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BIFURCATION, A-PRIORI BOUND AND NEGATIVE SOLUTIONS FOR THE COMPLEX HESSIAN EQUATION*

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Abstract This paper establishes global bifurcation and eigenvalue results for the following complex k-Hessian equation

$$\begin{cases} S_k\left(u_{i\bar{j}}\right)=\lambda^k f(-u) & \text{in } B,\\ u=0 & \text{on } \partial B.\end{cases}$$

The existence/nonexistence, uniqueness and multiplicity of radially symmetric negative solutions are investigated. Moreover, a-priori bound of radially symmetric negative solutions is also obtained.

Keywords Bifurcation, a-priori bound, complex Hessian equation.

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1. Introduction

In this paper, we study the following complex k-Hessian equation

$$\begin{cases} S_k\left(u_{i\bar{j}}\right)=\lambda^k f(-u) & \text{in } B,\\ u=0 & \text{on } \partial B,\end{cases} \tag{1.1}$$

where the complex k-Hessian operator $S_k\left(u_{i\bar{j}}\right)$ is the k-th symmetric polynomial of eigenvalues of the complex Hessian matrix $\left(u_{i\bar{j}}\right)=\left(\frac{\partial^2 u}{\partial z^i\partial \bar{z}^j}\right)$, B is the unit ball of $\mathbb{C}^N$ with $N\geq 1$, $k\in\{1,\ldots,N\}$, λ is a nonnegative parameter and $f:[0,+\infty)\to[0,+\infty)$ is a continuous function with $f(s)>0$ for $s>0$.

Notice that when $k=N$, the complex k-Hessian equation is reduced to the complex Monge-Ampère equation, which have been studied by many famous mathematicians to obtain the existence, uniqueness, regularity and the qualitative properties of solutions; for example [3,4,7,9,13,14,17–20,31] and the references therein. Meanwhile, some mathematicians have also got several celebrated results for the

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complex k-Hessian equation; see [5, 21, 29, 32] and the references therein. In [8, 28], the authors studied the existence of the real k-Hessian equation. J. Sánchez [26] studied asymptotic behavior of solutions of a k-Hessian evolution equation. The authors of [23] investigated the existence for a k-Hessian equation involving supercritical growth. The authors of [2] studied the existence for Polyharmonic k-Hessian equations in $\mathbb{R}^N$. But so far we haven't seen any systematic investigations on the complex k-Hessian equation using bifurcation method. Here in this paper we study the existence/nonexistence, uniqueness and multiplicity of radially symmetric negative solutions of the complex k-Hessian equation (1.1) by bifurcation method.

Let $r = |z|$, $C_n^m = n!/((n-m)!m!)$ be the combinatorial constant. It is well known that a radially symmetric solution of problem (1.1) is equivalent to a solution of the following problem

$$\begin{cases} \left(r^{2N-k}\left(w'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f(-w), \ r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{1.2}$$

A solution to problem (1.2) is a function of $C^2[0,1]$ that satisfies problem (1.2).

It is easy to verify that any solution of problem (1.2) is negative and strictly increasing. Problem (1.2) can be transformed into the following equivalent problem if we make $v = -w$,

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f(v), \ r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{1.3}$$

Decompose f into $f(s) = s^k + g(s)$, we can set up a global bifurcation result for problem (1.3),

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \left(v^k + g(v)\right), \ r \in (0,1), \\ v'(0) = v(1) = 0, \end{cases} \tag{1.4}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbb{R}_+ = [0, +\infty)$, and $\lim_{s\to 0^+} g(s)/s^k = 0$. Obviously, problem (1.4) always admits the trivial solution $v \equiv 0$.

To study problem (1.4), we need consider the following eigenvalue problem

$$\begin{cases} -\left(r^{2N-p+1}\left|v'\right|^{p-2}v'\right)' = \lambda^{p-1} \frac{N2^{k+1}}{C_N^k} r^{2N-1}|v|^{p-2}v, \ r \in (0,1), \\ v'(0) = v(1) = 0 \end{cases} \tag{1.5}$$

for any $p \in [2, N+1]$. For problem (1.5), we have the following theorem.

Theorem 1.1. *Problem (1.5) possesses a unique eigenvalue $\lambda = \lambda_1(p)$ such that the corresponding eigenfunctions have one sign, which is unique up to a multiplication. Moreover, $\lambda_1(p)$ is minimal, isolated and continuous with respect to p.*

In particular, taking $p = k+1$, it follows from Theorem 1.1 that the following eigenvalue problem

$$\begin{cases} S_k\left(u_{i\bar{j}}\right) = \lambda^k(-u)^k & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

has an eigenvalue $\lambda_1 := \lambda_1(k+1)$ such that the corresponding eigenfunction is negative and radially symmetric, which has pointed out in [21, Remark 1] without proof.

Let X be the Banach space $C[0,1]$ endowed with the norm $\|v\| = \max_{r\in[0,1]} |v(r)|$, $X^+ = \{v \in X : v \geq 0\}$ with the deduced norm of X and P^+ be the set of functions in X^+ which are positive in $[0,1)$. Also, set $K^+ = \mathbb{R} \times P^+$ under the product topology.

For problem (1.4), we have the following theorem.

Theorem 1.2. *The pair $(\lambda_1, 0)$ is a bifurcation point of problem (1.4). Moreover, the associated bifurcation branch $\mathcal{C}$ is unbounded and such that $\mathcal{C} \subseteq (K^+ \cup \{(\lambda_1, 0)\})$.*

Note that problem (1.5) is linear when $p = 2$. So by virtue of the index formula of an isolated zero [12] and the invariance of the Leray-Shauder degree under a compact homotopy, we can establish an index jumping result for problem (1.5). Then by this index jumping result, we can prove an index jumping result involving problem (1.4) which guarantees $(\lambda_1, 0)$ being a bifurcation point of problem (1.4). This is the reason why we introduce problem (1.5).

If $\lim_{s\to+\infty} f(s)/s^k = +\infty$, we call that f is superlinear. From now on, when f is superlinear, we always assume that f satisfies the following subcritical growth condition

$$|f(s)| \leq C\left(1 + |s|^p\right)$$

for some $p \in (k, k_*]$ and positive constant C, where

$$k_* = \begin{cases} \frac{2k(N+1)+1-k^2}{2N-2k} & \text{if } k < N, \\ +\infty & \text{if } k = N. \end{cases}$$

Note that problem (1.1) has no nontrivial non-positive solution (see [21, Theorem 2]) if $\lambda = 1$ and $f(s) = s^q$ with $q \geq k^*$ and

$$k^* = \begin{cases} \frac{(N+1)k}{N-k} & \text{if } k < N, \\ +\infty & \text{if } k = N, \end{cases}$$

where k^* is the critical exponent for the complex k-Hessian operator which was determined in [21]. It is easy to see that $k < k_* \leq k^*$. And then for convenience, we call k_* the *lower critical exponent* for the k-Hessian operator.

Now we have the following a-priori bound.

Theorem 1.3. *Assume that f is superlinear and satisfies the lower subcritical growth condition. Given a compact set $\Lambda \subset [0, +\infty)$, let u be any radially symmetric negative solution of problem* (1.1) *with $\lambda \in \Lambda$. Then there exists a constant C, independent of u, such that $\|u\| \leq C$.*

We use the blow up method introduced by Gidas and Spruck [15] in combination with the Liouville-type Theorems in [27] to prove Theorem 1.3. And then from Theorems 1.1–1.3, we get the intervals of the parameter λ, which guarantee the existence/nonexistence of single or multiple radially symmetric negative solutions of problem (1.1). To obtain the uniqueness, we propose an identity which is called *complex Hessian identity*. Then by Implicit Function Theorem, under some more strict assumptions on f, we show that the radially symmetric negative solution branch of problem (1.1) can be a smooth curve and that the radially symmetric negative solution is decreasing with respect to λ.

The rest of this paper is arranged as follows. In Section 2, we present the proof of Theorem 1.1 and an index jumping result. Section 3 is mainly devoted to the proofs of Theorems 1.2–1.3. Meanwhile, we also introduce a complex Hessian identity and establish a Sturm type comparison result in the same section. In Section 4, we find the intervals for the parameter λ which ensure existence/nonexistence of single or multiple radially symmetric negative solutions for problem (1.1) under some suitable assumptions on f. In the last Section, under some more strict assumptions on f, we show the uniqueness of radially symmetric negative solutions for problem (1.1).

2. Proof of Theorem 1.1

Set $E = \{v \in C^1[0,1] : v(1) = 0\}$. Let W_p be the real Banach space obtained by completing E under the following norm

$$\|v\|_p = \left(\int_0^1 r^{2N+1-p} \left|v'\right|^p \, dr\right)^{\frac{1}{p}}.$$

Then, by [24, Example 6.8], we have the following Sobolev type inequality.

Proposition 2.1. *There exists a constant C such that*

$$C\left(\int_0^1 r^{2N+1-p} \left|v'\right|^p \, dr\right)^{1/p} \geq \left(\int_0^1 r^{2N-1} |v|^q \, dr\right)^{1/q}$$

for any $v \in W_p$, where $q \in [1, +\infty)$.

For $q \geq 1$, define

$$L^q\left(r^{2N-1}\, dr\right) = \left\{v \in L^1(0,1) : \left(\int_0^1 r^{2N-1} |v|^q \, dr\right)^{1/q}\right\} < +\infty.$$

Then by Proposition 2.1, the following embedding result holds.

Proposition 2.2. *W_p is continuously embedded in $L^q\left(r^{2N-1}\, dr\right)$ for all $1 \leq q < +\infty$, further the embedding is compact for $q < p^*$. where*

$$p^* = \begin{cases} \frac{Np}{N-p+1} & \text{if } p < N+1, \\ +\infty & \text{if } p = N+1. \end{cases}$$

Proof. The continuous embedding is the direct corollary of Proposition 2.1. So it is sufficient to show the compact embedding. Let $S \subseteq W_p$ be a set bounded by a positive constant C. For any $v \in S$, we have that

$$\begin{aligned} |v(r+t) - v(r)| &\leq \int_r^{r+t} \left|v'(\tau)\right| \, d\tau \\ &\leq C\left(\int_0^1 r^{2N+1-p} \left|v'\right|^p \, dr\right)^{1/p} \left|(r+t)^{\frac{2(p-1)-2N}{p-1}} - r^{\frac{2(p-1)-2N}{p-1}}\right|^{\frac{p-1}{p}}. \end{aligned}$$

If $p = N + 1$, then

$$|v(r+t) - v(r)| = 0.$$

Thus,

$$\int_0^1 r^{2N-1}|v(r+t)-v(r)|^q\,dr \le C\int_0^1 r^{2N-1}\left|(r+t)^{\frac{2(p-1)-2N}{p-1}} - r^{\frac{2(p-1)-2N}{p-1}}\right|^{\frac{q(p-1)}{p}}\,dr = 0,$$

from which we get that S is relatively compact in $L^q\left(r^{2N-1}\,dr\right)$.

If $p < N + 1$, in view of $q < p^*$ and the Lebesgue dominated convergence theorem, we obtain that

$$\int_0^1 r^{2N-1}\left|(r+t)^{\frac{2(p-1)-2N}{p-1}} - r^{\frac{2(p-1)-2N}{p-1}}\right|^{\frac{q(p-1)}{p}}\,dr \to 0$$

as $t \to 0$. Note that the limit should be understood as one-side limit when $r = 0$ or $r = 1$. We thus have that

$$\int_0^1 r^{2N-1}|v(r+t) - v(r)|^q\,dr \to 0$$

as $t \to 0$ uniformly in $v \in S$, which shows that S is relatively compact in $L^q\left(r^{2N-1}\,dr\right)$. □

Moreover, we give the following inclusion relations.

Proposition 2.3. *$W_{p_1} \subseteq W_{p_2}$ and $L^{p_1}\left(r^{2N-1}\,dr\right) \subseteq L^{p_2}\left(r^{2N-1}\,dr\right)$ hold for any p_1, $p_2 \in [2, N+1]$ with $p_2 \le p_1$.*

Proof. For any $v \in W_{p_1}$, letting $\Omega = \{r \in (0,1) : |v'| \le 1\}$, we obtain that

$$\begin{aligned}\int_0^1 r^{2N+1-p_2}\left|v'\right|^{p_2}\,dr &= \int_\Omega r^{2N+1-p_2}\left|v'\right|^{p_2}\,dr + \int_{(0,1)\setminus\Omega} r^{N+1-p_2}\left|v'\right|^{p_2}\,dr\\ &\le 1 + \int_{(0,1)\setminus\Omega} r^{2N+1-p_2}\left|v'\right|^{p_2}\,dr\\ &\le 1 + \int_{(0,1)\setminus\Omega} r^{2N+1-p_1}\left|v'\right|^{p_1}\,dr < +\infty.\end{aligned}$$

Thus $u \in W_{p_2}$. The second inclusion can be proved similarly. □

We call $v \in W_p$ the generalized solution of problem (1.5) if for any $\phi \in W_p$,

$$\int_0^1 r^{2N+1-p}\left|v'\right|^{p-2}v'\phi'\,dr = \lambda^{p-1}\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^{p-2}v\phi\,dr \tag{2.1}$$

holds. The following result is the regularity of the generalized solution.

Proposition 2.4. *Suppose v be any generalized solution of problem (1.5). Then $v \in C^2[0,1]$ satisfies problem (1.5) in the classical sense.*

Proof. We first show that v is bounded. We define

$$\Phi(z) = \begin{cases} z^s - 1 & \text{for } z \in [1,\rho],\\ az + b & \text{for } z > \rho\end{cases}$$

for $s \geq 1$ and $\rho \geq 1$, where a, b are two constants such that $\Phi \in C^1[1, +\infty)$. It is clear that $|\Phi'(t)|$ is increasing. Set $v^+ = \max\{v, 0\}$, and choose $\int_1^{v^++1} |\Phi'(t)|^p \, dt :=$ φ as a test function. Substituting φ into (2.1) we can get that

$$\int_0^1 r^{2N+1-p} \left| \frac{d}{dr} \Phi\left(v^+ + 1\right) \right|^p dr \leq C \int_0^1 r^{2N-1} |v|^{p-2} v \left|\Phi'\left(v^+ + 1\right)\right|^p v^+ \, dr$$

by virtue of the monotonicity of $|\Phi'(t)|$. Taking a fixed $\beta > p$, by Proposition 2.1, 2.3 and the above inequality, we obtaind that

$$\left(\int_0^1 \Phi^\beta\left(v^+ + 1\right) r^{2N-1} \, dr\right)^{1/\beta} \leq C \left(\int_0^1 \left(\Phi'\left(v^+ + 1\right)\left(v^+ + 1\right)\right)^p r^{2N-1} \, dr\right)^{1/p}$$

for some positive constant C. Let $\rho \to +\infty$, then

$$\left\|v^+ + 1\right\|_{L^{s\kappa p}(r^{2N-1}\, dr)} \leq C(s) \left\|v^+ + 1\right\|_{L^{sp}(r^{2N-1}\, dr)},$$

where $\kappa = \beta/p > 1$. Set $s = \kappa^m$, $m \geq 1$, an iteration yields

$$\sup v^+ \leq C(s) \left(1 + \|v\|_{L^p(r^{2N-1}\, dr)}\right).$$

Similarly, if set $v^- = -\min\{v, 0\}$, we can also show the above estimate for v^-. It means that v is bounded.

For η being smooth, take $\varphi = \int_r^1 \eta(t) \, dt$ as a test function in (2.1). We obtain that

$$-\int_0^1 r^{2N+1-p} \left|v'\right|^{p-2} v'\eta(r) \, dr = \lambda^{p-1} \frac{N2^{k+1}}{C_N^k} \int_0^1 \eta(r) \int_0^r \left(\tau^{2N-1}|v|^{p-2}v\right) d\tau \, dr$$

by an integration by parts. And

$$-r^{2N+1-p} \left|v'\right|^{p-2} v' = \lambda^{p-1} \frac{N2^{k+1}}{C_N^k} \int_0^r \left(\tau^{2N-1}|v|^{p-2}v\right) d\tau \text{ a.e.}$$

The fact that v is bounded shows that $\int_0^r \left(\tau^{2N-1}|v|^{p-2}v\right) d\tau$ is continuous. Next, by some simple calculations, we get that $v \in C^2[0, 1]$, $v'(0) = \lim_{r\to 0^+} v'(r) = 0$ and that v satisfies problem (1.5). □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Define the functional

$$J(v) = \int_0^1 \frac{1}{p} r^{2N+1-p} \left|v'\right|^p \, dr - \lambda^{p-1} \frac{N2^{k+1}}{pC_N^k} \int_0^1 r^{2N-1} |v|^p \, dr$$

on W_p. It is easy to see that the critical points of J are the generalized solutions of problem (1.5). Taking

$$f_1(v) = \int_0^1 \frac{1}{p} r^{2N+1-p} \left|v'\right|^p \, dr \text{ and } f_2(v) = \frac{N2^{k+1}}{pC_N^k} \int_0^1 r^{2N-1}|v|^p \, dr,$$

we consider the eigenvalue problem

$$A(v) = \mu B(v), \tag{2.2}$$

where $A = \partial f_1$ and $B = \partial f_2$ denote the sub-differential of f_1 and f_2, respectively.

According to Proposition 2.1, for any $v \in W_p$ with $v \not\equiv 0$ and some positive constant c, we have that

$$\frac{f_1(v)}{f_2(v)} \geq c. \tag{2.3}$$

Set $\mu_1(p) = \inf_{v \in W_p, v \not\equiv 0} f_1(v)/f_2(v)$, then equation (2.2) has no nontrivial solution for $\mu \in (0, \mu_1(p))$.

We use the abstract results of [16] to prove the desired conclusions. Let $W_p = V$ and $\Omega = (0,1)$, and denote by $\Phi(V)$ the family of all proper lower semi-continuous convex functions φ from V into $(-\infty, +\infty]$. Next we verify the conditions (A0)–(A4) of [16].

Clearly $f_1(0) = f_2(0) = 0$. Then f_1 and f_2 are proper functions. It is easy to see that f_1 and f_2 are lower semi-continuous. Furthermore, we get f_1 and f_2 are strictly convex functions by Lemma 2.27 of [1]. Thus f_1, $f_2 \in \Phi(V)$. Since $D(f_1) = D(f_2) = V$ and $V \subset L^1_{\text{loc}}(0,1)$ are clear, then we have verified condition (Al). For $v \neq 0$, let $R(v) = f_2(v)/f_1(v)$. The fact that R is even gives that $R(|v|) \geq R(v)$ for all $v \in V$. We can easily get that $f_1(v) \geq 0$ for all $v \in V$, and $f_1(v) = 0$ is equivalent to $v = 0$. Inequality (2.3) means that there exists $u \in V$ such that $u \neq 0$ and $R(u) = \sup\{R(v); v \in V, v \neq 0\}$. It follows that condition (A2) is verified. Let $\alpha = p$, we have $f_i(tv) = t^\alpha f_i(v)$ for all $v \in V^+ = \{w \in V; w(r) \geq 0 \text{ a.e. } r \in (0,1)\}$, $\forall t > 0$, $i = 1,2$. Naturally condition (A3) is verified. Set $(u \vee w)(r) = \max\{u(r), w(r)\}$, $(u \wedge w)(r) = \min\{u(r), w(r)\}$, $I_1 = \{r \in [0,1] : u(r) \geq w(r)\}$, and $I_2 = \{r \in [0,1] : u(r) < w(r)\}$ for any u, $v \in V^+$, then

$$\begin{aligned}
f_1(u \vee w) + f_1(u \wedge w) &= \int_0^1 \frac{1}{p} r^{2N+1-p} \left|(u \vee w)'(r)\right|^p dr + \int_0^1 \frac{1}{p} r^{2N+1-p} \left|(u \wedge w)'(r)\right|^p dr \\
&= \int_{I_1} \frac{1}{p} r^{2N+1-p} \left|u'\right|^p dr + \int_{I_2} \frac{1}{p} r^{2N+1-p} \left|w'\right|^p dr \\
&\quad + \int_{I_1} \frac{1}{p} r^{2N+1-p} \left|w'\right|^p dr + \int_{I_2} \frac{1}{p} r^{2N+1-p} \left|u'\right|^p dr \\
&= \int_0^1 \frac{1}{p} r^{2N+1-p} \left|u'\right|^p dr + \int_0^1 \frac{1}{p} r^{2N+1-p} \left|w'\right|^p dr \\
&= f_1(u) + f_1(w).
\end{aligned}$$

Similarly, $f_2(u \vee w) + f_2(u \wedge w) = f_2(u) + f_2(w)$ holds. Condition (A4) is therefore verified. The last one is condition (A0). For $\varphi_p(s) = |s|^{p-2}s$, we have that

$$r^{2N+1-p} \varphi_p(v') = -\mu \frac{N 2^{k+1}}{C_N^k} \int_0^r \tau^{2N-1} v^p \, d\tau < 0 \tag{2.4}$$

for the nonnegative nontrivial solution v of equation (2.2). So $v(r) > 0$, $r \in [0,1)$. According to Proposition 2.4, $v \in C(0,1) \cap L^\infty(0,1)$. Final condition (A0) is verified.

From Theorem I in [16], we know that $\mu_1(p)$ is simple, that is equation (2.2) has an one-sign solution which is unique up to multiplication by a number. Meanwhile, $w \in C^1[0,1]$ and $w'(1) < 0$ for every positive solution w of equation (2.2) on the basis of (2.4) and Proposition 2.4. Further B is a monotone operator. From [16, Theorem II], we obtain that equation (2.2) has a positive solution if and only if $\mu = \mu_1(p)$. Therefore, for $\lambda \in (0, \lambda_1(p))$ with $\lambda_1(p) = \mu_1^{1/(p-1)}$, problem (1.5) has no nontrivial solution, but has an one-sign solution if and only if $\lambda = \lambda_1(p)$. And $\lambda_1(p)$ is simple.

Next we show that $\lambda_1(p)$ is the unique eigenvalue in $(0, \delta_p)$ for some $\delta_p > \lambda_1(p)$. It is enough to show that $\lambda_1(p)$ is right-isolated. By contradiction, we assume that there is a sequence of eigenvalues $\lambda_n \in (\lambda_1(p), \delta_p)$ that converges to $\lambda_1(p)$. Let v_n be the corresponding eigenfunctions to λ_n. Set

$$\psi_n = \frac{v_n}{\left(\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}\left|v_n\right|^p\,dr\right)^{\frac{1}{p}}}.$$

Then $\|\psi_n\|_p^p = \lambda_n^{p-1}$. Thus $\psi_n \in W_p$ is bounded. It follows that there is a subsequence, still denoted by ψ_n, such that $\psi_n \rightharpoonup \psi$ as $n \to +\infty$ for some $\psi \in W_p$. Since f_1 is sequentially weakly lower semi-continuous, then

$$\int_0^1 r^{2N+1-p}\left|\psi'\right|^p\,dr \le \liminf_{n\to+\infty}\int_0^1 r^{2N+1-p}\left|\psi_n'\right|^p\,dr = \lambda_1^{p-1}(p).$$

As $n \to +\infty$, we have that

$$1 = \frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}\left|\psi_n\right|^p\,dr \to \frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}\left|\psi\right|^p\,dr$$

by Proposition 2.2. So one has that

$$\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|\psi|^p\,dr = 1.$$

We thus obtain that

$$\int_0^1 r^{2N+1-p}\left|\psi'\right|^p\,dr = \mu_1(p).$$

It follows that ψ must be either positive or negative in $(0,1)$. Let us say without loss of generality that $\psi > 0$ in $(0,1)$, then we get $\psi_n \ge 0$ for n large enough, which is a contradiction.

Finally, we prove that the eigenvalue function $\lambda_1 : [2, N+1] \to \mathbb{R}$ is continuous. To do that we just have to prove that $\mu_1(p) : [2, N+1] \to \mathbb{R}$ is continuous. And this can be obtained by the variational characterization of $\mu_1(p)$ that

$$\mu_1(p) = \sup\left\{\mu > 0 : \frac{\mu N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^p\,dr \le \int_0^1 r^{2N+1-p}\left|v'\right|^p\,dr \text{ for } v \in E\right\}. \tag{2.5}$$

Suppose $\{p_n\}_{n=1}^{+\infty}$ is a sequence in $[2, N+1]$, and $\{p_n\}_{n=1}^{+\infty}$ converges to $p \in [2, N+1]$. Then

$$\mu_1(p_n)\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^{p_n}\,dr \le \int_0^1 r^{2N+1-p_n}\left|v'\right|^{p_n}\,dr$$

for any $v \in E$ by (2.5). The application of the Lebesgue dominated convergence theorem gives that

$$\limsup_{n\to+\infty}\mu_1(p_n)\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^p\,dr \le \int_0^1 r^{2N+1-p}\left|v'\right|^p\,dr. \tag{2.6}$$

Combining (2.6) and (2.5) we get

$$\limsup_{n\to+\infty}\mu_1(p_n) \le \mu_1(p).$$

Next we prove

$$\liminf_{n\to+\infty}\mu_1(p_n)\geq\mu_1(p). \tag{2.7}$$

Suppose $\{p_m\}_{m=1}^{+\infty}$ is a subsequence of $\{p_n\}_{n=1}^{+\infty}$ that satisfies $\lim\limits_{m\to+\infty}\mu_1(p_m)=\liminf\limits_{n\to+\infty}\mu_1(p_n)$. Choose $\varepsilon_0>0: p-\varepsilon_0>1$, and $p-\varepsilon<p_m<p+\varepsilon<p^*$ for each $0<\varepsilon<\varepsilon_0$ and $m\in\mathbb{N}$ large enough. Select $v_m\in W_{p_m}$ so that

$$\int_0^1 r^{2N+1-p_m}\left|v_m'\right|^{p_m}\,dr=1 \tag{2.8}$$

and

$$\int_0^1 r^{2N+1-p_m}\left|v_m'\right|^{p_m}\,dr=\mu_1(p_m)\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}\left|v_m\right|^{p_m}\,dr.$$

Then $\{v_m\}_{m=1}^{+\infty}\in W_{p_m}$ is bounded. It follows from Proposition 2.3 that v_m is bounded in $W_{p-\varepsilon}$ for m large enough. Up to a subsequence, assume $v_m\rightharpoonup v$ in $W_{p-\varepsilon}$ as $m\to+\infty$. By Proposition 2.2, we have $v_m\to v$ in $L^{p+\varepsilon}\left(r^{2N-1}\,dr\right)$ as $m\to+\infty$. By the Minkowski inequality, we have that

$$\int_0^1 r^{2N-1}\left|v_m\right|^{p}\,dr\to\int_0^1 r^{2N-1}\left|v\right|^{p}\,dr.$$

Thus $\|v_m\|_{p_m}^{p_m}\to\|v\|_p^p$ as $m\to+\infty$. Further,

$$\liminf_{n\to+\infty}\mu_1(p_n)\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^p\,dr=1. \tag{2.9}$$

Using (2.8), we have that

$$\|v\|_{p-\varepsilon}^{p-\varepsilon}\leq\liminf_{m\to+\infty}\|v_m\|_{p-\varepsilon}^{p-\varepsilon}\leq 1.$$

Let $\varepsilon\to0^+$. We can obtain that

$$\|v\|_p\leq 1 \tag{2.10}$$

by using the Fatou Lemma, immediately $v\in W_p$. By (2.9) and (2.10),

$$\liminf_{n\to+\infty}\mu_1(p_n)\frac{N2^{k+1}}{C_N^k}\int_0^1 r^{2N-1}|v|^p\,dr\geq\int_0^1 r^{2N+1-p}\left|v'\right|^p\,dr,$$

which means (2.7). Therefore

$$\lim_{n\to+\infty}\mu_1(p_n)=\mu_1(p). \tag{2.11}$$

This proves the continuity of eigenvalue function $\lambda_1:[2,N+1]\to\mathbb{R}$. □

Define the map $T_p: X\to X$ as follows.

$$T_pv=\int_1^r\varphi_{p'}\left(\frac{N2^{k+1}}{C_N^k}s^{p-2N-1}\int_s^0\tau^{2N-1}\varphi_p(v)\,d\tau\right)ds,\ 0\leq r\leq 1,$$

where $p' = p/(p-1)$. By the Arzelà-Ascoli Theorem, we can easily prove that T_p is continuous and compact. Now we can write problem (1.5) as the equivalent problem $v = \lambda T_p v$. Let

$$\lambda_2(p) = \inf\left\{\lambda > \lambda_1(p) : \lambda \text{ is an eigenvalue of problem (1.5)}\right\}.$$

By Theorem 1.1, we see that $\lambda_1(p) < \lambda_2(p)$.

Lemma 2.1. *For any interval $[a,b]$ that belongs to $[2, N+1]$, there exists $\delta > 0$ such that problem (1.5) has no eigenvalue in $(\lambda_1(p), \lambda_1(p)+\delta]$ for all $p \in [a,b]$.*

Proof. Instead, assume that there are sequences $\{p_n\}_{n=1}^{+\infty}$ in $[2, N+1]$, $\{\lambda_n\}_{n=1}^{+\infty}$ in $\mathbb{R}_+$, and $\{u_n\}_{n=1}^{+\infty}$ in $X \setminus \{0\}$ that satisfy $\lim_{n\to+\infty} p_n = p \in [2, N+1]$, $\lambda_n > \lambda_1(p_n)$, $\lim_{n\to+\infty}(\lambda_n - \lambda_1(p_n)) = 0$, and

$$u_n = \lambda_n T_{p_n}(u_n),\ n \in \mathbb{N}.$$

Now set $w_n = u_n / \|u_n\|$. Then

$$w_n = \lambda_n \int_1^r \varphi_{p_n'}\left(\frac{N2^{k+1}}{C_N^k} s^{p_n-2N-1} \int_s^0 \tau^{2N-1}\varphi_{p_n}(w_n)\, d\tau\right) ds := \lambda_n T(w_n),\ 0 \le r \le 1.$$

It is not difficult to show that $T(w_n)$ is completely continuous via the Arzelà-Ascoli theorem. So, up to a subsequence, we have that $w_n \to w$ in X as $n \to +\infty$. It follows from Theorem 1.1 that

$$w = \lambda_1(p) \int_1^r \varphi_{p'}\left(\frac{N2^{k+1}}{C_N^k} s^{p-2N-1} \int_s^0 \tau^{2N-1}\varphi_p(w)\, d\tau\right) ds,\ 0 \le r \le 1.$$

Thus w must have one sign in $(0,1)$. Therefore, u_n has one sign for n large enough, which contradicts the conclusions of Theorem 1.1. □

Clearly, for arbitrary R-ball $B_R(0)$ and $\lambda \in (0, \lambda_1(p)+\delta) \setminus \{\lambda_1(p)\}$, the Leray-Schauder degree $\deg(I - \lambda T_p, B_R(0), 0)$ is well defined, where δ is given as Lemma 2.1. We end this section by showing an index jumping result, which will be used later.

Theorem 2.1. *For any $R > 0$ and $p \in [2, N+1]$, there is*

$$\deg(I - \lambda T_p, B_R(0), 0) = \begin{cases} 1, & \text{if } \lambda \in (0, \lambda_1(p)), \\ -1, & \text{if } \lambda \in (\lambda_1(p), \lambda_1(p)+\delta). \end{cases}$$

Proof. First consider the case of $\lambda > \lambda_1(p)$. Note that $\lambda_1(p)$ is continuous, then we have a continuous function $\chi : [2, N+1] \to \mathbb{R}$ and $q \in [2, N+1]$ make $\lambda_1(q) < \chi(q) < \lambda_1(q) + \delta$ and $\lambda = \chi(p)$ by Lemma 2.1. Let

$$d(q) = \deg(I - \chi(q)T_q, B_R(0), 0).$$

Then $d(2) = -1$ from the fact of T_2 is compact and linear according to [12, Theorem 8.10]. If we define

$$G(q, v) = \chi(q)T_q(v).$$

Then $G : [2, N+1] \times X \to X$ is completely continuous by the Arzelà-Ascoli theorem. Now we use the invariance of the Leray-Shauder degree under a compact homotopy and get $d(q) \equiv$ *constant*, $q \in [2, N+1]$. Further $\deg(I - \lambda T_p, B_R(0), 0) = d(2) = -1$. The proof of the case of $\lambda < \lambda_1(p)$ is exactly the same, so we omit it. □

3. Proofs of Theorems 1.2–1.3

Let $T_g : X^+ \to X^+$ be as follows.

$$T_g v(r) = \int_1^r \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} \left(v^k + g(v)\right) \, d\tau \right)^{1/k} ds, \ 0 \leq r \leq 1.$$

Then T_g is completely continuous. Write problem (1.4) as the equivalent equation $v = \lambda T_g v$. We can see that if v is a fixed point of λT_g in X^+, then v belongs to $C^2[0,1]$ and is a solution to problem (1.4) in the classical sense.

Lemma 3.1. *Suppose (λ, v) is a solution to problem (1.4) in $\mathbb{R} \times X^+$, and v has a double zero. Then $v \equiv 0$.*

Proof. The desired conclusion can be obtained directly from the expressions of $-v'(r)$ and $v(r)$ using the monotonicity of v. □

Consider the problem

$$\begin{cases} \left(r^{2N-k} \left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} v^k, \ r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{3.1}$$

Taking $p = k+1$ in Section 2, we know that problem (3.1) possesses the unique principal eigenvalue $\lambda_1(k+1) := \lambda_1$ which is positive, simple and isolated. Define $T_k : X^+ \to X^+$ by

$$T_k v = \int_1^r \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} v^k \, d\tau \right)^{1/k} ds, \ 0 \leq r \leq 1.$$

We can easily know that $I - \lambda T_k$ is a completely continuous vector field in X^+. Thus for arbitrary R-ball $B_R(0)$ of X^+ and $\lambda \in (0, \lambda_1 + \delta] \setminus \{\lambda_1\}$, the Leray-Schauder degree $\deg\left(I - \lambda T_k, B_R(0), 0\right)$ is well defined, where δ can be found in Lemma 2.1.

Lemma 3.2. *For $\lambda \in (0, \lambda_1 + \delta] \setminus \{\lambda_1\}$ and any $R > 0$, we have*

$$\deg\left(I - \lambda T_k, B_R(0), 0\right) = \begin{cases} 1, & \text{if } \lambda \in (0, \lambda_1), \\ -1, & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta]. \end{cases}$$

Proof. For any $v \in X^+$, it is clear that $T_k(v) = T_{p-1}(v)$. we can get the desired conclusion right away as long as we take $p = k+1$ in Theorem 2.1. □

Next, we present a Rabinowitz type global bifurcation result [25]. Let E be a real Banach space with the norm $\|\cdot\|$, $\mathcal{O}$ be an open subset of $\mathbb{R} \times E$, $\mathrm{pr}_E(\mathcal{O})$ be the projection of $\mathcal{O}$ on E and $\mathrm{pr}_{\mathbb{R}}(\mathcal{O})$ be the projection of $\mathcal{O}$ on $\mathbb{R}$. Consider the following equation

$$u = L(\lambda)u + H(\lambda, u) := G(\lambda, u), \tag{3.2}$$

where λ varies in $\mathrm{pr}_{\mathbb{R}}\left(\overline{\mathcal{O}}\right)$, the map $\lambda \to L(\lambda)$ is continuous, $L(\cdot) : \mathrm{pr}_E\left(\overline{\mathcal{O}}\right) \to \mathrm{pr}_E\left(\overline{\mathcal{O}}\right)$ is a homogeneous completely continuous operator and $H : \overline{\mathcal{O}} \to E$ is compact with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals in $\overline{\mathcal{O}}$. Let

$$\mathscr{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ satisfies equation (3.2) and } u \not\equiv 0\ \}}^{\mathcal{O}}.$$

μ is called an eigenvalue of

$$u = L(\mu)u, \ u \in E \tag{3.3}$$

if there exists $\varphi \in E \setminus \{0\}$ such that $\varphi = L(\mu)\varphi$. Let Σ denote the set of real eigenvalues of equation (3.3). So the Leray-Schauder degree $\deg(I - L(\lambda), B_R(0), 0)$ is well defined for arbitrary r-ball $B_R(0)$ in $\mathcal{O}$ and $\lambda \notin \Sigma$. By an arguments similar to that of [11, Lemma 2.1] with obvious changes, we can show the following result.

Lemma 3.3. *If $\mu \in pr_{\mathbb{R}}(\mathcal{O}) \cap \Sigma$ such that the Leray-Schauder degree $\deg(I - L(\lambda), B_R(0))$ changes when λ passes μ, then $\mathscr{S}$ possesses a maximal subcontinuum $\mathcal{C}_\mu \subset \overline{\mathcal{O}}$ such that $(\mu, 0) \in \mathcal{C}_\mu$ and one of the following three properties is satisfied by $\mathcal{C}_\mu$:*

(i) $\mathcal{C}_\mu$ is unbounded in $\overline{\mathcal{O}}$;

(ii) meets $\partial\mathcal{O} \setminus \{(\mu, 0)\}$;

(iii) meets $(\overline{\mu}, 0)$, where $\overline{\mu} \in pr_{\mathbb{R}}\left(\overline{\mathcal{O}}\right) \cap \Sigma$ with $\overline{\mu} \neq \mu$.

Now, we give the proofs of Theorems 1.2–1.3.

Proof of Theorem 1.2. For any $v \in X^+$, set $H(v) = T_g(v) - T_k(v)$. Problem (1.4) is therefore equivalent to

$$v = \lambda T_k v + \lambda H(v).$$

Define

$$\widetilde{g}(u) = \max_{0 \le s \le u} |g(s)|.$$

Then $\widetilde{g}$ is nondecreasing with respect to u and

$$\lim_{u \to 0^+} \frac{\widetilde{g}(u)}{u^k} = 0. \tag{3.4}$$

By (3.4), we have that

$$\frac{|g(v)|}{\|v\|^k} \le \frac{\widetilde{g}(v)}{\|v\|^k} \le \frac{\widetilde{g}(\|v\|)}{\|v\|^k} \to 0 \text{ as } \|v\| \to 0, \tag{3.5}$$

which implies $H(v) = o(\|v\|)$ near $v = 0$ in X^+.

By Lemmas 3.2–3.3, we obtain that $(\lambda_1, 0)$ is a bifurcation point of problem (1.4) and the associated bifurcation branch $\mathcal{C}$ in $\mathbb{R} \times X^+$ whose closure contains $(\lambda_1, 0)$ is either unbounded or contains a pair $\left(\overline{\lambda}, 0\right)$ where $\overline{\lambda}$ is another eigenvalue of problem (3.1). For any $(\lambda, v) \in \mathcal{C}$, Lemma 3.1 implies that either $v \equiv 0$ or $v > 0$ in (0,1). We claim that the first alternatives is the only possibility. Suppose by contradiction that there exists $(\lambda_n, v_n) \to \left(\overline{\lambda}, 0\right)$ when $n \to +\infty$ with $(\lambda_n, v_n) \in \mathcal{C}$ and $v_n \not\equiv 0$. Set $w_n = v_n / \|v_n\|$, then w_n solves the problem

$$w = \lambda_n \int_1^r \left(\frac{N 2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} \left(w^k + \frac{g(v_n)}{\|v_n\|^k} \right) d\tau \right)^{1/k} ds.$$

From the fact that T_g is compact, we get some convenient subsequence subsequence $w_n \to w_0$ as $n \to +\infty$. By (3.5), one can get that $\left(\overline{\lambda}, w_0\right)$ solves problem (3.1) with $\|w_0\| = 1$. From Theorem 1.1, we know that w_0 must change its sign. This contradicts $v_n \in P^+$. So we have $\mathcal{C} \subseteq (K^+ \cup \{\lambda_1, 0\})$. $\square$

Proof of Theorem 1.3. We argue by contradiction. Let v_n be a sequence of positive solutions to problem (1.3) with $\lambda = \lambda_n$ such that $\lambda_n \to \lambda_*$ and $\|v_n\| \to +\infty$ as $n \to +\infty$. Define

$$w_n(r) = \frac{1}{M_n} v_n \left(r M_n^{\frac{k-p}{2k}} \right),$$

where $M_n = \|v_n\| = v_n(0)$. Then, by some elementary calculations, we can show that

$$\begin{aligned} w_n(r) &= \lambda_n \int_{M_n^{\frac{p-k}{2k}}}^{r} \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} \frac{f(v_n)}{M_n^p} \, d\tau \right)^{1/k} ds \\ &= \lambda_n \int_{M_n^{\frac{p-k}{2k}}}^{r} \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} \frac{f(v_n)}{v_n^p} w_n^p \, d\tau \right)^{1/k} ds, \ r \in \left[0, M_n^{\frac{p-k}{2k}}\right]. \end{aligned} \tag{3.6}$$

From the lower subcritical growth condition, we see that there exist $p \in (k, k_*]$ and positive constant c such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^p} = c.$$

It implies that

$$\lim_{n \to +\infty} \frac{f(v_n)}{v_n^p} = c. \tag{3.7}$$

By (3.6) and (3.7), we get that for some constant M, which is independent of n,

$$\begin{aligned} 0 \le -w_n' &= \lambda_n \left(\frac{N2^{k+1}}{C_N^k} r^{k-2N} \int_0^r \tau^{2N-1} \frac{f(v_n)}{v_n^p} w_n^p \, d\tau \right)^{1/k} \\ &\le \lambda_n \left(\frac{N2^{k+1}}{C_N^k} \int_0^r \tau^{k-1} \frac{f(v_n)}{v_n^p} w_n^p \, d\tau \right)^{1/k} \\ &\le M. \end{aligned}$$

For any $R > 0$, it can be seen that $\|w_n\|_{C^1[0,R]}$ is uniformly bounded. Thus there is a subsequence, which we still denote by w_n, such that $w_n \to w$ in $C[0, R]$. We can easily see $w(0) = 1$. By an argument similar to that of [15], we get that

$$w(r) = \lambda_* \int_{+\infty}^{r} \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} c w^p \, d\tau \right)^{1/k} ds, \ 0 \le r < +\infty,$$

i.e.

$$-\left(r^{2N-k} |w'|^{k-1} w' \right)' = c \lambda_*^k \frac{N2^{k+1}}{C_N^k} r^{2N-k} w^p.$$

So w is a nontrivial solution of

$$-\Delta_{k+1} u = c \lambda_*^k \frac{N2^{k+1}}{C_N^k} u^p \ \text{ in } \mathbb{R}^{2N-k+1}, \tag{3.8}$$

where $\Delta_m u = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u \right)$ is the well-known m-Laplace operator, $m > 1$. While, by Theorem I and I′ of [27], we know that equation (3.8) has only the trivial solution $u \equiv 0$ when $p \le k_*$. So we deduce a contradiction. $\square$

Lemma 3.4. *Suppose u_i are two solutions of the problems*

$$\begin{cases} \left(r^{2N-k}\left(-u'\right)^k\right)' = b_i u^k, \ i=1,2, \\ u'(0)=u(1)=0, \end{cases} \tag{3.9}$$

respectively when $b_i \in C[0,1](i=1,2)$ satisfy $b_2(r) \geq b_1(r) > 0$ for $r \in (0,1)$. If $u_1 \neq 0$ in $(0,1)$, then either $\exists\, \tau \in (0,1) : u_2(\tau) = 0$, or $b_2 \equiv b_1$ and $\exists\, \mu \neq 0 : u_2 \equiv \mu u_1$.

Proof. We know $u_1 \neq 0$ in $(0,1)$. If $u_2 \neq 0$ in $(0,1)$, and without losing generality to assume that $u_1 > 0$, $u_2 > 0$ in $(0,1)$. Then for $i=1,2$, we have

$$u_i' = -\left(r^{k-2N}\int_0^r b_i(\tau)u_i^k\, d\tau\right)^{1/k} < 0$$

and

$$u_i = \int_r^1 \left(s^{k-2N}\int_0^s b_i(\tau)u_i^k\, d\tau\right)^{1/k} ds$$

from (3.9). Clearly u_1 and u_2 are strictly decreasing in $(0,1]$. Thus $u_i(0) > 0$ and $u_i'(r) < 0, r \in (0,1]$, $i=1,2$.

Set $w = b_2 - b_1$, by some calculation, it's easy to get that

$$\begin{aligned} &\left(\frac{u_1^{k+1}r^{2N-k}\left(-u_2'\right)^k}{u_2^k} - u_1 r^{2N-k}\left(-u_1'\right)^k\right)' \\ =&w u_1^{k+1} + r^{2N-k}\left(\left(-u_1'\right)^{k+1} + k\left(\frac{-u_1u_2'}{u_2}\right)^{k+1} - (k+1)u_1^k u_1'\left(\frac{-u_2'}{u_2}\right)^k\right). \end{aligned} \tag{3.10}$$

Integrate both sides of the above inequality from 0 to 1 and we have

$$\begin{aligned} &\int_0^1 \left(\frac{u_1^{k+1}r^{2N-k}\left(-u_2'\right)^k}{u_2^k} - u_1 r^{2N-k}\left(-u_1'\right)^k\right)' dr \\ =&\int_0^1 \left(w u_1^{k+1} + r^{2N-k}\left(\left(-u_1'\right)^{k+1} + k\left(\frac{-u_1u_2'}{u_2}\right)^{k+1} - (k+1)u_1^k u_1'\left(\frac{-u_2'}{u_2}\right)^k\right)\right) dr. \end{aligned} \tag{3.11}$$

Note that the left-hand side of (3.11) is equal to

$$L := \lim_{r\to 1^-} \frac{u_1^{k+1}r^{2N-k}\left(-u_2'\right)^k}{u_2^k}.$$

In view of the L'Hospital rule, one has that

$$\begin{aligned} L =& \lim_{r\to 1^-} \frac{u_1^{k+1}r^{2N-k}\left(-u_2'\right)^k}{u_2^k} = \lim_{r\to 1^-} \frac{(k+1)u_1^k u_1' r^{2N-k}\left(-u_2'\right)^k + u_1^{k+1}\left(r^{2N-k}\left(-u_2'\right)^k\right)'}{k u_2^{k-1}u_2'} \\ =& \lim_{r\to 1^-} \frac{(k+1)u_1^k u_1' r^{2N-k}\left(-u_2'\right)^k + u_1^{k+1}b_2 u_2^k}{k u_2^{k-1}u_2'} \end{aligned}$$

$$= \lim_{r\to 1^-} \frac{(k+1)u_1^k u_1' r^{2N-k} \left(-u_2'\right)^k}{k u_2^{k-1} u_2'} + \lim_{r\to 1^-} \frac{u_1^{k+1} b_2 u_2^k}{k u_2^{k-1} u_2'}$$
$$= \lim_{r\to 1^-} \frac{(k+1)u_1' r^{2N-k} \left(-u_2'\right)^k}{k u_2'} \lim_{r\to 1^-} \frac{u_1^k}{u_2^{k-1}}.$$

Now we can show $L = 0$. For $k = 1$, $L = 0$ holds. If $k = 2$, $L = 0$ follows from

$$\lim_{r\to 1^-} \frac{u_1^k}{u_2^{k-1}} = \lim_{r\to 1^-} \frac{k u_1'}{(k-1)u_2'} \lim_{r\to 1^-} \frac{u_1^{k-1}}{u_2^{k-2}}.$$

Further we can continue this process $i - 2$ times to get $L = 0$ for any $k = i$, $i \in \{3, \dots, N\}$.

As discussed above, the right-hand side of (3.11) is equal to zero, too. By using the Young inequality we can get that

$$(-u_1')^{k+1} + k\left(\frac{-u_1 u_2'}{u_2}\right)^{k+1} - (k+1)u_1^k u_1' \left(\frac{-u_2'}{u_2}\right)^k \geq 0,$$

and the equals sign above is true if and only if

$$\left(\frac{-u_1'}{u_1}\right)^{k+1} = \left(\frac{-u_2'}{u_2}\right)^{k+1}.$$

And then naturally we get that there is a constant $\mu \neq 0$ that $u_2 \equiv \mu u_1$ and $b_2 \equiv b_1$. □

The identity (3.10) will be used later. For simplicity, we call it *complex Hessian identity*. As a corollary of Lemma 3.4, we obtain the following Sturm type comparison lemma.

Lemma 3.5. *Suppose $u_i (i = 1, 2)$ be two solutions of problem (3.9), respectively when $b_i \in C[0, 1] (i = 1, 2)$ satisfy $b_2 \geq b_1 > 0$ and $b_2 \not\equiv b_1$ on $[0, 1]$. If $u_1 \neq 0$ in $(0, 1)$, then u_2 has at least one zero in $(0, 1)$.*

4. Negative solutions

Define the map $T_f : X^+ \to X^+$ as follows.

$$T_f v(r) = \int_1^r \left(\frac{N 2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} f(v)\, d\tau\right)^{1/k} ds, \ 0 \leq r \leq 1.$$

Then T_f is completely continuous. Then problem (1.3) can be equally rewritten as as $v = \lambda T_f v$. Let $f_0, f_\infty \in \mathbb{R}_+$ be such that

$$f_0^k = \lim_{s\to 0^+} f(s)/s^k \text{ and } f_\infty^k = \lim_{s\to +\infty} f(s)/s^k.$$

We firstly give the nonexistence results.

Lemma 4.1. *If $\exists\, \rho > 0$ satisfying*

$$f(s)/s^k \geq \rho$$

for any $s > 0$. Then $\exists\, \xi_* > 0$ such that $\forall\, \lambda \in (\xi_*, +\infty)$, problem (1.1) has no radially symmetric negative solution.

Proof. By contradiction, assume that v is a positive solution of problem (1.3). Then we have that

$$\left(r^{2N-k}\left(-v'(r)\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \frac{f\left(v\right)}{v^k} v^k.$$

By Lemma 3.5, we have $\lambda \leq \lambda_1/\rho^{1/k}$. □

Lemma 4.2. *If* $\exists\, \varrho > 0$ *satisfying*

$$f(s)/s^k \leq \varrho$$

for any $s > 0$*. Then* $\exists\, \eta_* > 0$ *such that* $\forall\, \lambda \in (0, \eta_*)$*, problem* (1.1) *has no radially symmetric negative solution.*

Proof. Assume the opposite that v is a positive solution of problem (1.1). If we set $w = v/\left\|v\right\|$, then

$$1 = \|w\| = \lambda \left\| \int_1^r \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1} \left(\frac{f\left(v\right)}{\left\|v\right\|^k} \right) d\tau \right)^{1/k} ds \right\| \leq 2 \left(\frac{\varrho}{C_N^k} \right)^{1/k} \lambda,$$

which implies that $\lambda \geq 1/\left(2\left(C_N^k/\varrho\right)^{1/k}\right)$. □

Theorem 4.1. *If* f_0*,* $f_\infty \in (0, +\infty)$ *and* $f_\infty \neq f_0$*, then for any* $\lambda \in \left(\min\left\{\lambda_1/f_\infty, \lambda_1/f_0\right\},\ \max\left\{\lambda_1/f_0, \lambda_1/f_\infty\right\}\right)$*, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. We just need to show that problem (1.3) has at least one solution v such that it is positive in $[0, 1)$. Letting $\zeta(s) = f(s) - f_0^k s^k$, then we have $\lim_{s\to 0^+} \zeta(s)/s^k = 0$. By Theorem 1.2, we know that there is an unbounded continuum $\mathcal{C}$ emanating from $(\lambda_1/f_0, 0)$ such that

$$\mathcal{C} \subseteq \left(\{(\lambda_1/f_0, 0)\} \cup \left(\mathbb{R} \times P^+\right)\right).$$

It suffices to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. Suppose $(\lambda_n, v_n) \in \mathcal{C}$ with $\lambda_n + \|v_n\| \to +\infty$ as $n \to +\infty$. Because $(0, 0)$ is the only solution of problem (1.3), in view of Lemma 4.2, we have that $\mathcal{C} \cap (\{0\} \times X^+) = \emptyset$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Our assumptions imply that there is a positive constant ϱ such that $f\left(v_n\right)/v_n^k \geq \varrho$ for any $r \in (0, 1)$. By Lemma 4.1, there exists a constant M such that $\lambda_n \in (0, M]$ for $n \in \mathbb{N}$ large enough. Therefore $\|v_n\| \to +\infty$ as $n \to +\infty$.

Let $\xi(s) = f(s) - f_\infty^k s^k$. We have $\lim_{s\to+\infty} \xi(s)/s^k = 0$.
Set $\widetilde{\xi}(v) = \max_{0\leq s\leq v} |\xi(s)|$. We have $\widetilde{\xi}$ is nondecreasing.
Set $\overline{\xi}(v) = \max_{v/2\leq s\leq v} |\xi(s)|$, then

$$\lim_{v\to+\infty} \overline{\xi}(v)/v^k = 0 \ \text{ and } \ \widetilde{\xi}(v) \leq \widetilde{\xi}\left(v/2\right) + \overline{\xi}(v).$$

Thus $\lim_{v\to+\infty} \widetilde{\xi}(v)/v^k = 0$. Furthermore, we have that

$$\left|\xi\left(v_n\right)\right| / \left\|v_n\right\|^k \leq \widetilde{\xi}\left(\left|v_n\right|\right) / \left\|v_n\right\|^k \leq \widetilde{\xi}\left(\left\|v_n\right\|\right) / \left\|v_n\right\|^k \to 0 \tag{4.1}$$

as $n \to +\infty$. Divide the equation

$$\left(r^{2N-k}\left(-v_n'\right)^k\right)' - \lambda_n^k f_\infty^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} v_n^k = \lambda_n^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \xi\left(v_n\right)$$

by $\|v_n\|^k$ and set $\overline{v}_n = v_n/\|v_n\|$. Because $\overline{v}_n$ is bounded in X^+, up to a subsequence, we have $\overline{v}_n \rightharpoonup \overline{v}$ for some $\overline{v} \in X^+$ as $n \to +\infty$. Then

$$\left(r^{2N-k}\left(-\overline{v}'\right)^k\right)' - \overline{\lambda}^k f_\infty^k \frac{N2^{k+1}}{C_N^k} r^{2N-1}\overline{v}^k = 0,$$

by the continuity and compactness of T_f, where $\overline{\lambda} = \lim_{n\to+\infty} \lambda_n$, choosing a subsequence and relabeling it if necessary. It is obvious that $\overline{\lambda} = \lambda_1/f_\infty$. Finally we get that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. □

Theorem 4.2. *If $f_0 \in (0, +\infty)$ and $f_\infty = 0$, then for any $\lambda \in (\lambda_1/f_0, +\infty)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. It is sufficient to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(+\infty, +\infty)$. Firstly we prove that $\mathcal{C}$ is unbounded in the direction of X^+. Suppose, by contradiction, that there exists $M > 0$ such that $\|v\| \le M$ for any $(\lambda, v) \in \mathcal{C} \setminus \{(\lambda_1/f_0, 0)\}$. From $f(s) > 0$ for $s > 0$, $f_0 \in (0, +\infty)$ and $\|v\| \le M$, we get that $f(v)/v^k \ge \rho$ for some positive constant ρ and any $r \in (0, 1)$. By Lemma 4.1, $\mathcal{C}$ is also bounded in the direction of λ, which is a contradiction. Suppose, on the contrary, that there exists μ such that $(\mu, 0)$ is a blow up point and $\mu < +\infty$. Then there exists a sequence $\{(\lambda_n, v_n)\}$ such that $\lim_{n\to+\infty} \lambda_n = \mu$ and $\lim_{n\to+\infty} \|v_n\| = +\infty$ as $n \to +\infty$. Then $w_n := v_n/\|v_n\|$ satisfies the following equation

$$w = \lambda_n \int_1^r \left(\frac{N2^{k+1}}{C_N^k} s^{k-2N} \int_s^0 \tau^{2N-1}\left(\frac{f(v_n)}{\|v_n\|^k}\right) d\tau\right)^{1/k} ds.$$

Similar to that of (4.1), we can get that $\lim_{n\to+\infty} f(v_n(r))/\|v_n\|^k = 0$. Since T_f is compact, then we obtain that for some convenient subsequence $w_n \to w_0$ as $n \to +\infty$. So $w_0 \equiv 0$, and that contradicts $\|w_0\| = 1$. □

Theorem 4.3. *If $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$, then for any $\lambda \in (0, \lambda_1/f_0)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. In view of the proof of Theorem 4.1, it suffices to show that $\mathcal{C}$ joins $(\lambda_1/f_0, 0)$ to $(0, +\infty)$. Lemma 4.1 implies that $\mathcal{C}$ is bounded in the parameter direction. By Theorem 1.3, we know that the unique blow up point of $\mathcal{C}$ is at $\lambda = 0$. □

Theorem 4.4. *If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, then for any $\lambda \in (\lambda_1/f_\infty, +\infty)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. If (λ, v) solves problem (1.3) with $\|v\| \not\equiv 0$, then we can get

$$\begin{cases} \left(r^{2N-k}\left(-w'(r)\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \frac{f(v)}{\|v\|^{2k}}, \ r \in (0,1), \\ w'(0) = w(1) = 0 \end{cases} \tag{4.2}$$

through dividing problem (1.3) by $\|v\|^{2k}$ and setting $w = v/\|v\|^2$. Let

$$\widetilde{\xi}(w) = \begin{cases} \|w\|^{2k}\xi\left(w/\|w\|^2\right) & \text{if } w \ne 0, \\ 0 & \text{if } w = 0, \end{cases}$$

where ξ is given in the proof of Theorem 4.1.

Now we can write problem (4.2) equably as

$$\begin{cases} \left(r^{2N-k}\left(-w'(r)\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1}\left(f_\infty^k w^k + \widetilde{\xi}(w)\right), \ r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{4.3}$$

By (4.1), we see that $\lim_{\|w\|\to 0} \widetilde{\xi}(w)/\|w\| = 0$. For problem (4.3), the same to the proof of Theorem 1.2, we can get an unbounded continuum $\mathcal{D}$, which emanates from $(\lambda_1/f_\infty, 0)$ such that

$$\mathcal{D} \subseteq \left(\{(\lambda_1/f_\infty, 0)\} \cup \left(\mathbb{R} \times P^+\right)\right).$$

By the inversion $w \to w/\|w\|^2 = v$, we obtain a continuum $\mathcal{C}$ of solutions to problem (1.3) emanating from $(\lambda_1/f_\infty, +\infty)$. $\mathcal{C}$ is either unbounded in the direction of λ, or it meets $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. In combination with $f_0 = 0$, just like the discussion of Theorem 4.2, we know that it is impossible that $\mathcal{C}$ meets $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. Thus $\mathcal{C}$ is unbounded in the direction of λ.

Next we only need to claim that $\mathcal{C}$ joins to $(+\infty, 0)$. Suppose on the contrary that there exists a positive constant ρ and $(\lambda_n, v_n) \in \mathcal{C}$ such that $\lambda_n \to +\infty$ as $n \to +\infty$ and $\|v_n\| \geq \rho$ for any $n \in \mathbb{N}$. Note that v_n is strictly decreasing on $[0,1]$. Select a $\tau_n \in (0,1]$ such that $v_n(x) \geq \rho/2$ for any $x \in [0, \tau_n]$. Combined with $f_\infty \in (0, +\infty)$, it is true that there is a positive constant δ such that $f(v_n)/v_n^k \geq \delta$ for any $x \in (0, \tau_n)$ and all $n \in \mathbb{N}$. Suppose (γ_1, φ_1) is the principal eigenpair of

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} v^k, \ r \in (0, \tau_n), \\ v'(0) = v(\tau_n) = 0. \end{cases}$$

Then for n large enough and any $x \in (0, \tau_n)$, we have

$$\lambda_n^k \frac{f(v_n)}{v_n^k} > \gamma_1^k.$$

According to Lemma 3.5, we know that v_n has at least one zero in $(0, \tau_n)$ for sufficiently large n. This contradiction completes the proof. □

Theorem 4.5. *If $f_0 = 0$ and $f_\infty = 0$, then for any $\lambda \in (\lambda_*, +\infty)$, there exists $\lambda_* > 0$ such that problem* (1.1) *has at least two radially symmetric negative solutions.*

Proof. Let

$$f^n(s) = \begin{cases} s^k/n^k, & s \in [0, 1/n], \\ \left(f(2/n) - 1/n^{2k}\right) ns + 2/n^{2k} - f(2/n), & s \in (1/n, 2/n), \\ f(s), & s \in [2/n, +\infty) \end{cases}$$

and consider

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f^n(v), \ r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases}$$

Firstly $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = 1/n$ and $f_\infty^n = f_\infty = 0$. From Theorem 4.2, we have a sequence of unbounded continua $\mathcal{C}_n$ emanating from $(n\lambda_1, 0)$ and joining to $(+\infty, +\infty) := z_*$. Let $\mathcal{C} = \limsup_{n\to+\infty} \mathcal{C}_n$. For any $(\lambda, v) \in \mathcal{C}$, the definition of superior limit (see [30]) shows that there exists a sequence $(\lambda_n, v_n) \in \mathcal{C}_n$ such that $(\lambda_n, v_n) \to (\lambda, v)$ as $n \to +\infty$. Then a continuity argument shows that v is a solution of problem (1.3).

By Proposition 2 of [11], for each $\epsilon > 0$ there exists an N_0 such that for every $n > N_0$, $\mathcal{C}_n \subset V_\epsilon(\mathcal{C})$, where $V_\epsilon(\mathcal{C})$ denote the ϵ-neighborhood of $\mathcal{C}$. It follows that

$$(n\lambda_1, +\infty) \subseteq \mathrm{Proj}\,(\mathcal{C}_n) \subseteq \mathrm{Proj}\,(V_\epsilon(\mathcal{C})),$$

where $\mathrm{Proj}\,(\mathcal{C}_n)$ is the projection of $\mathcal{C}_n$ on $\mathbb{R}$. So $(n\lambda_1 + \epsilon, +\infty) \subseteq \mathrm{Proj}\,(\mathcal{C})$. Further $\mathcal{C} \setminus \{\infty\} \neq \emptyset$.

Let

$$S_1 = \{(+\infty, u) : 0 < \|u\| < +\infty\}.$$

For any fixed $n \in \mathbb{N}$, we claim that $\mathcal{C}_n \cap S_1 = \emptyset$. Otherwise, there exists a sequence $(\lambda_m, v_m) \in \mathcal{C}_n$ such that $(\lambda_m, v_m) \to (+\infty, v_*) \in S_1$ with $\|v_*\| < +\infty$. It follows that $\|v_m\| \le M_n$ for some constant $M_n > 0$. It implies that $f^n(v_m)/v_m \ge \delta_n$ for some positive constant δ_n and all $m \in \mathbb{N}$. From Lemma 4.1 we have that $v_m \equiv 0$ for m large enough, which contradicts the fact of $\|v_*\| > 0$. Thus $\left(\cup_{n=1}^{+\infty}\mathcal{C}_n\right) \cap S_1 = \cup_{n=1}^{+\infty}(\mathcal{C}_n \cap S_1) = \emptyset$. Since $\mathcal{C} \subseteq \left(\cup_{n=1}^{+\infty}\mathcal{C}_n\right)$, one has that $\mathcal{C} \cap S_1 = \emptyset$. Set

$$S_2 := \{(\lambda, +\infty) : 0 \le \lambda < +\infty\}.$$

For any fixed $n \in \mathbb{N}$, by $f_\infty = 0$ and an argument similar to that of Theorem 4.2, we have that $\mathcal{C}_n \cap S_2 = \emptyset$. Then reasoning as the above, we have that $\mathcal{C} \cap S_2 = \emptyset$. Hence, $\mathcal{C} \cap (S_1 \cup S_2) = \emptyset$. Taking $z^* = (+\infty, 0)$, we have $z^* \in \liminf_{n\to+\infty} \mathcal{C}_n$ with $\|z^*\|_{\mathbb{R}\times X^+} = +\infty$. Therefore, we obtain that $\mathcal{C} \cap \{\infty\} = \{z_*, z^*\}$.

The compactness of T_f implies that $\left(\cup_{n=1}^{+\infty}\mathcal{C}_n\right) \cap \overline{B}_R$ is pre-compact. So Lemma 3.1 of [10] implies that $\mathcal{C}$ is connected. By an argument similar to that of Theorem 4.2, we can show that $\mathcal{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$. Now the desired conclusion can be deduced from the global structure of $\mathcal{C}$ immediately. □

Theorem 4.6. *If $f_0 = 0$ and $f_\infty = +\infty$, then for any $\lambda \in (0, +\infty)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. In view of Theorem 1.3, by an argument similar to that of Theorem 4.5 and the conclusion of Theorem 4.3, we can obtain the desired conclusion. □

Theorem 4.7. *If $f_0 = +\infty$ and $f_\infty = 0$, then for any $\lambda \in (0, +\infty)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. Let

$$f^n(s) = \begin{cases} n^k s^k, & s \in [0, 1/n], \\ (f(2/n) - 1)\,ns + 2 - f(2/n), & s \in (1/n, 2/n), \\ f(s), & s \in [2/n, +\infty) \end{cases}$$

and consider

$$\begin{cases} \left(r^{2N-k}(-v')^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f^n(v), \ r \in (0,1), \\ v'(0) = v(1) = 0. \end{cases} \tag{4.4}$$

Firstly $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = n$ and $f_\infty^n = f_\infty = 0$. We have from Theorem 4.2 that there is a sequence of unbounded continua $\mathcal{C}_n$, which emanate from $(\lambda_1/n, 0)$ and join to $(+\infty, +\infty) := z_*$.

Set $z^* = (0,0)$, then one has $z^* \in \liminf_{n\to+\infty} \mathcal{C}^n$. Since T_f is compact, we have that $\left(\cup_{n=1}^{+\infty}\mathcal{C}_n\right) \cap \overline{B}_R$ is pre-compact. By Lemma 3.5 of [11], $\mathcal{C} = \limsup_{n\to+\infty} \mathcal{C}_n$ is connected and $z_*, z^* \in \mathcal{C}$. The conclusion is proved. □

Theorem 4.8. *If $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, then for any $\lambda \in (0, \lambda_1/f_\infty)$, problem* (1.1) *has at least one radially symmetric negative solution.*

Proof. By combining the conclusion of Theorem 4.1 and the same argument of Theorem 4.7, the required conclusion can be obtained. □

Theorem 4.9. *If $f_0 = +\infty$ and $f_\infty = +\infty$, then for any $\lambda \in (0, \lambda^*)$, there exists $\lambda^* > 0$ such that problem* (1.1) *has at least two radially symmetric negative solutions.*

Proof. Consider problem (4.4) again. By Theorem 4.3, there exists a sequence of unbounded continua $\mathcal{C}_n$, which bifurcate from $(\lambda_1/n, 0)$ and join to $(0, +\infty)$. Reasoning as that of Theorem 4.7, we have that $\mathcal{C} := \limsup_{n\to+\infty} \mathcal{C}_n$ is connected, which joins $(0,0)$ to $(0,+\infty)$. Next we show that $\mathcal{C} \cap ((0,+\infty) \times \{0\}) = \emptyset$. If there exists a sequence $\{(\lambda_n, v_n)\}$ with $v_n \in P^+$ such that $\lim\limits_{n\to+\infty} \lambda_n = \mu > 0$ and $\lim\limits_{n\to+\infty} \|v_n\| = 0$ as $n \to +\infty$. By Lemma 4.1 we have $\mu < +\infty$. And $f_0 = +\infty$ implies that

$$\lambda_n^k \frac{f(v_n)}{v_n^k} > \lambda_1^k \text{ for any } r \in (0,1)$$

and n large enough. From Lemma 3.5, we know that v_n must change its sign for n large enough, which is a contradiction. □

Example 4.1. Consider the specific case of $f(s) = s^q$ with $q \in (0, k_*]$. According to Theorem 4.6 or Theorem 4.7, we have that problem (1.1) has at least one radially symmetric negative solution for any $\lambda \in (0, +\infty)$. It is just the corresponding conclusion of [21, Theorem 2] when $\lambda = 1$. Therefore our Theorem 4.1–4.9 improve and extend the corresponding results of [21] except the case of $q \in (k_*, k^*)$.

5. Uniqueness

We get from Theorem 1.2 a bifurcation branch $\mathcal{C}$. Now we study its local structure near $(\lambda_1, 0)$. Let $\mathbb{X} = \mathbb{R} \times X^+$, $\Phi(\lambda, v) = v - \lambda T_g(v)$ and

$$\mathcal{S} = \overline{\{(\lambda, v) \in \mathbb{X} : \Phi(\lambda, v) = 0, v \neq 0\}}^{\mathbb{X}}.$$

For $\lambda \in \mathbb{R}$ and $0 < s < +\infty$, define an open neighborhood of $(\lambda_1, 0)$ in $\mathbb{X}$ as follows.

$$\mathbb{B}_s(\lambda_1, 0) = \{(\lambda, v) \in \mathbb{X} : \|v\| + |\lambda - \lambda_1| < s\}.$$

Let X_0 be a closed subset of X satisfying $X = \operatorname{span}\{\psi_1\} \oplus X_0$, where ψ_1 is an eigenfunction corresponding to λ_1 with $\|\psi_1\| = 1$. According to the Hahn-Banach theorem, we have $l \in X^*$ satisfying

$$l(\psi_1) = 1 \text{ and } X_0 = \{v \in X : l(v) = 0\},$$

where X^* denotes the dual space of X. Note that $X^+ \subseteq X$, so $l \in (X^+)^*$. Then

$$K^+_{\varepsilon,\eta} = \{(\lambda, v) \in \mathbb{X} : |\lambda - \lambda_1| < \varepsilon, l(v) > \eta\|v\|\}$$

is well defined for any $0 < \varepsilon < +\infty$ and $0 < \eta < 1$.

Similar to that of [22, Lemma 6.4.1], we can show the following lemma.

Lemma 5.1. *Let $\eta \in (0,1)$, there is $\delta_0 > 0$ such that for each $\delta : 0 < \delta < \delta_0$, it holds that*

$$((\mathcal{S} \setminus \{(\lambda_1, 0)\}) \cap \mathbb{B}_\delta(\lambda_1, 0)) \subset K^+_{\varepsilon,\eta}.$$

And there exist $s \in \mathbb{R}$ and a unique $y \in X_0$ such that

$$v = s\psi_1 + y \text{ and } s > \eta\|v\|$$

for each $(\lambda, v) \in (\mathcal{S} \setminus \{(\lambda_1, 0)\}) \cap (\mathbb{B}_\delta(\lambda_1, 0))$. Further, $\lambda = \lambda_1 + o(1)$ and $y = o(s)$ as $s \to 0^+$ for these solutions (λ, v).

The following are the main results of this section.

Theorem 5.1. *Let $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $f(s)/s^k$ is decreasing for $s > 0$. Then*

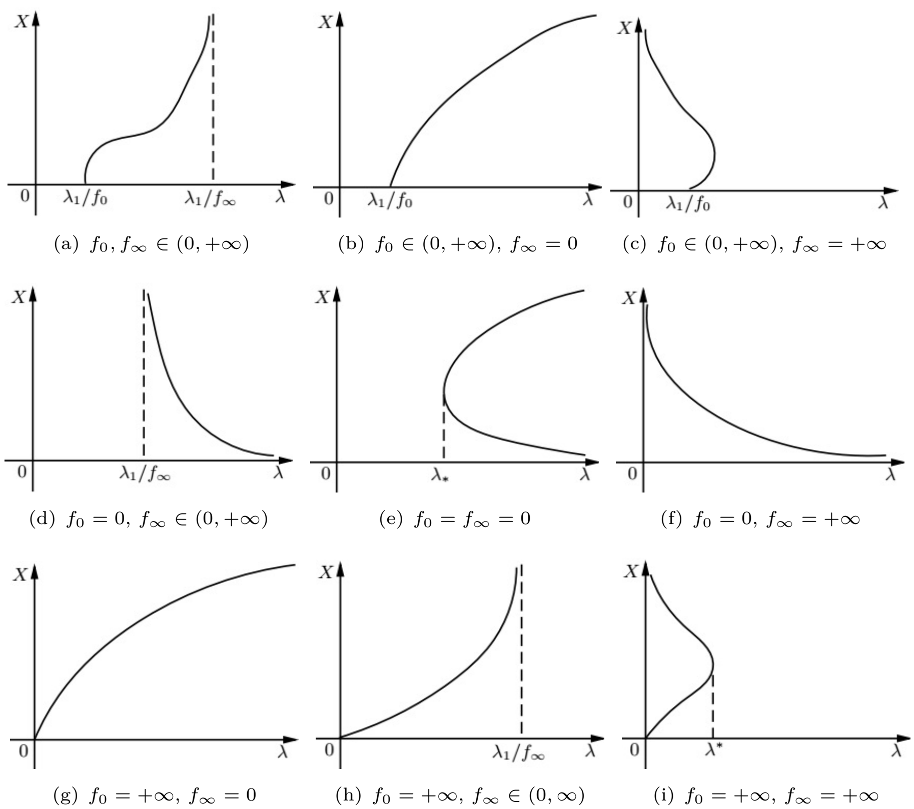

Figure 1. Bifurcation diagrams of Theorems 4.1–4.9 and Theorem 5.1.

(i) if $f_0 \in (0,+\infty)$ and $f_\infty = 0$, problem (1.1) *has a unique radially symmetric negative solution for any $\lambda \in (\lambda_1/f_0,+\infty)$ and has only the trivial radially symmetric solution for any $\lambda \in (0,\lambda_1/f_0]$;*

(ii) if $f_0 \in (0,+\infty)$ and $f_\infty \in (0,+\infty)$ with $f_\infty \neq f_0$, problem (1.1) *has a unique radially symmetric negative solution for any $\lambda \in (\lambda_1/f_0, \lambda_1/f_\infty)$ and has only the trivial radially symmetric solution for any $\lambda \in (0,\lambda_1/f_0] \cup [\lambda_1/f_\infty,+\infty)$;*

(iii) if $f_0 = +\infty$ and $f_\infty \in (0,+\infty)$, problem (1.1) *has a unique radially symmetric negative solution for any $\lambda \in (0,\lambda_1/f_\infty)$ and has only the trivial radially symmetric solution for any $\lambda \in [\lambda_1/f_\infty,+\infty]$;*

(iv) if $f_0 = +\infty$ and $f_\infty = 0$, problem (1.1) *has a unique radially symmetric negative solution for any $\lambda \in (0,+\infty)$.*

Let

$$Y = \left\{v \in C^2(0,1) : v'(0) = v(1) = 0\right\}.$$

For any $\phi \in Y$ and positive solution v of problem (1.3), we can calculate that the linearized eigenvalue problem of (1.3) at the direction ϕ is

$$\begin{cases} \left(-\phi' r^{2N-k}\left(-v'\right)^{k-1}\right)' - \lambda^k \frac{N2^{k+1}}{kC_N^k} r^{2N-1} f'(v)\phi = \frac{\mu}{k}\phi, \ r \in (0,1), \\ \phi'(0) = \phi(1) = 0. \end{cases} \tag{5.1}$$

Next is the introduction to the stability property of solutions.

Suppose v is a solution of problem (1.3). The linear stability of v can be determined by the linearized eigenvalue problem (5.1). If all eigenvalues of problem (5.1) are positive, then we call v is *stable*, otherwise we call it *unstable*.

The *Morse index* $M(v)$ of v is defined as the number of negative eigenvalues of problem (5.1). Call v is *degenerate* if 0 is an eigenvalue of problem (5.1), otherwise it is *non-degenerate*.

Lemma 5.2. *If f satisfies the conditions of Theorem 5.1, then any solution v of problem (1.3) is stable and non-degenerate, and their Morse index are $M(v) = 0$.*

Proof. Let v be a solution of problem (1.3), and let (μ_1, φ_1) be the corresponding principal eigenpair of problem (5.1) with $\varphi_1 > 0$ in $(0,1)$. Notice that v and φ_1 satisfy

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^{k}\right)' - \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f(v) = 0, \ r \in (0,1), \\ v'(0) = v(1) = 0 \end{cases} \tag{5.2}$$

and

$$\begin{cases} \left(-\varphi_1' r^{2N-k}\left(-v'\right)^{k-1}\right)' - \lambda^k \frac{N2^{k+1}}{kC_N^k} r^{2N-1} f'(v)\varphi_1 = \frac{\mu_1}{k}\varphi_1, \ r \in (0,1), \\ \varphi_1'(0) = \varphi_1(1) = 0. \end{cases} \tag{5.3}$$

By multiplying the first equation of problem (5.3) by $-v$, and the first equation of problem (5.2) by $-\varphi_1$, subtracting and then integrating, we get that

$$\mu_1 \int_0^1 \varphi_1 v \, dr = \frac{\lambda^k N 2^{k+1}}{C_N^k} \int_0^1 r^{2N-1} \varphi_1 \left(k f(v) - f'(v) v\right) \, dr.$$

Note that $f'(s) \leq kf(s)/s$ and $f'(s) \not\equiv kf(s)/s$ for $s > 0$, and we know $v > 0$ and $\varphi_1 > 0$ in $(0,1)$, so $\mu_1 > 0$. It follows that v is stable. $\square$

Next we give the proof of Theorem 5.1.

Proof of Theorem 5.1. (i) Review $\mathcal{C}$ in Theorem 4.2, we know from Lemma 5.1 that $\mathcal{C}$ near $(\lambda_1, 0)$ is a curve $(\lambda(s), v(s)) = (\lambda_1 + o(1), s\psi_1 + o(s))$. Let $F : \mathbb{R} \times X^+ \to X^+$ be defined as

$$F(\lambda, v) = \left(r^{2N-k}\left(-v'\right)^k\right)' - \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} f(v).$$

It follows from Lemma 5.2 that any solution v of problem (1.3) is stable. So, at any solution (λ^*, v^*), we can use the Implicit Function Theorem to $F(\lambda, v) = 0$, and all the solutions of $F(\lambda, v) = 0$ near (λ^*, v^*) are on a curve $(\lambda, v(\lambda))$ with $|\lambda - \lambda^*| \le \varepsilon$ for some small $\varepsilon > 0$. Hence, the unbounded continuum $\mathcal{C}$ is a curve.

We claim that any solution u_λ is decreasing with respect to λ. That is if $\left(\lambda^1, v_1\right)$ and $\left(\lambda^2, v_2\right)$ are two positive solution pairs of problem (1.3) and $\lambda^1 < \lambda^2$, we need show $v_1 \le v_2$. Suppose, on the contrary, that $\exists\, r_0 \in [0,1) : v_1(r_0) > v_2(r_0)$. Set I is the connected component of $\{r \in [0,1) : v_1(r) > v_2(r)\}$ containing r_0.

For the case of $0 \in I$, there exists $r_1 > 0$ such that $I = [0, r_1)$. Using the complex Hessian identity, we can get that

$$\int_0^{r_1} \left(\frac{v_1^{k+1} r^{2N-k}\left(-v_2'\right)^k}{v_2^k} - v_1 r^{2N-k}\left(-v_1'\right)^k\right)' dr = L_1 + L_2,$$

where

$$L_1 = \int_0^{r_1} \left(\lambda_2^k \frac{f(v_2)}{v_2^k} - \lambda_1^k \frac{f(v_1)}{v_1^k}\right) \frac{N2^{k+1}}{C_N^k} r^{2N-1} v_1^{k+1}\, dr$$

and

$$L_2 = \int_0^{r_1} r^{2N-k}\left(\left(-v_1'\right)^{k+1} + k\left(\frac{-v_1 v_2'}{v_2}\right)^{k+1} - (k+1) v_1^k v_1' \left(\frac{-v_2'}{v_2}\right)^k\right) dr.$$

Further

$$\begin{aligned}
&\int_0^{r_1} \left(\frac{v_1^{k+1} r^{2N-k}\left(-v_2'\right)^k}{v_2^k} - v_1 r^{2N-k}\left(-v_1'\right)^k\right)' dr \\
&= v_1(r_1)\, r_1^{2N-k}\left(\left(-v_2'(r_1)\right)^k - \left(-v_1'(r_1)\right)^k\right) \\
&\le 0. \qquad (5.4)
\end{aligned}$$

It follows that $L_1 + L_2 \le 0$, then $L_2 \ge 0$ from the Young inequality. Combine the monotonicity of $f(s)/s^k$ and our assumptions, we get that $L_1 > 0$, which is a contradiction.

For the case of $1 \in \overline{I}$, there exists $r_2 > 0$ such that $I = (r_2, 1)$. Using the complex Hessian identity again, we get that

$$\int_{r_2}^{1} \left(\frac{v_1^{k+1} r^{2N-k}\left(-v_2'\right)^k}{v_2^k} - v_1 r^{2N-k}\left(-v_1'\right)^k\right)' dr = L_1' + L_2', \qquad (5.5)$$

where

$$L_1' = \int_{r_2}^{1} \left(\lambda_2^k \frac{f(v_2)}{v_2^k} - \lambda_1^k \frac{f(v_1)}{v_1^k} \right) \frac{N2^{k+1}}{C_N^k} r^{2N-1} v_1^{k+1}\, dr$$

and

$$L_2' = \int_{r_2}^{1} r^{2N-k} \left((-v_1')^{k+1} + k \left(\frac{-v_1 v_2'}{v_2} \right)^{k+1} - (k+1) v_1^k v_1' \left(\frac{-v_2'}{v_2} \right)^k \right) dr.$$

We can see from the argument of Lemma 3.4 that the left-hand side of (5.5) is equal to

$$-v_1(r_2)\, r_2^{2N-k} \left((-v_2'(r_2))^k - (-v_1'(r_2))^k \right) \le 0.$$

Then a contradiction like the one above emerges.

For the case where $\overline{I}$ does not contain 0 and 1, there exist $r_3,\ r_4 \in (0,1)$ such that $I = (r_3, r_4)$. We use the complex Hessian identity once again and obtain that

$$\int_{r_3}^{r_4} \left(\frac{v_1^{k+1} r^{2N-k} (-v_2')^k}{v_2^k} - v_1 r^{2N-k} (-v_1')^k \right)' dr = L_1'' + L_2'',$$

where

$$L_1'' = \int_{r_3}^{r_4} \left(\lambda_2^k \frac{f(v_2)}{v_2^k} - \lambda_1^k \frac{f(v_1)}{v_1^k} \right) \frac{N2^{k+1}}{C_N^k} r^{2N-1} v_1^{k+1}\, dr$$

and

$$L_2'' = \int_{r_3}^{r_4} r^{2N-k} \left((-v_1')^{k+1} + k \left(\frac{-v_1 v_2'}{v_2} \right)^{k+1} - (k+1) v_1^k v_1' \left(\frac{-v_2'}{v_2} \right)^k \right) dr.$$

Further

$$\begin{aligned}
&\int_{r_3}^{r_4} \left(\frac{v_1^{k+1} r^{2N-k} (-v_2')^k}{v_2^k} - v_1 r^{2N-k} (-v_1')^k \right)' dr \\
&= r^{2N-k} \left(\frac{v_1^{k+1} (-v_2')^k}{v_2^k} - v_1 (-v_1')^k \right)(r_4) - r^{2N-k} \left(\frac{v_1^{k+1} (-v_2')^k}{v_2^k} - v_1 (-v_1')^k \right)(r_3) \\
&= v_1(r_4)\, r_4^{2N-k} \left((-v_2')^k (r_4) - (-v_1')^k (r_4) \right) - v_1(r_3)\, r_3^{2N-k} \left((-v_2')^k (r_3) - (-v_1')^k (r_3) \right) \\
&\le 0.
\end{aligned}$$

Then reasoning as the above, we can obtain a contradiction again.

Next we prove the uniqueness of radially symmetric negative of problem (1.1). Suppose on the contrary that there exist two solutions v_1 and v_2 with $v_1 \in \mathcal{C}$ for $\lambda \in (\lambda_1/f_0, +\infty)$. for $\epsilon > 0$, take $(\lambda - \varepsilon, v_{\lambda-\varepsilon})$, $(\lambda + \varepsilon, v_{\lambda+\varepsilon}) \in \mathcal{C}$, then $v_{\lambda\pm\varepsilon} \to v_1$ as $\varepsilon \to 0$. By the monotonicity of v_2 with respect to λ, we get $v_{\lambda-\varepsilon} \le v_2 \le v_{\lambda+\varepsilon}$. Then $v_2 = v_1$.

Through the same discussion as above, we know that problem (1.1) with $\lambda = \lambda_1/f_0$ has only the trivial solution. So next we only need to show that problem (1.1) has only the trivial solution for any $\lambda \in (0, \lambda_1/f_0)$. Suppose, by contradiction, that problem (1.1) has a radially symmetric negative solution u for some $\lambda \in (0, \lambda_1/f_0)$. Then

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \frac{f(v)}{v^k} v^k, \in (0,1), \\ v'(0) = v(1) = 0, \end{cases}$$

where $v = -u$. From $\lambda < \lambda_1/f_0$, it is clear that

$$\lambda^k \frac{f(v)}{v^k} < \lambda_1^k.$$

According to Lemma 3.5, ψ_1 changes its sign. This is a contradiction.

(ii) If u is a radially symmetric negative solution of problem (1.1) for a $\lambda \in (\lambda_1/f_\infty, +\infty)$, then

$$\begin{cases} \left(r^{2N-k}\left(-v'\right)^k\right)' = \lambda^k \frac{N2^{k+1}}{C_N^k} r^{2N-1} \frac{f(v)}{v^k} v^k, \in (0,1), \\ v'(0) = v(1) = 0. \end{cases}$$

Since $\lambda > \lambda_1/f_\infty$, we have that

$$\lambda^k \frac{f(v)}{v^k} > \lambda_1^k.$$

By Lemma 3.5, v changes its sign, which is a contradiction. The next proof is the same as that of (i).

(iii) From (ii), we know that problem (1.1) has only the trivial solution for any $\lambda \in [\lambda_1/f_\infty, +\infty)$. The existence can be got from Theorem 4.8. The uniqueness can be obtained similarly as that of (i).

(iv) In view of Theorem 4.7 and the argument of (i), the desired conclusion can be obtained immediately. $\square$

See Figure 1 for bifurcation diagrams of Theorems 4.1–4.9 and Theorem 5.1.

For Example 4.1 with $q \in (0, k)$, Theorem 5.1 implies that problem (1.1) has a unique radially symmetric negative solution, which improves the corresponding results of [21, Theorem 2] even in the case of $\lambda = 1$, where only the existence was proved. In addition, if $q \in (k, k_*)$, Theorem 4.6 combining with Corollary 4.5 of [29] shows that problem (1.1) has a unique radially symmetric negative solution.

Competing interests

The author declare that no conflict of interest exits in the submission of this manuscript.

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