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CONTROLLABILITY AND HYERS–ULAM STABILITY OF IMPULSIVE SECOND ORDER ABSTRACT DAMPED DIFFERENTIAL SYSTEMS*

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Abstract In this paper, we consider system of damped second order abstract impulsive differential equations to investigate its controllability and Hyers–Ulam stability. For our results about the controllability, we utilized the theory of strongly continuous cosine families of linear operators combined with Sadovskii fixed point theorem. In addition, different types of Hyers–Ulam stability is established with the help of Grönwall's type inequality and Lipschitz conditions. At last, we give an example of damped wave equation which outline the application of our principle results.

Keywords Damped differential equation, impulsive system, controllability, β–Hyers–Ulam–Rassias stability.

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1. Introduction

The theory of differential equations (DEs) with impulses has been well utilized in mathematical modeling. In real life problems, there are numerous procedures and phenomena that are characterized by the fact that at certain occasions they experienced sudden changes in their states. These procedures are exposed to short-term perturbations and is known as impulsive effects in the system. In recent years, the theory of DEs with impulses has been investigated by many author's like Samoilenko and Perestyuk [20], Lakshmikantham *et al.* [9], Rogovchenko [16] and Wang *et al.* [24].

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In control theory, controllability of systems is a scientific problem which consists of determining the control parameters, which steers the solutions of the system from its initial state to final state. In the last few decades, controllability received an increasing interest. Several authors examined the controllability of impulsive systems for both instantaneous and non-instantaneous impulses e.g., one can see the work of Shubov *et al.* [21], Qin *et al.* [15] and Park *et al.* [14]. Particularly, concerning the damped DEs of first and second order systems, we recommend the efforts of Arthi and Balachandran [1], Lin and Tanaka [8] and Hernandez *et al.* [5].

In 1940, Ulam posed a problem about the stability of homomorphisms, in his talk at University of Wisconsin [23]. He asked: does there exists a relationship between the exact and approximate homomorphisms, from a group Θ_1 to a metric group Θ_2. After one year, Hyers [4] solved the problem over Banach spaces. In 1978, Rassias [17] gave more extension to the idea of Hyers, where the bound for the norm of Cauchy difference was found in more general form. This concept of stability is termed as Ulam–Hyers–Rassias (UHR) stability. After that, researchers gave their contributions in this vast area of stabilities, for the different functional equations, with various methodologies. Interested readers on the mentioned topic are referred to [6, 7, 18, 22, 31, 32].

Ulam's type stability (UTS) of DEs with impulses was discussed in 2012, by Wang *et al.* [25]. They utilized the idea of bounded interval with impulses and examined UTS for nonlinear impulsive DEs of first order. For more information and approaches about the UTS of impulsive DEs, we suggest [3, 10, 13, 26–30].

In 2018, Muslim *et al.* [11] discussed the stability of second order nonlinear DE with non–instantaneous impulses, using the integral Grönwall's inequality, of the form:

$$\begin{cases} \Theta'' = A\Theta + \zeta(\omega, \Theta(\omega), \Theta(\alpha(\Theta(\omega), \omega))), \ \omega \in (s_k, \omega_{k+1}), \ \ k = 1, 2, \ldots, m, \\ \Theta(\omega) = I_k^1(\omega, \Theta(\omega_k^-)), \ \omega \in (\omega_k, s_k], \ k = 1, 2, \ldots, m, \\ \Theta'(\omega) = J_k^2(\omega, \Theta(\omega_k^-)), \ \omega \in (\omega_k, s_k], \ k = 1, 2, \ldots, m, \\ \Theta(0) = \Theta_0, \quad \Theta'(0) = \Theta_1. \end{cases} \tag{1.1}$$

Motivated from the work done in [1, 11, 25], we investigated the controllability and UTS of the impulsive second order abstract damped equation with control parameter u of the form:

$$\begin{cases} \Theta'' = A\Theta + B\Theta' + Du(\omega) + \zeta(\omega, \Theta(\omega), \Theta(\alpha(\omega))), \ \omega \in I = [0, \rho], \ \omega \neq \omega_k, \\ \Delta\Theta(\omega_k) = I_k(\Theta(\omega_k)), \ k = 1, 2, \ldots, n, \\ \Delta\Theta'(\omega_k) = J_k(\Theta(\omega_k)), \ k = 1, 2, \ldots, n, \\ \Theta(0) = \Theta_0, \quad \Theta'(0) = \Theta_1, \end{cases} \tag{1.2}$$

where A, B and D are linear bounded operators (LBOs) on Banach space $\daleth$, such that A is the infinitesimal generator ($\mathcal{IG}$) of a strongly continuous cosine functions $(C(\omega))_{\omega \in R}$. The control function $u(\cdot)$ is taken from $L^2(I, U)$, a Banach space of admissible control functions with U. Also $\alpha(\cdot)$, $I_k(\cdot)$, $J_k(\cdot)$ and $\zeta(\cdot)$ are appropriate functions and the symbol Δ represents the jump of the function.

This paper is organized as follows: In the first and second sections, we provide introduction, basic notations and definitions which is required for the main results. In the third and fourth sections, we give the main results of controllability and HU stability for system (1.2), respectively. In last section, we consider a damped wave equation and check the applicability of our main results.

2. Basic Notions

In this section, we give some definitions and remarks which can be utilized in our fundamental results. Throughout this paper $(\daleth, \|\cdot\|)$ is a Banach space and $A : D(A) \subset \daleth \to \daleth$ is the $\mathcal{IG}$ of $(\mathcal{C}(\omega))_{\omega\in\mathbb{R}}$ of LBOs on $\daleth$. We signify $(\mathcal{S}(\omega))_{\omega\in\mathbb{R}}$ the sine functions identified with $(\mathcal{C}(\omega))_{\omega\in\mathbb{R}}$, which is portrayed by $\mathcal{S}(\omega)x = \int_0^\omega \mathcal{C}(\tau)xd\tau$, for $x \in \daleth$ and $\omega \in \mathbb{R}$. In addition $\mathcal{M}$ and $\mathcal{N}$ are positive constants to such an extent that $\|\mathcal{C}(\omega)\| \le \mathcal{M}$ and $\|\mathcal{S}(\omega)\| \le \mathcal{N}$, for each $\omega \in I$. The notation E represent the space of the vectors $x \in \daleth$ for which $C(\cdot)x$ is of class C^1.

Definition 2.1 ([5]). A function $\Theta(\cdot)$ is said to be a mild solution of (1.2) if

$$\begin{aligned}\Theta(\omega) =&\mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \int_0^\omega \mathcal{S}(\omega-\tau)B\Theta_\tau(\tau)d\tau + \int_0^\omega \mathcal{S}(\omega-\tau)[Du(\tau)\\ &+ \zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))]d\tau + \sum_{\omega_k<\omega}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))\\ &+ \sum_{\omega_k<\omega}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k)),\ \ \omega\in I.\end{aligned}$$

Replace ω by ρ, we get the following

$$\begin{aligned}\Theta(\rho) =& \mathcal{C}(\rho)\Theta_0 + \mathcal{S}(\rho)\Theta_1 + \int_0^\rho \mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau + \int_0^\omega \mathcal{S}(\rho-\tau)[Du(\tau)\\ &+ \zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))]d\tau + \sum_{\omega_k<\rho}\mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k))\\ &+ \sum_{\omega_k<\rho}\mathcal{S}(\rho-\omega_k)J_k(\Theta(\omega_k))\\ \Rightarrow\ \ &\Theta_1 - \mathcal{C}(\rho)\Theta_0 - \mathcal{S}(\rho)\Theta_1 - \int_0^\rho \mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau - \int_0^\rho \mathcal{S}(\rho-\tau)\\ &\times\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau - \sum_{\omega_k<\rho}\mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k))\\ &- \sum_{\omega_k<\rho}\mathcal{S}(\rho-\omega_k)J_k(\Theta(\omega_k)) = \int_0^\rho \mathcal{S}(\rho-\tau)Du(\tau)d\tau,\end{aligned}\tag{2.1}$$

where $y_1 = \Theta(\rho)$.

In the sequel, we consider the following presumptions :

$[C1]$: The function $f : I \times \daleth^2 \to \daleth$ satisfies the following conditions :

- $f(\omega,\cdot,\cdot) : \daleth\times\daleth \to \daleth$ is continuous a.e. $\omega \in I$. For every $x, y \in \daleth$, the function $f(\cdot, x, y) : I \to \daleth$ is strongly measurable.
- There is a function $m \in L^1(I, [0,\infty))$ and non-decreasing function $Y \in C([0,\infty),(0,\infty))$ such that, for all $\omega \in I$ and every $x, y \in \daleth$,

$$\big\|f(\omega,x,y)\big\| \le m(\omega)Y(\big\|x\big\| + \big\|y\big\|).$$

- For each $\omega \in I$, the function $f(\omega, \cdot, \cdot) : \beth \times \beth \to \beth$ is completely continuous.

$[C2]$: The operator $D : U \to \beth$ is continuous and the linear operator
$Y : L^2(I, U) \to \beth$, defined by

$$Yu = \int_0^\rho \mathcal{S}(\rho - \tau) Du(\tau) d\tau,$$

has a bounded invertible operator Y^{-1} which takes the values from $L^2(I, U)/\ker Y$ such that, $\|D\| \leq M_i$ and $\|y^{-1}\| \leq M_j$, where M_i and M_j are positive constants, $i, j = 1, 2, \ldots, n$.

$[C3]$: The function $\alpha : I \to I$ is continuous and $\alpha(\omega) \leq \omega$ for every $\omega \in I$.

$[C4]$: The maps $I_k, J_k : \beth \to \beth, k = 1, 2, \ldots, n$ are completely continuous and there exist non-decreasing functions $\mu_k, \nu_k : [0, \infty) \to (0, \infty)$, such that

$$|I_k(x)\| \leq \mu_k(\|x\|) \quad \text{and} \quad |J_k(x)\| \leq \nu_k(\|x\|), \quad \text{for all} \;\; x \in \beth.$$

$[C5]$: The inequality $\left(\mathcal{N} M_1 + \sum_{k=1}^n \mathcal{M} L_{I_k} + \mathcal{N} L_{J_k}\right) < 1$ holds.

Definition 2.2. Let $\mathcal{V}$ be a vector space over some field K.
A function $\| \cdot \|_\beta : \mathcal{V} \to [0, \infty)$ is called $\beta-$norm if:
(a) : $\|\mu\|_\beta = 0$ if and only if $\mu = 0$,
(b) : $\|c\mu\|_\beta = |c|^\beta \|\mu\|_\beta$ for each $c \in K$ and $u \in \mathcal{V}$,
(c) : $\|\mu + \nu\|_\beta \leq \|\mu\|_\beta + \|\nu\|_\beta$.
Then $(\mathcal{V}, \| \,.\, \|_\beta)$ is known as $\beta-$normed space.

To discuss UTS of the given system, we need some conditions that can be used in our results. The conditions are:
$[H_1]$: A is the $\mathcal{IG}$ of $(\mathcal{C}(\omega))_\omega \in R$.
$[H_2]$: $\zeta : I \times \beth^2 \to \beth$ is a continuous function and their exists a positive constant $L_\zeta(\omega)$ such that:

$$\begin{aligned} &\left\|\zeta(\omega, \Upsilon(\omega), \Upsilon(a(\omega))) - \zeta(\omega, \Theta(\omega), \Theta(a(\omega)))\right\| \\ \leq &L_\zeta(\omega)(\left\|\Upsilon(\omega) - \Theta(\omega)\right\| + \left\|\Upsilon(\alpha(\omega)) - \Theta(a(\omega))\right\|). \end{aligned}$$

$[H_3]$: The functions $I_k, J_k : \beth \to \beth$ are continuous and there exist positive constants L_I and L_J such that:

$$\begin{aligned} \left\|I_k(\mu) - I_k(\nu)\right\| &\leq L_I \left\|\mu - \nu\right\|, \\ \left\|J_k(\mu) - J_k(\nu)\right\| &\leq L_J \left\|\mu - \nu\right\|. \end{aligned}$$

$[H_4]$: There exists a nondecreasing function $\phi \in PC(I, S)$ with $\phi(\omega) \geq 0$ and a constant c_ϕ such that,

$$\int_0^\omega \phi(\tau) d\tau \leq c_\phi \phi(\omega),$$

for each $\omega \in I$ and $\tau \in S$.

Lemma 2.1 (Lemma 3.1, [12])**.** *Let us assume that all the assumptions listed in Lemma 3.1 are fulfilled. Then the operator*

$$\Lambda\gamma(\omega) = \int_0^\omega \mathcal{S}(\omega - \tau)[f(\tau, \gamma(\tau)) + (Du_\gamma)(\tau)] d\tau, \quad \omega \in [0, \rho],$$

is completely continuous.

Lemma 2.2 (Sadoskii Lemma [19]). *Assume that ζ is a condensing operator on $\daleth$. If $\zeta(A) \subset A$ is closed, bounded and convex set of $\daleth$, then ζ has a fixed point in A.*

Lemma 2.3 (Grönwall's Lemma [2]). *For any $\omega \geq 0$ with*

$$\Theta(\omega) \leq q(\omega) + \int_0^\omega p(\tau)\Theta(\tau)d\tau + \sum_{0<\omega_k<\omega} \gamma_k \Theta(\omega_k^-),$$

where $x, p, q \in PC(I, \mathbb{R}^+), q$ is nondecreasing and $\gamma > 0$, we have:

$$\Theta(\omega) \leq q(\omega)(1+\gamma_k)^k \exp(\int_0^\omega p(\tau)d\tau), \quad \forall\ \omega \in \mathbb{R}^+, \tag{2.2}$$

where $k \in M$.

3. Controllability

In the following segment, we establish the results for controllability of impulsive second order damped problem (1.2).

Definition 3.1. The system listed in (1.2) is controllable on the interval I, if for each $\xi \in D(A)$, $\eta \in E$ and $\Phi_1 \in \daleth$, there exists a control $u \in L^2(I, U)$, such that the mild solution $\Phi(\omega)$ of (1.2) satisfies $\Phi(\rho) = y_1$.

Theorem 3.1. *Assume that the presumptions $[C1]$–$[C5]$ as well as $[H_1]$–$[H_3]$ are fulfilled. Then the system (1.2) is controllable on I, provided that*

$$(1+\rho\mathcal{N}M_iM_j)\Big[\mathcal{N}M_1 + \liminf_{l\to\infty} \frac{\mathcal{N}Y(2l)}{l}\int_0^\rho m(\tau)d\tau + (V\mathcal{M}L_I + \mathcal{N}L_J)\Big] < 1.$$

Proof. Consider the space $\mathcal{Z} = PC([0,\rho] : \daleth)$ endowed with the uniform convergence topology. Using the condition $[C2]$, for an arbitrary function $\Theta(\cdot)$, the control parameter is defined as:

$$\begin{aligned} u(\omega) =& Y^{-1}\big[y_1 - \mathcal{C}(\rho)\Theta_0 - \mathcal{S}(\rho)\Theta_1 - \int_0^\rho \mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau - \int_0^\rho \mathcal{S}(\rho-\tau) \\ & \times \zeta(\tau, \Theta(\tau), \Theta(\alpha(\tau)))d\tau - \sum_{\omega_k<\rho} \mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k)) \\ & - \sum_{\omega_k<\rho} \mathcal{S}(\rho-\omega_k)J_k(\Theta(\omega_k))](\omega). \end{aligned}$$

Using the above control function, we will show that Φ has a fixed point, where $\Phi : \mathcal{Z} \to \mathcal{Z}$ is defined by

$$\begin{aligned} \Phi x(\omega) =& \mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \int_0^t \mathcal{S}(\omega-\tau)B\Theta_\tau(\tau)d\tau + \int_0^t \mathcal{S}(\omega-\tau)DY^{-1} \\ & \times \big[y_1 - \mathcal{C}(\rho)\Theta_0 - \mathcal{S}(\rho)\Theta_1 - \int_0^\rho \mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau - \int_0^\rho \mathcal{S}(\rho-\tau) \end{aligned}$$

$$\times\,\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau+\int_0^t\mathcal{S}(\omega-\tau)\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))$$
$$-\sum_{\omega_k<\rho}\mathcal{S}(\rho-\omega_k)J_k(\Theta(\omega_k))\big](\psi)d\psi-\sum_{\omega_k<\rho}\mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k))]d\tau$$
$$+\sum_{\omega_k<\omega}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))+\sum_{\omega_k<\omega}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k)),\;\;\omega\in I.$$

According to Sadovskii Lemma, this fixed point is then a mild solution of the system (1.2). Clearly, $(\phi x)(\rho)=y_1$ which implies that u steers (1.2) from initial state Θ_0 to y_1 in time ρ, provided if we can get a fixed point of the operator Φ then (1.2) is controllable. From the presumptions, clearly Φ is continuous and is well defined.

Next, we ensure that there exists a positive number l, such that $\Phi(B_l(0,PC))\subset B_l(0,PC)$. If this assertion is false, at that point for each $l>0$, there exist $y_1\in B_l(0,PC)$, $k=0,\dots,n$ and $\omega_l\in[\omega_k,\omega_{k+1}]$ such that $\|\Phi x_l(\omega_l)\|>l$. Consequently,

$$\begin{aligned}
l<{}&\big\|\Phi x_l(\omega_l)\big\|\\
={}&\big\|\mathcal{C}(\omega)\Theta_0+\mathcal{S}(\omega)\Theta_1+\int_0^\omega\mathcal{S}(\omega-\tau)B\Theta_\tau(\tau)d\tau+\int_0^\omega\mathcal{S}(\omega-\tau)DY^{-1}\\
&\times\big[y_1-\mathcal{C}(\rho)\Theta_0-\mathcal{S}(\rho)\Theta_1-\int_0^\rho\mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau-\int_0^\rho\mathcal{S}(\rho-\tau)\\
&\times\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau-\sum_{\omega_k<\rho}\mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k))-\sum_{\omega_k<\rho}\mathcal{S}(\rho-\omega_k)\\
&\times J_k(\Theta(\omega_k))\big](\psi)d\psi+\int_0^\omega\mathcal{S}(\omega-\tau)\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau\\
&+\sum_{\omega_k<\omega}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))+\sum_{\omega_k<\omega}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k))\big\|\\
\le{}&\mathcal{M}\|\Theta_0\|+\mathcal{N}\|\Theta_1\|+\mathcal{N}M_1\int_0^t\|\Theta_\tau(\tau)d\tau\|+\rho\mathcal{N}M_iM_j\big[\|y_1\|+\mathcal{M}\|\Theta_0\|\\
&+\mathcal{N}\|\Theta_1\|+\mathcal{N}M_1\int_0^\rho\|\Theta_\tau(\tau)d\tau\|+\mathcal{N}\int_0^\rho m(\tau)Y[\|\Theta(\tau)\|+\|\Theta(\alpha(\tau))\|]d\tau\\
&+\mathcal{M}\sum_{\omega_k<\rho}\|I_k(\Theta(\omega_k))\|+\mathcal{N}\sum_{\omega_k<\rho}\|J_k(\Theta(\omega_k))\|\big]d\psi+\mathcal{N}\int_0^\omega m(\tau)Y[\|\Theta(\tau)\|\\
&+\|\Theta(\alpha(\tau))\|]d\tau+\mathcal{M}\sum_{\omega_k<\omega}\|I_k(\Theta(\omega_k))\|+\mathcal{N}\sum_{\omega_k<\omega}\|J_k(\Theta(\omega_k))\|\\
\le{}&\mathcal{M}\|\Theta_0\|+\mathcal{N}\|\Theta_1\|+\mathcal{N}M_1l+\rho\mathcal{N}M_iM_j\big[\|y_1\|+\mathcal{M}\|\Theta_0\|+\mathcal{N}\|\Theta_1\|\\
&+\mathcal{N}M_1l+\mathcal{N}Y(2l)\int_0^\rho m(\tau)d\tau+[(\mathcal{M}L_I+\mathcal{N}L_J)l+\mathcal{M}\|I_k(0)\|+\mathcal{N}\|J_k(0)\|]\big]\\
&+\mathcal{N}Y(2l)\int_0^\rho m(\tau)d\tau+\big[(\mathcal{M}L_I+\mathcal{N}L_J)l+\mathcal{M}\|I_k(0)\|+\mathcal{N}\|J_k(0)\|\big],\\
\Rightarrow\;\; l\le{}&\mathcal{M}\|\Theta_0\|+\mathcal{N}\|\Theta_1\|+\mathcal{N}\mathcal{M}_1l+b\mathcal{N}\mathcal{M}_iM_j\big[\|y_1\|+\mathcal{M}\|\Theta_0\|+\mathcal{N}\|\Theta_1\|\\
&+\mathcal{N}M_1l+\mathcal{N}Y(2l)\int_0^b m(s)ds+[(\mathcal{M}L_I+\mathcal{N}L_J)l+\mathcal{M}\|I_k(0)\|+\mathcal{N}\|J_k(0)\|]\big]\\
&+\mathcal{N}Y(2l)\int_0^b m(s)ds+[(\mathcal{M}L_I+\mathcal{N}L_J)l+\mathcal{M}\|I_k(0)\|+\mathcal{N}\|J_k(0)\|],
\end{aligned}$$

$$\Rightarrow\ 1 \leq (1+\rho\mathcal{N}M_iM_j)\Big[\mathcal{N}M_1 + \liminf_{l\to\infty}\frac{\mathcal{N}Y(2l)}{l}\int_0^\rho m(\tau)d\tau + (\mathcal{M}L_I + \mathcal{N}L_J)\Big].$$

This contradicts our presumption.

Let l be a positive number such that $\Phi(B_l(0,PC)) \subset B_l(0,PC)$. All together to demonstrate that $\Phi : B_l(0,PC) \to B_l(0,PC)$ is a condensing map. We introduce the decomposition $\Phi = \Phi_1 + \Phi_2$, where

$$\begin{aligned}\Phi_1\Theta(\omega) =& \mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \int_0^\omega \mathcal{S}(\omega-\tau)B\Theta_\tau(\tau)d\tau + \sum_{\omega_k<\omega}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))\\ &+ \sum_{\omega_k<\omega}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k)),\\ \Phi_2\Theta(\omega) =& \int_0^\omega \mathcal{S}(\omega-\tau)[Du(\tau) + \zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))]d\tau.\end{aligned}$$

Now

$$\begin{aligned}\|Du(\tau)\| =& \Big\|DY^{-1}\big[y_1 - \mathcal{C}(\rho)\Theta_0 - \mathcal{S}(\rho)\Theta_1 - \int_0^\rho \mathcal{S}(\rho-\tau)\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau\\ &- \int_0^\rho \mathcal{S}(\rho-\tau)B\Theta_\tau(\tau)d\tau - \sum_{\omega_k<\rho}\mathcal{C}(\rho-\omega_k)I_k(\Theta(\omega_k))\\ &- \sum_{\omega_k<\rho}\mathcal{S}(\rho-\omega_k)J_k(\Theta(\omega_k))\big]\Big\|\\ \leq& M_iM_j\big[\|y_1\| + \mathcal{M}\|\Theta_0\| + \mathcal{N}\|\Theta_1\| + \mathcal{N}M_1\int_0^\rho \Theta_\tau(\tau)d\tau + \mathcal{N}\int_0^\rho m_l(\tau)d\tau\\ &+ \mathcal{M}\sum_{i=1}^n \mu_i\|\Theta(\omega_i)\| + \mathcal{N}\sum_{i=1}^n \nu_i\|\Theta(\omega_i)\|\big]\\ \leq& M_iM_j\big[\|y_1\| + \mathcal{M}\|\Theta_0\| + \mathcal{N}\|\Theta_1\| + \mathcal{N}M_1 l + \mathcal{N}\int_0^\rho m_l(\tau)d\tau\\ &+ \sum_{i=1}^n l[\mathcal{M}\mu_i + \mathcal{N}\nu_i]\big] = M_0.\end{aligned}$$

Here by applying a similar strategy which is referenced in Lemma 2.2. From the presumptions $[C1]$, $[C2]$ and $[C3]$, we derived that Φ_2 is completely continuous on $B_l(0,PC)$. Next, we need to show that Φ_1 is contraction on $B_l(0,PC)$. For this let $x_1, x_2 \in B_l(0,PC)$, we have

$$\begin{aligned}\|\Phi_1\xi - \Phi_1\eta\| =& \Big\|\mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \sum_{k=1}^n \mathcal{C}(\omega-\omega_k)I_k(\xi(\omega_k)) + \int_0^\omega \mathcal{S}(\omega-\tau)B\xi_\tau(\tau)d\tau\\ &+ \sum_{k=1}^n \mathcal{S}(\omega-\omega_k)J_k(\xi(\omega_k)) - \big(\mathcal{C}(\omega)\Theta_0 + \int_0^\omega \mathcal{S}(\omega-\tau)B\eta_\tau(\tau)d\tau\\ &+ \mathcal{S}(\omega)\Theta_1 + \sum_{k=1}^n \mathcal{C}(\omega-\omega_k)I_k(\eta(\omega_k)) + \sum_{k=1}^n \mathcal{S}(\omega-\omega_k)J_k(\eta(\omega_k))\big)\Big\|\\ \leq& \mathcal{N}M_1\|\xi-\eta\| + \mathcal{M}\sum_{k=1}^n L_{I_k}\|\xi-\eta\| + \mathcal{N}\sum_{k=1}^n L_{J_k}\|\xi-\eta\|\end{aligned}$$

$$\leq \big(\mathcal{N}M_1 + \sum_{k=1}^{n} \mathcal{M}L_{I_k} + \mathcal{N}L_{J_k}\big)\big\|\xi - \eta\big\|.$$

Hence Φ_1 is contraction and $\Phi(\cdot)$ is a condensing operator on $B_l(0, PC)$.

Now, from Lemma 2.2, Φ has a fixed point in PC. This implies that any fixed point of Φ is a mild solution of the system (1.2). Thus (1.2) is controllable on I. □

4. Ulam's type stability

Hernandez *et al.* [5] found the solution of the system:

$$\begin{cases} \Theta'' = A\Theta + B\Theta' + \zeta(\omega, \Theta(\omega), \Theta(\alpha(\omega))), \omega \in I, \omega \neq \omega_k,\ k = 1, 2, \ldots, n, \\ \Delta\Theta(\omega_k) = I_k(\Theta(\omega_k)),\ k = 1, 2, \ldots, n, \\ \Delta\Theta'(\omega_k) = J_k(\Theta(\omega_k)),\ k = 1, 2, \ldots, n, \\ \Theta(0) = \Theta_0, \quad \Theta'(0) = \Theta_1, \end{cases} \tag{4.1}$$

in the form

$$\begin{aligned} \Theta(\omega) =& \mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \int_0^\omega \mathcal{S}(\omega - \tau)B\Theta_\tau(\tau)d\tau + \sum_{\omega_k < \omega} \mathcal{C}(\omega - \omega_k)I_k(\Theta(\omega_k)) \\ &+ \int_0^\omega \mathcal{S}(\omega - \tau)\zeta(\tau, \Theta(\tau), \Theta(\alpha(\tau)))d\tau + \sum_{\omega_k < \omega} \mathcal{S}(\omega - \omega_k)J_k(\Theta(\omega_k)),\ \ \omega \in I. \end{aligned} \tag{4.2}$$

Let $\epsilon > 0, \psi \geq 0$ and $\phi \in PC(I, R^+)$ be the nondecreasing functions. We consider the following inequalities

$$\begin{cases} \big\|\Upsilon'' - A\Upsilon(\omega) - B\Upsilon' - \zeta(\omega, \Upsilon(\omega), \Upsilon(\alpha(\omega)))\big\| \leq \epsilon,\ \ \omega \in I \\ \big\|\Delta\Upsilon(\omega_k) - I_k(\Upsilon(\omega_k))\big\| \leq \epsilon,\ \ \omega \neq \omega_k \\ \big\|\Delta\Upsilon_t(\omega_k) - J_i(\Upsilon(\omega_k))\big\| \leq \epsilon,\ \ \omega \neq \omega_k \end{cases} \tag{4.3}$$

and

$$\begin{cases} \big\|\Upsilon''(\omega) - A\Upsilon(\omega) - B\Upsilon'(\omega) - \zeta(\omega, \Upsilon(\omega), \Upsilon(\alpha(\omega)))\big\| \leq \epsilon\phi(\omega),\ \ \omega \in I \\ \big\|\Delta\Upsilon(\omega_k) - I_k(\Upsilon(\omega_k))\big\| \leq \epsilon\psi,\ \ \omega \neq \omega_k \\ \big\|\Delta\Upsilon_t(\omega_k) - J_k(\Upsilon(\omega_k))\big\| \leq \epsilon\psi,\ \ \omega \neq \omega_k. \end{cases} \tag{4.4}$$

Remark 4.1. It is direct consequence of inequality (4.3) that a function $\Upsilon \in \mathcal{Z}$ is solution of the inequality (4.3), if and only if there are $\mathcal{G} \in C^2(I, \beth)$, $g_1 \in C(I, \beth)$ and $g_2 \in C^1(I, \beth)$ such that:

$$\begin{cases} \|\mathcal{G}(\omega)\| \leq \epsilon, \|g_1(\omega)\| \leq \epsilon \ \text{ and } \ \|g_2(\omega)\| \leq \epsilon, \quad \omega \in I. \\ \Upsilon''(\omega) = A\Upsilon(\omega) + B\Upsilon'(\omega) + \zeta(\omega, \Upsilon(\omega), \Upsilon(\alpha(\omega))) + \mathcal{G}(\omega), \\ \omega \in I, \omega \neq \omega_k,\ k = 1, 2, \ldots, n, \\ \Upsilon(0) = \Theta_0 + \mathcal{G}(\omega), \Upsilon'(0) = \Theta_1 + \mathcal{G}(\omega), \\ \Delta\Upsilon(\omega_k) = I_k(\Upsilon(\omega_k)) + g_1(\omega_k),\ k = 1, 2, \ldots, n, \\ \Delta\Upsilon'(\omega_k) = J_k(\Upsilon(\omega_k)) + g_2(\omega_k),\ k = 1, 2, \ldots, n. \end{cases} \tag{4.5}$$

Remark 4.2. A function $\Upsilon \in \mathcal{Z}$ is solution of the inequality (4.4) if and only if there are $\mathcal{G} \in C^2(I, \beth)$, $g_1 \in C(I, \beth)$ and $g_2 \in C^1(I, \beth)$ such that:

$$\begin{cases} \|\mathcal{G}(\omega)\| \leq \epsilon\phi(\omega), \|g_1(\omega)\| \leq \epsilon\psi \ \ \textit{and} \ \ \|g_2(\omega)\| \leq \epsilon\psi, \quad \omega \in I. \\ \Upsilon''(\omega) = A\Upsilon(\omega) + B\Upsilon'(\omega) + \zeta(\omega, \Upsilon(\omega), \Upsilon(\alpha(\omega))) + \mathcal{G}(\omega), \ \ \omega \in I, \omega \neq \omega_k, \\ \Upsilon(0) = \Theta_0 + \mathcal{G}(\omega), \Upsilon'(0) = \Theta_1 + \mathcal{G}(\omega), \\ \Delta\Upsilon(\omega_k) = I_k(\Upsilon(\omega_k)) + g_1(\omega_k), \ \ k = 1, 2, \ldots, n. \\ \Delta\Upsilon_\omega(\omega_k) = J_k(\Upsilon(\omega_k)) + g_2(\omega_k), \ \ k = 1, 2, \ldots, n. \end{cases} \tag{4.6}$$

Definition 4.1. The system (4.1) is HU stable if there exists $\vartheta(K_1, L_1, L_\zeta) > 0$ such that for each $\epsilon > 0$ and for each solution $\Upsilon \in \mathcal{Z}$ of the inequality (4.3), there is a solution $\Theta \in \mathcal{Z}$ of Eq.(4.1), such that

$$\big\|\Upsilon(\omega) - \Theta(\omega)\big\| \leq \vartheta(K_1, L_1, L_\zeta)\epsilon, \quad \omega \in I. \tag{4.7}$$

Definition 4.2. The equation (4.1) is HUR stable with regard to (ϕ, ψ) if there exists $\vartheta(K_1, L_1, L_\zeta, \phi) > 0$ such that, for each $\epsilon > 0$ and for every solution $\Upsilon \in \mathcal{Z}$ of the inequality (4.4), there is a solution $\Theta \in \mathcal{Z}$ of Eq.(4.1), such that

$$\big\|\Upsilon(\omega) - \Theta(\omega)\big\| \leq \vartheta(K_1, L_1, L_\zeta, \phi)\epsilon(\phi(\omega) + \omega\psi), \quad \omega \in I. \tag{4.8}$$

Definition 4.3. The equation (4.1) is β–HUR stable with regard to (ϕ^β, ψ^β) if there exists $\vartheta(K_1, L_1, L_\zeta, \phi, \psi) > 0$ such that for each $\epsilon > 0$, and for every solution $\Upsilon \in \mathcal{Z}$ of the inequality (4.4), there is a solution $\Theta \in \mathcal{Z}$ of Eq.(4.1), such that

$$\big\|\Upsilon(\omega) - \Theta(\omega)\big\| \leq \vartheta(K_1, L_1, L_\zeta, \phi, \psi)\epsilon(\phi(\omega) + \omega\psi), \quad \omega \in I. \tag{4.9}$$

Theorem 4.1. *If assumptions $[H_1]$–$[H_3]$ are fulfilled, then the Eq.(4.1) is HU stable with respect to ϵ.*

Proof. On the basis of Remark 4.1, we can say that solution of the system (4.5) is:

$$\begin{aligned} \Upsilon(\omega) = {} & \mathcal{C}(\omega)(\Theta_0 + \mathcal{G}(\omega)) + \mathcal{S}(\omega)(\Theta_1 + \mathcal{G}(\omega)) + \int_0^\omega \mathcal{S}(\omega - \tau)B\Upsilon'(\tau)d\tau \\ & + \int_0^\omega \mathcal{S}(\omega - \tau)[\zeta(\tau, \Upsilon(\tau), \Upsilon(\alpha(\tau))) + \mathcal{G}(\tau)]d\tau + \sum_{\omega > \omega_k} \mathcal{C}(\omega - \omega_k)[I_k(\Upsilon(\omega_k)) \\ & + g_1(\omega_k)] + \sum_{\omega > \omega_k} \mathcal{S}(\omega - \omega_k)[J_k(\Upsilon(\omega_k)) + g_2(\omega_k)], \quad \omega \in I. \end{aligned}$$

Let Υ be the solution of inequality (4.3). Then for every $\omega \in I$, we obtain

$$\begin{aligned} & \Big\|\Upsilon(\omega) - \mathcal{C}(\omega)\Theta_0 - \mathcal{S}(\omega)\Theta_1 - \int_0^\omega \mathcal{S}(\omega - \tau)B\Upsilon'(\tau)d\tau - \sum_{\omega > \omega_k} \mathcal{C}(\omega - \omega_k)I_k(\Upsilon(\omega_k)) \\ & - \int_0^\omega \mathcal{S}(\omega - \tau)\zeta(\tau, \Upsilon(\tau), \Upsilon(\alpha(\tau)))d\tau - \sum_{\omega > \omega_k} \mathcal{S}(\omega - \omega_k)J_k(\Upsilon(\omega_k))\Big\| \\ \leq {} & \epsilon(\mathcal{M} + \mathcal{N}) + \epsilon\mathcal{N}\int_0^\omega d\tau + \epsilon\mathcal{M}\omega + \epsilon\mathcal{N}\omega \end{aligned}$$

$$\leq \epsilon(M_0 + 2\mathcal{N}\omega + \mathcal{M}\omega).$$

Therefore, for each $\omega \in I$, we get

$$\begin{aligned}\|\Upsilon(\omega) - \Theta(\omega)\| =& \|\Upsilon(\omega) - \mathcal{C}(\omega)\Theta_0 - \mathcal{S}(\omega)\Theta_1 - \int_0^\omega \mathcal{S}(\omega-\tau)B\Theta'(\tau)d\tau \\ & - \int_0^\omega \mathcal{S}(\omega-\tau)\zeta(\tau, \Theta(\tau), \Theta(\alpha(\tau)))d\tau - \sum_{\omega>\omega_k} \mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k)) \\ & - \sum_{\omega>\omega_k} \mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k))\| \\ \leq& \epsilon(M_0 + 2\mathcal{N}t + \mathcal{M}t) + \mathcal{N}\|B\| \| \int_0^\omega (\Upsilon'(\tau) - \Theta'(\tau))d\tau \| \\ & + \mathcal{N} \int_0^\omega L_\zeta(\tau)[\|\Upsilon(\tau) - \Theta(\tau)\| + \|\Upsilon(a(\tau)) - \Theta(\alpha(\tau))\|]d\tau \\ & + \mathcal{M} \sum_{\omega>\omega_k} L_K \|\Upsilon(\omega_k) - \Theta(\omega_k)\| + \mathcal{N} \sum_{\omega>\omega_k} L_J \|\Upsilon(\omega_k) - \Theta(\omega_k)\|.\end{aligned}$$

This implies

$$\begin{aligned}\|\Upsilon(\omega) - \Theta(\omega)\| \leq& \epsilon\left(\frac{M_0 + 2\mathcal{N}\omega + \mathcal{M}\omega}{1 - \mathcal{N}\|B\|}\right) \\ & + \frac{\mathcal{N}}{1 - \mathcal{N}\|B\|} \int_0^\omega L_\zeta(\tau)\Big[\|\Upsilon(\tau) - \Theta(\tau)\| + \|\Upsilon(\alpha(\tau)) - \Theta(\alpha(\tau))\|\Big]d\tau \\ & + \frac{2M_1}{1 - \mathcal{N}\|B\|} \sum_{\omega>\omega_k} L_1 \|\Upsilon(\omega_k) - \Theta(\omega_k)\|,\end{aligned}$$

where $M_1 = \max\{\mathcal{M}, \mathcal{N}\}$ and $L_1 = \max\{L_K, L_J\}$.

Now, using Lemma 2.3, we get

$$\begin{aligned}\|\Upsilon(\omega) - \Theta(\omega)\| \leq& \epsilon\big[\frac{M_0 + 2\mathcal{N}\omega + \mathcal{M}\omega}{1 - \mathcal{N}\|B\|}\big]\big[1 + \frac{2M_1}{1 - \mathcal{N}\|B\|}L_1\big]^m \exp\Big(\frac{\mathcal{N}}{1 - \mathcal{N}\|B\|} \\ & \times \int_0^\omega L_\zeta(\tau)d\tau\Big) \\ \leq& \vartheta(K_1, L_1, L_\zeta)\epsilon,\end{aligned}$$

where,

$$\begin{aligned}\vartheta(K_1, L_1, L_\zeta) =& \big[\frac{M_0 + 2\mathcal{N}\omega + \mathcal{M}\omega}{1 - \mathcal{N}B}\big]\big[1 + \frac{2M_1}{1 - \mathcal{N}\|B\|}L_1\big]^m \exp\big(\frac{\mathcal{N}}{1 - \mathcal{N}\|B\|} \\ & \times \int_0^\omega L_\zeta(\tau)d\tau\big).\end{aligned}$$

Hence Eq.(4.1) is HU stable. □

Theorem 4.2. *If assumptions* $[H_1]$–$[H_4]$ *are fulfilled, then the Eq.*(4.1) *is HUR stable with regard to* (ϕ, ψ).

Proof. Let Υ be a solution of the inequality (4.4) and Θ be the unique solution of the system (4.1), which is given in (4.2). On the basis of Remark 4.2, we have the solution of the system (4.6) as:

$$\begin{aligned}\Upsilon(\omega) = &\mathcal{C}(\omega)(\Theta_0 + \mathcal{G}(\omega)) + \mathcal{S}(\omega)(\Theta_1 + \mathcal{G}(\omega)) + \int_0^\omega \mathcal{S}(\omega-\tau)B\Upsilon'(\tau)d\tau\\ &+ \int_0^\omega \mathcal{S}(\omega-\tau)\big[\zeta(\tau,\Upsilon(\tau),\Upsilon(\alpha(\tau))) + \mathcal{G}(\tau)\big]d\tau\\ &+ \sum_{\omega>\omega_k} \mathcal{C}(\omega-\omega_k)[I_k(\Upsilon(\omega_k))+g_1(\omega_k)] \sum_{\omega>\omega_k} \mathcal{S}(\omega-\omega_k)[J_k(\Upsilon(\omega_k))+g_2(\omega_k)].\end{aligned}$$

Let Υ be solution of (4.4). Then for every $\omega \in I$, we obtain

$$\begin{aligned}&\big\|\Upsilon(\omega) - \mathcal{C}(\omega)\Theta_0 - \mathcal{S}(\omega)\Theta_1 - \int_0^\omega \mathcal{S}(\omega-\tau)B\Upsilon'(\tau)d\tau\\ &- \int_0^\omega \mathcal{S}(\omega-\tau)\zeta(\tau,\Upsilon(\tau),\Upsilon(\alpha(\tau)))d\tau - \sum_{\omega>\omega_k}\mathcal{C}(\omega-\omega_k)I_k(\Upsilon(\omega_k))\\ &- \sum_{\omega>\omega_k}\mathcal{S}(\omega-\omega_k)J_k(\Upsilon(\omega_k))\big\|\\ \leq& M_0\epsilon\phi(\omega) + \mathcal{N}\epsilon\int_0^\omega \phi(\tau)d\tau + \mathcal{M}\omega\epsilon\psi + \mathcal{N}\omega\epsilon\psi\\ \leq& \epsilon(\phi(\omega) + \omega\psi)(M_0 + \mathcal{N}C_\phi).\end{aligned}$$

Thus for every $\omega \in I$, we get

$$\begin{aligned}&\big\|\Upsilon(\omega) - \Theta(\omega)\big\|\\ =&\big\|\Upsilon(\omega) - \mathcal{C}(\omega)\Theta_0 - \mathcal{S}(\omega)\Theta_1 - \int_0^\omega \mathcal{S}(\omega-\tau)B\Theta'(\tau)d\tau\\ &- \int_0^\omega \mathcal{S}(\omega-\tau)\zeta(\tau,\Theta(\tau),\Theta(\alpha(\tau)))d\tau - \sum_{\omega>\omega_k}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))\\ &- \sum_{\omega>\omega_k}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k))\big\|\\ \leq&\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi) + \mathcal{N}B\int_0^\omega \big\|(\Upsilon'(\tau) - \Theta'(\tau))\big\|d\tau\\ &+ \mathcal{N}\int_0^\omega \big\|(\zeta(\tau,\Upsilon(\tau),\Upsilon(a(\tau))) - \Theta(\tau,\Theta(\tau),\Theta(\alpha(\tau))))\big\|d\tau\\ &+ \mathcal{M}\sum_{\omega>\omega_k}\big\|I_k(\Upsilon(\omega_k)) - I_k(\Theta(\omega_k))\big\| + \mathcal{N}\sum_{\omega>\omega_k}\big\|J_k(\Upsilon(\omega_k)) - J_k(\Theta(\omega_k))\big\|.\end{aligned}$$

This implies

$$\begin{aligned}&\big\|\Upsilon(\omega) - \Theta(\omega)\big\|\\ \leq&\epsilon\frac{(\phi(\omega)+t\psi)(M_0+\mathcal{N}C_\phi)}{1-\mathcal{N}\|B\|} + \frac{2M_1}{1-\mathcal{N}\|B\|}\sum_{\omega>\omega_k} L_1\big\|\Upsilon(\omega_k) - \Theta(\omega_k)\\ &+ \frac{\mathcal{N}}{1-\mathcal{N}\|B\|}\int_0^\omega L_\zeta(\tau)\big[\big\|\Upsilon(\tau)-\Theta(\tau)\big\| + \big\|\Upsilon(\alpha(\tau)) - \Theta(\alpha(\tau))\big\|\big]d\tau\big\|,\end{aligned}$$

where $M_1 = \max\{\mathcal{M},\mathcal{N}\}$, and $L_1 = \max\{L_K, L_J\}$.

Now, using Lemma 2.3, we get

$$\begin{aligned}&\big\|\Upsilon(\omega)-\Theta(\omega)\big\|\\ \leq&\epsilon(\phi(\omega)+\omega\psi)\frac{(M_0+\mathcal{N}C_\phi)}{1-\mathcal{N}\|B\|}\big[1+\frac{2M_1}{1-\mathcal{N}\|B\|}L_1\big]^m\exp\big(\frac{\mathcal{N}}{1-\mathcal{N}\|B\|}\int_0^\omega L_\zeta(\tau)d\tau\big)\\ \leq&\vartheta(K_1,L_1,L_\zeta,\phi)\epsilon(\phi(\omega)+\omega\psi),\end{aligned}$$

where,

$$\vartheta(K_1,L_1,L_\zeta,\phi)=\frac{(M_0+\mathcal{N}C_\phi)}{1-\mathcal{N}\|B\|}\big[1+\frac{2M_1}{1-\mathcal{N}\|B\|}L_1\big]^m\exp\big(\frac{\mathcal{N}}{1-\mathcal{N}\|B\|}\int_0^\omega L_\zeta(\tau)d\tau\big).$$

□

Hence system (4.1) is HUR stable with respect to (ϕ,ψ).

Theorem 4.3. *If assumptions* $[H_1]$–$[H_4]$ *and Definition 2.2 are fulfilled, then Eq.*(4.1) *is* $\beta-HUR$ *stable with respect to* (ϕ^β,ψ^β).

Proof. Let Υ be a solution of the inequality (4.4) and Θ be a unique solution of the system (4.1), which is given in (4.2). On the basis of Remark 4.2 the solution of the system (4.4) is

$$\begin{aligned}\Upsilon(\omega)=&\,\mathcal{C}(\omega)(\Theta_0+\mathcal{G}(\omega))+\mathcal{S}(\omega)(\Theta_1+\mathcal{G}(\omega))+\int_0^\omega\mathcal{S}(\omega-\tau)B\Upsilon'(\tau)d\tau\\ &+\int_0^\omega\mathcal{S}(\omega-\tau)\big[\zeta(\tau,\Upsilon(\tau),\Upsilon(\alpha(\tau)))+\mathcal{G}(\tau)\big]d\tau\\ &+\sum_{\omega>\omega_k}\mathcal{C}(\omega-\omega_k)[I_k(\Upsilon(\omega_k))+g_1(\omega_k)]\sum_{\omega>\omega_k}\mathcal{S}(\omega-\omega_k)[J_k(\Upsilon(\omega_k))+g_2(\omega_k).\end{aligned}$$

Thus for every $\omega\in I$, we get

$$\begin{aligned}&\big\|\Upsilon(\omega)-\Theta(\omega)\big\|^\beta\\ =&\big\|\Upsilon(\omega)-\mathcal{C}(\omega)\Theta_0-\mathcal{S}(\omega)\Theta_1-\int_0^\omega\mathcal{S}(\omega-\tau)B\Theta'(\tau)d\tau\\ &-\int_0^\omega\mathcal{S}(\omega-\tau)\zeta(s,\Theta(\tau),\Theta(\alpha(\tau)))d\tau-\sum_{\omega>\omega_k}\mathcal{C}(\omega-\omega_k)I_k(\Theta(\omega_k))\\ &-\sum_{\omega>\omega_k}\mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k))\|^\beta\\ \leq&\big[\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi)\big]^\beta+(\mathcal{N}\|B\|\int_0^\omega(\|\Upsilon'(\tau)-\Theta'(\tau))\|d\tau)^\beta\\ &+(\mathcal{N}\int_0^\omega L_\zeta(\tau)\big[\|\Upsilon(\tau)-\Theta(\tau)\|+\|\Upsilon(\alpha(\tau))-\Theta(\alpha(\tau))\|\big]d\tau)^\beta\\ &+(\mathcal{M}\sum_{\omega>\omega_k}L_K\|\Upsilon(\omega_k)-\Theta(\omega_k)\|)^\beta+(\mathcal{N}\sum_{\omega>\omega_k}L_J\|\Upsilon(\omega_k)-\Theta(\omega_k)\|)^\beta.\end{aligned}$$

This implies

$$\big\|\Upsilon(\omega)-\Theta(\omega)\big\|^\beta$$

$$\leq [\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi)]^\beta + (\mathcal{N}\int_0^\omega L_\zeta(\tau)[\|\Upsilon(\tau)-\Theta(\tau)\| + \|\Upsilon(\alpha(\tau)) - \Theta(\alpha(\tau))\|]d\tau)^\beta + (2M_1\sum_{\omega>\omega_k} L_1\|\Upsilon(\omega_k)-\Theta(\omega_k)\|)^\beta,$$

where $M_1 = \max\{\mathcal{M},\mathcal{N}\}$, $L_1 = \max\{L_K, L_J\}$ and $|\mathcal{N}\|B\||^\beta < 1$.

Thus,

$$\|\Upsilon(\omega)-\Theta(\omega)\| \leq 3^{\frac{1}{\beta}-1}\Big[[\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi)] + (\mathcal{N}\int_0^\omega L_\zeta(\tau)[\|\Upsilon(\tau)-\Theta(\tau)\| + \|\Upsilon(\alpha(\tau))-\Theta(\alpha(\tau))\|]d\tau) + (2M_1\sum_{\omega>\omega_k} L_1\|\Upsilon(\omega_k)-\Theta(\omega_k)\|)\Big].$$

By using the relation,

$$(x+y+z)^\beta \leq 3^{\beta-1}(x^\beta+y^\beta+z^\beta),$$

where $x, y, z \geq 0$, and $\beta > 1$.

Now using Lemma 2.3,

$$\begin{aligned}
\|\Upsilon(\omega)-\Theta(\omega)\| &\leq 3^{\frac{1}{\beta}-1}\Big[[\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi)]\big[1+3^{\frac{1}{\beta}-1}2M_0L_1\big]^m \\
&\quad \times \exp(3^{\frac{1}{\beta}-1}\mathcal{N}\int_0^\omega L_\zeta(\tau)d\tau)\Big] \\
\Rightarrow \ \|\Upsilon(\omega)-\Theta(\omega)\|^\beta &\leq 3^{1-\beta}\big[\epsilon(\phi(\omega)+\omega\psi)(M_0+\mathcal{N}C_\phi)\big]^\beta\big[1+3^{\frac{1}{\beta}-1}2M_0L_1\big]^{m\beta} \\
&\quad \times \exp(3^{\frac{1}{\beta}-1}\mathcal{N}\int_0^\omega L_\zeta(\tau)d\tau)^\beta \\
&\leq 3^{1-\beta}\big[\epsilon^\beta(\phi(\omega)+\omega\psi)^\beta(M_0+\mathcal{N}C_\phi)^\beta\big]\big[1+3^{\frac{1}{\beta}-1}2M_0L_1\big]^{m\beta} \\
&\quad \times \exp(3^{\frac{1}{\beta}-1}\beta\mathcal{N}\int_0^\omega L_\zeta(\tau)d\tau) \\
&\leq \vartheta(K_1, L_1, L_\zeta, \phi, \psi)\epsilon^\beta(\phi^\beta(\omega)+\omega^\beta\psi^\beta),
\end{aligned}$$

where

$$\vartheta(K_1, L_1, L_\zeta, \phi, \psi) = 3^{1-\beta}(M_0+\mathcal{N}C_\phi)^\beta\big[1+3^{\frac{1}{\beta}-1}2M_0L_1\big]^{m\beta}\exp(3^{\frac{1}{\beta}-1}\beta\mathcal{N}\int_0^\omega L_\zeta(\tau)d\tau).$$

This completes the proof. □

Similarly, if we take the following system,

$$\begin{cases}
\Theta''(\omega) = A\Theta(\omega) + B\Theta'(\omega) + \zeta(\omega, \Theta(\omega), \Theta(a(\omega)), \Theta'(\omega), \Theta'(b(\omega))), \ \omega \in I, \ \omega \neq \omega_k, \\
\Theta(0) = \Theta_0, \Theta'(0) = \Theta_1, \\
\Delta\Theta(\omega_k) = I_k(\Theta(\omega_k), \Theta'(\bar{\omega_k})), \ k = 1, 2, \ldots, n, \\
\Delta\Theta'(\omega_k) = J_k(\Theta(\omega_k), \Theta'(\bar{\omega_k})), k = 1, 2, \ldots, n.
\end{cases} \tag{4.10}$$

The solution of the system (4.10), see [5], is:

$$\Theta(\omega) = \mathcal{C}(\omega)\Theta_0 + \mathcal{S}(\omega)\Theta_1 + \int_0^\omega \mathcal{S}(\omega-\tau)B\Theta'(\tau)d\tau + \int_0^\omega \mathcal{S}(\omega-\tau)$$

$$\times \zeta(\tau, \Theta(\tau), \Theta(a(\tau)), \Theta'(\tau), \Theta'(b(\tau)))d\tau + \sum_{\omega>\omega_k} \mathcal{C}(\omega-\omega_k)$$
$$\times I_k(\Theta(\omega_k), \Theta'(\bar{\omega_k})) + \sum_{\omega>\omega_k} \mathcal{S}(\omega-\omega_k)J_k(\Theta(\omega_k), \Theta'(\bar{\omega_k})), \quad \omega \in I.$$

To prove its UTS such as HU, HUR and β–HUR stability. We proceeds the same procedure as derived for the system (4.1).

5. Application

In this portion we investigate strongly damped wave equation for HUR stability.

Example 5.1.

$$\begin{cases} \Theta''+\eta(-\Delta)\Theta'+\gamma(-\Delta)\Theta=+\zeta(\omega,\Theta(\omega),\Theta(a(\omega)),\ \omega\in I,\ \omega\neq\omega_i,\ i=1,2,\dots,n,\\ \Theta(0)=\Theta_0,\ \ \Theta'(0)=\Theta_1,\\ \Theta'(\omega_i^+,x)=\Theta'(\omega_i^-,x)+I_i(\omega_i),\ \ i=1,2,\dots,n \text{ and } \ x\in\mathcal{W}, \end{cases} \tag{5.1}$$

in the space $\mathcal{Z} = D((-\Delta) \times L_2(\mathcal{W})$, $\Theta = (0,1) \times I$, $\mathcal{W}$ is bounded domain in R^N, $N \geq 1$, Θ_0 and Θ_1 are positive numbers and $\zeta, I_k \in C((0,1) \times R \times R; R), k = 1, 2, \dots, m$. For main space, if $z = (w, v)^T = (\Theta, \Theta')^T$, then we have

$$\|z\| = \sqrt{\int_{\mathcal{W}} (\|(-\Delta)w\|^2 + \|v\|^2 dx}, \quad \text{for all} \quad z \in \mathcal{Z} = D((-\Delta)) \times L_2(\mathcal{W}).$$

Let $\beth = L_2(\mathcal{W}) = L_2(\mathcal{W}, R)$ and consider the linear operator $A : D(A) \subset \beth \to \beth$ defined by $A\phi = -\Delta\phi$, where $D(A) = H^2(\mathcal{W}, R) \bigcap H_0^1(\mathcal{W}, R)$. The fractional power space $\beth^r$ are give by

$$\beth^r = D(A^r) = \{x \in \beth : \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 < \infty\},\ r \geq 0,$$

with

$$\|x\|_r = \|A^r x\| = \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2,\ \ x \in \beth^r.$$

Therefore, the abstract form of system (5.1) is as under

$$\begin{cases} \Theta''+\eta A\Theta'+\gamma A\Theta=\zeta^e(\omega,\Theta(\omega),\Theta(a(\omega))),\ \omega\neq\omega_k;\\ \Theta(\tau)=\Theta_0,\ \ \Theta'(\tau)=\Theta_1,\\ \Theta'(\omega_k^+)=\Theta_t(\omega_k^-)+I_k(\omega_k),\ k=1,2,\dots,m; \end{cases} \tag{5.2}$$

for all $x \in \mathcal{W}$, $k = 1, 2, \dots, m$, $L_k^e : (0,1) \times \mathcal{Z} \times U \to \beth$ and $\zeta^e : (0,1) \times C([-r, 0], \mathcal{Z}) \times U \to \beth$ are defined by $I_k^e(\omega_k, \Theta, \Theta(a))(x) = I_k^e(\omega_k, \Theta(x), \Theta(a(x)))$, $\zeta^e(\omega, \Theta_0, \Theta_1)(x) = \zeta^e(\omega, \Theta_0(x), \Theta_1(x))$, with the change of variable $\Theta' = z$, we can write the second order system (5.2) as a first order system of ordinary DE with impulses and delay in the space $\mathcal{Z} = \beth \times \beth$, as follows

$$\begin{cases} z'=Bz+F(\omega,z(\omega),z(a(\omega))),\ z\in\mathcal{Z},\ \omega\neq\omega_k;\\ z(0)=z_0,\\ z(\omega_k^+)=z(\omega_k^-)+J_k(z(\omega_k)),\ k=1,2,\dots,m; \end{cases} \tag{5.3}$$

where,

$$B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and $J_k : (0,1) \times \mathcal{Z} \to \mathcal{Z}$, $F : ((0,1) \times I, \mathcal{Z}) \to \mathcal{Z}$ are defined by

$$F(\omega, \Phi) = \begin{bmatrix} 0 \\ \zeta^e(\omega, \phi, \psi) \end{bmatrix}, \ J_k(\omega) = \begin{bmatrix} 0 \\ I_k^e(\omega) \end{bmatrix}.$$

We take, $F(\omega, z(\omega), z(a(\omega))) = \frac{|z(\omega)|}{(15+e^{\omega})}$ and $J_k(y(\omega_k)) = \frac{1}{20(e^{\omega_k}+|z(\omega_k)|)}$. The only mild solution of (1.3) is

$$\begin{cases} z(\omega) = T(\omega)\Phi(0) + \int_0^t T(\omega-\tau)\dfrac{|z(\omega)|}{(15+e^t)}d\tau + \sum\limits_{0<\omega_k<\omega} T(\omega-\omega_k)\dfrac{1}{20(e^{\omega_k}+|z(\omega_k)|)}; \\ z(\tau) = \Phi(\tau). \end{cases} \tag{5.4}$$

After the application of conditions (H_2) and (H_3), we found $L_F = \frac{1}{15} > 0$ and $L_{I_k} = \frac{1}{20} > 0$, such that

$$\max\{\frac{1}{15}, \frac{1}{20}\} < 1.$$

Thus (5.3) has only one solution. Next, we provide an approximation of (5.3). Let $e^{\omega} > 0$. Then

$$\begin{cases} |y' - By - F(\omega, y(\omega), y(a(\omega)))| \le e^{\omega}, \quad y \in \mathcal{Z}, \ \omega \ne \omega_k; \\ |y(\tau) - \Phi(\tau)| \le e^{\omega}, \\ |y(\omega_k^+) - y(\omega_k^-) - J_k(y(\omega_k))| \le 5, \ k = 1, 2, \ldots, m. \end{cases} \tag{5.5}$$

Let $h(\omega) \in C(R\backslash\{\omega_k\})$ and $h(\omega_k)$, $k = 1, 2, \ldots, m$. Then we have $h(\omega) \le e^{\omega}$, $\omega \in R\backslash\{\omega_k\}$ and $h(\omega_k) \le 5$. Thus (5.5) yields

$$\begin{cases} y' = By + F(\omega, y(\omega), y(a(\omega))) + h(\omega), \ y \in \mathcal{Z}, \ \omega \ne \omega_k; \\ y(\tau) = \phi(\tau) + h(\omega), \\ y(\omega_k^+) = y(\omega_k^-) + J_k(y(\omega_k)) + h(\omega_k), \ k = 1, 2, \ldots, m. \end{cases} \tag{5.6}$$

Hence system (5.6) has the following solution

$$\begin{cases} y(\omega) = T(\omega)\Big(y(0) + h(0)\Big) + \in \omega_0^t T(\omega - \tau)\Big(\dfrac{|y(\omega)|}{(15+e^t)} + h(\tau)\Big)d\tau \\ \quad + \sum\limits_{0<\omega_k<\omega} T(\omega - \omega_k)\Big(\dfrac{1}{20(e^{\omega_k} + |y(\omega_k)|)} + h(\omega_k)\Big), \ \omega \in (0,1); \\ y(\omega) = \Phi(\omega) + h(\omega), \ \omega \in I. \end{cases} \tag{5.7}$$

Now, we proceed to the main result, the HUR stability. So, for $\omega \in I$

$$\begin{aligned} \|y(\omega) - z(\omega)\| \le \Bigg\| & y(\omega) - T(\omega)\Phi(0) - \int_0^{\omega} T(\omega - \tau)\frac{|z(\omega)|}{(15+e^{\omega})}d\tau \\ & - \sum_{0<\omega_k<\omega} T(\omega - \omega_k)\frac{1}{20(e^{\omega_k} + |z(\omega_k)|)}\Bigg\|. \end{aligned}$$

This yields

$$\|y(\omega) - z(\omega)\| \leq M(m + c_e)(5 + e^{\omega}) + M \int_0^{\omega} \|y(\tau) - z(\tau)\| d\tau + \sum_{k=0}^{m} M \left(\|y(\omega_k) - z(\omega_k)\|\right).$$

From the above expression, we get

$$\big\|y - z\big\| \leq M(m + c_e)(5 + e^{\omega}) + M \big\|y - z\big\| + M\, m \big\|y - z\big\|,$$

which yields

$$\big\|y - z\big\| \leq \frac{M(m + c_e)}{1 - M - M\, m}(5 + e^{\omega}).$$

Thus, the wave equation (5.1) is HUR stable with respect to $(5, e^{\omega})$.

6. Conclusion

In this article, we established the controllability and HUS of damped second order abstract impulsive DEs over a Banach space $\beth$. The result is obtained with the help of Sadovskii fixed point theorem and with the theory of cosine family of operators. Moreover, with Gronwall's integral inequality and strong Lipschitz conditions we derived different types of stability *i.e.*, Hyers–Ulam, Hyers–Ulam–Rassias and β–Hyers–Ulam–Rassias stability.

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