Journal of Applied Analysis and Computation
Volume 11, Number 3, June 2021, 1208–1221
Website:http://www.jaac-online.com
DOI:10.11948/20200049

A STUDY OF GENERALIZED CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH STEILTJES-TYPE FRACTIONAL INTEGRAL BOUNDARY CONDITIONS VIA FIXED-POINT THEORY

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Abstract In this paper, we discuss the existence of solutions for generalized Caputo fractional differential equations and inclusions equipped with Steiltjes-type fractional integral boundary conditions via fixed-point theory. Examples are constructed for illustrating the obtained results.

Keywords Fractional differential equations and inclusions, Caputo derivative, fractional integral, existence, fixed point.

MSC(2010) 26A33, 34A60, 34B15, 34K05.

1. Introduction

During the last two decades, fractional-order differential and integral operators played a key role in improving the mathematical modeling of several real world phenomena. It has been mainly due to the nonlocal nature of such operators, which can trace the past history of the phenomenon under investigation. An overwhelming interest has been shown in the subject of fractional calculus due to its applications in many scientific and engineering disciplines. Examples include chaotic synchronization [32], dynamical networks [33], co-infection of malaria and HIV/AIDS [5], HIV-immune system with memory [12], continuum mechanics [26], financial economics [13], etc. On the other hand, for theoretical background of fractional differential equations, for instance, see [1, 20].

The topic of fractional order boundary value problems has also been extensively studied by many researchers. Now the literature on this topic is much enriched and contains a variety of results involving classical, nonlocal and integral boundary conditions. Nonexistence of positive solutions for a system of coupled fractional boundary value problems was discussed in [17]. A boundary value problem involving a nonlocal boundary condition characterized by a linear functional was studied in [15]. Smooth solutions to a mixed-order fractional differential system were derived in [14]. For some recent results on fractional differential equations with multi-point and integral boundary conditions, see [27, 29, 31]. More recently, in [7, 18], the

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authors analyzed boundary value problems involving Caputo and Riemann-Liouville fractional derivatives respectively.

Differential inclusions find useful applications in the processes like optimization of financial issues [21], control systems [22], synchronization of fractional order systems [11], etc. Influenced by the importance of differential inclusions, several researchers turned to the study of fractional differential inclusions. In [3, 19], the authors investigated semilinear fractional order differential and integro-differential inclusions respectively. Some recent results on fractional differential inclusions with nonlocal boundary conditions can be found in [8,9]. Existence of infinitely many solutions for a fractional differential inclusion with oscillatory potential was discussed in [34]. In a recent paper [2], the existence of solutions for an inclusions problem involving both Caputo and Hadamard fractional derivatives was studied.

In [4], the authors applied monotone iterative method to prove the existence of extremal solutions for a Caputo-type generalized fractional differential equation supplemented with Steiltjes-type fractional integral boundary conditions given by

$$\begin{cases} ({}_c^\rho D_{0+}^\alpha y)(t) = f(t, y(t)), \quad t \in J := [0, T], \\ y(0) = \lambda (I_{0+;H}^\gamma y)(T) + \kappa, \quad t^{1-\rho} \frac{dy}{dt}|_{t=0} = 0, \ \lambda, \kappa \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}_c^\rho D_{0+}^\alpha$ denotes the Caputo-type generalized fractional derivative of order $1 < \alpha \leq 2$, $\rho > 0$, defined by

$$({}_c^\rho D_{0+}^\alpha y)(t) = \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-1}} \Big(\Big(s^{1-\rho} \frac{d}{ds} \Big)^2 y \Big)(s) ds,$$

$f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $(I_{0+;H}^\gamma y)(T)$ is the fractional integral of y with respect to the function $H : [0, T] \to \mathbb{R}$, given by

$$(I_{0+;H}^\gamma y)(T) = \frac{1}{\Gamma(\gamma)} \int_0^T \frac{H'(s) y(s)}{[H(T) - H(s)]^{1-\gamma}} ds,$$

$H(t)$ is increasing and monotone function on $(0, T]$ and possesses a continuous derivative $H'(t)$ on $(0, T)$. For details of Caputo-type generalized fractional derivatives, see [4] and the references cited therein.

This objective of the present work is to enrich the existence theory for the problem (1.1) and its inclusions (multivalued) analogue. In [4], the authors used the concept of lower and upper solutions together with the monotone iterative technique to prove the existence of extremal solutions for the problem (1.1). In this article, we present two types of sufficient conditions for existence of solutions for the problem (1.1) by applying Krasnoselskii's fixed point theorem and the contraction mapping principle. Then we introduce and investigate the multivalued analogue (inclusions case) of the problem (1.1) given by

$$\begin{cases} ({}_c^\rho D_{0+}^\alpha y)(t) \in F(t, y(t)), \quad t \in J := [0, T], \\ y(0) = \lambda (I_{0+;H}^\gamma y)(T) + \kappa, \quad t^{1-\rho} \frac{dy}{dt}|_{t=0} = 0, \end{cases} \tag{1.2}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$). The existence result for the problem (1.2) with convex values of F is derived by applying nonlinear alternative of Leray-Schauder type, while the existence of solutions for the problem (1.2) with nonconvex valued right hand side is proved by means of a fixed point theorem for contractive multivalued maps due to Covitz and Nadler.

The rest of the paper is arranged as follows. In Section 2, we prove the existence and uniqueness results for the (1.1), while Section 3 contains the existence results for the problem (1.2). Some interesting observations are presented in the last section.

2. Existence and uniqueness results for the problem (1.1)

Let us start this section by defining what we mean by a solution of the problem (1.1).

Definition 2.1. A function $y \in C([0,T],\mathbb{R})$ is said to be a solution of (1.1) if y satisfies the equation $({}^{\rho}_{c}D^{\alpha}_{0+}y)(t) = f(t,y(t))$, on $[0,T]$, and the conditions

$$y(0) = \lambda(I^{\gamma}_{0^+;H}y)(T) + \kappa, \quad t^{1-\rho}\frac{dy}{dt}|_{t=0} = 0.$$

As argued in [4], we define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathcal{G}y)(t) = {}^{\rho}I^{\alpha}_{0+}f(t,y(t)) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0+}f(s,y(s))](T) + \kappa\Big\}, \tag{2.1}$$

where $\mathcal{C} = C(J,\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$.

For computational convenience, we set

$$\Lambda = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T)-H(0)]^{\gamma}, \tag{2.2}$$

$$\Omega = 1 - \frac{\lambda}{\Gamma(\gamma+1)}[H(T)-H(0)]^{\gamma} \neq 0. \tag{2.3}$$

In our first existence result for the problem (1.1), we make use of Krasnoselskii's fixed point theorem [23], which is stated below.

Lemma 2.1 (Krasnoselskii's fixed point theorem)**.** *Let $\mathcal{S}$ be a closed convex and nonempty subset of a Banach space E. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be the operators such that (i) $\mathcal{Q}_1x + \mathcal{Q}_2y \in \mathcal{S}$ whenever $u, v \in \mathcal{S}$; (ii) $\mathcal{Q}_1$ is compact and continuous; and (iii) $\mathcal{Q}_2$ is a contraction mapping. Then there exists $\omega \in \mathcal{S}$ such that $\omega = \mathcal{Q}_1\omega + \mathcal{Q}_2\omega$.*

Theorem 2.1. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function and satisfies the following conditions:*

(A_1) *there exists a positive constant k such that*

$$|f(t,u) - f(t,v)| \le k\|u-v\|, \quad \textit{for } t \in J \textit{ and every } u, v \in \mathbb{R};$$

(A_2) *there exists a continuous function $\Psi \in C([0,T],\mathbb{R}^+)$ such that*

$$|f(t,u)| \le \Psi(t), \quad \forall (t,u) \in J \times \mathbb{R}.$$

Then the problem (1.1) has at least one solution on J, provided that

$$k\Big(\frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T)-H(0)]^{\gamma}\Big)<1. \tag{2.4}$$

Proof. Let us fix $r > \|\Psi\|\Lambda + \frac{|\kappa|}{|\Omega|}$, where $\|\Psi\| = \sup_{t\in[0,T]}|\Psi(t)|$, and consider $B_r = \{y \in \mathcal{C} : \|y\| \le r\}$. Define operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_r as follows:

$$(\mathcal{G}_1 y)(t) = {}^{\rho}I^{\alpha}_{0^+} f(t,y(t)), \quad (\mathcal{G}_2 y)(t) = \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+} f(s,y(s))](T)+\kappa\Big\}.$$

For $x, y \in B_r$, we find that

$$\begin{aligned}\|\mathcal{G}_1 x + \mathcal{G}_2 y\| &\le \sup_{t\in J}\left\{{}^{\rho}I^{\alpha}_{0^+}|f(t,x(t))| + \frac{1}{|\Omega|}\Big\{|\lambda| I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}|f(s,y(s))|](T)+|\kappa|\Big\}\right\}\\ &\le \|\Psi\|\left\{\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T)-H(0)]^{\gamma}\right\} + \frac{|\kappa|}{|\Omega|}\\ &\le \|\Psi\|\Lambda + \frac{|\kappa|}{|\Omega|} < r.\end{aligned}$$

Thus, $\mathcal{G}_1 x + \mathcal{G}_2 y \in B_r$. Now, for $x, y \in \mathcal{C}$ and for each $t \in J$, we obtain

$$\begin{aligned}\|\mathcal{G}_2 x - \mathcal{G}_2 y\| &\le \sup_{t\in J}\left\{\frac{1}{|\Omega|}\Big\{|\lambda| I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}|f(s,x(s)) - f(s,y(s))|](T)\Big\}\right\}\\ &\le k\Big(\frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T)-H(0)]^{\gamma}\Big)\|x-y\|,\end{aligned}$$

which, together with the condition (2.4), implies that $\mathcal{G}_2$ is a contraction. Continuity of f implies that the operator $\mathcal{G}_1$ is continuous. Also, $\mathcal{G}_1$ is uniformly bounded on B_r as

$$\|\mathcal{G}_1 y\| \le \|\Psi\|\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

Finally, we prove the compactness of the operator $\mathcal{G}_1$. Let $\sup_{(t,y)\in J\times B_r}|f(t,y)| = \bar{f} < \infty$. Then, for $t_1, t_2 \in J$, $t_1 < t_2$, we have

$$\begin{aligned}|(\mathcal{G}_1 y)(t_2) - (\mathcal{G}_1 y)(t_1)| \le& \Big|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\Big[\int_0^{t_1} s^{\rho-1}[(t_2^{\rho}-s^{\rho})^{\alpha-1} - (t_1^{\rho}-s^{\rho})^{\alpha-1}]f(s,y(s))ds\\ &+ \int_{t_1}^{t_2} s^{\rho-1}(t_2^{\rho}-s^{\rho})^{\alpha-1} f(s,y(s))ds\Big]\Big|\\ \le& \frac{\bar{f}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big\{|t_2^{\rho\alpha} - t_1^{\rho\alpha}| + 2(t_2^{\rho}-t_1^{\rho})^{\alpha}\Big\},\end{aligned}$$

which is independent of y and tends to zero as $t_2 \to t_1$. Thus $\mathcal{G}_1$ is equicontinuous. So $\mathcal{G}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_1$ is compact on B_r. Thus all the assumptions of Lemma 2.1 are satisfied. So the conclusion of Lemma 2.1 implies that the problem (1.1) has at least one solution on J. □

Our next result deals with the uniqueness of solutions for the problem (1.1) and relies on Banach's fixed point theorem.

Theorem 2.2. *If the assumption* (A_1) *is satisfied, then the problem (1.1) has a unique solution on* $[0,T]$ *if*

$$k\Lambda < 1, \tag{2.5}$$

where Λ *is given by (2.2).*

Proof. Consider the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ defined by (2.1) and set $\sup_{t\in[0,T]} |f(t,0)| = M$. Choosing $\bar{r} \geq \frac{\Lambda M + \frac{|\kappa|}{|\Omega|}}{1-k\Lambda}$, we show that $\mathcal{G}B_{\bar{r}} \subset B_{\bar{r}}$, where $B_{\bar{r}} = \{y \in C([0,T],\mathbb{R}) : \|y\| \leq \bar{r}\}$. For $y \in B_{\bar{r}}$, using (A_1), we get

$$\begin{aligned}
|(\mathcal{G}y)(t)| \leq& {}^{\rho}I_{0+}^{\alpha}[|f(t,y(t)) - f(t,0)| + |f(t,0)|] \\
&+ \frac{1}{|\Omega|}\Big\{|\lambda| I_{0+;H}^{\gamma}\Big[{}^{\rho}I_{0+}^{\alpha}[|f(s,y(s)) - f(s,0)| + |f(s,0)|]\Big](T) + |\kappa|\Big\} \\
\leq& (k\bar{r} + M)\Big(\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T) - H(0)]^{\gamma}\Big) + \frac{|\kappa|}{|\Omega|} \\
\leq& (k\bar{r} + M)\Lambda + \frac{|\kappa|}{|\Omega|} \leq \bar{r},
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$, yields $\|\mathcal{G}y\| \leq \bar{r}$. This shows that $\mathcal{G}$ maps $B_{\bar{r}}$ into itself. In order to show that the operator $\mathcal{G}$ is a contraction, let $y, z \in C([0,T],\mathbb{R})$. Then, for each $t \in [0,T]$, we obtain

$$\begin{aligned}
&|(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| \\
\leq& {}^{\rho}I_{0+}^{\alpha}|f(t,y(t)) - f(t,z(t))| + \frac{1}{|\Omega|}\Big\{|\lambda| I_{0+;H}^{\gamma}[{}^{\rho}I_{0+}^{\alpha}|f(s,y(s)) - f(s,z(s))|](T)\Big\} \\
\leq& k\|y - z\|\Big(\frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T) - H(0)]^{\gamma}\Big) \\
\leq& k\Lambda\|y - z\|.
\end{aligned}$$

Taking the norm of the above inequality for $t \in [0,T]$, we obtain $\|\mathcal{G}y - \mathcal{G}z\| \leq k\Lambda\|y - z\|$. Thus, in view of the condition (2.5), it follows that the operator $\mathcal{G}$ is a contraction. Hence the operator $\mathcal{G}$ has a unique fixed point by Banach's contraction principle, which corresponds to a unique solution of the problem (1.1). The proof is completed. $\square$

Remark 2.1. Setting $H(s) = s$ in the problem (1.1) , the Steiltjes-type fractional integral boundary condition takes the form:

$$y(0) = \lambda(I_{0+}^{\gamma} y)(T) := \lambda \frac{1}{\Gamma(\gamma)} \int_0^T \frac{y(s)}{(T-s)^{1-\gamma}} ds. \tag{2.6}$$

In this case, the value of Λ becomes

$$\Lambda = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha+\gamma}\Gamma(\rho\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\rho\alpha+\gamma+1)|\Omega|}$$

and the operator (2.1) modifies to the form:

$$(\mathcal{G}y)(t) = {}^{\rho}I_{0+}^{\alpha} f(t,y(t)) + \frac{1}{\Omega}\Big\{\lambda I_{0+}^{\gamma}[{}^{\rho}I_{0+}^{\alpha} f(s,y(s))](T) + \kappa\Big\},$$

where

$$\Omega = 1 - \frac{\lambda T^{\gamma}}{\Gamma(\gamma+1)}.$$

Remark 2.2. Letting $\gamma \to 1$ in the problem (1.1), the integral boundary conditions reduce to

$$y(0) = \lambda \int_0^T H'(s)y(s)ds + \kappa, \quad t^{1-\rho}\frac{dy}{dt}|_{t=0} = 0, \tag{2.7}$$

which is a special case of the following problem with H being a monotonically increasing function on $[0,T]$:

$$\begin{cases} ({}^{\rho}_{c}D^{\alpha}_{0+}y)(t) = f(t, y(t)), \quad t \in [0,T], \\ y(0) = \lambda \int_0^T y(s)dH(s) + \kappa, \quad t^{1-\rho}\frac{dy}{dt}|_{t=0} = 0. \end{cases} \tag{2.8}$$

In this case, the value of Λ is

$$\Lambda = \frac{T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)|\Omega|}(H(T) - H(0))$$

and the operator (2.1) takes the form:

$$(\mathcal{G}y)(t) = {}^{\rho}I^{\alpha}_{0+}f(t, y(t)) + \frac{1}{\Omega}\Big\{\lambda \int_0^T [{}^{\rho}I^{\alpha}_{0+}f(s, y(s))]H'(s)ds + \kappa\Big\},$$

where

$$\Omega = 1 - \lambda(H(T) - H(0)).$$

Example 2.1. Consider the following problem

$$\begin{cases} ({}^{1/2}_{\ c}D^{5/3}_{0^+}y)(t) = \dfrac{1}{20(t+1)}\Big(\dfrac{|y(t)|+2}{|y(t)|+1} + \cos t\Big), \ t \in [0,2], \\ y(0) = 3/4\ (I^{2/3}_{0^+;H}y)(2) + 1/4, \quad t^{1/2}\frac{dy}{dt}|_{t=0} = 0, \end{cases} \tag{2.9}$$

where $\rho = 1/2$, $\alpha = 5/3$, $\gamma = 2/3$, $\lambda = 3/4$, $\kappa = 1/4$, $T = 2$. Let us take $H(t) = t^3 + t$ and note that $H(t)$ is monotonically increasing function on $[0,2]$ and $H'(t) = 3t^2 + 1$ is continuous function on $(0,2)$. Using the given data, we find that $|\Omega| = 2.856227946$, $\Lambda = 8.835875493$, where Ω and Λ are respectively given by (2.3) and (2.2). Clearly $f(t,y)$ is continuous and satisfies the conditions (A_1) and (A_2) with $k = 1/20$ and $\Psi = \frac{2+\cos t}{20(t+1)}$. Also $k\Big(\frac{|\lambda|T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|}[H(T) - H(0)]^{\gamma}\Big) \approx 0.2538053922 < 1$. Thus all the conditions of Theorem 2.1 are satisfied and consequently the problem (2.9) with $H(t) = t^3 + t$ has at least one solution on $[0,2]$.

Furthermore, the hypothesis of Theorem 2.2 is also satisfied as $k = 1/20$ and $k\Lambda \approx 0.4417937746 < 1$. So, by the conclusion of Theorem 2.2, we deduce that the problem (2.9) with $H(t) = t^3 + t$ has a unique solution on $[0,2]$.

3. Existence results for the problem (1.2)

This section is devoted to the existence of solutions for the problem (1.2).

Definition 3.1. A function $y \in C([0,T], \mathbb{R})$ possessing its Caputo-type generalized derivative of order α is said to be a solution of the initial value problem (1.2) if

$$y(0) = \lambda(I^{\gamma}_{0^+;H}y)(T) + \kappa, \ t^{1-\rho}\frac{dy}{dt}|_{t=0} = 0,$$

and there exists a function $v \in L^1([0,1],\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on $[0,T]$ and

$$y(t) = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}v(s)](T) + \kappa\Big\}. \tag{3.1}$$

For each $y \in C([0,T],\mathbb{R})$, let the set of selections of F be defined by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [0,T]\}.$$

3.1. The Carathéodory case

In this subsection, we consider the case when F has convex values and is of Carathéodory type, and prove an existence result for the problem (1.2) by applying nonlinear alternative of Leray-Schauder type [16].

Theorem 3.1. *Assume that*

(B_1) $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory, where* $\mathcal{P}_{cp,c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y$ *is compact and convex*$\}$;

(B_2) *there exists a continuous nondecreasing function* $\Psi : [0,\infty) \to (0,\infty)$ *and a function* $\Phi \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|x| : x \in F(t,y)\} \le \Phi(t)\Psi(\|y\|) \text{ for each } (t,y) \in [0,T]\times\mathbb{R};$$

(B_3) *there exists a constant* $K > 0$ *such that*

$$\frac{K}{\Psi(K)\Big({}^{\rho}I^{\alpha}_{0^+}\Phi(T) + \frac{|\lambda|}{|\Omega|}I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}\Phi(s)](T)\Big) + \frac{|\kappa|}{|\Omega|}} > 1.$$

Then the problem (1.2) has at least one solution on $[0,T]$.

Proof. Introduce an operator $\mathcal{F} : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,T],\mathbb{R}))$ to transform the problem (1.2) into a fixed point problem as follows:

$$\mathcal{F}(y) = \{h \in C([0,T],\mathbb{R}) : h(t) = (\mathcal{G}y)(t)\}, \tag{3.2}$$

where

$$(\mathcal{G}y)(t) = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}v(s)](T) + \kappa\Big\},\ v \in S_{F,y}.$$

It is obvious that the fixed points of $\mathcal{F}$ are solutions of the problem (1.2).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative [16] in several steps.

Step 1. $\mathcal{F}(y)$ *is convex for each* $y \in C([0,T],\mathbb{R})$.

This step is obvious since $S_{F,y}$ is convex (F has convex values). Indeed, if h_1, h_2 belongs to $\mathcal{F}(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that, for each $t \in [0,T]$, we have

$$h_i(t) = {}^{\rho}I^{\alpha}_{0^+}v_i(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}v_i(s)](T) + \kappa\Big\},\ \ i = 1,2.$$

Let $0 \le \sigma \le 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned}&[\sigma h_1 + (1-\sigma)h_2](t)\\ =&{}^{\rho}I^{\alpha}_{0^+}[\sigma v_1(s) + (1-\sigma)v_2(s)](t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}(\sigma v_1(s) + (1-\sigma)v_2(s))](T) + \kappa\Big\}.\end{aligned}$$

Since F has convex values ($S_{F,y}$ is convex), therefore, $\sigma h_1 + (1-\sigma)h_2 \in \mathcal{F}(y)$.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.

For a positive number r, let $B_r = \{y \in C([0,T],\mathbb{R}) : \|y\| \le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{F}(y)$, $y \in B_r$, there exists $v \in S_{F,y}$ such that

$$h(t) = {}^{\rho}I^{\alpha}_{0+}v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}v(s)](T) + \kappa\Big\}.$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned}|h(t)| &\le {}^{\rho}I^{\alpha}_{0+}|v(t)| + \frac{1}{|\Omega|}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}|v(s)|](T) + |\kappa|\Big\} \\ &\le \Psi(\|y\|)\Big({}^{\rho}I^{\alpha}_{0+}\Phi(T) + \frac{|\lambda|}{|\Omega|}I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}\Phi(s)](T)\Big) + \frac{|\kappa|}{|\Omega|}.\end{aligned}$$

Thus

$$\|h\| \le \Psi(r)\Big({}^{\rho}I^{\alpha}_{0+}\Phi(T) + \frac{|\lambda|}{|\Omega|}I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}\Phi(s)](T)\Big) + \frac{|\kappa|}{|\Omega|} := \ell.$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, and let $y \in B_r$. Then

$$\begin{aligned}&|h(t_2) - h(t_1)| \\ \le& \Big|{}^{\rho}I^{\alpha}_{0+}v(t_2) - {}^{\rho}I^{\alpha}_{0+}v(t_1)\Big| \\ \le& \frac{\rho^{1-\alpha}\Psi(r)}{\Gamma(\alpha)}\left|\int_0^{t_1}\Big[\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_1^{\rho}-s^{\rho})^{1-\alpha}}\Big]\Phi(s)ds + \int_{t_1}^{t_2}\frac{s^{\rho-1}}{(t_2^{\rho}-s^{\rho})^{1-\alpha}}\Phi(s)ds\right|.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $y \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{F} : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\mathcal{F}$ is u.s.c. Since $\mathcal{F}$ is completely continuous, it is enough to establish that it has a closed graph.

Step 4. $\mathcal{F}$ has a closed graph.

Let $y_n \to y_*, h_n \in \mathcal{F}(y_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(y_*)$. Associated with $h_n \in \mathcal{F}(y_n)$, there exists $v_n \in S_{F,y_n}$ such that for each $t \in [0,T]$,

$$h_n(t) = {}^{\rho}I^{\alpha}_{0+}v_n(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}v_n(s)](T) + \kappa\Big\}.$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = {}^{\rho}I^{\alpha}_{0+}v_*(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}v_*(s)](T) + \kappa\Big\}.$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$v \mapsto \Theta v(t) = {}^{\rho}I^{\alpha}_{0+}v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}v(s)](T) + \kappa\Big\}.$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$, so it follows by closed graph operator theorem [24] that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, therefore, we have

$$h_*(t) = {}^{\rho}I^{\alpha}_{0+}v_*(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0+;H}[{}^{\rho}I^{\alpha}_{0+}v_*(s)](T) + \kappa\Big\}, \text{ for some } \ v_* \in S_{F,y_*}.$$

Step 5. We show there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $y \notin \mu\mathcal{F}(y)$ for any $\mu \in (0,1)$ and all $y \in \partial U$.

Let $\mu \in (0,1)$ and $y \in \mu\mathcal{F}(y)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0,T]$, we have

$$y(t) = {}^{\rho}I^{\alpha}_{0^+}v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}v(s)](T) + \kappa\Big\}.$$

As in the second step, one can obtain

$$\begin{aligned}|y(t)| &\le {}^{\rho}I^{\alpha}_{0^+}|v(t)| + \frac{1}{|\Omega|}\Big\{|\lambda| I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}|v(s)|](T) + |\kappa|\Big\}\\ &\le \Psi(\|y\|)\Big({}^{\rho}I^{\alpha}_{0^+}\Phi(T) + \frac{|\lambda|}{|\Omega|}I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}\Phi(s)](T)\Big) + \frac{|\kappa|}{|\Omega|},\end{aligned}$$

which implies that

$$\frac{\|y\|}{\Psi(\|y\|)\Big({}^{\rho}I^{\alpha}_{0^+}\Phi(T) + \frac{|\lambda|}{|\Omega|}I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}\Phi(s)](T)\Big) + \frac{|\kappa|}{|\Omega|}} \le 1.$$

In view of (B_3), there exists K such that $\|y\| \neq K$. Let us set

$$U = \{y \in C(J,\mathbb{R}) : \|y\| < K\}.$$

Note that the operator $\mathcal{F} : \overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is a compact multivalued map, u.s.c. with convex closed values. From the choice of U, there is no $y \in \partial U$ such that $y \in \mu\mathcal{F}(y)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [16], we deduce that $\mathcal{F}$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.2) . This completes the proof. □

3.2. The Lipschitz case

In this subsection we prove the existence of solutions for the problem (1.2) with nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [10].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see [21]), where $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$.

Definition 3.2. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called

(a) $\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for each } x, y \in X;$$

(b) a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.

Lemma 3.1 ([10]). *Let (X, d) be a complete metric space and $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ is closed$\}$. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.2. *Assume that:*

(B_4) *$F(\cdot, y) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$, where $\mathcal{P}_{cp}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y$ is compact$\}$;*

(B_5) *$H_d(F(t, y), F(t, \bar{y})) \leq \omega(t)|y - \bar{y}|$ for almost all $t \in [0, T]$ and $y, \bar{y} \in \mathbb{R}$ with $\omega \in C([0, T], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq \omega(t)$ for almost all $t \in [0, T]$.*

Then the problem (1.2) has at least one solution on $[0, T]$ if $\|\omega\|\Lambda < 1$, where Λ is given in (2.2).

Proof. It follows by the assumption (B_4) that the set $S_{F,y}$ is nonempty for each $y \in C([0, T], \mathbb{R})$. So F has a measurable selection (see Theorem III.6 [6]). Now we show that the operator $\mathcal{F}$ defined by (3.2) satisfies the assumptions of Lemma 3.1. Firstly we show that $\mathcal{F}(y) \in \mathcal{P}_{cl}((C[0, T], \mathbb{R}))$ for each $y \in C([0, T], \mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in \mathcal{F}(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0, T], \mathbb{R})$. Then $u \in C([0, T], \mathbb{R})$ and there exists $v_n \in S_{F,y_n}$ such that, for each $t \in [0, T]$,

$$u_n(t) = {}^{\rho}I^{\alpha}_{0^+} v_n(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+} v_n(s)](T) + \kappa\Big\}.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0, T], \mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in [0, T]$, we have

$$u_n(t) \to u(t) = {}^{\rho}I^{\alpha}_{0^+} v(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+} v(s)](T) + \kappa\Big\}.$$

Hence, $u \in \mathcal{F}(y)$.

Next we show that the operator $\mathcal{F}$ is a contraction. So let $y, \bar{y} \in C([0, T], \mathbb{R})$ and $h_1 \in \mathcal{F}(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0, T]$,

$$h_1(t) = {}^{\rho}I^{\alpha}_{0^+} v_1(t) + \frac{1}{|\Omega|}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+} v_1(s)](T) + \kappa\Big\}.$$

By (B_5), we have

$$H_d(F(t, y), F(t, \bar{y})) \leq \omega(t)|y(t) - \bar{y}(t)|.$$

So, there exists $m \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - m| \leq \omega(t)|y(t) - \bar{y}(t)|, \quad t \in [0, T].$$

Define $U : [0, T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{m \in \mathbb{R} : |v_1(t) - m| \leq \omega(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{y}(t))$ is measurable (Proposition III.4 [6]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t, \bar{y}(t))$. So $v_2(t) \in F(t, \bar{y}(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \leq \omega(t)|y(t) - \bar{y}(t)|$.

For each $t \in [0, T]$, let us define

$$h_2(t) = {}^{\rho}I^{\alpha}_{0^+} v_2(t) + \frac{1}{\Omega}\Big\{\lambda I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+} v_2(s)](T) + \kappa\Big\}.$$

Then

$$|h_1(t) - h_2(t)| \leq {}^{\rho}I^{\alpha}_{0^+}|v_1(t) - v_2(t)| + \frac{|\lambda|}{|\Omega|}\Big\{I^{\gamma}_{0^+;H}[{}^{\rho}I^{\alpha}_{0^+}|v_1(s) - v_2(s)|](T)\Big\}$$

$$\leq \|\omega\| \Big[\frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|} [H(T)-H(0)]^\gamma \Big] \|y-\bar{y}\|.$$

Hence

$$\|h_1 - h_2\| \leq \|\omega\| \Big[\frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|} [H(T)-H(0)]^\gamma \Big] \|y-\bar{y}\|.$$

Analogously, interchanging the roles of y and $\overline{y}$, we can obtain

$$H_d(\mathcal{F}(y), \mathcal{F}(\bar{y})) \leq \|\omega\| \Big[\frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{|\lambda| T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)\Gamma(\gamma+1)|\Omega|} [H(T)-H(0)]^\gamma \Big] \|y-\bar{y}\|.$$

So $\mathcal{F}$ is a contraction when $\|\omega\|\Lambda < 1$. Therefore, it follows by Covitz and Nadler theorem [10] that $\mathcal{F}$ has a fixed point y which is a solution of (1.2). This completes the proof. □

Example 3.1. Let us consider the following boundary value problem

$$\begin{cases} ({}^{1/2}_{c}D^{5/3}_{0^+} y)(t) \in F(t, y(t)), \quad t \in J := [0,2], \\ y(0) = 3/4 \; (I^{2/3}_{0^+;H} y)(2) + 1/4, \;\; t^{1/2} \frac{dy}{dt}|_{t=0} = 0, \end{cases} \tag{3.3}$$

where the data are the same as in example 3.1 and $F(t, y(t))$ will be fixed later.

Let us take $H(t) = t^3 + 1$ and $H'(t) = 3t^2$ is continuous function on $(0, 2)$. Using the given data, we find that $|\Omega| = 2.323196503$, $\Lambda = 9.137895027$, where Ω and Λ are respectively given by (2.3) and (2.2).

For illustrating Theorem 3.1, we take

$$F(t, y) = \Big[\frac{(t+1)}{2\sqrt{900+t}} \Big(\sin y + \frac{|y|}{|y|+1} \Big) \; , \; \frac{e^{-t}\cos t}{250} \Big(\frac{|y|}{4|y|+1} + \frac{1}{8} \Big) \Big]. \tag{3.4}$$

It is easy to check that $F(t, y)$ is L^1-Carathéodory. In view of (B_2) we find that $\Phi(t) = \dfrac{(1+t)}{60}$, $\Psi(\|y\|) = ||y|| + 1$. By condition (B_3), we get $K > 0.2728856315$. Thus all the conditions of Theorem 3.1 are satisfied and consequently, there exists at least one solution for the problem (3.3) with $F(t, y)$ given by (3.4) on $[0, 2]$.

In order to demonstrate the application of Theorem 3.2, let us choose

$$F(t, y) = \Big[\frac{(t+1)}{60} \Big(\sin y + e^{-t} \Big) \; , \; \frac{e^{-t}\cos t}{250} \Big(\frac{|y|}{|y|+1} + \frac{1}{8} \Big) \Big]. \tag{3.5}$$

Clearly

$$H_d(F(t, x), F(t, \bar{x})) \leq \frac{(t+1)}{60} \|y - \bar{y}\|.$$

Letting $\omega(t) = \dfrac{(t+1)}{60}$, it is easy to check that $d(0, F(t, 0)) \leq \omega(t)$ holds for almost all $t \in [0, 2]$ and that $\|\omega\|\Lambda \approx 0.4568947514 < 1$. As the hypotheses of Theorem 3.2 are satisfied, we conclude that the problem (3.3) with $F(t, y)$ given by (3.5) has at least one solution on $[0, 2]$.

4. Conclusions

We have presented the sufficient criteria for the existence and uniqueness of solutions for a generalized Caputo fractional differential equation supplemented with Steiltjes-type fractional integral boundary conditions. As a second problem, we introduced an inclusions problem involving generalized Caputo fractional derivative and Steiltjes-type fractional integral, and proved the existence of its solutions for convex and nonconvex values of the multivalued map involved in the problem. Remarks 2.1 and 2.2 present interesting observations about the obtained results. As an outcome of the present work, one can anticipate that it may be useful in generalizing the Feynman and Wiener path integrals [25] and anomalous diffusion [28, 30], etc. In future, we plan to extend the present study to the case of integro-multipoint boundary conditions. We also intend to formulate and investigate the problems of sequential fractional differential equations and Langevin equation of different fractional order involving generalized Caputo fractional derivatives and Steiltjes-type fractional integral boundary conditions.

Acknowledgments. This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under grant no. (RG-45-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support. We also thank the reviewer for his/her useful remarks on our work.

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