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STABILITY ANALYSIS BETWEEN THE HYBRID STOCHASTIC DELAY DIFFERENTIAL EQUATIONS WITH JUMPS AND THE EULER-MARUYAMA METHOD*

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Abstract The aim of this paper is to concern with the mean square exponential stability equivalence between the hybrid stochastic delay differential equations with jumps and the Euler-Maruyama method (EM-method). Precisely, under the global Lipschitz condition, it is shown that a stochastic delay differential equation with Markovian switching and jumps (SDDEwMJ) is mean square exponentially stable if and only if for some sufficiently small step size, its EM-method is mean square exponentially stable. Based on such a result, the mean square exponential stability of a SDDEwMJ can be investigated by the careful numerical simulations in practice without resorting to Lyapunov functions. Moreover, a numerical example is provided to confirm the obtained results.

Keywords Mean square stability, stochastic delay differential equations, Euler-Maruyama method, stability equivalence, Markovian switching, jumps.

MSC(2010) 60H10, 93E15, 65L20, 60H35.

1. Introduction

A great many stochastic differential systems depend not only on the present states but also on the past states. For such systems, stochastic delay differential equations (SDDEs) are often used to describe them, and which have been widely developed and are applicable to biological systems, genetic regulatory networks, chemical engineering systems and control, etc. [1, 4, 5, 17, 28]. To our knowledge, a Brownian motion is a continuous stochastic process, however, some systems may suffer from the jump type abrupt perturbations and the phenomenon of discontinuous random pulse excitation. In such cases, incorporating jumps into SDDEs seems to be necessary, and it is therefore valuable to discuss the SDDEs with jumps [7, 21, 26]. When SDDEs with jumps encounter abrupt changes in their structure and parameters,

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SDDEs with Markovian switching and jumps can be applied to model them. This kind of models are more realistic models and the research of them have aroused a great deal of attention [12, 16, 27, 29, 32].

In the study of stochastic differential equations (SDEs), stability analysis has received a great deal of attention, see [6, 11, 17, 25, 31, 33] and the reference therein. A classical and powerful technique for investigating the stochastic stability is the Lyapunov functions method. A natural problem: how do we study the stochastic stability when an appropriate Lyapunov function is not found? Using the numerical simulation to study the stochastic stability may be an alternative technique, and many results on the stability of the numerical methods for the SDEs have been obtained, see [2, 20, 22, 24] and the reference therein. We therefore should consider whether the stochastic stability between the underlying equation and its numerical solution are equivalent. If it is positive, then we can investigate the stochastic stability of the underlying equation via its careful numerical simulations.

For SDEs, Higham et al. [8] showed the mean square exponential stability of SDEs and that of the numerical methods for sufficiently small step sizes are equivalent; Later, the authors in [9] showed that a linear scalar SDE is almost surely exponentially stable if and only if its EM-method is almost surely exponentially stable with small enough step sizes; Mao [19] proved that under the global Lipschitz condition, the almost sure exponential stability of SDEs is shared with that of the stochastic theta method. Liu et al. [15] established a exponential stability equivalence theorem between the neutral SDDEs and the EM-method in the sense of mean square. Recently, for SDEs driven by G-Brownian motion, Yang and Li [30] showed under some appropriate conditions, the p-th($p \in (0, 1)$) moment exponential stability between the equation and its stochastic theta method are equivalent. Deng et al. [3] presented the mean square exponential stability equivalence between the SDDEs driven by G-Brownian motion and the EM-method. For SDEs with Markovian switching, Higham et al. [10] revealed that under the global Lipschitz condition, the EM-method and the back EM-method can preserve the mean square exponential stability of the corresponding equations. Pang et al. [23] proved that the EM-method can capture the almost sure and the p-th moment exponential stability for a linear scalar SDEs with Markovian switching. However, researchers in the above literature did not address the problem is that if a numerical method applied to a SDE with Markovian switching is stochastically stable, then the underlying equation is stochastically stable. Moreover, because of the complexity of the Markovian switching and jumps, there is so far no literature on the stability equivalence between a SDDEwMJ and its numerical methods.

Inspired by the aforementioned works, this paper establishes the mean square exponential stability equivalence theorem between a SDDEwMJ and its EM-method. Based on such a result, one can investigate the mean square exponential stability of a SDDEwMJ by the numerical method in the absence of an appropriate Lyapunov function.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, we devote to present the main results. In Section 4, to show the effectiveness of the obtained theory, an illustrative example is provided. Finally, we end this paper with a brief conclusion.

2. Preliminaries

Throughout the paper, the essential notations are given as follows. Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the integer part of a. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space, and $|x|$ represents the Euclidean norm of a vector x. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|\varphi\| = \sup_{\{-\tau\leq\theta\leq 0\}} |\varphi(\theta)|$. Let $BC([-\tau, 0]; \mathbb{R}^n)$ represent all bounded functions defined on the $C([-\tau, 0]; \mathbb{R}^n)$. $L^2_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^n)$ is the family of the square integrable $\mathcal{F}_t$-measurable functions defined on the $BC([-\tau, 0]; \mathbb{R}^n)$. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, \infty)$, we define $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ as a $C([-\tau, 0]; \mathbb{R}^n)$-valued stochastic process. For $\psi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, define $\|\psi\|^2_{\mathbb{E}} = \sup_{\theta\in[-\tau,0]} \mathbb{E}|\psi(\theta)|^2$. Denote $\{r(t)\}_{t\geq 0}$ by a right-continuous Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite state space $S = \{1, \cdots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } \ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } \ i = j, \end{cases}$$

where $\Delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j. One can refer to Mao and Yuan [20] for the properties of $r(t)$.

Consider the following SDDEwMJ of the form

$$\begin{aligned} dx(t) =& f(x(t), x(t-\tau), r(t))dt + g(x(t), x(t-\tau), r(t))d\omega(t) \\ &+ h(x(t), x(t-\tau), r(t))dN(t), \ t \geq 0 \end{aligned} \tag{2.1}$$

with the initial value $x_0 = \eta = \{\eta(\theta) : -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and $r(0) = i_0 \in S$, where $f, g, h : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$. $\omega(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda > 0$. $\tilde{N}(t) = N(t) - \lambda t$ represents its corresponding compensated Poisson process. Moreover, $\omega(t)$, $N(t)$ and $r(t)$ are assumed to be mutually independent. For the purpose of the stability study, it is assumed that $f(0, 0, i) = g(0, 0, i) = h(0, 0, i) = 0$ for $\forall i \in S$. The following hypothesis is further assumed to be satisfied.

(**H**) Assume that f, g, h satisfy the global Lipschitz condtion, namely, there are constants $K_i > 0 (i = 1, 2, 3)$ such that for any $i \in S$ and $x_j, y_j \in \mathbb{R}^n (j{=}1{,}2)$,

$$\begin{aligned} &|f(x_1, y_1, i) - f(x_2, y_2, i)|^2 \leq K_1(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &|g(x_1, y_1, i) - g(x_2, y_2, i)|^2 \leq K_2(|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ &|h(x_1, y_1, i) - h(x_2, y_2, i)|^2 \leq K_3(|x_1 - x_2|^2 + |y_1 - y_2|^2). \end{aligned} \tag{2.2}$$

Remark 2.1. Notice that $h \equiv 0$ or $r(t) \equiv i_0$ in Eq. (2.1), then it becomes the SDDEs with Markovian switching and SDDEs with jumps, which have been well studied in [7,14,18,20]. According to the proof of the existence and uniqueness of the solution to SDDEs with Markovian switching and SDDEs with jumps, analogously, one can confirm that Eq. (2.1) admits a unique global solution $x(t) = x(t; 0, \eta)$ on $t \geq -\tau$ with the condition (**H**). The proof is standard, we therefore omit it.

Definition 2.1. The SDDEwMJ (2.1) is said to be mean square exponentially stable if there exist a pair of positive constants α and M such that for any initial value $\eta \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $\mathbb{E}|x(t; 0, \eta)| \leq M\|\eta\|^2_{\mathbb{E}} e^{-\alpha t}$, $\forall t \geq 0$.

In the next, we will discuss the EM-method applied to Eq. (2.1). Before discussing it, the following lemma will be used.

Lemma 2.1 ([20, pp.112]). *Given $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k \geq 0$. Then $\{r_k^\Delta, k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with one-step transition probability matrix.*

$$p(\Delta) = (p_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}. \tag{2.3}$$

Given a step size $\Delta > 0$, the discrete Markov chain $\{r_k^\Delta,\ k = 0, 1, 2, \ldots\}$ can be simulated as follows: Compute the one-step transition probability matrix by (2.3). Let $r_0^\Delta = i_0$ and generate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Define

$$r_1^\Delta = \begin{cases} i_1, & \text{if } i_1 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} p_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} p_{i_0,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} p_{i_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{i=1}^{0} p_{i_0,j}(\Delta) = 0$ as usual. Generate independently a new random number ξ_2 which is again uniformly distributed in $[0, 1]$ and then define

$$r_2^\Delta = \begin{cases} i_2, & \text{if } i_2 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} p_{r_1^\Delta,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} p_{r_1^\Delta,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} p_{r_1^\Delta,j}(\Delta) \leq \xi_2. \end{cases}$$

Repeating this procedure, a trajectory of $\{r_k^\Delta, k = 0, 1, 2, \ldots\}$ can be generated. This procedure can be carried out independently to obtain more trajectories. After explaining how to simulate the discrete Markov chain $\{r_k^\Delta, k = 0, 1, 2, \ldots\}$, we can now give the EM-method for Eq. (2.1). Take the step size Δ as $\Delta = \tau/m$ for some positive integer m. Let $t_k = k\Delta$ for $k \geq -m$ and $k \in \mathbb{Z}$ ($\mathbb{Z}$ is a set of integers), the discrete EM-solution for Eq. (2.1) is defined by setting initial value $(y_0, r_0^\Delta) = (x_0, i_0)$ and performing

$$\begin{aligned} y(t_{k+1}) =& y(t_k) + f(y(t_k), y(t_{k-m}), r_k^\Delta)\Delta + g(y(t_k), y(t_{k-m}), r_k^\Delta)\Delta\omega_k \\ &+ h(y(t_k), y(t_{k-m}), r_k^\Delta)\Delta N_k, k \geq 0, \end{aligned} \tag{2.4}$$

where $\Delta\omega_k = \omega(t_{k+1}) - \omega(t_k)$, $\Delta N_k = N(t_{k+1}) - N(t_k)$ and $r_k^\Delta = r(t_k)$. Set $y(t_k) = \eta(t_k), -m \leq k \leq 0$. For $t \in [t_k, t_{k+1})$, define $z(t) = y(t_k)$ with the initial $z(t) = \eta(t)$ on $[-\tau, 0]$ and $\bar{r}(t) = r_k^\Delta$. We then define the continuous time EM-solution as follows.

$$\begin{aligned} y(t) =& \eta(0) + \int_0^t f(z(s), z(s-\tau), \bar{r}(s))ds + \int_0^t g(z(s), z(s-\tau), \bar{r}(s))d\omega(s) \\ &+ \int_0^t h(z(s), z(s-\tau), \bar{r}(s))dN(s), t \geq 0, \end{aligned} \tag{2.5}$$

and $y(t) = \eta(t)$ for $-\tau \leq t \leq 0$. Now we give the definition of the mean square exponential stability for the continuous time EM-method.

Definition 2.2. Given a step size Δ, if there are constants N and β such that for any initial value $\eta \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $\mathbb{E}|y(t; 0, \eta)|^2 \leq N\|\eta\|^2_{\mathbb{E}} e^{-\beta t}$, $\forall t \geq 0$, then the continuous EM-method applied to Eq. (2.1) is said to be mean square exponentially stable.

3. Main results

3.1. Approximation

In this section, we show that the continuous EM-solution $y(t)$ converges to the exact solution $x(t)$. To facilitate the discussion, from now on we will denote $x(t) = x(t;0,\eta)$, $y(t) = y(t;0,\eta)$ and $z(t) = z(t;0,\eta)$.

Theorem 3.1. *If the solution to Eq.* (2.1) *is the $x(t) = x(t;0,\eta)$ with the initial value η, then*

$$\sup_{-\tau\le t\le T+\tau} \mathbb{E}|x(t)-y(t)|^2 \le H(T+\tau)\|\eta\|_{\mathbb{E}}^2\Delta,\ \forall T>0, \tag{3.1}$$

where $H(T+\tau) = [2\bar{H}_1(T+\tau)H_2(T+\tau)(T+\tau)+\bar{H}_2(T+\tau)]e^{2\bar{H}_1(T+\tau)(T+\tau)}$, $\bar{H}_1(\cdot)$, $\bar{H}_2(\cdot)$ and $H_2(\cdot)$ are defined below.

Since the proof of Theorem 3.1 is complex, we divide it into the following lemmas.

Lemma 3.1. *If* (**H**) *is true, then for $\forall T>0$,*

$$\sup_{-\tau\le t\le\tau+T} \mathbb{E}|y(t)|^2 \vee \sup_{-\tau\le t\le\tau+T} \mathbb{E}|x(t)|^2 \le H_1(T+\tau)\|\eta\|_{\mathbb{E}}^2, \tag{3.2}$$

where

$$H_1(T+\tau) = 4\big[1+[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\tau\big]e^{8[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3](T+\tau)}.$$

Proof. For $0\le t\le T+\tau$, it follows from (2.5), Hölder inequality, (**H**) and $N(t) = \tilde{N}(t) - \lambda t$ that

$$\begin{aligned}
\mathbb{E}|y(t)|^2 \le& 4\mathbb{E}|\eta(0)|^2 + 4t\mathbb{E}\int_0^t |f(z(s),z(s-\tau),\bar{r}(s))|^2ds + 4\mathbb{E}\int_0^t |g(z(s),z(s-\tau),\bar{r}(s))|^2ds \\
&+ 8\lambda\mathbb{E}\int_0^t |h(z(s),z(s-\tau),\bar{r}(s))|^2ds + 8\lambda^2 t\mathbb{E}\int_0^t |h(z(s),z(s-\tau),\bar{r}(s))|^2ds \\
\le& 4\mathbb{E}|\eta(0)|^2 + 4[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\int_0^t (\mathbb{E}|z(s)|^2+\mathbb{E}|z(s-\tau)|^2)ds \\
\le& 4\left[1+[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\tau\right]\|\eta\|_{\mathbb{E}}^2 \\
&+ 8[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\int_0^t \mathbb{E}|z(s)|^2ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{0\le s\le t}\mathbb{E}|y(s)|^2 \le& 4\left[1+[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\tau\right]\|\eta\|_{\mathbb{E}}^2 \\
&+ 8[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\int_0^t \mathbb{E}(\sup_{0\le r\le s}|y(r)|^2)ds.
\end{aligned}$$

It follows from Gronwall inequality that one obtains

$$\begin{aligned}
\sup_{0\le s\le t}\mathbb{E}|y(s)|^2 \le& 4\left[1+[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\tau\right] \\
&\times e^{8[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]t}\|\eta\|_{\mathbb{E}}^2.
\end{aligned}$$

One further gains

$$\sup_{0\le t\le T+\tau}\mathbb{E}|y(t)|^2 \le 4\left[1+[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3]\tau\right] \times e^{8[(K_1+2\lambda^2K_3)(T+\tau)+K_2+2\lambda K_3](T+\tau)}\|\eta\|_{\mathbb{E}}^2.$$

For $-\tau\le t\le 0$, it is obvious that $\mathbb{E}|y(s)|^2\le\|\eta\|_{\mathbb{E}}^2$. Therefore, one can acquire $\sup_{-\tau\le t\le\tau+T}\mathbb{E}|y(t)|^2\le H_1(T+\tau)\|\eta\|_{\mathbb{E}}^2$. Analogously, one can obtain $\sup_{-\tau\le t\le\tau+T}\mathbb{E}|x(t)|^2\le H_1(T+\tau)\|\eta\|_{\mathbb{E}}^2$. The proof is complete. □

Lemma 3.2. *If (**H**) is true, then for $\forall T>0$,*

$$\mathbb{E}|y(t)-z(t)|^2\le H_2(T+\tau)\|\eta\|_{\mathbb{E}}^2\Delta, \forall t\in[-\tau,\tau+T], \tag{3.3}$$

where

$$H_2(T+\tau)=6[K_2+2\lambda K_3+(K_1+2\lambda^2K_3)\tau]H_1(T+\tau).$$

Proof. For any $t\in[0,\tau+T]$, there exists an integer k such that $t\in[t_k,t_{k+1})\subset[0,\tau+T]$. By (**H**) and Hölder inequality, one obtains

$$\begin{aligned}\mathbb{E}|y(t)-z(t)|^2 \le&3(t-t_k)\mathbb{E}\int_{t_k}^t|f(z(s),z(s-\tau),\bar r(s))|^2ds\\ &+3\mathbb{E}\int_{t_k}^t|g(z(s),z(s-\tau),\bar r(s))|^2ds\\ &+3\mathbb{E}\left|\int_{t_k}^t f(z(s),z(s-\tau),\bar r(s))dN(s)\right|^2\\ \le&3[K_2+2\lambda K_3+(K_1+2\lambda^2K_3)\Delta]\int_{t_k}^{t_{k+1}}(\mathbb{E}|z(s)|^2+\mathbb{E}|z(s-\tau)|^2)ds.\end{aligned}$$

It follows from $m\Delta=\tau(m\ge1)$ and Lemma 3.1 that one gains

$$\mathbb{E}|y(t)-z(t)|^2 \le6[K_2+2\lambda K_3+(K_1+2\lambda^2K_3)\tau]H_1(T+\tau)\|\eta\|_{\mathbb{E}}^2\Delta. \tag{3.4}$$

For $-\tau\le t\le0$, $\mathbb{E}|y(t)-z(t)|^2=0$, (3.4) is also acquired. Therefore, (3.3) is true. The proof is complete. □

Lemma 3.3. *Under the condition (**H**), for any $t\in[0,t+\tau]$, one can get the following estimation*

$$\begin{aligned}&\mathbb{E}\int_0^t|f(x(s),x(s-\tau),r(s))-f(z(s),z(s-\tau),\bar r(s))|^2ds\\ \le&4K_1\int_0^t\mathbb{E}|x(s)-z(s)|^2ds\\ &+16K_1(\max_{1\le i\le N}(-r_{ii})+o(1))H_1(T+\tau)(T+\tau)\|\eta\|_{\mathbb{E}}^2\Delta.\end{aligned}\tag{3.5}$$

Proof. Let $j=\lfloor(T+\tau)/\Delta\rfloor$. Then, by the condition (**H**), one obtains

$$\begin{aligned}&\mathbb{E}\int_0^t|f(x(s),x(s-\tau),r(s))-f(z(s),z(s-\tau),\bar r(s))|^2ds\\ \le&2\mathbb{E}\int_0^t|f(x(s),x(s-\tau),r(s))-f(z(s),z(s-\tau),r(s))|^2ds\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\int_0^t |f(z(s),z(s-\tau),r(s)) - f(z(s),z(s-\tau),\bar{r}(s))|^2 ds\\
\leq & 2K_1\int_0^t (\mathbb{E}|x(s)-z(s)|^2 + \mathbb{E}|x(s-\tau)-z(s-\tau)|^2)ds\\
&+ 2\mathbb{E}\int_0^t |f(z(s),z(s-\tau),r(s)) - f(z(s),z(s-\tau),\bar{r}(s))|^2 ds\\
\leq & 4K_1\int_0^t \mathbb{E}|x(s)-z(s)|^2 ds\\
&+ 2\sum_{i=0}^{j}\mathbb{E}\int_{t_i}^{t_{i+1}} |f(z(t_i),z(t_{i-m}),r(s)) - f(z(t_i),z(t_{i-m}),r(t_i))|^2 ds.
\end{aligned} \tag{3.6}
$$

For

$$
\begin{aligned}
&\mathbb{E}\int_{t_i}^{t_{i+1}} |f(z(t_i),z(t_{i-m}),r(s)) - f(z(t_i),z(t_{i-m}),r(t_i))|^2 ds\\
\leq & 2\mathbb{E}\int_{t_i}^{t_{i+1}} \left[|f(z(t_i),z(t_{i-m}),r(s))|^2 + |f(z(t_i),z(t_{i-m}),r(t_i))|^2\right] I_{\{r(s)\neq r(t_i)\}} ds\\
\leq & 4K_1\mathbb{E}\int_{t_i}^{t_{i+1}} (|z(t_i)|^2 + |z(t_{i-m})|^2) I_{\{r(s)\neq r(t_i)\}} ds\\
\leq & 4K_1\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\mathbb{E}[(|z(t_i)|^2 + |z(t_{i-m})|^2) I_{\{r(s)\neq r(t_i)\}}|r(t_i)]\right] ds.
\end{aligned}
$$

According to the estimation (4.16) in [20], one further obtains

$$
\begin{aligned}
&\mathbb{E}\int_{t_i}^{t_{i+1}} |f(z(t_i),z(t_{i-m}),r(s)) - f(z(t_i),z(t_{i-m}),r(t_i))|^2 ds\\
\leq & 4K_1\int_{t_i}^{t_{i+1}} \mathbb{E}\left[\mathbb{E}[(|z(t_i)|^2 + |z(t_{i-m})|^2)|r(t_i)]\mathbb{E}[I_{\{r(s)\neq r(t_i)\}}|r(t_i)]\right] ds\\
\leq & 4K_1(\max_{1\leq i\leq s}(-r_{ii})\Delta + o(\Delta))\int_{t_i}^{t_{i+1}} \mathbb{E}(|z(t_i)|^2 + |z(t_{i-m})|^2) ds.
\end{aligned}
$$

Applying Lemma 3.1, one gets

$$
\begin{aligned}
&\mathbb{E}\int_0^t |f(x(s),x(s-\tau),r(s)) - f(z(s),z(s-\tau),\bar{r}(s))|^2 ds\\
\leq & 4K_1(\max_{1\leq i\leq s}(-r_{ii})\Delta + o(\Delta))\sum_{i=0}^{j}\int_{t_i}^{t_{i+1}} \mathbb{E}(|z(t_i)|^2 + |z(t_{i-m})|^2) ds\\
\leq & 8K_1(\max_{1\leq i\leq s}(-r_{ii}) + o(1))(T+\tau)H_1(T+\tau)\|\eta\|_{\mathbb{E}}^2\Delta.
\end{aligned} \tag{3.7}
$$

Substituting (3.7) into (3.6), the assertion (3.5) can be obtained. The proof is complete. □

Remark 3.1. Analogously, under the condition (**H**), one can also gain

$$
\mathbb{E}\int_0^t |g(x(s),x(s-\tau),r(s)) - g(z(s),z(s-\tau),\bar{r}(s))|^2 ds
$$

$$
\begin{aligned}
\le & 4K_2 \int_0^t \mathbb{E}|x(s) - z(s)|^2 ds \\
& + 16K_2 (\max_{1 \le i \le N} (-r_{ii}) + o(1)) H_1(T+\tau)(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta, \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E} \int_0^t |h(x(s), x(s-\tau), r(s)) - h(z(s), z(s-\tau), \bar{r}(s))|^2 ds \\
\le & 4K_3 \int_0^t \mathbb{E}|x(s) - z(s)|^2 ds \\
& + 16K_3 (\max_{1 \le i \le N} (-r_{ii}) + o(1)) H_1(T+\tau)(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta. \qquad (3.9)
\end{aligned}
$$

Proof of Theorem 3.1. For any $t \in [0, T+\tau]$, it follows from (2.1) and (2.5) that

$$
\begin{aligned}
& \mathbb{E}|x(t) - y(t)|^2 \\
\le & 3t\mathbb{E} \int_0^t |f(x(s), x(s-\tau), r(s)) - f(z(s), z(s-\tau), \bar{r}(s))|^2 ds \\
& + 3\mathbb{E} \int_0^t |g(x(s), x(s-\tau), r(s)) - g(z(s), z(s-\tau), \bar{r}(s))|^2 ds \\
& + 6\lambda(1+\lambda t)\mathbb{E} \int_0^t |h(x(s), x(s-\tau), r(s)) - h(z(s), z(s-\tau), \bar{r}(s))|^2 ds. \qquad (3.10)
\end{aligned}
$$

Substituting (3.5), (3.8) and (3.9) into (3.10) that one gains

$$
\begin{aligned}
\mathbb{E}|x(t) - y(t)|^2 \le & \bar{H}_1(T+\tau) \int_0^t \mathbb{E}|x(s) - z(s)|^2 ds + \bar{H}_2(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta \\
\le & 2\bar{H}_1(T+\tau) \int_0^t \mathbb{E}|x(s) - y(s)|^2 ds + 2\bar{H}_1(T+\tau) \int_0^t \mathbb{E}|y(s) - z(s)|^2 ds \\
& + \bar{H}_2(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta,
\end{aligned}
$$

where $\bar{H}_1(T+\tau) = 12[(K_1 + 2\lambda^2 K_3)(T+\tau) + K_2 + 2\lambda K_3]$ and

$$
\bar{H}_2(T+\tau) = 48[(K_1 + 2\lambda^2 K_3)(T+\tau) + K_2 + 2\lambda K_3](\max_{1 \le i \le N} (-r_{ii}) + o(1)) H_1(T+\tau)(T+\tau).
$$

By (3.3), one gets

$$
\begin{aligned}
& \mathbb{E}|x(t) - y(t)|^2 \\
\le & \bar{H}_1(T+\tau) \int_0^t \mathbb{E}|x(s) - z(s)|^2 ds + \bar{H}_2(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta \\
\le & 2\bar{H}_1(T+\tau) \int_0^t \mathbb{E}|x(s) - y(s)|^2 ds + 2\bar{H}_1(T+\tau)(T+\tau) H_2(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta \\
& + \bar{H}_2(T+\tau) \|\eta\|_{\mathbb{E}}^2 \Delta. \qquad (3.11)
\end{aligned}
$$

The assertion (3.1) follows from the Gronwall inequality. $\square$

3.2. Stability equivalence

In this section, we prove that the mean square exponential stability of the continuous EM-solution $y(t)$ is equivalent to that of the exact solution $x(t)$.

Theorem 3.2. *Assume that the SDDEwMJ* (2.1) *is mean square exponentially stable and satisfies* $\mathbb{E}|x(t)|^2 \leq M\|\eta\|_{\mathbb{E}}^2 e^{-\alpha t}$, $\forall t \geq 0$. *Under the condition (**H**), if there exists a step size* Δ *such that*

$$2H(2T-\tau)\Delta + 2Me^{-\alpha(T-\tau)} \leq e^{-\frac{1}{2}\alpha T}, \tag{3.12}$$

then the EM-method is mean square exponentially stable and fulfills

$$\mathbb{E}|y(t)|^2 \leq H_1(T-\tau)e^{\frac{1}{2}\alpha T}\|\eta\|_{\mathbb{E}}^2 e^{-\frac{1}{2}\alpha t}, \tag{3.13}$$

where $H_1(\cdot)$ *is defined as the same as in Lemma 3.1.*

Proof. Let $T = 5(\tau + \lfloor \log(2M)/\alpha \rfloor)$. Then $2Me^{-\alpha(T-\tau)} \leq e^{-\frac{3}{4}\alpha T}$. It follows from Theorem 3.1 that one gains

$$\begin{aligned}\sup_{T-\tau\leq t\leq 2T-\tau} \mathbb{E}|y(t)|^2 &\leq 2\sup_{T-\tau\leq t\leq 2T-\tau} \mathbb{E}|x(t)-y(t)|^2 + 2\sup_{T-\tau\leq t\leq 2T-\tau} \mathbb{E}|x(t)|^2 \\ &\leq 2H(2T-\tau)\|\eta\|_{\mathbb{E}}^2\Delta + 2M\|\eta\|_{\mathbb{E}}^2 e^{-\alpha(T-\tau)}.\end{aligned}$$

One can choose a sufficiently small step size Δ^* such that $2H(2T-\tau)\Delta + 2Me^{-\alpha(T-\tau)} \leq e^{-\frac{1}{2}\alpha T}$ for any $\Delta < \Delta^*$. Therefore,

$$\begin{aligned}\sup_{T-\tau\leq t\leq 2T-\tau} \mathbb{E}|y(t)|^2 \leq e^{-\frac{1}{2}\alpha T}\|\eta\|_{\mathbb{E}}^2 &\leq e^{-\frac{1}{2}\alpha T}\sup_{-\tau\leq t\leq 0} \mathbb{E}|y(t)|^2 \\ &\leq e^{-\frac{1}{2}\alpha T}\sup_{-\tau\leq t\leq \tau} \mathbb{E}|y(t)|^2.\end{aligned} \tag{3.14}$$

By the flow property of the continuous time approximate solution $y(t)$, for $y(t) = y(t; jT, y_{jT})$ $(j = 0, 1, 2, \ldots)$, we repeat the above procedure, one can get

$$\begin{aligned}\sup_{(j+1)T-\tau\leq t\leq (j+2)T-\tau} \mathbb{E}|y(t)|^2 &\leq e^{-\frac{1}{2}\alpha T}\sup_{jT-\tau\leq t\leq jT+\tau} \mathbb{E}|y(t)|^2 \\ &\leq e^{-\frac{1}{2}\alpha T}\sup_{jT-\tau\leq t\leq (j+1)T-\tau} \mathbb{E}|y(t)|^2 \\ &\leq e^{-\frac{1}{2}\alpha(j+1)T}\sup_{-\tau\leq t\leq T-\tau} \mathbb{E}|y(t)|^2.\end{aligned} \tag{3.15}$$

It follows from Lemma 3.1 that one can further gains

$$\begin{aligned}\sup_{(j+1)T-\tau\leq t\leq (j+2)T-\tau} \mathbb{E}|y(t)|^2 &\leq e^{-\frac{1}{2}\alpha(j+1)T}H_1(T-\tau)\|\eta\|_{\mathbb{E}}^2 \\ &\leq H_1(T-\tau)e^{\frac{1}{2}\alpha T}\|\eta\|_{\mathbb{E}}^2 e^{-\frac{1}{2}\alpha t}.\end{aligned} \tag{3.16}$$

Using Lemma 3.1 again, one gets

$$\sup_{-\tau\leq t\leq T-\tau} \mathbb{E}|y(t)|^2 \leq H_1(T-\tau)\|\eta\|_{\mathbb{E}}^2 \leq H_1(T-\tau)e^{\frac{1}{2}\alpha T}\|\eta\|_{\mathbb{E}}^2 e^{-\frac{1}{2}\alpha t}. \tag{3.17}$$

In summary, for $\forall t \geq 0$, according to the (3.16) and (3.17) that one can obtain

$$\mathbb{E}|y(t)|^2 \leq H_1(T-\tau)e^{\frac{1}{2}\alpha T}\|\eta\|_{\mathbb{E}}^2 e^{-\frac{1}{2}\alpha t}.$$

The proof is therefore complete. $\square$

Theorem 3.3. *Assume that the EM-method on the SDDEwMJ* (2.1) *is mean square exponentially stable and satisfies* $\mathbb{E}|y(t)|^2 \le N\|\eta\|^2_{\mathbb{E}} e^{-\beta t}$, $\forall t \ge 0$. *Under the condition (***H***), if there exists a step size* Δ *such that*

$$2H(2T-\tau)\Delta + 2Ne^{-\beta(T-\tau)} \le e^{-\frac{1}{2}\beta T}, \tag{3.18}$$

then the SDDEwMJ (2.1) *is mean square exponentially stable and satisfies*

$$\mathbb{E}|x(t)|^2 \le H_1(T-\tau)e^{\frac{1}{2}\beta T}\|\eta\|^2_{\mathbb{E}} e^{-\frac{1}{2}\beta t}, \tag{3.19}$$

where $H_1(\cdot)$ *is also defined as the same as in Lemma 3.1.*

Proof. Let $T = 5(\tau + \lfloor \log(2N)/\beta \rfloor)$. Then $2Ne^{-\beta(T-\tau)} \le e^{-\frac{3}{4}\beta T}$. According to Theorem 3.1, one derives

$$\begin{aligned}\sup_{T-\tau\le t\le 2T-\tau} \mathbb{E}|x(t)|^2 &\le 2 \sup_{T-\tau\le t\le 2T-\tau} \mathbb{E}|x(t)-y(t)|^2 + 2 \sup_{T-\tau\le t\le 2T-\tau} \mathbb{E}|y(t)|^2 \\ &\le 2H(2T-\tau)\|\eta\|^2_{\mathbb{E}}\Delta + 2N\|\eta\|^2_{\mathbb{E}} e^{-\beta(T-\tau)}.\end{aligned}$$

One can choose a small enough step size Δ^{**} such that $2H(2T-\tau)\Delta + 2Ne^{-\beta(T-\tau)} \le e^{-\frac{1}{2}\beta T}$ for any $\Delta < \Delta^{**}$. Therefore,

$$\begin{aligned}\sup_{T-\tau\le t\le 2T-\tau} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}\beta T}\|\eta\|^2_{\mathbb{E}} &\le e^{-\frac{1}{2}\beta T} \sup_{-\tau\le t\le 0} \mathbb{E}|x(t)|^2 \\ &\le e^{-\frac{1}{2}\beta T} \sup_{-\tau\le t\le \tau} \mathbb{E}|x(t)|^2.\end{aligned} \tag{3.20}$$

The remaining proof is similar to Theorem 3.2, thus one can get the required result. The proof is complete. □

According to Theorems 3.2 and 3.3, one can derive the following equivalence theorem.

Theorem 3.4. *Under the condition (***H***), the SDDEwMJ* (2.1) *is mean square exponentially stable if and only if for the sufficiently small step size* Δ*, the EM-method applied to this equation is mean square exponentially stable.*

Remark 3.2. It follows from Theorem 3.4 that one can study the mean square exponential stability of a SDDEwMJ by the EM-method in the absence of an appropriate Lyapunov function under some conditions. EM-method is a classical numerical method. Based on the result in Theorem 3.4, it can be extended to other numerical methods(e.g. Backward EM-method, θ-method), one therefore can investigate the stability equivalence between the SDDEwMJs and other numerical methods. Moreover, the equations considered in [3, 8, 9, 15, 19, 30] are either the SDEs or the SDDEs, results obtained in this paper generalize the results in these papers.

4. A numerical experiment

In this section, an example is provided to illustrate the obtained results. Let $\omega(t)$ be a scalar Brownian motion and $N(t)$ be a Poisson process with intensity $\lambda = 1$. Let $r(t)$ be a right-continuous Markov chain taking values $S = \{1, 2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Of course, $\omega(t)$, $r(t)$ and $N(t)$ are assumed to be independent of each other.

Consider the following SDDEwMJ of the form

$$\begin{aligned} dx(t) =& f(x(t), x(t-\tau), r(t))dt + g(x(t), x(t-\tau), r(t))d\omega(t) \\ &+ h(x(t), x(t-\tau), r(t))dN(t),\ t \geq 0, \end{aligned} \tag{4.1}$$

with initial value $x_0 = 1$, where $\tau = 1$,

$$\begin{aligned} &f(x(t), x(t-\tau), 1) = -x(t), \quad f(x(t), x(t-\tau), 2) = -2x(t) + \frac{1}{8}\sin(x(t-\tau)), \\ &g(x(t), x(t-\tau), 1) = \frac{1}{4}\sin(x(t-\tau)), \quad g(x(t), x(t-\tau), 2) = \frac{1}{8}x(t-\tau), \\ &h(x(t), x(t-\tau), 1) = -\frac{1}{4}\sin(x(t)) - \frac{1}{16}x(t-\tau), \\ &h(x(t), x(t-\tau), 2) = -\frac{1}{16}\sin(x(t)). \end{aligned}$$

Obviously, $f(0,0,i) = g(0,0,i) = h(0,0,i) = 0$ for any $i \in S$. It is easy to check that the condition ($\mathbf{H}$) is satisfied with $K_1 = 8$, $K_2 = 1/16$, $K_3 = 1/8$. Also, one can compute that

$$\begin{aligned} &\text{if} \quad i=1, \ \ 2x(t)(-x(t)) + \frac{1}{16}\sin^2(x(t-\tau)) \leq -2x(t)^2 + \frac{1}{16}x^2(t-\tau), \\ &\text{if} \quad i=2, \ \ 2x(t)(-2x(t)+\frac{1}{8}\sin(x(t-\tau)))+\frac{1}{64}x^2(t-\tau) \leq -\frac{31}{8}x^2(t)+\frac{9}{64}x^2(t-\tau). \end{aligned} \tag{4.2}$$

Then, it follows from Theorem 3.1 in [13] that one can similarly obtain the (4.1) is mean square exponentially stable.

On the other hand, based on the EM-method (2.4), we will show the mean square stability of numerical solution and calculate the Lyapunov exponent of Eq. (4.1) with the initial value $x_0 = 1$ and $r_0 = 1$, respectively. Based on the idea from [7, Chapter 4], one can approximate the zero-one jump law through the acceptance-rejection method under small time step-size Δ. To do this, we set $\Delta = 0.1$. All the figures are drawn by the mean square data coming from 500 sample paths, that is, $\mathbb{E}|y(t_k)|^2 \approx \frac{1}{500}\sum_{i=1}^{500} |y^i(t_k)|^2$. The corresponding figures are shown in Fig. 1 and Fig. 2.

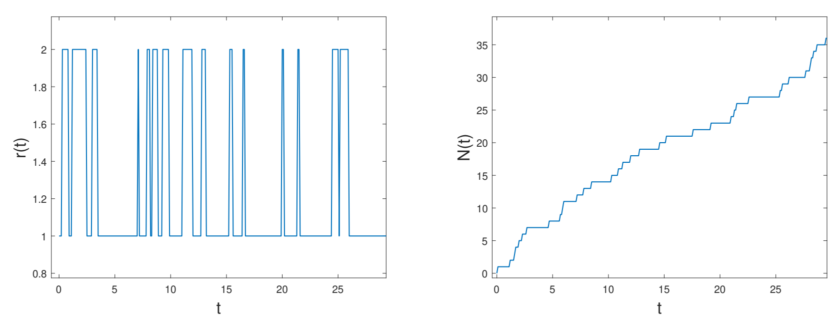

Figure 1. The numerical simulation of the $r(t)$ (left) and the $N(t)$ (right).

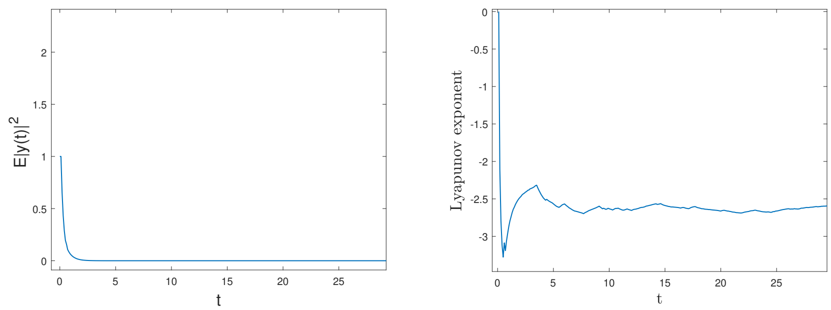

Figure 2. The mean square curves of numerical solution (left) and the Lyapunov exponent (right) of Eq. (4.1).

5. Conclusion

In this research, for hybrid stochastic delay differential equations with jumps, we investigate the equivalence between the mean square exponential stability of the underlying equation and the proposed numerical method. This bridge the gap between the exponential stability of the exact and numerical solutions for hybrid stochastic delay differential equations with jumps. It is well known that Lyapunov functions method is the classical and powerful technique in the study of stochastic stability, but in some cases, it is very difficult to construct an appropriate Lyapunov function. In the absence of Lyapunov function method, the theory established in this paper enables us to study mean square exponential stability of hybrid stochastic delay differential equations with jumps using the numerical method, without resorting to Lyapunov functions technique. Therefore, we can now carry out careful numerical simulations using the EM-method with the sufficiently small step size to simulate the solutions of hybrid stochastic delay differential equations with jumps, so as to study its stability. In the future, we will continue to study the equivalence between the asymptotical stability of the neutral hybrid stochastic delay differential equations with jumps under the non-Lipschitz condition.

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