-Hopf bifurcation point. Finally, Some numerical simulations are given to illustrate our theoretical results.

Keywords Predator-prey, Turing instability, Hopf bifurcation, Turing-Hopf bifurcation.

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1. Introduction

Ecosystem plays an important role in nature. The relationship between predators and prey has been extensively studied in ecosystem [1, 14, 17, 19, 25]. The functional response has a strong influence on the development of populations [6,12,13, 22]. One kind of functional response is prey-dependent type, such as Holling I-III type functional responses [7]. Another kind of functional response is predator-dependent type, such as Beddington-DeAngelis [2], Crowley-Martin [3], Hassel-Varley [8]. Skalski and Gilliam [16] suggested that Beddington-DeAngelis functional response is suitable for the case that predator feeding rate becomes independent of predator density at high prey density. The Beddington-DeAngelis functional response is with the following form

$$f(x, y) = \frac{bx}{1 + k_1 x + k_2 y},$$

where x and y are prey and predator densities, respectively. b and k_1 describe the effects of capture rate and handling time, respectively. n is the birth rate of the predator. $k_2 y$ represents the interaction between predators and prey.

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A number of natural mechanisms can generate the Beddington-DeAngelis functional response [5, 15, 18, 26], and it is worthy to study because of its rich dynamic properties. Zhang et al. [26] investigated a discrete prey-predator system with harvesting of both species and Beddington-DeAngelis functional response. They established that the system undergoes flip bifurcation and Hopf bifurcation by using the center manifold theorem and bifurcation theory. In [5], the authors proposed a intraguild predation model with predator interference and Beddington-DeAngelis functional response. They mainly considered Hopf bifurcation and zero-Hopf bifurcation of the model. In [24], Yan and Zhang considered the following model

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + ru\left(1 - \frac{u}{K}\right) - \frac{\alpha uv}{a+bu+cv} \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v + v\left(-d + \frac{mu}{a+bu+cv}\right) \\ u_x\left(x,t\right) = v_x\left(x,t\right) = 0, x \in \partial\Omega, t > 0 \\ u_x\left(x,0\right) = u_0\left(x\right) \geq 0, v_x\left(x,0\right) = v_0\left(x\right) \geq 0, x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are prey and predator densities at the location x and time t respectively. d_1 and d_2 are diffusion coefficients of prey and predator respectively. r represents the intrinsic growth rate. K is carrying capacity of prey in the absence of predation. $\frac{\alpha uv}{a+bu+cv}$ is the Beddington-DeAngelis response, and $\frac{mu}{a+bu+cv}$ is predator's growth rate. All parameters are positive. Yan and Zhang mainly discussed the locally (globally) asymptotic stability and Turing instability of the positive constant steady state [24]. Based on the model (1.1), Xu and Fu considered a new model with the density-dependent death rate for the predator [23]. They mainly studied the stability and Turing instability of positive equilibrium. They also investigated the nonexistence (existence) of nonconstant positive steady state.

In predator-prey model, Turing bifurcation and Hopf bifurcation are two important research contents. Tian et al. [21] studed a Leslie-Gower predator-prey model with Beddington-DeAngelis functional response. They mainly considered stability, Turing instability and the Hopf bifurcation, and showed the existence of Hopf bifurcation, steady state solution and Turing-Hopf bifurcation via numerical simulations. Jiang and Tang [9] mainly studied the stability and Hopf bifurcation in a diffusive delayed predator-prey model with herd behavior and prey harvesting. Djilali [4] mainly investigated the existence of Hopf bifurcation and Turing bifurcation, and showed the existence of Turing-Hopf bifurcation point.

As far as we know, there is few work to systematic analyze the Turing-Hopf bifurcation for the model (1.1). Hence, we will give a complete and rigorous analysis of the dynamics including the existence of Turing bifurcation, Hopf bifurcation and Turing-Hopf bifurcation in this paper. We also give the normal form of Turing-Hopf bifurcation, and some numerical simulations to show the rich dynamic phenomena of the model (1.1).

The rest section of this paper is arranged. In Sect.2, we investigate the existence of the positive equilibrium of system (2.1). In Sect.3, we discuss a series of bifurcations. In Sect.4, we get the normal form of Turing-Hopf bifurcation. In Sect.5, some numerical simulations are given to verify our previous results. In Sect. 6, we give a conclusion.

2. Equilibrium Analysis

We denote as $\beta = \frac{\alpha}{r}, \delta = \frac{d}{m}$, system (1.1) can be rewritten as

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + ru\left(1 - \frac{u}{K} - \frac{\beta v}{a+bu+cv}\right), x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v + mv\left(-\delta + \frac{u}{a+bu+cv}\right), x \in \Omega, t > 0, \\ u_x(x,t) = v_x(x,t) = 0, x \in \partial\Omega, t > 0, \\ u_x(x,0) = u_0(x) \geq 0, v_x(x,0) = v_0(x) \geq 0, x \in \overline{\Omega}. \end{cases} \tag{2.1}$$

In this paper, we choose newman boundary condition, and consider the space is one-dimensional $\Omega = (0, l\pi)$, where $l > 0$ for convenience.

The first work is to analyze the existence of coexisting equilibrium, the system without diffusion terms is given as follows

$$\begin{cases} ru\left(1 - \frac{u}{K} - \frac{\beta v}{a+bu+cv}\right) = 0, \\ mv\left(-\delta + \frac{u}{a+bu+cv}\right) = 0. \end{cases} \tag{2.2}$$

It is easy to get that the system (2.1) has boundary equilibria $(0,0)$ and $(K,0)$. But we mainly focus on the coexisting equilibrium of the system (2.1). Now we assume (u_*, v_*) is coexisting equilibrium of the system (2.1) and discuss the existence of (u_*, v_*). From the second equation of (2.2), we get $u = \frac{-a\delta - cv\delta}{-1+b\delta}$. Submitting it into the first equation of (2.2), we obtain

$$h(v) = c^2\delta v^2 + v\left[2ac\delta + cK(-1+b\delta) + K\beta(-1+b\delta)^2\right] + aK(-1+b\delta) + a^2\delta. \tag{2.3}$$

If $b\delta > 1$, then $h(0) = a^2\delta + aK(-1+b\delta) > 0$, and the symmetrical axis of $h(v)$ is

$$h_0 = -\frac{2ac\delta + cK(-1+b\delta) + K\beta(-1+b\delta)^2}{2c^2\delta} < 0.$$

This indicates that system (2.1) has no coexisting equilibrium. Therefore $b\delta < 1$ holds, there exist two cases as follow.

1. If $h(0) < 0$, we can get $\delta(a + bK) < K$, system (2.1) has a unique coexisting equilibrium.

2. If $h(0) > 0$, we can get the symmetrical axis $h_0 < 0$ of $h(v)$, which means system (2.1) has no coexisting equilibrium.

Based on the above analysis, we can get the following lemma.

Lemma 2.1. *When* $\delta < \min\left\{\frac{1}{b}, \frac{K}{a+bK}\right\}$, *the system* (2.1) *has a unique coexisting equilibrium* (u_*, v_*).

3. Bifurcation analysis

In the work [24], Yan and Zhang have discussed the local stability and Turing instability of the coexisting equilibrium (u_*, v_*). For the sake of completeness and the convenience of analysis, we still do the following analysis process.

In this paper, we mainly do some bifurcation analysis. Define the real-valued Sobolev space

$$X := \left\{(u,v)^T : u, v \in H^2(0, l\pi),\ (u_x, v_x)|_{x=0,l\pi} = 0\right\}$$

and its complexification is

$$X_{\mathbb{C}} := X \oplus iX = \{x_1 + ix_2 |\ x_1, x_2 \in X\}.$$

The linearlization of (2.1) near (u_*, v_*) has the form:

$$\dot{U}(t) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \Delta U(t) + \begin{pmatrix} a_1 & -a_2 \\ mb_1 & -mb_2 \end{pmatrix} U(t), \tag{3.1}$$

where

$$\begin{aligned} a_1 &= ru_*\left(-\tfrac{1}{K} + \tfrac{bv_*\beta}{(a+bu_*+cv_*)^2}\right), \quad a_2 = -\tfrac{ru_*(a+bu_*)\beta}{(a+bu_*+cv_*)^2}, \\ b_1 &= \tfrac{mv_*(a+cv_*)}{(a+bu_*+cv_*)^2}, \quad b_2 = -\tfrac{cmu_*v_*}{(a+bu_*+cv_*)^2}. \end{aligned} \tag{3.2}$$

Then the characteristic equation of Eq. (3.1) is given by

$$\lambda y - D\triangle y - Ly = 0, \qquad \text{for some} \ \ y \in \mathrm{dom}(D\triangle)\backslash\{0\}, \tag{3.3}$$

where

$$\mathrm{dom}(D\triangle) = \{(u,v) \in X |\ \ \partial_\nu u(t,x) = \partial_\nu v(t,x) = 0,\ x = 0, l\pi\}.$$

It is well known that the operator $u \mapsto \triangle u$ with $\partial_\nu u = 0$ at 0 and $l\pi$ has eigenvalues $-z_n$ ($z_n = n^2/l^2$, $n \in N_0$) with corresponding eigenfunctions $\cos \frac{nx}{l}$. Let

$$\phi = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{nx}{l}$$

be an eigenfunction for $\triangle + L$ with eigenvalue λ. Hence, we gain the characteristic equation at $E_*(u_*, v_*)$, that is

$$\lambda^2 - t_n\lambda + \theta_n = 0, \quad n \in \mathbb{N}_0, \tag{3.4}$$

where

$$\begin{cases} t_n = a_1 - mb_2 - (d_1 + d_2)\, z_n, \\ \theta_n = d_1d_2z_n^2 + (b_2d_1m - a_1d_2)\, z_n + m\,(a_2b_1 - a_1b_2)\,, \end{cases} \tag{3.5}$$

and the eigenvalues are given by

$$\lambda_{1,2}^{(n)}(r) = \frac{t_n \pm \sqrt{t_n^2 - 4\theta_n}}{2}, \qquad n \in \mathbb{N}_0 \equiv \mathbb{N} \cup 0. \tag{3.6}$$

Then, we can get the Theorem 3.1 as follows.

Theorem 3.1. *If $a_1 < 0$, then the equilibrium $E_*(u_*, v_*)$ is locally asymptotically stable for system* (2.1).

Proof. Notice that $a_1 < 0$ which imply that $t_0 = a_1 - mb_2 < 0$ and $\theta_0 = m\,(a_2b_1 - a_1b_2) > 0$, then the eigenvalues of (3.4) have negative real parts. Hence, this theorem has proved. □

3.1. Turing instability

To investigate Turing instability, we make the following hypothesis

$$(\mathbf{H_1}) \qquad a_1 - mb_2 < 0,$$
$$(\mathbf{H_2}) \qquad a_2b_1 - a_1b_2 > 0.$$

The equilibrium $E_*(u_*, v_*)$ is locally asymptotically stable for the system (2.1) without diffusion ($d_1 = d_2 = 0$) under hypothesis ($\mathbf{H_1}$) and ($\mathbf{H_2}$).

For $\theta_n = d_1d_2z_n^2 + (b_2d_1m - a_1d_2)\, z_n + m\,(a_2b_1 - a_1b_2)$, we can get the symmetrical axis $z_0 = \frac{a_1d_2 - mb_2d_1}{2d_1d_2}$.

Divide the parameter m into the following two cases.

$$\textbf{Case 1:}\quad m \geq \frac{a_1d_2}{b_2d_1}, \quad \text{or} \quad m < \frac{a_1d_2}{b_2d_1} \text{ and } (b_2d_1m - a_1d_2)^2 - 4d_1d_2\,(a_2b_1 - a_1b_2) < 0.$$
$$\textbf{Case 2:}\quad m < \frac{a_1d_2}{b_2d_1}, \quad \text{and} \quad (b_2d_1m - a_1d_2)^2 - 4d_1d_2\,(a_2b_1 - a_1b_2) > 0. \tag{3.7}$$

If $z_0 > 0$ which means $a_1d_2 - mb_2d_1 > 0$, we gain the parameter $0 < m < \frac{a_1d_2}{b_2d_1}$. It also implies $a_1 > 0$.

Theorem 3.2. *Suppose* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold. Then for system* (2.1)*, the following statements are true.*

(i) In **Case 1***, the equilibrium $E_*(u_*, v_*)$ is locally asymptotically stable;*

(ii) In **Case 2***, there does not exists a z_k ($k \in \mathbb{N}_0$) such that $\theta_k < 0$, then $E_*(u_*, v_*)$ is locally asymptotically stable;*

(iii) In **Case 2***, there exists a z_k ($k \in \mathbb{N}$) such that $\theta_k < 0$, then $E_*(u_*, v_*)$ is Turing unstable.*

Proof. Notice that, ($\mathbf{H_1}$) implies that $t_n < 0$ ($n \in \mathbb{N}_0$). ($\mathbf{H_1}$) implies that $\theta_n > 0$ ($n \in \mathbb{N}_0$) when **Case 1** holds. Then the eigenvalues of (3.4) have negative real parts, implying statement (i) is true. Similarly, statement (ii) is true. If parameters in **Case 2**, and there exist a $k \in \mathbb{N}$ such that $\theta_k < 0$, then the eigenvalues of (3.4) have positive real part $\lambda_1^{(k)}$. This implies that $E_*(u_*, v_*)$ is unstable for system (2.1). Then statement (iii) is proved. □

3.2. Hopf bifurcation

In this section, we make some Hopf bifurcation analysis. Denote

$$m = m_n := \frac{a_1 - (d_1 + d_2)\, z_n}{b_2}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

And $\theta_0(m_0) > 0$ when hypothesis ($\mathbf{H_2}$) holds. Then, we have the following conclusion of Hopf bifurcation.

Theorem 3.3. *Suppose hypothesis* ($\mathbf{H_2}$) *holds. When $m = m_n$, the system* (2.1) *undergoes a Hopf bifurcation at equilibrium $E_*(u_*, v_*)$ for $0 \leq n \leq n^* - 1$. In addition, the bifurcating periodic solution is spatially homogeneous when $m = m_0$ and spatially non-homogeneous when $m = m_n$ for $1 \leq n \leq n^* - 1$.*

($n^* = \max\{k \in |\theta_n(m_n) > 0 \;\text{ and }\; m_n > 0 \;\text{ for }\; n = 0, 1, \cdots, k-1.$)

Proof. When $m = m_n$, $t_n(m_n) = 0$ and $\theta_n(m_n) > 0$ for $0 \le n \le n^* - 1$, implying that (3.4) has purely imaginary. Let

$$\lambda_n(s) = \alpha_n(m) \pm i\omega_n(m), \quad n = 0, 1, \cdots, n^* - 1$$

be the roots of Eq. (3.4) satisfying

$$\alpha_n(m_n) = 0, \quad \omega_n(m_n) = \sqrt{\theta_n(m_n)}.$$

Then, when m is near m_n

$$\alpha_n(m) = \frac{t_n(m)}{2}, \quad \omega_n(m) = \sqrt{\theta_n - \alpha_n^2(m)}.$$

In (3.5), we obtain

$$\alpha_n'(m_n) = -\frac{b_2}{2} < 0. \tag{3.9}$$

This implies that the transversal condition is satisfied at each m_n, $n = 0, 1, 2, \cdots, n^* - 1$. This completes the proof. □

3.3. Turing-Hopf bifurcation

In this section, we suppose hypothesis $(\mathbf{H_2})$ always holds. From Theorem 3.3, we know that when $m^* := a_1/b_2$, system (2.1) undergoes hopf bifurcation, and the bifurcating periodic solution is spatially homogeneous.

Due to the Turing-Hopf bifurcation is spatially codimension-2 bifurcation, it may produce rich dynamic phenomenon [11, 20]. When system undergoes Turing-Hopf bifurcation, the following conditions need to be satisfied.
(1) There are a pair of simple purely imaginary roots $\pm i\omega$ for Eq. (3.4) when $n = 0$;
(2) There is a simple zero root $\lambda = 0$ for Eq. (3.4) when $n = j \in \mathbb{N}$.

We can get a series of Turing bifurcation curves d_2^k:

$$d_2^k = \frac{b_2 d_1 m z_k - (a_2 b_1 - a_1 b_2)\, m}{z_k\,(-a_1 + d_1 z_k)}, \qquad \mathbb{S} = \left\{k \in \mathbb{N}_0 | \, d_2^k > 0\right\}. \tag{3.10}$$

There exist $k_* \in \mathbb{N}$ such that $d_2^{k_*} = \frac{b_2 d_1 z_{k_*} - (a_2 b_1 - a_1 b_2)}{z_{k_*}(-a_1 + d_1 z_{k_*})} m^* = \min\limits_{k \in \mathbb{S}} \frac{b_2 d_1 z_k - (a_2 b_1 - a_1 b_2)}{z_k(-a_1 + d_1 z_k)} m^*$.

Based on the above analysis, we give out following theorem about Turing-Hopf bifurcation.

Theorem 3.4. *Suppose* $(\mathbf{H_2})$ *holds. For system* (2.1)*, the following statements are true.*

(i) If $\mathbb{S} = \emptyset$*, system* (2.1) *does not undergo Turing-Hopf bifurcation.*

(ii) If $\mathbb{S} \ne \emptyset$*, the equilibrium* $E_*(u_*, v_*)$ *is locally asymptotically stable for* $(m, d_2) \in \{(m, d_2) \mid m > m^*,\ 0 < d_2 < \frac{b_2 d_1 z_{k_*} - (a_2 b_1 - a_1 b_2)}{z_{k_*}(-a_1 + d_1 z_{k_*})} m\}$*, and system* (2.1) *undergoes Turing-Hopf bifurcation at the point* $(m, d_2) = (m^*, d_2^{k_*})$*.*

Proof. In $m - d_2$ plane, the Hopf bifurcation curve is

$$\mathcal{H}_0: \ m = m^*.$$

We define the Turing bifurcation curves as follows:

$$\mathcal{L}_k: \ d_2^k = \frac{b_2 d_1 z_k - (a_2 b_1 - a_1 b_2)}{z_k\left(-a_1 + d_1 z_k\right)} m, \ k \in \mathbb{S}.$$

1.If $\mathbb{S} = \emptyset$, then $\mathcal{L}_k$ and $\mathcal{H}_0$ have no intersection in the first quadrant, so system (2.1) does not undergo Turing-Hopf bifurcation.

2.If $\mathbb{S} \neq \emptyset$, it's clear that $t_n < 0$ and $\theta_n > 0$ for $n \in \mathbb{N}_0$, implying that the equilibrium $E_*(u_*, v_*)$ is locally asymptotically stable. For $(m, d_2) \in \{(m, d_2) \mid m > m^*, \ 0 < d_2 < \frac{b_2 d_1 z_{k_*} - (a_2 b_1 - a_1 b_2)}{z_{k_*}(-a_1 + d_1 z_{k_*})} m$, the Turing bifurcation curves $\mathcal{L}_{k_*}$ intersects with Hopf bifurcation curve $\mathcal{H}_0$ at Turing-Hopf bifurcation point $(m^*, d_2^{k_*})$, with real parts of all other eigenvalues of Eq. (3.4) $(n \neq 0, k_*)$ being negative. Moreover, suppose $\lambda_1(m) = \alpha_1(m) + i\beta_1(m)$ with $\alpha_1(m^*) = 0$, $\beta_1(m^*) = \omega > 0$, and $\lambda_2(m) = \alpha_2(m) + i\beta_2(m)$ with $\alpha_2(m^*) = 0$, $\beta_2(m^*) = 0$, then the transversality conditions are as follows:

$$\begin{aligned} \frac{dRe(\lambda_1(m))}{dm} \Big|_{m=\frac{a_1}{b_2}, \mathcal{H}_0} &= -\frac{b_2}{2} < 0, \\ \frac{dRe(\lambda_2(m))}{dm} \Big|_{m=\frac{a_1}{b_2}, \mathcal{L}_{k_*}} &= \frac{b_2 d_1 z_n + (a_2 b_1 - a_1 b_2)}{T_n} < 0. \end{aligned}$$

This completes the proof. □

4. Normal forms for Turing-Hopf bifurcation

In this section, we calculate normal forms of Turing-Hopf bifurcation for system (2.1) at equilibrium (u_*, v_*). We introduce μ_1 and μ_2 as perturbation parameters, and let $m = m^* + \mu_1$ and $d_2 = d_2^{k_*} + \mu_2$, so that when $\mu_1 = 0$, $\mu_2 = 0$, the system (2.1) can undergo Turing-Hopf bifurcation. Then the reaction-diffusion system (2.1) can be transformed into

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + ru\left(1 - \frac{u}{K} - \frac{\beta v}{a + bu + cv}\right), \\ \frac{\partial v(x,t)}{\partial t} = \left(d_2^{k_*} + \mu_1\right) \Delta v + \left(m^* + \mu_2\right) v \left(-\delta + \frac{u}{a + bu + cv}\right). \end{cases} \tag{4.1}$$

In this paper, we always let (u_*, v_*) is the coexistence equilibrium for system (4.1). We apply the generic formulas developed from Jiang [10], and consider the transformations $\bar{u} = u - u_*$, $\bar{v} = v - v_*$ and drop the bars, then system (4.1) can be rewritten as

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + r\left(u + u_*\right)\left(1 - \frac{u + u_*}{K} - \frac{\beta(v + v_*)}{a + b(u + u_*) + c(v + v_*)}\right), \\ \frac{\partial v(x,t)}{\partial t} = \left(d_2^{k_*} + \mu_1\right) \Delta v + \left(m^* + \mu_2\right)\left(v + v_*\right)\left(-\delta + \frac{u + u_*}{a + b(u + u_*) + c(v + v_*)}\right). \end{cases} \tag{4.2}$$

Thus, according to Jiang [30], for system (4.2) we have,

$$D(\mu) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2^{k_*} + \mu_1 \end{pmatrix}, L(\mu) = \begin{pmatrix} a_1 & -a_2 \\ \left(m^* + \mu_1\right) b_1 & -\left(m^* + \mu_2\right) b_2 \end{pmatrix},$$

$$F(\phi,\mu)=\begin{pmatrix} r\left(\phi_1+u_*\right)\left(1-\frac{\phi_1+u_*}{K}-\frac{\beta(\phi_2+v_*)}{a+b(\phi_1+u_*)+c(\phi_2+v_*)}\right)-a_1\phi_1+a_2\phi_2 \\ \left(m^*+\mu_2\right)\left(\phi_2+v_*\right)\left(-\delta+\frac{\phi_1+u_*}{a+b(\phi_1+u_*)+c(\phi_2+v_*)}\right)-\left(m^*+\mu_1\right)b_1+\left(m^*+\mu_2\right)b_2 \end{pmatrix}, \tag{4.3}$$

where $\phi=(\phi_1,\phi_2)^T\in X$.

Then, we have

$$D(0)=\begin{pmatrix} d_1 & 0 \\ 0 & d_2^{k_*} \end{pmatrix}, D_1(\mu)=\begin{pmatrix} 0 & 0 \\ 0 & 2\mu_1 \end{pmatrix},$$

$$L(0)=\begin{pmatrix} a_1 & -a_2 \\ m^*b_1 & -m^*b_2 \end{pmatrix}, L_1(\mu)=\begin{pmatrix} 0 & 0 \\ 2\mu_2b_1 & -2\mu_2b_2 \end{pmatrix},$$

$$Q(\phi,\psi)=\begin{pmatrix} \alpha_{11}\phi_1\psi_1+\alpha_{12}(\phi_1\psi_2+\psi_1\phi_2)+\alpha_{13}\phi_2\psi_2 \\ \alpha_{21}\phi_1\psi_1+\alpha_{22}(\phi_1\psi_2+\psi_1\phi_2)+\alpha_{23}\phi_2\psi_2 \end{pmatrix},$$

$$C(\phi,\psi,\upsilon)=\begin{pmatrix} \beta_{11}\phi_1\psi_1\upsilon_1+\beta_{12}(\phi_1\psi_1\upsilon_2+\phi_1\psi_2\upsilon_1+\phi_2\psi_1\upsilon_1)+\beta_{13}(\phi_1\psi_2\upsilon_2+\phi_2\psi_1\upsilon_2+\phi_2\psi_2\upsilon_1)+\beta_{14}\phi_2\psi_2\upsilon_2 \\ \beta_{21}\phi_1\psi_1\upsilon_1+\beta_{22}(\phi_1\psi_1\upsilon_2+\phi_1\psi_2\upsilon_1+\phi_2\psi_1\upsilon_1)+\beta_{23}(\phi_1\psi_2\upsilon_2+\phi_2\psi_1\upsilon_2+\phi_2\psi_2\upsilon_1)+\beta_{24}\phi_2\psi_2\upsilon_2 \end{pmatrix},$$

with

$$\alpha_{11}=2r\left(\frac{v_*\left(ab+bcv_*\right)\beta}{\left(a+bu_*+cv_*\right)^3}-\frac{1}{K}\right),\quad \alpha_{12}=-\frac{r\beta\left(a^2+abu_*+acv_*+2bcu_*v_*\right)}{\left(a+bu_*+cv_*\right)^3},$$

$$\alpha_{13}=\frac{2rc\beta u_*\left(a+bu_*\right)}{\left(a+bu_*+cv_*\right)^3},\quad \alpha_{21}=-\frac{2mv_*b\left(a+cv_*\right)}{\left(a+bu_*+cv_*\right)^3},$$

$$\alpha_{22}=-\frac{cmv_*\left(a-bu_*+cv_*\right)}{\left(a+bu_*+cv_*\right)^3},\quad \alpha_{23}=\frac{2c^2mv_*u_*}{\left(a+bu_*+cv_*\right)^3},$$

$$\beta_{11}=-\frac{6br\beta v_*\left(ab+bcv_*\right)}{\left(a+bu_*+cv_*\right)^4},\quad \beta_{12}=-\frac{2\left(-a^2br\beta-ab^2ru\beta-2b^2cruv\beta+bc^2rv^2\beta\right)}{\left(a+bu+cv\right)^4},$$

$$\beta_{13}=\frac{2r\beta\left(a^2c-b^2cu_*^2+ac^2v_*+2bc^2u_*v_*\right)}{\left(a+bu_*+cv_*\right)^4},\quad \beta_{14}=-\frac{6c^2r\beta u_*\left(a+bu_*\right)}{\left(a+bu_*+cv_*\right)^4},$$

$$\beta_{21}=\frac{6b^2mv_*\left(a+cv_*\right)}{\left(a+bu_*+cv_*\right)^4},\quad \beta_{22}=-\frac{2cmv\left(-2ab+b^2u_*-2bcv_*\right)}{\left(a+bu_*+cv_*\right)^4},$$

$$\beta_{23}=\frac{2c^2mv_*\left(a-2bu_*+cv_*\right)}{\left(a+bu_*+cv_*\right)^4},\quad \beta_{24}=-\frac{6c^3mv_*u_*}{\left(a+bu_*+cv_*\right)^4},$$

and $\phi=(\phi_1,\phi_2)^T,\psi=(\psi_1,\psi_2)^T,\upsilon=(\upsilon_1,\upsilon_2)^T\in X$. The corresponding characteristic matrices are given by

$$\Gamma_k(\lambda)=\begin{pmatrix} \lambda+d_1z_k-a_1 & a_2 \\ -m^*b_1 & \lambda+d_2^{k_*}z_k+m^*b_2 \end{pmatrix},\qquad k\in\mathbb{N}.$$

Obviously, $\lambda=\pm i\omega$ with $\omega=\sqrt{m^*\left(a_2b_1-a_1b_2\right)}$, are eigenvalues of $\mathbb{D}_0(\lambda)$, and $\lambda=0$ is a simple eigenvalue for $\Gamma_{k_*}(\lambda)$, with other eigenvalues having negative real parts, according to Theorem 3.4. Then, we have

$$\phi_1 = \begin{pmatrix} 1 \\ \frac{a_1 - d_1 z_{k_*}}{a_2} \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} \frac{-m^* b_1}{(a_2 b_1 - a_1 b_2)m^* + (b_2 d_1 m^* - a_1 d_2) z_{k_*} + d_1 d_2 z_{k_*}^2} \\ \frac{a_1 - d_1 z_{k_*}}{(a_2 b_1 - a_1 b_2)m^* + (b_2 d_1 m^* - a_1 d_2) z_{k_*} + d_1 d_2 z_{k_*}^2} \end{pmatrix}^T,$$

$$\phi_2 = \begin{pmatrix} 1 \\ \frac{a_1 - i\omega}{a_2} \end{pmatrix}, \psi_2 = \begin{pmatrix} \frac{a_2 b_1 m^*}{a_2 b_1 m^* + (\omega + a_1 i)^2} \\ \frac{a_2 (i\omega - a_1)}{a_2 b_1 m^* + (\omega + a_1 i)^2} \end{pmatrix}^T.$$

Therefore, $\Phi = (\phi_1, \phi_2, \bar{\phi}_2)$ and$\Psi = (\psi_1, \psi_2, \bar{\psi}_2)^T$ satisfying $\Phi\Psi = I_3$, where I_3 is identity matrix. By [10], the following parameters can be computed.

$$a_1(\mu) = \frac{1}{2}\psi_1 \left(L_1(\mu)\phi_1 - \mu_{k_*} D_1(\mu)\phi_1\right), \quad a_{200} = a_{011} = b_{110} = 0,$$

$$b_2(\mu) = \frac{1}{2}\psi_2 (L_1(\mu)\phi_2 - 0 D_1(\mu)\phi_2),$$

$$a_{300} = \frac{1}{4}\psi_1 C_{\phi_1\phi_1\phi_1} + \frac{1}{\omega}\psi_1 \mathrm{Re}[iQ_{\phi_1\phi_2}\psi_2] Q_{\phi_1\phi_1} + \psi_1 Q_{\phi_1 (h_{200}^0 + \frac{1}{\sqrt{2}} h_{200}^{2k_*})},$$

$$a_{111} = \psi_1 C_{\phi_1\phi_2\bar{\phi}_2} + \frac{2}{\omega}\psi_1 \mathrm{Re}[iQ_{\phi_1\phi_2}\psi_2] Q_{\phi_2\bar{\phi}_2} + \psi_1 (Q_{\phi_1 (h_{011}^0 + \frac{1}{\sqrt{2}} h_{011}^{2k_*})} + Q_{\phi_2 h_{101}^{k_*}} + Q_{\bar{\phi}_2 h_{110}^{k_*}}),$$

$$\begin{aligned} b_{210} = {} & \frac{1}{2}\psi_2 C_{\phi_1\phi_1\phi_2} + \frac{1}{2i\omega}\psi_2 (2Q_{\phi_1\phi_1}\psi_1 Q_{\phi_1\phi_2} + (-Q_{\phi_2\phi_2}\psi_2 + Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2) Q_{\phi_1\phi_1}) \\ & + \psi_2 (Q_{\phi_1 h_{110}^{k_*}} + Q_{\phi_2 h_{200}^0}), \end{aligned}$$

$$\begin{aligned} b_{021} = {} & \frac{1}{2}\psi_2 C_{\phi_2\phi_2\bar{\phi}_2} + \frac{1}{4i\omega}\psi_2 \left(\frac{2}{3} Q_{\bar{\phi}_2\bar{\phi}_2}\bar{\psi}_2 Q_{\phi_2\phi_2} + (-2Q_{\phi_2\phi_2}\psi_2 + 4Q_{\phi_2\bar{\phi}_2}\bar{\psi}_2) Q_{\phi_2\bar{\phi}_2}\right) \\ & + \psi_2 (Q_{\phi_2 h_{011}^0} + Q_{\bar{\phi}_2 h_{020}^0}), \end{aligned}$$

where

$$\begin{aligned} h_{200}^0 = & -\frac{1}{2} L^{-1}(0) Q_{\phi_1\phi_1} + \frac{1}{2\omega i} (\phi_2\psi_2 - \bar{\phi}_2\bar{\psi}_2) Q_{\phi_1\phi_1}, \\ h_{200}^{2k_*} = & -\frac{1}{2\sqrt{2}} \left[L(0) + \mathrm{diag}(-4\mu_{k_*}, -4 d_{k_*}\mu_{k_*})\right]^{-1} Q_{\phi_1\phi_1}, \\ h_{011}^0 = & - L^{-1}(0) Q_{\phi_2\bar{\phi}_2} + \frac{1}{\omega i} (\phi_2\psi_2 - \bar{\phi}_2\bar{\psi}_2) Q_{\phi_2\bar{\phi}_2}, \\ h_{020}^0 = & \frac{1}{2} \left[2i\omega I - L(0)\right]^{-1} Q_{\phi_2\phi_2} - \frac{1}{2\omega i} \left(\phi_2\psi_2 + \frac{1}{3}\bar{\phi}_2\bar{\psi}_2\right) Q_{\phi_2\phi_2}, \\ h_{110}^{k_*} = & \left[i\omega I - (L(0) - \mathrm{diag}(-\mu_{k_*}, -d_{k_*}\mu_{k_*}))\right]^{-1} Q_{\phi_1\phi_2} - \frac{1}{\omega i}\phi_1\psi_1 Q_{\phi_1\phi_2}, \\ h_{002}^0 = & \overline{h_{020}^0}, \quad h_{101}^{k_*} = \overline{h_{110}^{k_*}}, \quad h_{011}^{2k_*} = 0. \end{aligned}$$

By [10], we get the following normal form restricted on center manifold up to order 3 for reaction-diffusion system (2.1).

$$\begin{cases} \dot{z}_1 = a_1(\mu) z_1 + a_{200} z_1^2 + a_{011} z_2 \bar{z}_2 + a_{300} z_1^3 + a_{111} z_1 z_2 \bar{z}_2 + h.o.t., \\ \dot{z}_2 = i\omega z_2 + b_2(\mu) z_2 + b_{110} z_1 z_2 + b_{210} z_1^2 z_2 + b_{021} z_2^2 \bar{z}_2 + h.o.t., \\ \dot{\bar{z}}_2 = -i\omega \bar{z}_2 + \bar{b}_2(\mu) \bar{z}_2 + \bar{b}_{110} z_1 \bar{z}_2 + \bar{b}_{210} z_1^2 \bar{z}_2 + \bar{b}_{021} z_2 \bar{z}_2^2 + h.o.t. \end{cases} \tag{4.4}$$

Let's make the cylindrical coordinate transformation $z_1 = r$, $z_2 = \rho\cos\theta - \mathrm{i}\rho\sin\theta$, the normal form for Eq. (4.4) can be written in real coordinates form :

$$\begin{cases} \dot{r} = a_1(\mu)r + a_{300}r^3 + a_{111}r\rho^2, \\ \dot{\rho} = \mathrm{Re}(b_2(\mu))\rho + \mathrm{Re}(b_{210})\rho r^2 + \mathrm{Re}(b_{021})\rho^3. \end{cases} \tag{4.5}$$

5. Numerical simulations

In order to verify our previous conclusions, numerical simulation is carried out here. Taking $\beta = 0.8, K = 10, b = 0.7, a = 0.2, c = 0.2, r = 5, \delta = 1, d_1 = 1$, we have

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \Delta u + 5u\left(1 - \frac{u}{10} - \frac{0.8v}{0.2+0.7u+0.2v}\right), & x \in (0, 4\pi),\ t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2\Delta v + mv\left(-1 + \frac{u}{0.2+0.7u+0.2v}\right), & x \in (0, 4\pi),\ t > 0. \end{cases} \tag{5.1}$$

By calculation, we obtain $(u_*, v_*) = (2.0, 2.0)$ is a unique coexisting equilibrium. And $a_1 = 1.8$, $a_2 = 3.2$, $b_1 = 0.3$, $b_2 = 0.2$, then hypothesis $(\mathbf{H_2})$ holds. In addition, $\mathbb{S} = \{1\,2\,3\,4\,5\}$, $m^* = 9.0$, $k_* = 4$, $d_2^{k_*} = 9.0$. The Hopf bifurcation curve in $m - d_2$ plane is

$$\mathcal{H}_0 :\ m = m^* = 9.0.$$

The Turing bifurcation curves are

$$\mathcal{L}_k :\ d_2^k = \frac{b_2 d_1 z_k - (a_2 b_1 - a_1 b_2)}{z_k\,(-a_1 + d_1 z_k)} m,\ k \in \mathbb{S}.$$

The normal form restricted on center manifold for reaction-diffusion system (5.1) at Turing-Hopf singularity is

$$\begin{cases} \dot{z}_1 = \left(1.1259 \times 10^{14}\mu_1 - 1.1259 \times 10^{14}\mu_2\right) z_1 - 4.2567 \times 10^{14} z_1^3 \\ \qquad - 1.0045 \times 10^{15} z_1 z_2 \bar{z}_2 + h.o.t., \\ \dot{z}_2 = 2.3238 i z_2 + (-0.1000 + 0.12916\mathrm{i})\,\mu_2 z_2 + (-0.1953 + 0.1250\mathrm{i})\, z_1^2 z_2 \\ \qquad + (-0.1335 + 0.1957\mathrm{i})\, z_2^2 \bar{z}_2 + h.o.t., \\ \dot{\bar{z}}_2 = -2.3238 i \bar{z}_2 + (-0.1000 - 0.12916\mathrm{i})\,\mu_2 \bar{z}_2 + (-0.1953 - 0.1250\mathrm{i})\, z_1^2 \bar{z}_2 \\ \qquad + (-0.1335 - 0.1957\mathrm{i})\, z_2 \bar{z}_2^2 + h.o.t. \end{cases}$$

Then we have

$$\begin{cases} \dot{r} = \left(1.1259 \times 10^{14}\mu_1 - 1.1259 \times 10^{14}\mu_2\right) r - 4.2567 \times 10^{14} r^3 - 1.0045 \times 10^{15} r\rho^2, \\ \dot{\rho} = -0.100\mu 2\rho - 0.1953\rho r^2 - 0.1335\rho^3. \end{cases} \tag{5.2}$$

Considering $\rho > 0$, from [10], system (5.4) has coexistence equilibrium A_0; spatially inhomogeneous steady states $A_1^{\pm}$; spatially homogeneous periodic solution

A_2; spatially inhomogeneous periodic solutions $A_3^{\pm}$. We calculate them as follows.

$$
\begin{aligned}
&A_0 = (0,0),\\
&A_1^{\pm} = \left(\pm\sqrt{0.2645\,(1.0000\mu_1 - 1.0000\mu_2)}, 0\right), \quad \text{for} \quad \mu_1 < 1.0000\mu_2\\
&A_2 = (0, \sqrt{-0.1146\mu_1}), \quad \text{for} \quad \mu_1 > 0,\\
&A_3^{\pm} = \left(\pm\sqrt{-0.1079\,(1.0000\mu_1 + 5.6821\mu_2)}, \sqrt{0.1578\,(1.0000\mu_1 + 0.9357\mu_2)}\right),\\
&\text{for} \quad 1.0000\mu_1 + 5.6821\mu_2 < 0, \quad \text{and} \quad 1.0000\mu_1 + 0.9357\mu_2 > 0.
\end{aligned}
\tag{5.3}
$$

Then, we can obtain the following critical bifurcation curves

$$
\begin{aligned}
&\mathcal{H}_0 : \mu_2 = 0, \ \mathcal{T} : \mu_2 = 1.0000\mu_1\\
&\mathcal{T}_1 : \mu_2 = -0.1760\mu_1, \quad \mu_1 \geq 0,\\
&\mathcal{T}_2 : \mu_2 = -1.0687\mu_1, \quad \mu_1 \geq 0.
\end{aligned}
\tag{5.4}
$$

From Fig.1 (right), we choose the first intersection point (m, d_2) of Turing curve $\mathcal{L}_k$ and Hopf curve $\mathcal{H}_0$ as the Turing-Hopf bifurcation point $(m, d_2) = (m^*, d_2^{k_*})$. Therefore, system (5.1) undergoes Turing-Hopf bifurcation at the point $(m^*, d_2^{k_*}) = (9.0, 9.0)$. Then, we can get the bifurcation diagram, see Fig.1(left).

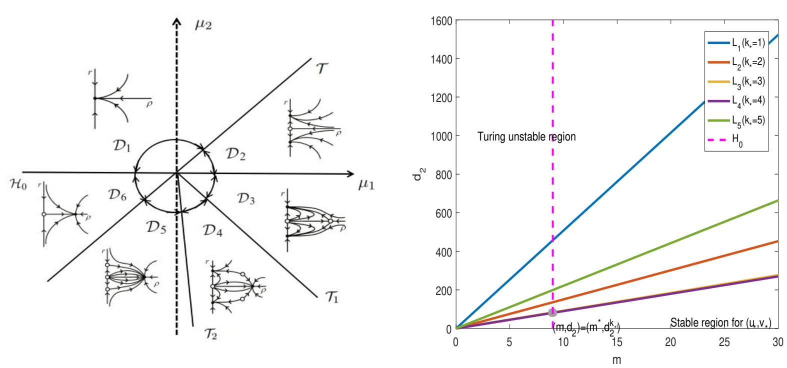


Figure 1. Left:the bifurcation diagram. Right:Stable region for (u_*, v_*), Turing-Hopf bifurcation point $(m^*, d_2^{k_*})$ in $m - d_2$ plane.

For each region, we obtain the following conclusions.

Proposition 5.1. *By bifurcation curves $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$, the parameter plane (μ_1, μ_2) is divided into six regions. For every region, the system* (2.1) *can show different dynamic phenomenon. We get the following main results:*

1. *When $(\mu_1, \mu_2) \in D_1$, the equilibrium (u_*, v_*) of the system* (2.1) *is asymptotically stable (see Fig.2).*

2. *When $(\mu_1, \mu_2) \in D_2$, there exist a pair of stable spatially inhomogeneous steady states (see Fig.3).*

3. *When $(\mu_1, \mu_2) \in D_3$, the system* (2.1) *has a pair of stable spatially inhomogeneous steady states and an unstable spatially homogeneous periodic solution (see Fig.4).*

4. *When* $(\mu_1, \mu_2) \in D_4$*, the system* (2.1) *has a stable spatially homogeneous periodic solution, a pair of stable spatially inhomogeneous steady states, and a pair of unstable spatially inhomogeneous periodic solutions for system* (2.1) *(see Fig.5).*

5. *When* $(\mu_1, \mu_2) \in D_5$*, there are a stable spatially homogeneous periodic solution, and also has a pair of unstable spatially inhomogeneous steady states (see Fig.6).*

6. *When* $(\mu_1, \mu_2) \in D_6$*, the system* (2.1) *only has a stable spatially homogeneous periodic solution (see Fig.7).*

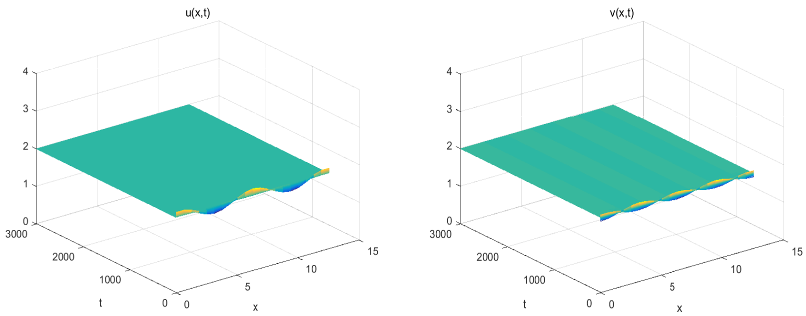
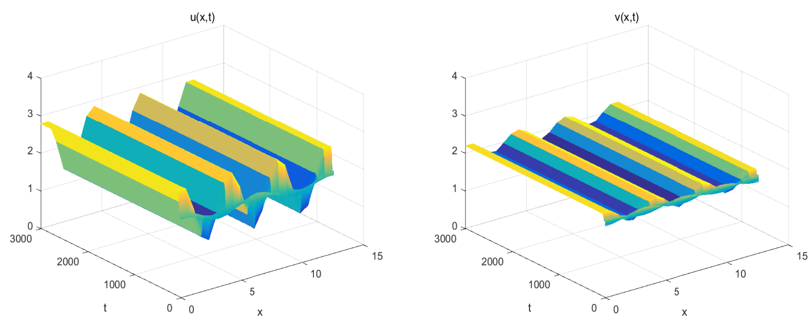
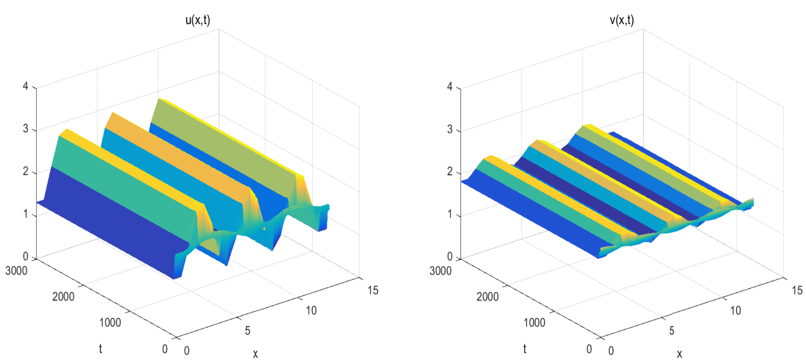


Figure 2. For $(\mu_1, \mu_2) = (0.1, 1.0) \in D_1$, the positive equilibrium (u_*, v_*) of the system (2.1) is asymptotically stable.

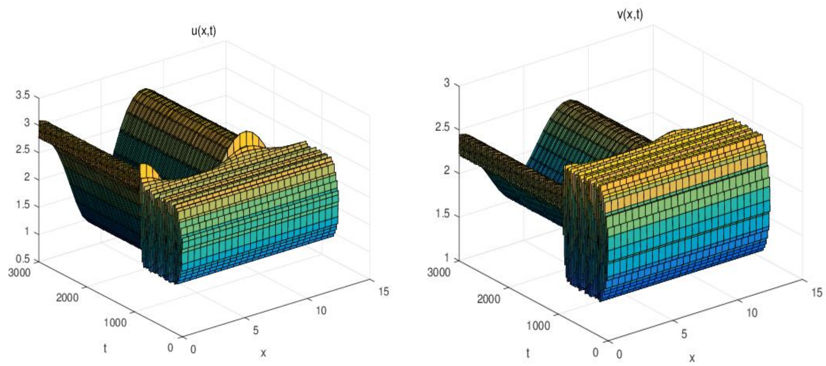


(a) The initial values are $u(x,0) = 2.0 + 0.1\cos(4x)$, $v(x,0) = 2.0 - 0.1\cos(4x)$.

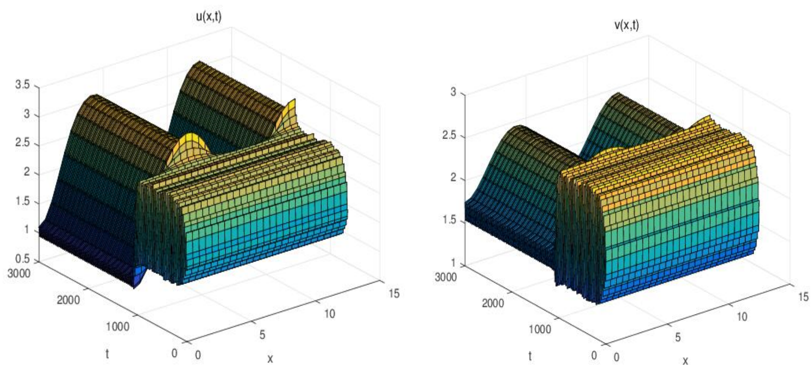


(b) The initial values are $u(x,0) = 2.0 - 0.1\cos(4x)$, $v(x,0) = 2.0 + 0.1\cos(4x)$.

Figure 3. For $(\mu_1, \mu_2) = (1.0, 0.1) \in D_2$, there is a pair of stable spatially inhomogeneous steady states.



(a) The initial values are $u\,(x,0) = 2.0 + 0.1, v\,(x,0) = 2.0 - 0.1$.



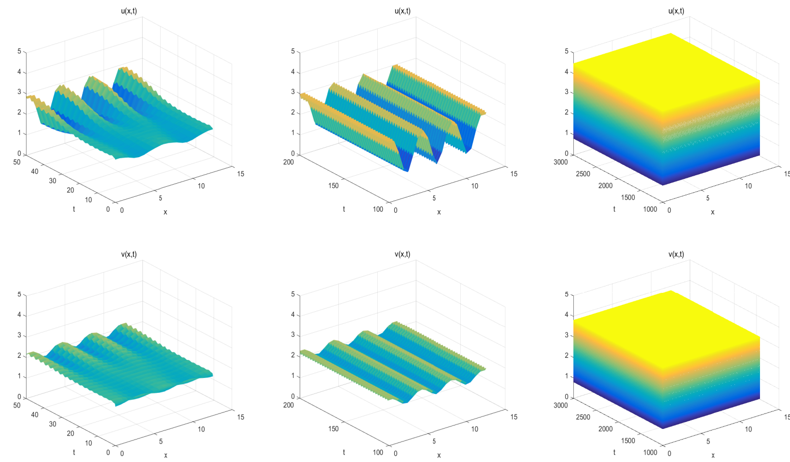
(b) The initial values are $u\,(x,0) = 2.0 - 0.1, v\,(x,0) = 2.0 + 0.1$

Figure 4. For $(\mu_1, \mu_2) = (1.0, -0.1) \in D_3$, the system (2.1) has a pair of stable spatially inhomogeneous steady states and an unstable spatially homogeneous periodic solution.

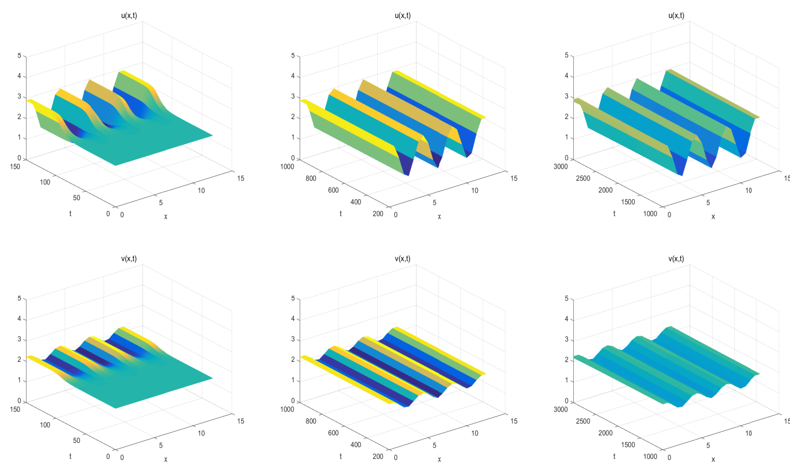
6. Conclusion

In this article, we discuss a diffusive predator-prey model with Beddington-DeAngelis functional response. Our main work is to do some bifurcation analysis including Turing bifurcation, Hopf bifurcation, and Turing-Hopf bifurcation.

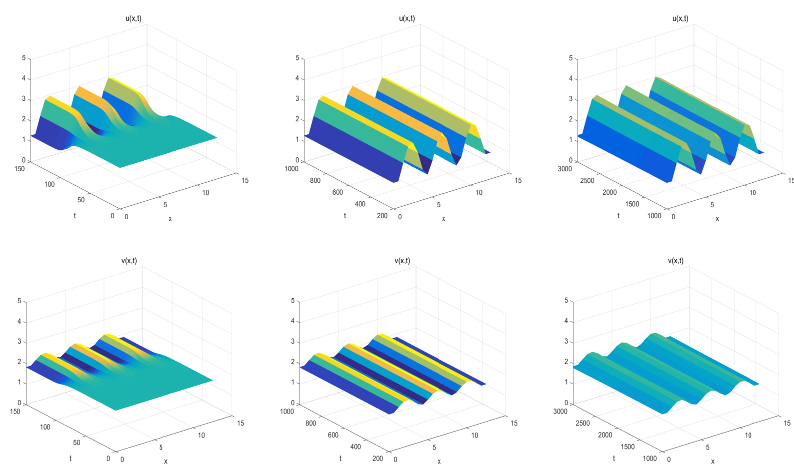
At first, we analysis the existence of the equilibrium. Then, we get the conditions about Turing instability of the system (2.1). Under hypothesis ($\mathbf{H_2}$) holds, the system (2.1) can undergo a Hopf bifurcation at $E_*(u_*, v_*)$ when $m = m_n$, for $0 \leq n \leq n^* - 1$. In addition, we choose m, d_2 as bifurcation parameters, then system (2.1) undergoes Turing-Hopf bifurcation at the point $(m, d_2) = (m^*, d_2^{k_*})$. We also compute the normal form of Turing-Hopf bifurcation, and get some bifurcation curves which divide bifurcation diagram into six regions. For every region, the system (2.1) can show different dynamic phenomena.



(a) The initial values are $u\left(x,0\right)=2.0+0.1\cos\left(4x\right), v\left(x,0\right)=2.0-0.1\cos\left(4x\right)$.



(b) The initial values are $u\left(x,0\right)=2.0+0.0001\cos\left(4x\right), v\left(x,0\right)=2.0-0.00001\cos\left(4x\right)$.



(c) The initial values are $u\left(x,0\right)=2.0-0.0001\cos\left(4x\right), v\left(x,0\right)=2.0+0.0001\cos\left(4x\right)$.

Figure 5. When $(\mu_1,\mu_2)=(1.0,-0.2)\in D_4$, there exists a stable spatially homogeneous periodic solution and a pair of stable spatially inhomogeneous steady states, as well as a pair of unstable spatially inhomogeneous periodic solutions.

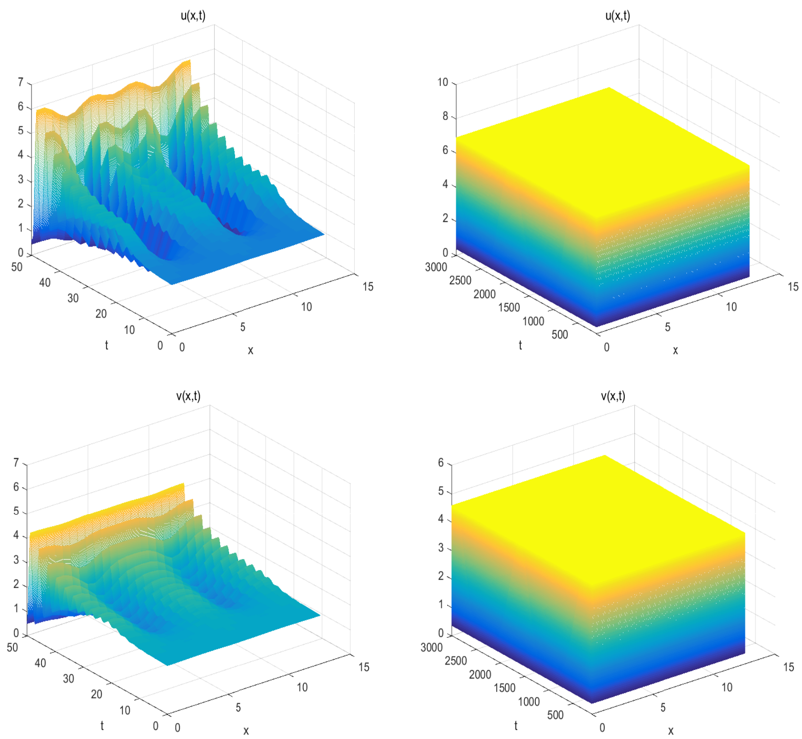


Figure 6. For $(\mu_1, \mu_2) = (1.0, -2.0) \in D_5$, the system (2.1) has a stable spatially homogeneous periodic solution and a pair of unstable spatially inhomogeneous steady states. The initial values are $u(x, 0) = 2.0 + 0.2\cos(4x)$, $v(x, 0) = 2.0 - 0.2\cos(4x)$.

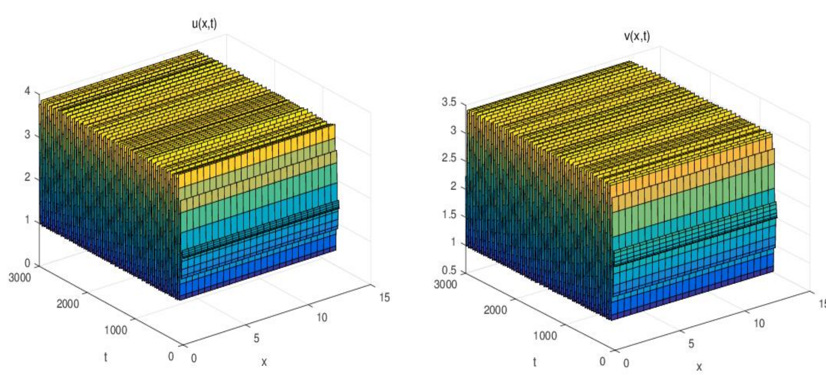


Figure 7. When $(\mu_1, \mu_2) = (-1.0, -0.5) \in D_6$, there is a stable spatially homogeneous periodic solution. The initial values are $u(x, 0) = 2.0 + 0.01$, $v(x, 0) = 2.0 - 0.01$.

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