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NEW INSIGHTS ON BIFURCATION IN A FRACTIONAL-ORDER DELAYED COMPETITION AND COOPERATION MODEL OF TWO ENTERPRISES*

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Abstract Over the past decades, many authors establish various kinds of competition and cooperation models of two enterprises to analyze the dynamic interaction. However, they are only concerned with integer-order differential equation models, while the reports on fractional-order ones are very rare. In this article, based on the earlier studies, we propose a new fractional-order delayed competition and cooperation model of two enterprises. Letting the delay be bifurcation parameter and analyzing the corresponding characteristic equation of involved model, we establish some new sufficient conditions to guarantee the stability and the existence of Hopf bifurcation of fractional-order delayed competition and cooperation model of two enterprises. The research indicates that different delays have different effect on the stability and Hopf bifurcation of involved model. The impact of the fractional order on the stability and Hopf bifurcation of involved model is displayed. To check the correctness of theoretical analysis, we implement some computer simulations.

Keywords Fractional order competition and cooperation model, enterprise, Hopf bifurcation, stability, delay.

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1. Introduction

Recently, the research on competition and cooperation among enterprises has been an important active field and has attracted the great attention of many scholars. Correctly handling competition and cooperation among enterprises plays a crucial

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role in mastering the management and decision-making of enterprises. In order to grasp the internal operation mechanism among different enterprises, a lot of scholars are striving to establish the the competition and cooperation model among enterprises and investigate their dynamical behavior. Many excellent results have been reported. In 2012, Xu [50] studied the following continuous competition and cooperation model of two enterprises:

$$\begin{cases} \dot{y}_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t)}{a_1} - \dfrac{r_1(y_2(t)-\beta_2)}{a_2}\right], \\ \dot{y}_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t)}{a_2} + \dfrac{r_2(y_1(t)-\beta_1)}{a_1}\right], \\ y_1(0) \geq 0, y_2(0) \geq 0, \end{cases} \tag{1.1}$$

where $y_1(t)$ and $y_2(t)$ denote for the output of enterprise $\mathcal{A}$ and enterprise $\mathcal{B}$, respectively; $\alpha_i(i = 1, 2)$ stands for the intrinsic growth rate of enterprise $\mathcal{A}$ and enterprise $\mathcal{B}$, respectively; $a_i(i = 1, 2)$ represents the natural market carrying capacity of enterprise $\mathcal{A}$ and enterprise $\mathcal{B}$; r_1 represents the consumption coefficient of the enterprise with the output $y_2(t)$ to the one with the output $y_1(t)$ and r_2 represents the transformation coefficient of the enterprise with the output $y_1(t)$ to the one with the output $y_2(t)$; β_1 and β_2 are the initial outputs of enterprise $\mathcal{A}$ and enterprise $\mathcal{B}$, respectively. With the help of the coincidence degree theory, Xu [50] discussed the existence of periodic solutions of model (1.1). Liao et al. [27] thought that the duration of output for enterprises has important effect on the dynamics of competition and cooperation model of two enterprises and then he established the following delayed competition and cooperation model of two enterprises:

$$\begin{cases} \dot{y}_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon)}{a_1} - \dfrac{r_1(y_2(t-\varepsilon)-\beta_2)}{a_2}\right], \\ \dot{y}_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon)}{a_2} + \dfrac{r_2(y_1(t-\varepsilon)-\beta_1)}{a_1}\right], \\ y_1(0) = \phi_1(t), y_2(0) = \phi_2(t), t \in [-\varepsilon, 0], \end{cases} \tag{1.2}$$

where ε denotes the time delay of interior of two enterprises and $\phi_1(t), \phi_2(t) \in C([-\varepsilon, 0], , R)$. Applying the normal theory and center manifold theorem, Liao et al. obtained the sufficient conditions to assure the stability and the existence of Hopf bifurcation of (1.2) and the explicit algorithms to determine the direction, period and stability of Hopf bifurcation for model (1.2). Considering the effect of different delays on the dynamics of (1.2), in 2014, Liao et al. [26] modified the model (1.2) as follows:

$$\begin{cases} \dot{y}_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon_1)}{a_1} - \dfrac{r_1(y_2(t-\varepsilon_2)-\beta_2)}{a_2}\right], \\ \dot{y}_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon_1)}{a_2} + \dfrac{r_2(y_1(t-\varepsilon_2)-\beta_1)}{a_1}\right], \\ y_1(0) = \phi_1(t), y_2(0) = \phi_2(t), t \in [-\varepsilon, 0]. \end{cases} \tag{1.3}$$

Applying the Hopf bifurcation theory of delayed differential equations, Liao et al. [26] discussed the effect of two different delays on the bifurcation behavior of system (1.3). In 2016, Li et al. [22] proposed the following delayed competition and cooperation model of two enterprises with four delays:

$$\begin{cases} \dot{y}_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon_1)}{a_1} - \dfrac{r_1(y_2(t-\varepsilon_2)-\beta_2)}{a_2}\right], \\ \dot{y}_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon_3)}{a_2} + \dfrac{r_2(y_1(t-\varepsilon_4)-\beta_1)}{a_1}\right], \\ y_1(0) = \phi_1(t), y_2(0) = \phi_2(t), t \in [-\varepsilon, 0]. \end{cases} \tag{1.4}$$

By assuming $\varepsilon_1 = \varepsilon_3 = 0$ and $\varepsilon_2 + \varepsilon_4 = \varepsilon$, Li et al. [22] discussed the Hopf bifurcation issue of system (1.4). For more relative topic, one can see [3, 22, 25, 26, 30, 53, 61].

As is known to us, the fractional calculus is a generalization of ordinary differentiation and integration [10, 12, 19, 21, 24, 36, 38, 40, 42, 43, 60, 62–64]. However, the fractional calculus attracts little attention due to its complexity and the lack of practical background. Up to the last few decades, the fractional calculus has gained extensive applications in numerous areas of science and engineering such as mechanics, chemistry, viscoelasticity, biology, physics, finance and so on [4, 59]. Many authors argued that in many cases, it is more reasonable to model the natural world by fractional-order differential equations than integer-order ones since fractional-order differential equations give a better description of the memory and hereditary nature of various materials and processes. Recently, many researchers pay much attention to the dynamical behavior of fractional-order differential systems and outstanding achievements have been made. We refer the readers to [2, 18, 32, 41]. In particular, some fruits about Hopf bifurcation phenomenon of fractional-order differential models have also been reported. For example, Rakkiyappan et al. [35] investigated the Hopf bifurcation of fractional-order complex-valued neural networks, Abdelouahab et al. [1] analyzed the Hopf bifurcation and chaos for a fractional-order hybrid optical model, Huang and Cao [14] discussed the bifurcation behavior of fractional order neural networks with leakage delays, Deshpande et al. [8] focused on the Hopf bifurcation in a fractional order systems. In details, we refer the readers to [9, 13, 15–17, 33, 34, 44–48, 52].

Here it is worth pointing out that all the above publications about Hopf bifurcation of fractional-order differential models are mainly concerned with single or the sum of different delays. So far, there are no papers that focus on the impact of different delays on the Hopf bifurcation of involved systems. Not to speak of dealing with the impact of different delays on the Hopf bifurcation of fractional-order competition and cooperation model of two enterprises with different delays. In order to further reveal the effect of different delays on Hopf bifurcation of fractional-order competition and cooperation model of two enterprises and master the operation rules of enterprises effectively, we think that it is necessary to investigate the role of time delay in fractional-order competition and cooperation model.

Motivated by the analysis above, we modify (1.3) as a fractional-order delayed competition and cooperation model of two enterprises as follows:

$$\begin{cases} D^q y_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon_1)}{a_1} - \dfrac{r_1(y_2(t-\varepsilon_2)-\beta_2)}{a_2}\right], \\ D^q y_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon_1)}{a_2} + \dfrac{r_2(y_1(t-\varepsilon_2)-\beta_1)}{a_1}\right], \end{cases} \tag{1.5}$$

where $q \in (0, 1]$. All other variables and coefficients have the same meaning as those in (1.3).

The primary object of this manuscript is listed as follows: (**a**) seeking the sufficient conditions to ensure the stability and existence of Hopf bifurcation of model (1.5); (**b**) revealing the impact of two different delays on Hopf bifurcation of model (1.5); (**c**) illustrating the effect of fractional order on the stability and Hopf bifurcation of model (1.5).

The highlights of this manuscript consist of the following aspects:

- A new fractional-order competition and cooperation model of two enterprises with two delays, which can better depict the memory and hereditary nature of competition and cooperation of two enterprises, is established.
- A new set of sufficient criteria to ensure the stability and the existence of Hopf bifurcation of fractional-order competition and cooperation model of two enterprises with two delays are obtained. The influence of two different delays and fractional-order on the stability and Hopf bifurcation of (1.5) are revealed.
- So far, no author focuses on the Hopf bifurcation of fractional-order competition and cooperation model of two enterprises with two delays.
- The research ideas of this manuscript will provide an excellent reference to handle a lot of delayed fractional-order differential systems.

The rest of this manuscript is planned as follows. In Segment 2, some basic knowledge on fractional calculus is prepared. In Segment 3, the sufficient conditions to assure the stability and the existence of Hopf bifurcation of involved model are derived. The impact of two different delays on the stability and Hopf bifurcation of the involved competition and cooperation model of two enterprises is displayed. In Segment 4, computer simulations are carried out to verify the main findings. Segment 5 ends the manuscript.

Remark 1.1. In (1.3), we replace the first-order derivatives by Caputo fractional derivatives of order $q \in (0, 1]$. According to the idea of [23, 45], we have

$$\begin{cases} D^q y_1(t) = \alpha_1^q y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon_1)}{a_1} - \dfrac{r_1\gamma^{1-q}(y_2(t-\varepsilon_2) - \beta_2\gamma^{1-q})}{a_2}\right], \\ D^q y_2(t) = \alpha_2^q y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon_1)}{a_2} + \dfrac{r_2\gamma^{1-q}(y_1(t-\varepsilon_2) - \beta_1\gamma^{1-q})}{a_1}\right], \end{cases}$$

where γ is a fractional time constant. Let $\bar{\alpha}_1 = \alpha_1^q, \bar{\alpha}_2 = \alpha_2^q, \bar{r}_1 = r_1\gamma^{1-q}, \bar{r}_2 = r_2\gamma^{1-q}, \bar{\beta}_2 = \beta_2\gamma^{1-q}, \bar{\beta}_1 = \beta_1\gamma^{1-q}$ and still denote $\bar{\alpha}_1, \bar{\alpha}_2, \bar{r}_1, \bar{\beta}_2, \bar{\beta}_1, \bar{\beta}_2$ by $\alpha_1, \alpha_2, r_1, \beta_2, \beta_1, \beta_2$, respectively, then one has

$$\begin{cases} D^q y_1(t) = \alpha_1 y_1(t)\left[1 - \dfrac{y_1(t-\varepsilon_1)}{a_1} - \dfrac{r_1(y_2(t-\varepsilon_2) - \beta_2)}{a_2}\right], \\ D^q y_2(t) = \alpha_2 y_2(t)\left[1 - \dfrac{y_2(t-\varepsilon_1)}{a_2} + \dfrac{r_2(y_1(t-\varepsilon_2) - \beta_1)}{a_1}\right]. \end{cases}$$

For the detailed derivation process, one also can see [23, 45].

2. Basic knowledge

In this section, some related definitions and lemmas about fractional calculus are displayed.

Definition 2.1 ([31]). The fractional integral of order q for a function $g(\xi)$ is defined as follows:

$$\mathcal{I}^q g(\xi) = \frac{1}{\Gamma(q)} \int_{\xi_0}^{\xi} (\xi - s)^{q-1} g(s) ds,$$

where $\xi \geq \xi_0, q > 0$, and $\Gamma(s) = \int_0^\infty \xi^{s-1} e^{-\xi} d\xi$ denotes Gamma function.

Definition 2.2 ([31]). Let $g(\xi) \in C([\xi_0, \infty), R)$. Define the Caputo fractional-order derivative of order q of $g(\xi)$ as follows:

$$\mathcal{D}^q g(\xi) = \frac{1}{\Gamma(l-q)} \int_{\xi_0}^{\xi} \frac{g^{(l)}(s)}{(\xi - s)^{q-l+1}} ds,$$

where $\xi \geq \xi_0$ and l denotes a positive integer which satisfies $l - 1 \leq q < 1$. Furthermore, if $0 < q < 1$, then

$$\mathcal{D}^q g(v) = \frac{1}{\Gamma(1-q)} \int_{\xi_0}^{\xi} \frac{g^{'}(s)}{(\xi - s)^q} ds.$$

Definition 2.3 ([16]). For given the fractional order system:

$$\mathcal{D}^q u_i(t) = h_i(u_i(t)), i = 1, 2, \cdots, n, \tag{2.1}$$

where $q \in (0, 1], u_i(t) = (u_1(t), u_2(t), \cdots, u_n(t)), h_i(t) = (h_1(t), h_2(t), \cdots, h_n(t))$. $(u_1^*, u_2^*, \cdots, u_n^*)$ is said to be the equilibrium point if $h_i(u_i^*) = 0$.

Lemma 2.1 ([29]). *For the system $\mathcal{D}^q z = \mathcal{A}z, z(0) = z_0$ where $0 < q < 1, z \in R^n, \mathcal{A} \in R^{n\times n}$. Assume that $\lambda_i (i = 1, 2, \cdots, n)$ is the root of the characteristic equation of $\mathcal{D}^q z = \mathcal{A}z$. Then system $\mathcal{D}^q z = \mathcal{A}z$ is asymptotically stable $\Leftrightarrow$ $|arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2, \cdots, n)$. Moreover, this system is stable $\Leftrightarrow |arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2, \cdots, n)$ and those critical eigenvalues that satisfy $|arg(\lambda_i)| = \frac{q\pi}{2} (i = 1, 2, \cdots, n)$ have geometric multiplicity one.*

Lemma 2.2 ([7]). *For the given fractional-order system: $\mathcal{D}^q u(t) = \mathcal{G}_1 u(t) + \mathcal{G}_2 u(t-\varrho)$, where $u(t) = \phi(t), t \in [-\varrho, 0], q \in (0, 1], u \in R^n, \mathcal{G}_1, \mathcal{G}_2 \in R^{n\times n}, \varrho \in R^{+(n\times n)}$. Then the characteristic equation of the system is $\det |s^q \mathcal{I} - \mathcal{G}_1 - \mathcal{G}_2 e^{-s\varrho}| = 0$. If all the roots of the characteristic equation of the system have negative real roots, then the zero solution of the system is asymptotically stable.*

3. Impact of two different delays on Hopf bifurcation for model (1.5)

In this section, we will discuss the impact of two different delays on Hopf bifurcation for model (1.5).

Let $b_i = \frac{\alpha_i}{a_i} (i = 1, 2), c_1 = \frac{\alpha_1 r_1}{a_2}, c_2 = \frac{\alpha_2 r_2}{a_1}$, then system (1.5) becomes

$$\begin{cases} \mathcal{D}^q y_1(t) = y_1(t) \left[\alpha_1 - b_1 y_1(t - \varepsilon_1) - c_1 (y_2(t - \varepsilon_2) - \beta_2)^2\right], \\ \mathcal{D}^q y_2(t) = y_2(t) \left[\alpha_2 - b_2 y_2(t - \varepsilon_1) + c_2 (y_1(t - \varepsilon_2) - \beta_1)^2\right]. \end{cases} \tag{3.1}$$

Set $x_1(t) = y_1(t) - \beta_1, x_2(t) = y_2(t) - \beta_2$, then system (3.1) takes the form:

$$\begin{cases} \mathcal{D}^q x_1(t) = (x_1(t) + \beta_1)\left[d_1 - b_1 x_1(t-\varepsilon_1) - c_1 x_2^2(t-\varepsilon_2)\right], \\ \mathcal{D}^q x_2(t) = (x_2(t) + \beta_2)\left[d_2 - b_2 x_2(t-\varepsilon_1) + c_2 x_1^2(t-\varepsilon_2)\right]. \end{cases} \tag{3.2}$$

where $d_i = \alpha_i - b_i\alpha_i (i = 1, 2)$.

Now the assumption is given as follows:

(A1) $b_2^2\beta_1 > c_1 d_2^2$.

Lemma 3.1. *If (A1) holds true, then system (3.2) has a unique equilibrium point (x_{10}, x_{20}), which is locally asymptotically stable, where $x_{10} = \sqrt{\frac{b_2 x_{20} - d_2}{c_2}}$ and x_{20} satisfies the following condition:*

$$d_2 - b_2 x_{20} + \frac{c_2(d_1 - c_1 x_{20}^2)^2}{b_1^2} = 0.$$

Proof. In view of Lemma 2.1 of Liao et al. [27], we know that system (3.2) has a unique equilibrium point (x_{10}, x_{20}). Next we prove that the equilibrium point (x_{10}, x_{20}) is locally asymptotically stable.

The linear equation of (3.2) around the equilibrium point (x_{10}, x_{20}) is given by:

$$\begin{cases} \mathcal{D}^q x_1(t) = \mathcal{A}_1 x_1(t-\varepsilon_1) + \mathcal{B}_1 x_2(t-\varepsilon_2), \\ \mathcal{D}^q x_2(t) = \mathcal{A}_2 x_1(t-\varepsilon_2) + \mathcal{B}_2 x_2(t-\varepsilon_1), \end{cases} \tag{3.3}$$

where $\mathcal{A}_1 = -b_1(x_{10} + \beta_1), \mathcal{B}_1 = -2c_1(x_{10} + \beta_1)x_{20}, \mathcal{A}_2 = 2c_2(x_{20} + \beta_2)x_{10}, \mathcal{B}_2 = -b_2(x_{20} + \beta_2)$. The corresponding characteristic equation of (3.3) is given by

$$\det \begin{bmatrix} s^q - \mathcal{A}_1 e^{-s\varepsilon_1} & -\mathcal{B}_1 e^{-s\varepsilon_2} \\ -\mathcal{A}_2 e^{-s\varepsilon_2} & s^q - \mathcal{B}_2 e^{-s\varepsilon_1} \end{bmatrix}. \tag{3.4}$$

It follows from (3.4) that

$$s^{2q} - (\mathcal{A}_1 + \mathcal{B}_2)s^q e^{-s\varepsilon_1} + \mathcal{A}_1\mathcal{B}_2 e^{-2s\varepsilon_1} - \mathcal{A}_2\mathcal{B}_1 e^{-2s\varepsilon_2} = 0. \tag{3.5}$$

If $\varepsilon_1 = \varepsilon_2 = 0$. Then (3.5) takes the form:

$$\lambda^2 - (\mathcal{A}_1 + \mathcal{B}_2)\lambda + \mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1 = 0. \tag{3.6}$$

Obviously, $\mathcal{A}_1 + \mathcal{B}_2 < 0, \mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1 > 0$. Then all the roots λ_i of (3.6) satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2} (i = 1, 2)$ By Lemma 3.1, we can conclude that the equilibrium point (x_{10}, x_{20}) of (3.2) with $\varepsilon_1 = \varepsilon_2 = 0$ is locally asymptotically stable. The proof of Lemma 3.1 ends. □

Next we consider two cases:

Assume that $\varepsilon_1 = 0, \varepsilon_2 > 0$. Then (3.5) takes the form:

$$s^{2q} - (\mathcal{A}_1 + \mathcal{B}_2)s^q + \mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1 e^{-2s\varepsilon_2} = 0. \tag{3.7}$$

Assume that $s = i\phi = \phi\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ is a root of (3.7). Then

$$\begin{cases} \mathcal{A}_2\mathcal{B}_1\cos 2\phi\varepsilon_2 = \phi^{2q}\cos q\pi - (\mathcal{A}_1+\mathcal{B}_2)\phi^q\cos\dfrac{q\pi}{2} + \mathcal{A}_1\mathcal{B}_2, \\ \mathcal{A}_2\mathcal{B}_1\sin 2\phi\varepsilon_2 = -\phi^{2q}\sin q\pi + (\mathcal{A}_1+\mathcal{B}_2)\phi^q\sin\dfrac{q\pi}{2}. \end{cases} \tag{3.8}$$

It follows from (3.8) that

$$\phi^{4q} + \kappa_1\phi^{3q} + \kappa_2\phi^{2q} + \kappa_3\phi^q + \kappa_4 = 0, \tag{3.9}$$

where

$$\begin{aligned} \kappa_1 &= -2(\mathcal{A}_1+\mathcal{B}_2)\left(\cos q\pi\cos\frac{q\pi}{2} + \sin q\pi\sin\frac{q\pi}{2}\right), \\ \kappa_2 &= (\mathcal{A}_1+\mathcal{B}_2)^2 + 2\mathcal{A}_1\mathcal{B}_2\cos q\pi, \\ \kappa_3 &= -2\mathcal{A}_1\mathcal{B}_2(\mathcal{A}_1+\mathcal{B}_2)\cos q\pi\cos\frac{q\pi}{2}, \\ \kappa_4 &= -(\mathcal{A}_2+\mathcal{B}_1)^2. \end{aligned}$$

Denote

$$\chi(\phi) = \phi^{4q} + \kappa_1\phi^{3q} + \kappa_2\phi^{2q} + \kappa_3\phi^q + \kappa_4. \tag{3.10}$$

Consider that $\kappa_4 < 0$ and $\frac{d\chi(\phi)}{d\phi} > 0, \ \forall \ \phi > 0$, then Eq.(3.9) has at least one positive real root. Therefore Eq.(3.7) has at least one pair of purely roots.

Here we suppose that Eq.(3.9) has four positive real roots $\phi_l (l = 1,2,3,4)$. By (3.8), one gets

$$\varepsilon_{2l}^k = \frac{1}{2\phi_l}\left[\arccos\left(\frac{\phi^{2q}\cos q\pi - (\mathcal{A}_1+\mathcal{B}_2)\phi^q\cos\frac{q\pi}{2} + \mathcal{A}_1\mathcal{B}_2}{\mathcal{A}_2\mathcal{B}_1}\right) + 2k\pi\right], \tag{3.11}$$

where $k = 0,1,2,\cdots, l = 1,2,3,4$. Denote

$$\varepsilon_{20} = \min_{l=1,2,3,4}\{\varepsilon_{2l}^0\}, \phi_0 = \phi|_{\varepsilon_2=\varepsilon_{20}}. \tag{3.12}$$

Now the following hypothesis is given:
(A2) $\mathcal{K}_1\mathcal{L}_1 + \mathcal{K}_2\mathcal{L}_2 > 0$, where

$$\begin{aligned} \mathcal{K}_1 &= (\mathcal{A}_1+\mathcal{B}_2)q\phi_0^{q-1}\cos\frac{(q-1)\pi}{2} - 2q\phi_0^{2q-1}\cos\frac{(2q-1)\pi}{2}, \\ \mathcal{K}_2 &= (\mathcal{A}_1+\mathcal{B}_2)q\phi_0^{q-1}\sin\frac{(q-1)\pi}{2} - 2q\phi_0^{2q-1}\sin\frac{(2q-1)\pi}{2}, \\ \mathcal{L}_1 &= \mathcal{A}_2\mathcal{B}_1\phi_0\sin 2\phi_0\varepsilon_{20}, \\ \mathcal{L}_2 &= \mathcal{A}_2\mathcal{B}_1\phi_0\cos 2\phi_0\varepsilon_{20}. \end{aligned}$$

Lemma 3.2. *Assume that $s(\varepsilon_2) = \mu(\varepsilon_2) + i\phi(\varepsilon_2)$ is the root of (3.7) around $\varepsilon_2 = \varepsilon_{20}$ which satisfies $\mu(\varepsilon_{20}) = 0, \phi(\varepsilon_{20}) = \phi_0$, then Re $\left[\frac{ds}{d\varepsilon_2}\right]_{\varepsilon_2=\varepsilon_{20},\phi=\phi_0} > 0$.*

Proof. By (3.7), one has

$$\left(\frac{ds}{d\varepsilon_2}\right)^{-1} = \frac{(\mathcal{A}_1+\mathcal{B}_2)qs^{q-1} - 2qs^{2q-1}}{2\mathcal{A}_2\mathcal{B}_1 se^{-2s\varepsilon_2}} - \frac{\varepsilon_2}{s}, \tag{3.13}$$

then

$$\mathrm{Re}\left[\left(\frac{ds}{d\varepsilon_2}\right)^{-1}\right] = \mathrm{Re}\left[\frac{(\mathcal{A}_1+\mathcal{B}_2)qs^{q-1}-2qs^{2q-1}}{2\mathcal{A}_2\mathcal{B}_1 se^{-2s\varepsilon_2}}\right]. \tag{3.14}$$

Therefore

$$\begin{aligned}\mathrm{Re}\left[\left(\frac{ds}{d\varepsilon_2}\right)^{-1}\right]_{\varepsilon_2=\varepsilon_{20},\phi=\phi_0} &= \mathrm{Re}\left[\frac{(\mathcal{A}_1+\mathcal{B}_2)qs^{q-1}-2qs^{2q-1}}{2\mathcal{A}_2\mathcal{B}_1 se^{-2s\varepsilon_2}}\right]_{\varepsilon_2=\varepsilon_{20},\phi=\phi_0}\\ &= \frac{\mathcal{K}_1\mathcal{L}_1+\mathcal{K}_2\mathcal{L}_2}{\mathcal{L}_1^2+\mathcal{L}_2^2}.\end{aligned}$$

In view of (A2), we get

$$\mathrm{Re}\left[\left(\frac{ds}{d\varepsilon_2}\right)^{-1}\right]_{\varepsilon_2=\varepsilon_{20},\phi=\phi_0} > 0.$$

This ends the proof of Lemma 3.2. □

According to analysis above, one gets the following theorem.

Theorem 3.1. *For system (1.5), assume that $\varepsilon_1 = 0$ and (A1)and (A2) hold true, then the equilibrium point (x_{10}, x_{20}) is locally asymptotically stable when $\varepsilon_2 \in [0, \varepsilon_{20})$ and a Hopf bifurcation appears around the equilibrium point (x_{10}, x_{20}) for $\varepsilon_2 = \varepsilon_{20}$.*

Assume that $\varepsilon_1 > 0, \varepsilon_2 > 0$. Let $\varepsilon_2 \in [0, \varepsilon_{20})$. Choose the ε_1 as a bifurcation parameter. Eq. (3.5) can be written as follows:

$$\left(s^{2q}-\mathcal{A}_2\mathcal{B}_1e^{-2s\varepsilon_2}\right)-(\mathcal{A}_1+\mathcal{B}_2)s^qe^{-s\tau_1}+\mathcal{A}_1\mathcal{B}_2e^{-2s\varepsilon_1}=0. \tag{3.15}$$

According to (3.15), we have

$$\left(s^{2q}-\mathcal{A}_2\mathcal{B}_1e^{-2s\tau_2}\right)e^{s\varepsilon_1}-(\mathcal{A}_1+\mathcal{B}_2)s^q+\mathcal{A}_1\mathcal{B}_2e^{-s\varepsilon_1}=0. \tag{3.16}$$

Assume that $s = i\varphi = \varphi\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ is a root of (3.16). Then

$$\begin{cases}\mathcal{E}_1\cos\varphi\varepsilon_1+\mathcal{E}_2\sin\varphi\varepsilon_1=\mathcal{E}_3,\\ \mathcal{E}_2\cos\varphi\varepsilon_1+\mathcal{E}_4\sin\varphi\varepsilon_1=\mathcal{E}_5,\end{cases} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{E}_1 &= \varphi^{2q}\cos q\pi-\mathcal{A}_2\mathcal{B}_1\cos 2\varphi\varepsilon_2+\mathcal{A}_1\mathcal{B}_2,\\ \mathcal{E}_2 &= \varphi^{2q}\sin q\pi+\mathcal{A}_2\mathcal{B}_1\sin 2\varphi\varepsilon_2,\\ \mathcal{E}_3 &= (\mathcal{A}_1+\mathcal{B}_2)\varphi^q\cos\frac{q\pi}{2},\\ \mathcal{E}_4 &= \varphi^{2q}\cos q\pi-\mathcal{A}_2\mathcal{B}_1\cos 2\varphi\varepsilon_2-\mathcal{A}_1\mathcal{B}_2,\\ \mathcal{E}_5 &= (\mathcal{A}_1+\mathcal{B}_2)\varphi^q\sin\frac{q\pi}{2}.\end{aligned}$$

By (3.17), we have

$$\begin{cases}\cos\varphi\varepsilon_1=\dfrac{\mathcal{E}_3\mathcal{E}_4-\mathcal{E}_2\mathcal{E}_5}{\mathcal{E}_1\mathcal{E}_4-\mathcal{E}_2^2},\\ \sin\varphi\varepsilon_1=\dfrac{\mathcal{E}_1\mathcal{E}_5-\mathcal{E}_2\mathcal{E}_3}{\mathcal{E}_1\mathcal{E}_4-\mathcal{E}_2^2}.\end{cases} \tag{3.18}$$

By (3.18), one gets

$$(\mathcal{E}_3\mathcal{E}_4 - \mathcal{E}_2\mathcal{E}_5)^2 + (\mathcal{E}_1\mathcal{E}_5 - \mathcal{E}_2\mathcal{E}_3)^2 = (\mathcal{E}_1\mathcal{E}_4 - \mathcal{E}_2^2)^2. \tag{3.19}$$

Since

$$\begin{aligned}
(\mathcal{E}_3\mathcal{E}_4 - \mathcal{E}_2\mathcal{E}_5)^2 &= (\mu_1\varphi^{3q} + \mu_2\varphi^q)^2,\\
(\mathcal{E}_1\mathcal{E}_5 - \mathcal{E}_2\mathcal{E}_3)^2 &= (\mu_1\varphi^{3q} + \mu_3\varphi^q)^2,\\
(\mathcal{E}_1\mathcal{E}_4 - \mathcal{E}_2^2)^2 &= (\mu_4\varphi^{4q} + \mu_5\varphi^{2q} + \mu_6)^2,
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= (\mathcal{A}_1 + \mathcal{B}_2)\left(\cos q\pi \cos\frac{q\pi}{2} - \sin q\pi \sin\frac{q\pi}{2}\right),\\
\mu_2 &= (\mathcal{A}_1 + \mathcal{B}_2)\left[(\mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1\cos 2\varphi\varepsilon_2)\cos\frac{q\pi}{2} - \mathcal{A}_2\mathcal{B}_1\sin 2\varphi\varepsilon_2 \sin\frac{q\pi}{2}\right],\\
\mu_3 &= (\mathcal{A}_1 + \mathcal{B}_2)\left[(\mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1\cos 2\varphi\varepsilon_2)\sin\frac{q\pi}{2} - \mathcal{A}_2\mathcal{B}_1\sin 2\varphi\varepsilon_2 \cos\frac{q\pi}{2}\right],\\
\mu_4 &= \cos 2q\pi,\\
\mu_5 &= 2\mathcal{A}_2\mathcal{B}_1(\cos 2\varphi\varepsilon_2 \cos q\pi + \sin 2\varphi\tau_2 \sin q\pi),\\
\mu_6 &= \mathcal{A}_2^2\mathcal{B}_1^2\cos 4\varphi\varepsilon_2 - \mathcal{A}_2^2\mathcal{B}_1^2.
\end{aligned}$$

It follows from (3.19) that

$$\eta_1\varphi^{8q} + \eta_2\varphi^{6q} + \eta_3\varphi^{4q} + \eta_4\varphi^{2q} + \eta_5 = 0, \tag{3.20}$$

where

$$\begin{aligned}
\eta_1 &= \mu_4^2,\\
\eta_2 &= 2\mu_4\mu_5 - 2\mu_1^2,\\
\eta_3 &= \mu_5^2 - 2\mu_1\mu_2 - 2\mu_1\mu_3,\\
\eta_4 &= 2\mu_5\mu_6 - \mu_2^2 - \mu_3^2,\\
\eta_5 &= \mu_6^2.
\end{aligned}$$

Set

$$\rho(\varphi) = \eta_1\varphi^{8q} + \eta_2\varphi^{6q} + \eta_3\varphi^{4q} + \eta_4\varphi^{2q} + \eta_5 \tag{3.21}$$

and

$$\psi(\nu) = \eta_1\nu^8 + \eta_2\nu^6 + \eta_3\nu^4 + \eta_4\nu^2 + \eta_5. \tag{3.22}$$

Clearly, $\eta_1 > 0, \eta_5 > 0$. Then we can easily obtain the following results.

Lemma 3.3. *For Eq. (3.15), the following conclusions hold:*

(1) Assume that $\eta_i (i = 2, 3, 4) > 0$, then Eq. (3.15) has no root with zero real parts.
(2) Assume that there exists a constant $\zeta > 0$ such that $\psi'(\zeta) < 0$, then Eq. (3.15) has at least two pairs of purely imaginary roots.

Proof. (1) In view of $\eta_i > 0 (i = 1, 2, 3, 4, 5)$, then $\frac{d\rho(\varphi)}{d\varphi} > 0\ \forall\ \varphi > 0$ and $\rho(0) = \eta_5 > 0$. Hence Eq. (3.20) has no positive real root. Therefore Eq.(3.15) has no purely imaginary root. According to $\mathcal{A}_1\mathcal{B}_2 - \mathcal{A}_2\mathcal{B}_1 > 0$, $s = 0$ is not the root of Eq. (3.15). The proof of (1) ends.

(2) Since $\psi(0) = \eta_5 > 0, \psi(\vartheta_0) < 0 (\vartheta_0 > 0)$ and $\lim_{\vartheta\to+\infty} \frac{\psi(\vartheta)}{d\vartheta} = +\infty$, then there exist $\vartheta_{01} \in (0, \vartheta_0)$ and $\vartheta_{02} \in (\vartheta_0, +\infty)$ which satisfy $\psi(\vartheta_{01}) = \psi(\vartheta_{02}) = 0$. Then

(3.20) has at least two positive real roots. Thus (3.15) has at least two pairs of purely imaginary roots. The proof of (2) ends. □

Here we suppose that (3.20) has eight positive real roots $\varphi_l (l = 1, 2, 3, 4, 5, 6, 7, 8)$. By (3.18), one has

$$\varepsilon_{1l}^k = \frac{1}{\varphi_l}\left[\arccos\left(\frac{\mathcal{E}_3\mathcal{E}_4 - \mathcal{E}_2\mathcal{E}_5}{\mathcal{E}_1\mathcal{E}_4 - \mathcal{E}_2^2}\right) + 2k\pi\right], \tag{3.23}$$

where $k = 0, 1, 2, \cdots, l = 1, 2, 3, 4, 5, 6, 7, 8$. Set

$$\varepsilon_{10} = \min_{l=1,2,3,4,5,6,7,8}\{\varepsilon_{1l}^0\}, \varphi_0 = \varphi|_{\varepsilon_1=\varepsilon_{10}}. \tag{3.24}$$

The following hypothesis is given:

(A3) $\mathcal{X}_1\mathcal{Y}_1 + \mathcal{X}_2\mathcal{Y}_2 > 0$, where

$$\begin{aligned}
\mathcal{X}_1 = &\left[2q\varphi_0^{2q-1}\cos\frac{(2q-1)\pi}{2} + 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2\cos\varphi_0\varepsilon_2\right]\cos\varphi_0\varepsilon_{10}\\
&-\left[2q\varphi_0^{2q-1}\sin\frac{(2q-1)\pi}{2} - 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2\sin\varphi_0\varepsilon_2\right]\sin\varphi_0\varepsilon_{10}\\
&-(\mathcal{A}_1+\mathcal{B}_2)q\phi_0^{q-1}\cos\frac{(q-1)\pi}{2},\\
\mathcal{X}_2 = &\left[2q\varphi_0^{2q-1}\cos\frac{(2q-1)\pi}{2} + 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2\cos\varphi_0\tau_2\right]\sin\varphi_0\varepsilon_{10}\\
&-\left[2q\varphi_0^{2q-1}\sin\frac{(2q-1)\pi}{2} - 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2\sin\varphi_0\varepsilon_2\right]\cos\varphi_0\varepsilon_{10}\\
&-(\mathcal{A}_1+\mathcal{B}_2)q\phi_0^{q-1}\sin\frac{(q-1)\pi}{2},\\
\mathcal{Y}_1 = &\varphi_0[\mathcal{A}_1\mathcal{B}_2\sin\varphi_0\varepsilon_{10} - \cos\varphi_0\varepsilon_{10}(\varphi_0^{2q}\sin q\pi + \mathcal{A}_2\mathcal{B}_1\sin\varphi_0\varepsilon_2)\\
&-\sin\varphi_0\varepsilon_{10}(\varphi_0^{2q}\cos q\pi - \mathcal{A}_2\mathcal{B}_1\cos\varphi_0\varepsilon_2)],\\
\mathcal{Y}_2 = &\varphi_0[\mathcal{A}_1\mathcal{B}_2\cos\varphi_0\varepsilon_{10} - \cos\varphi_0\varepsilon_{10}(\varphi_0^{2q}\cos q\pi + \mathcal{A}_2\mathcal{B}_1\cos\varphi_0\varepsilon_2)\\
&-\sin\varphi_0\tau_{10}(\varphi_0^{2q}\sin q\pi - \mathcal{A}_2\mathcal{B}_1\sin\varphi_0\varepsilon_2)].
\end{aligned}$$

Lemma 3.4. *Assume that $s(\varepsilon_1) = \varrho(\varepsilon_1) + i\phi(\varepsilon_1)$ is the root of (3.7) around $\varepsilon_1 = \varepsilon_{10}$ which satisfies $\varrho(\varepsilon_{10}) = 0, \varphi(\varepsilon_{10}) = \varphi_0$, then Re $\left[\frac{ds}{d\varepsilon_1}\right]_{\varepsilon_1=\varepsilon_{10},\varphi=\varphi_0} > 0$.*

Proof. By (3.16), one gets

$$\left(\frac{ds}{d\varepsilon_1}\right)^{-1} = \frac{[2qs^{2q-1} + 2\mathcal{A}_2\mathcal{B}_1\tau_2 e^{-s\varepsilon_2}]e^{s\varepsilon_1} - (\mathcal{A}_1+\mathcal{B}_2)qs^{q-1}}{s[\mathcal{A}_1\mathcal{B}_2 e^{-s\tau_1} - e^{s\varepsilon_1}(s^{2q} - \mathcal{A}_2\mathcal{B}_1 e^{-s\varepsilon_2})]} - \frac{\varepsilon_1}{s}, \tag{3.25}$$

then

$$\text{Re}\left[\left(\frac{ds}{d\varepsilon_1}\right)^{-1}\right] = \text{Re}\left[\frac{(2qs^{2q-1} + 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2 e^{-s\varepsilon_2})e^{s\varepsilon_1} - (\mathcal{A}_1+\mathcal{B}_2)qs^{q-1}}{s(\mathcal{A}_1\mathcal{B}_2 e^{-s\varepsilon_1} - e^{s\varepsilon_1}(s^{2q} - \mathcal{A}_2\mathcal{B}_1 e^{-s\varepsilon_2}))}\right]. \tag{3.26}$$

Thus

$$\begin{aligned}
\text{Re}\left[\left(\frac{ds}{d\varepsilon_1}\right)^{-1}\right]_{\varepsilon_1=\varepsilon_{10},\phi=\phi_0} &= \text{Re}\left[\frac{(2qs^{2q-1} + 2\mathcal{A}_2\mathcal{B}_1\varepsilon_2 e^{-s\tau_2})e^{s\varepsilon_1} - (\mathcal{A}_1+\mathcal{B}_2)qs^{q-1}}{s(\mathcal{A}_1\mathcal{B}_2 e^{-s\tau_1} - e^{s\tau_1}(s^{2q} - \mathcal{A}_2\mathcal{B}_1 e^{-s\tau_2}))}\right]_\varepsilon_1 = \varepsilon_{10}, \varphi = \varphi_0\\
&= \frac{\mathcal{X}_1\mathcal{Y}_1 + \mathcal{X}_2\mathcal{Y}_2}{\mathcal{Y}_1^2 + \mathcal{Y}_2^2}.
\end{aligned}$$

In view of (A3), we get

$$\mathrm{Re}\left[\left(\frac{ds}{d\varepsilon_1}\right)^{-1}\right]_{\varepsilon_1=\varepsilon_{10},\varphi=\varphi_0} > 0.$$

This ends the proof of Lemma 3.4. □

According to the analysis above, one gets the following theorem.

Theorem 3.2. *For system (1.5), assume that $\varepsilon_2 \in [0, \varepsilon_{20})$ and (A1)and (A3) hold true, then the equilibrium point (x_{10}, x_{20}) is locally asymptotically stable for $\varepsilon_1 \in [0, \varepsilon_{10})$ and a Hopf bifurcation appears around the equilibrium point (x_{10}, x_{20}) when $\varepsilon_1 = \varepsilon_{10}$.*

Remark 3.1. In [5, 6, 11, 20, 37, 49, 51, 54–58], the authors considered the Hopf bifurcation of integer-order delayed models. In this paper, we investigate the stability and Hopf bifurcation of fractional-order competition and cooperation model of two enterprises with two different delays. All the obtained results and analysis methods [5, 6, 11, 20, 37, 49, 51, 54–58] can not be applied to (1.5) to obtain the stability and the existence of Hopf bifurcation for (1.5). In [22, 26–28, 50], the authors investigated the Hopf bifurcation of competition and cooperation model of two enterprises with delay, but they did not involve the fractional-order case. Based on these viewpoints, the fruit of this paper about the stability and the existence of Hopf bifurcation for (1.5) are completely new and an important supplement to some earlier works.

Remark 3.2. In [1, 8, 13–17, 33, 35, 39, 47, 48], the authors dealt with the Hopf bifurcation of fractional-order models. They did not consider the effect of different delays on the stability and Hopf bifurcation of involved models. Up to now, there are no results on the effect of different delays on Hopf bifurcation of involved fractional-order systems. From the analysis above, the obtained results of this paper is new.

Remark 3.3. Compared with the integer-order competition and cooperation model of two enterprises, the fractional-order competition and cooperation model of two enterprises can characterize memory property, history state, nonlocal effects of the output of two enterprises, which implies that the fractional-order competition and cooperation model of two enterprises have more advantage than the integer-order one.

Remark 3.4. In Theorem 3.1, we assume that $\varepsilon_1 = 0$ which implies the single case. In a similar way, we can deal with $\varepsilon_2 = 0$. Considering the practical meaning of competition and cooperation of two enterprises, we focus on the effect of double delays on Hopf bifurcation of fractional-order competition and cooperation model of two enterprises.

4. Computer simulations

Consider the following fractional-order model:

$$\begin{cases} \mathcal{D}^q x_1(t) = (x_1(t)+1)\left[0.3 - 0.2x_1(t-\varepsilon_1) - 0.1x_2^2(t-\varepsilon_2)\right], \\ \mathcal{D}^q x_2(t) = (x_2(t)+1)\left[0.3 - 0.5x_2(t-\varepsilon_1) + 0.2x_1^2(t-\varepsilon_2)\right]. \end{cases} \tag{4.1}$$

All the coefficients are same as those in Liao [26]. It is not difficult to see that system (4.1) has the equilibrium point $(1, 1)$. Let $\varepsilon_1 = 0, q = 0.83$. Then $\phi_0 = 0.6257$ and

$\varepsilon_{20} = 1.9939$. Then the hypotheses (A1) and (A2) of Theorem 3.1 are fulfilled. Figures 1–4 indicate that the equilibrium point $(1, 1)$ of system (4.1) is locally asymptotically stable when $\varepsilon_2 \in [0, \varepsilon_{20})$. Figures 5–8 manifest that system (1.1) becomes unstable, a Hopf bifurcation appears when $\varepsilon_2 \in [\varepsilon_{20}, +\infty)$. The relation of parameters q, ϕ_0 and ε_{20} of (4.1) is displayed in Table 1. One can see that the order can postpone the emergence of Hopf bifurcation (compared with Liao [3]). Next let $\varepsilon_2 = 1, q = 0.83$. Then $\varphi_0 = 0.7125$ and $\varepsilon_{10} = 1.1530$. Then the hypotheses (A1) and (A3) of Theorem 3.2 are fulfilled. Figures 9–12 reveal that the equilibrium point $(1, 1)$ of system (4.1) is locally asymptotically stable when $\varepsilon_1 \in [0, \varepsilon_{10})$. Figures 13–16 imply that system (1.1) becomes unstable, a Hopf bifurcation appears when $\varepsilon_1 \in [\varepsilon_{10}, +\infty)$. In Table 2, we have given the relation of q, φ_0 and ε_{10} of (4.1). It is not difficult to see that the order can make the Hopf bifurcation appear ahead of time compared with the integer-order ones (see [26]).

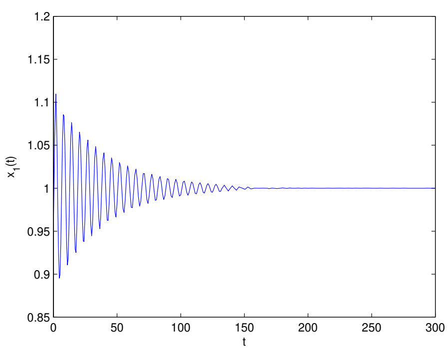

Figure 1. The relation of t-$x_1(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 1.8 < \varepsilon_{20} = 1.9939$.

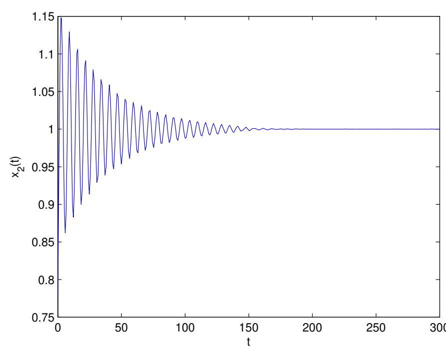

Figure 2. The relation of t-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 1.8 < \varepsilon_{20} = 1.9939$.

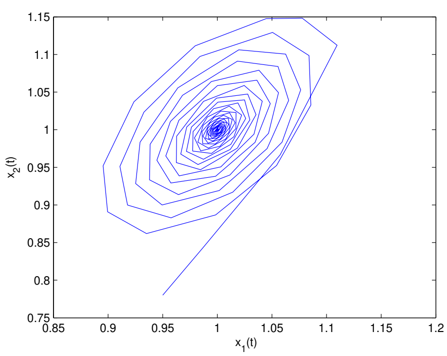

Figure 3. The relation of $x_1(t)$-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 1.8 < \varepsilon_{20} = 1.9939$.

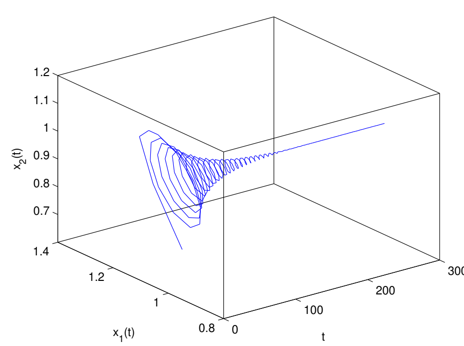

Figure 4. The relation of t-$x_1(t)$-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 1.8 < \varepsilon_{20} = 1.9939$.

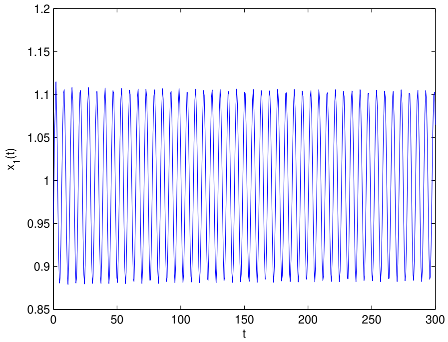

Figure 5. The relation of t-$x_1(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 2.2 > \varepsilon_{20} = 1.9939$.

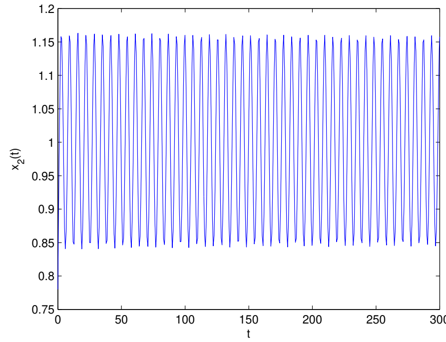

Figure 6. The relation of t-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 2.2 > \varepsilon_{20} = 1.9939$.

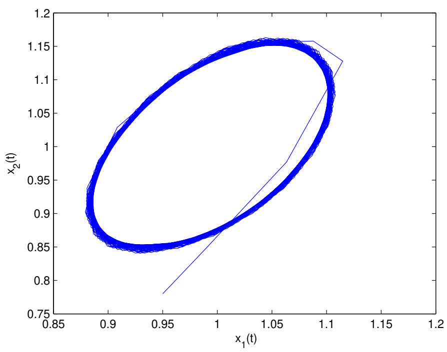

Figure 7. The relation of $x_1(t)$-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 2.2 > \varepsilon_{20} = 1.9939$.

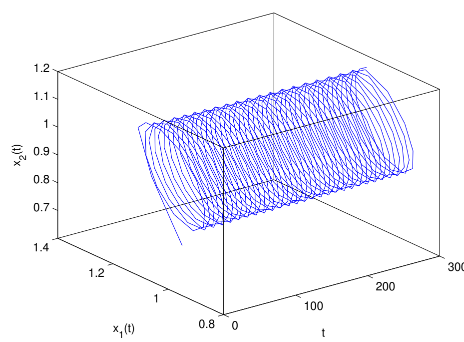

Figure 8. The relation of t-$x_1(t)$-$x_2(t)$ when $\varepsilon_1 = 0, \varepsilon_2 = 2.2 > \varepsilon_{20} = 1.9939$.

Table 1. The relation of q, ϕ_0 and ε_{20} of (4.1).

q	ϕ_0	ε_{20}
0.15	1.6771	0.3722
0.24	1.5134	0.5929
0.32	1.4267	0.7875
0.43	1.1093	1.0526
0.51	1.0132	1.2437
0.64	0.8937	1.5511
0.76	0.7348	1.8317
0.83	0.6257	1.9939
0.91	0.5546	2.1781

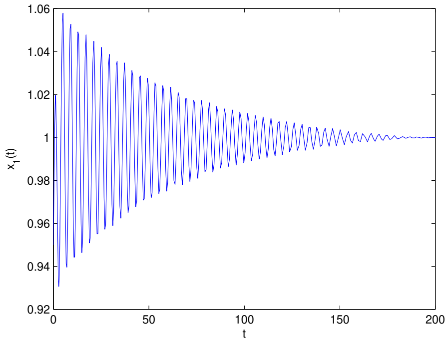

Figure 9. The relation of t-$x_1(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.0 < \varepsilon_{10} = 1.1530$.

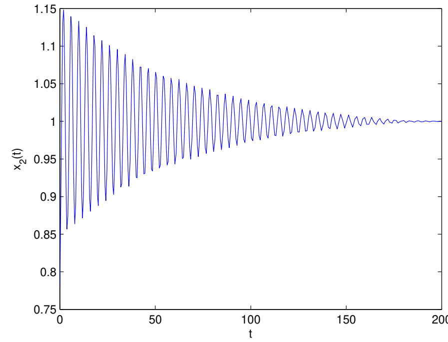

Figure 10. The relation of t-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.0 < \varepsilon_{10} = 1.1530$.

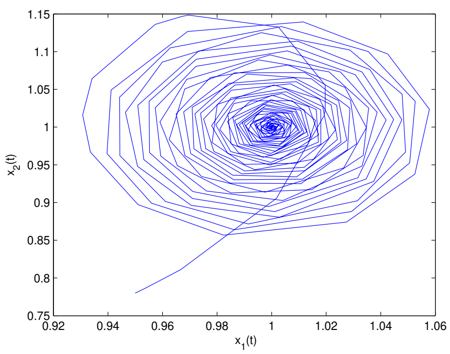

Figure 11. The relation of $x_1(t)$-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.0 < \varepsilon_{10} = 1.1530$.

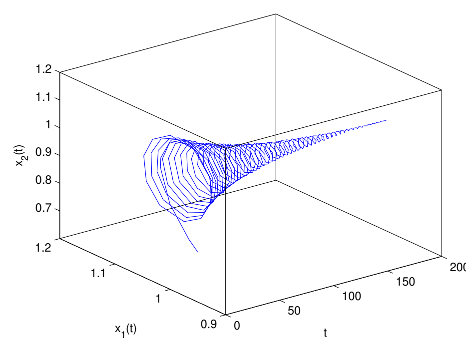

Figure 12. The relation of t-$x_1(t)$-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.0 < \varepsilon_{10} = 1.1530$.

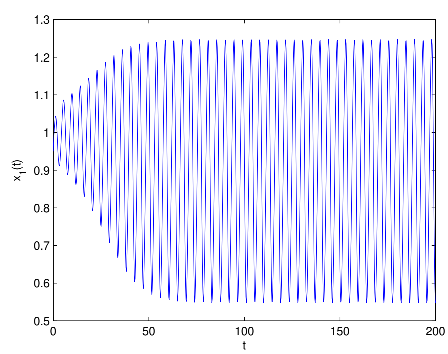

Figure 13. The relation of t-$x_1(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.2 > \varepsilon_{10} = 1.1530$.

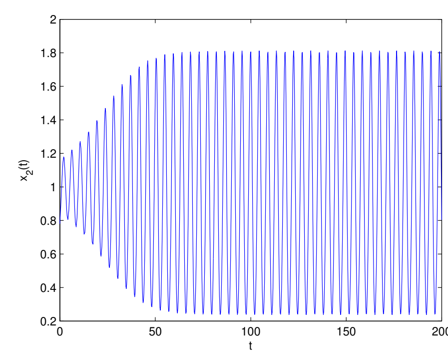

Figure 14. The relation of t-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.2 > \varepsilon_{10} = 1.1530$.

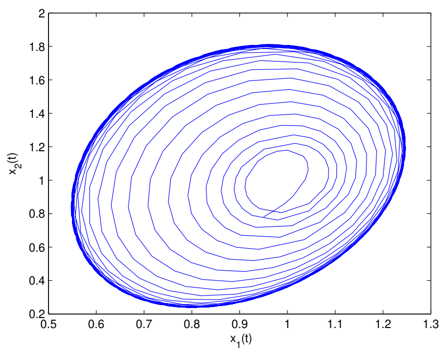

Figure 15. The relation of $x_1(t)$-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.2 > \varepsilon_{10} = 1.1530$.

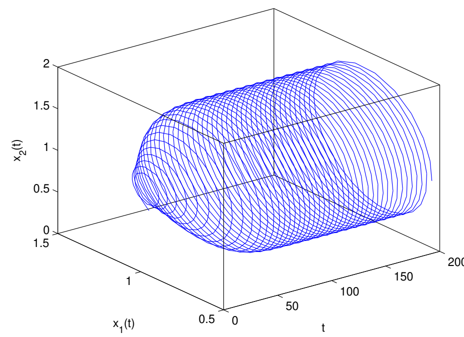

Figure 16. The relation of t-$x_1(t)$-$x_2(t)$ when $\varepsilon_2 = 1, \varepsilon_1 = 1.2 > \varepsilon_{10} = 1.1530$.

Table 2. The relation of q, φ_0 and ε_{10} of (4.1).

q	φ_0	ε_{10}
0.15	1.8722	0.2217
0.24	1.6309	0.3517
0.32	1.4877	0.4653
0.43	1.2732	0.6189
0.51	1.0833	0.7287
0.64	1.0642	0.9039
0.76	0.8751	1.0621
0.83	0.7125	1.1530
0.91	0.6718	1.2555

5. Conclusions

The competition and cooperation among different enterprises is an important aspect in production and management of enterprises. In the manuscript, based on earlier publications, we propose a new fractional-order competition and cooperation model of two enterprises with two different delays. By regarding two different delays as bifurcation parameter, we establish two sets of sufficient conditions to assure the the stability and the existence of Hopf bifurcation for involved competition and cooperation model of two enterprises. The investigation manifests that the two different delays have different effect on the stability and Hopf bifurcation of involved model. Also the relation of fractional-order and bifurcation point are displayed. The derived results have important theoretical significance and practical value in managing the production of enterprises. Besides, we point out that how to control the duration time of output for enterprises is an interesting issue. It involves the bifurcation control issue. We will study this aspect in the near future. In addition, we will try to investigate the effect of multiple delays on Hopf bifurcation for fractional-order delayed competition and cooperation model. Now we still can not deal with the direction and stability of Hopf bifurcation of fractional order differential systems due to the lack of theory.

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