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ON THE WELL-POSEDNESS OF THE STOCHASTIC 2D PRIMITIVE EQUATIONS*

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Abstract Under non-Lipschitz conditions for the external force term and noise term, the two-dimensional stochastic primitive equations are studied in this paper. Based on Galerkin method, iterative method and the moment estimations, we prove the existence and uniqueness of the solutions in a fixed probability space.

Keywords Primitive equations, existence and uniqueness, non-Lipschtiz conditions.

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1. Introduction

We consider the system of 2D viscous Primitive Equations (PE) for three dimensional Geophysical Fluid Dynamics in the two dimensional spatial domain [30]. It is well known that the system of Primitive Equations is derived from the Navier-Stokes equations, coupled with the thermodynamic equations and the diffusion equations, replacing vertical momentum balance with a simple static equation, and it is used as the fundamental model of meteorology and geophysical fluid dynamics [3, 28].

The mathematical study of primitive equations was started by Lions, Temam and Wang [24–26], where the notions of weak and strong solutions were defined and existence of weak solutions was proved. Since then, the well-posedness and regularity of strong solutions with different conditions have been studied, such as [2,14,16–21,29,30]. There exists an unresolved mathematical problem for viscous PE is about uniqueness of weak solutions, by introducing the notion of "z-weak" solution to 3D (or 2D) viscous PE, we have some results about uniqueness, see [18,20,23,32].

Due to the influence of external force and internal instability process, white noise driven random term was added to the basic control equations. Research [27] showed that these random terms meet the basic physical principles. In the past two decades, there are numerous works about the stochastic primitive equations, we mention some of them. For the well-posedness, regularity, random attractor and existence and regularity of invariant measures, we refer the reader to the papers [4–12, 15, 31]. For the deviation principles and small time asymptotics of the primitive

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equations, see [6, 7]. As well as we know, above results of the primitive equations are based on the Lipschitz conditions for the external force term and noise term. But, these conditions are so strong that they are not appropriate to reflect all problems. This paper will investigate 2D stochastic primitive equations with non-Lipschitz conditions which is a much weaker condition than Lipschitz one. The non-Lipschitz stochastic evolution equations have been considered by many authors, such as [1, 34]. Due to complex nonlinear term of primitive equations, the previous results cannot be applied here. Furthermore, it is known that the well-posedness of solutions to the stochastic non-Lipschitz Navier-Stokes equations was obtained by Taniguchi [33]. In this paper, we follow the lines of [33]. The difference between the primitive equations and the Navier-Stokes equations is that the nonlinear term of the primitive equations is more complicated, we want to fill the gap of well-posedness under non-Lipschitz conditions for primitive equations. The main work is to deal with nonlinear terms and study the well-posedness of solutions which closely related to the notion of z-weak solutions.

This paper is organized as follows. Firstly, the preliminaries are given in §2. We introduce the model, the related function spaces and some properties of operators. Some assumptions are also provided. Secondly, in §3, we construct the auxiliary stochastic primitive equations, in which external force term and noise term are determined, then using Galerkin method to prove the existence and uniqueness of the functions. On the basis of the conclusions in §3, combining iterative method and moment estimations, the existence and uniqueness of solutions on local time are obtained in §4. Finally, the global existence of solutions is considered in §5.

2. Preliminaries

We consider the following two dimensional stochastic primitive equations [10, 13]:

$$\partial_t u - \nu\Delta u + u\partial_x u + w\partial_z u + \partial_x p = f + g(t,u)\dot{W}(t), \tag{2.1}$$

$$\partial_x u + \partial_z w = 0, \tag{2.2}$$

in the bounded domain $\mathcal{M} = \{(x,z)|0 \le x \le L, -h \le z \le 0\}$, where L, h are constants. We denote by $(u,w), p$ the unknown the field of the flow and the pressure respectively. Note that p does not depend on the vertical variable z. In this paper, in order to focus main attention on the difficulties arising from the nonlinear term, we ignore the temperature and salinity equations.

The boundary is divided into the top $\Gamma_i = \{z = 0\}$, the bottom $\Gamma_b = \{z = -h\}$ and the sides $\Gamma_s = \{x = 0\} \cup \{x = L\}$. The following boundary conditions are proposed:

$$\begin{aligned} \text{on } \Gamma_s \quad &: \quad u = 0, \\ \text{on } \Gamma_i \cup \Gamma_b \quad &: \quad \partial_z u = 0, w = 0. \end{aligned}$$

Generally, we make on further assumptions (see [13]):

$$\int_{-h}^{0} f dz = 0, \int_{-h}^{0} g dz = 0, \int_{-h}^{0} u dz = 0.$$

From (2.2) we have $w(x,z) = -\int_{-h}^{z} \partial_x u(x,\tilde{z})d\tilde{z}$. We will be working on the Hilbert spaces:

$$H = \Big\{u \in L^2(\mathcal{M}) \mid \int_{-h}^{0} u dz = 0\Big\},$$
$$V = \Big\{u \in H^1(\mathcal{M}) \mid \int_{-h}^{0} u dz = 0, u = 0 \text{ on } \Gamma_s\Big\}.$$

These spaces are endowed with the L^2 and H^1 norms which we respectively denote by $|\cdot|$ and $|\cdot|_V$. The inner product and norms on H, V are given by $(u,v) = \int_{\mathcal{M}} uvdxdz$, and $|u| = (u,u)^{\frac{1}{2}}$, $|u|_V = (|u|^2 + \|u\|^2)^{\frac{1}{2}}$, where $u, v \in H$ and $\|u\| = |\nabla u| = (\nabla u, \nabla u)^{\frac{1}{2}}$. Let V^* be the dense and continuous embedding $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$ and denote by $\langle u, \psi\rangle$ the duality between $u \in V$ and $\psi \in V^*$. Consider an unbounded linear operator $A : D(A) \to H$ with $D(A) = V \cap H^2(\mathcal{M})$ and define

$$\langle Au, v\rangle = (\nabla u, \nabla v),\ \forall\, u, v \in D(A).$$

The Stokes-type operator A is self-adjoint and positive, with compact self-adjoint inverse. Next we address the nonlinear term. Take $\mathcal{W}(v) = -\int_{-h}^{z} \partial_x v(x,\tilde{z})d\tilde{z}$ and $B(u,v) = u\partial_x v + \mathcal{W}(u)\partial_z v$, where $u, v \in V$.

Define the bilinear operator $B(u,v) : V \times V \to V^*$ according to $\langle B(u,v), w\rangle = b(u,v,w)$, where $b(u,v,w) = \int_{\mathcal{M}} (u\partial_x vw + \mathcal{W}(u)\partial_z vw)d\mathcal{M}$. In the sequel, when no confusion arises, we denote by C a constant which may change from one line to the next one.

Lemma 2.1 (see [13,31])**.** *The trilinear forms b and B have the following properties. There exists a constant $C > 0$ such that*

$$|b(u,v,w)| \leq C\Big(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}} + |\partial_x u||\partial_z v||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}\Big),\quad u, v, w \in V, \tag{2.3}$$
$$b(u,v,v) = 0, \qquad u, v \in V, \tag{2.4}$$
$$\langle B(u,u), \partial_{zz} u\rangle = 0, \qquad u \in V. \tag{2.5}$$

For $u \in V$, define $E(u) = -Au - B(u)$. We obtain the monotonicity property of E.

Lemma 2.2 (see [31])**.** *Assume that $u, v \in V$, we have*

$$\langle E(u) - E(v), u - v\rangle + \frac{1}{2}\|u - v\|^2 \leq C|u - v|\|u - v\|\|v\| + C(1 + |\partial_z v|^4)|u - v|^2.$$

Let K be another separable Hilbert space with the inner product $(\cdot,\cdot)_K$. Let $L(K;H)$ denote the space of all bounded linear operators from K to H. Let $Q \in L(K;K)$ be a positive self-adjoint operator. Furthermore, $L_2^0(K;H)$ denotes the space of all $\xi \in L(K;H)$ such that $\xi\sqrt{Q}$ is a Hilbert-Schimdt operator and so $tr(\xi Q \xi^*) < \infty$. The norm is given by $|\xi|^2_{L_2^0} = |\xi\sqrt{Q}|^2_{HS} = tr(\xi Q \xi^*)$.

Let $(\Omega, P, \mathscr{F})$ be a complete probability space on which an increasing and right continuous family $(\mathscr{F}_t)_{t\in[0,\infty]}$ of complete sub-σ-algebra of $\mathscr{F}$ is defined. $\mathscr{F}_0$ contains all the null sets of $\mathscr{F}$. Let $e_n (n = 1, 2, 3...)$ be a complete orthonormal basis in K. We consider a K-valued cylindrical Wiener process $W(t)$ given by the following series:

$$W(t) = \sum_{n=1}^{\infty} \beta_n(t)\sqrt{Q}e_n, \quad t \geq 0, Q \in L(K;K).$$

Let u_0 be an $\mathscr{F}_0$-random variable. The stochastic 2D primitive equations can be rewritten in the abstract mathematical setting with an initial value $u(0) = u_0$ as follows:

$$du(t) + [\nu Au(t) + B(u(t))]dt = f(t, u(t))dt + g(t, u(t))dW(t). \tag{2.6}$$

In this paper we use the following conditions.

Assumption 1. There exist the functions $F_k(t,u), H_k(t,u) : R^+ \times R^+ \to R^+ (k = 1, 2)$ such that they are locally integrable in $t \geq 0$ for any fixed $u \geq 0$ and continuous, monotone nondecreasing in u for any fixed $t \geq 0$ with $F_k(t, 0) = H_k(t, 0) = 0$. The following inequalities are satisfied:

$$\mathbb{E}|f(t,u)|^{2k}_{V^*} + \mathbb{E}|g(t,u)|^{2k}_{L^0_2} \leq F_k(t, \mathbb{E}|u|^{2k}), \qquad u \in L^{2k}(\Omega; H), \tag{2.7}$$

$$\mathbb{E}|\partial_z f(t,u)|^{2k}_{V^*} + \mathbb{E}|\partial_z g(t,u)|^{2k}_{L^0_2} \leq H_k(t, \mathbb{E}|\partial_z u|^{2k}), \quad \partial_z u \in L^{2k}(\Omega; H). \tag{2.8}$$

Assumption 2. There exist the functions $G_k(t,u) : R^+ \times R^+ \to R^+ (k = 1, 2)$ which are locally integrable in $t \geq 0$ for any fixed $u \geq 0$ and continuous, monotone nondecreasing in u for any fixed $t \geq 0$ with $G_k(t, 0) = 0, k = 1, 2$. Furthermore, the functions $G_k (k = 1, 2)$ satisfy the following inequalities:

$$\mathbb{E}|f(t,u) - f(t,v)|^{2k}_{V^*} + \mathbb{E}|g(t,u) - g(t,v)|^{2k}_{L^0_2} \leq G_k(t, \mathbb{E}|u - v|^{2k}). \tag{2.9}$$

If for any given constants $C_k \geq 0$, non-negative functions $z_k(t)$ satisfy that $z_k(0) = 0$ and

$$z_k(t) \leq C_k \int_0^t G_k(s, z_k(s))ds, \quad k = 1, 2,$$

for all $t \in R^+$, then $z_k(t) = 0$ on R^+.

3. The existence and uniqueness of the auxiliary equations

Let $0 \leq t \leq T \leq 1$, we study the following stochastic differential equation:

$$\begin{aligned} u(t) = \ & u_0 + \int_0^t [-\nu Au(s) - B(u(s))]ds \\ & + \int_0^t f_*(s, \xi(s))ds + \int_0^t g_*(s, \xi(s))dW(s), \end{aligned} \tag{3.1}$$

with an initial $\Im_0$-random variable u_0, where $\xi(s)$ is a stochastic process, $f_* : [0, \infty] \times V \to V^*$ and $g_* : [0, \infty] \times H \to L^0_2(K; H)$ are both progressively measurable. Now we consider the Galerkin approximation to (3.1) as follows:

$$\begin{aligned} u_n(t) = P_n u_0 & + \int_0^t [-\nu Au_n(s) - P_n B(u_n(s))]ds \\ & + \int_0^t P_n f_*(s, \xi(s))ds + \int_0^t P_n g_*(s, \xi(s))dW(s). \end{aligned} \tag{3.2}$$

Assumption 3. Let $0 \le t \le T$ and $\xi(t)$ be a stochastic process and satisfy the following conditions:

$$\begin{aligned}
&f_*(t,\xi(t)) \in L^2([0,T]\times\Omega;V^*)\cap L^4([0,T]\times\Omega;V^*),\\
&g_*(t,\xi(t)) \in L^2([0,T]\times\Omega;L_2^0(K;H))\cap L^4([0,T]\times\Omega;L_2^0(K;H)),\\
&\partial_z f_*(t,\xi(t)) \in L^2([0,T]\times\Omega;V^*)\cap L^4([0,T]\times\Omega;V^*),\\
&\partial_z g_*(t,\xi(t)) \in L^2([0,T]\times\Omega;L_2^0(K;H))\cap L^4([0,T]\times\Omega;L_2^0(K;H)).
\end{aligned}$$

Note that for $\psi \in V$, the map $u_n \longmapsto \langle -\nu A u_n, \psi\rangle$ is globally Lipschitz, while using Lemma 2.1, the map $B(u_n)$ is locally Lipschitz. Furthermore, because $\xi(s)$ is a given function and f_*, g_* are unrelated with u_n. Hence by a well-posedness result for stochastic ordinary differential equations [22], there exists a solution $u_n(t)$ to (3.2) and satisfies

$$\begin{aligned}
&d\langle u_n,\psi\rangle + \langle \nu A u_n + P_n B(u_n),\psi\rangle dt = \langle P_n f_*,\psi\rangle dt + \langle P_n g_* dW(t),\psi\rangle,\\
&u_n(0) = P_n u_0.
\end{aligned}$$

We next establish some uniform a priori estimates on u_n (independent of n) in the following lemmas.

Lemma 3.1. *Let u_0 be an initial value with $\mathbb{E}|u_0|^{2p}, \mathbb{E}|\partial_z u_0|^{2p} < \infty$, $(p = 1,2)$. Suppose that Assumption 3 is satisfied. Then for the solution $u_n(t)$ to (3.2), there exists a constant $K_i > 0$ $(i = 1,2)$ such that*

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t}|u_n(s)|^{2p}\Big) + \mathbb{E}\int_0^t |u_n(s)|^{2(p-1)}\|u_n(s)\|^2 ds \le K_1,\\
&\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u_n(s)|^{2p}\Big) + \mathbb{E}\int_0^t |\partial_z u_n(s)|^{2(p-1)}\|\partial_z u_n(s)\|^2 ds < K_2,
\end{aligned}$$

uniformly in all $n \ge 1$.

Lemma 3.2. *Let u_0 be an initial value with $\mathbb{E}|u_0|^4 < \infty$. Then the solution to (3.2) satisfies*

$$\mathbb{E}\Big(\int_0^t \|u_n(s)\|^2 ds\Big)^2 \le C(|f_*|^2_{L^2(\Omega;L^2(0,T;V^*))}, |g_*|^2_{L^2(\Omega;L^2(0,T;L_2^0))})\mathbb{E}|u_0|^4.$$

Uniform estimates of Lemmas 3.1-3.2 are similar as the proofs of Lemmas 4.1-4.5, even more easier. We give only the sketch proofs of Lemma 3.1 for $|u_n(t)|^{2p}$ and Lemma 3.2.

Proof of Lemma 3.1 for $|u_n(t)|^{2p}$. Using Itô formula for $|u_n(s)|^{2p}$, we deduce that

$$\begin{aligned}
&|u_n(s)|^{2p} + 2p\nu\int_0^t \|u_n\|^2|u_n|^{2(p-1)}ds\\
=&|u_n(0)|^{2p} + 2p\int_0^t \langle P_n f_*, u_n\rangle |u_n|^{2(p-1)}ds + p(2p-1)\int_0^t |P_n g_*|^2_{L_2^0}|u_n|^{2(p-1)}ds\\
&+ 2p\int_0^t |u_n|^{2(p-1)}\langle u_n, P_n g_* dW(s)\rangle
\end{aligned}$$

$$=|u_n(0)|^{2p} + J_1 + J_2 + J_3. \tag{3.3}$$

For the deterministic term, we estimate

$$\begin{aligned}
J_1 \leq & C\int_0^t |P_n f_*|^2_{V^*}|u_n|^{2(p-1)}ds + \nu p\int_0^t \|u_n\|^2|u_n|^{2(p-1)}ds \\
\leq & C\Big(\sup_{s\in[0,t]}|u_n|^{2(p-1)}\Big)\int_0^t |P_n f_*|^2_{V^*}ds \\
& + \nu p\int_0^t \|u_n\|^2|u_n|^{2(p-1)}ds \\
\leq & \frac{1}{6}\sup_{s\in[0,t]}(|u_n(s)|^{2p}) + \nu p\int_0^t \|u_n\|^2|u_n|^{2(p-1)}ds + C|f_*|^{2p}_{L^2(0,t;V^*)}, \qquad (3.4)\\
J_2 \leq & C\int_0^t |P_n g_*|^2_{L_2^0}|u_n|^{2(p-1)}ds \\
\leq & C\Big(\sup_{s\in[0,t]}|u_n|^{2(p-1)}\Big)\int_0^t |P_n g_*|^2_{L_2^0}ds \\
\leq & \frac{1}{6}\sup_{s\in[0,t]}(|u_n(s)|^{2p}) + C|P_n g_*|^{2p}_{L^2(0,t;L_2^0)}. \qquad (3.5)
\end{aligned}$$

For the term J_3, we apply the Burkholder-Davis-Gundy inequality. This yields the following:

$$\begin{aligned}
J_3 \leq & C\mathbb{E}\sup_{s\in[0,t]}\Big|\int_0^t |u_n|^{2(p-1)}\langle u_n, P_n g_* dW(s)\rangle\Big| \\
\leq & C\mathbb{E}\Big(\int_0^t |u_n|^{2(p-1)}|P_n g_*|^2_{L_2^0}ds\Big)^{\frac{1}{2}} \\
\leq & \frac{1}{6}\sup_{s\in[0,t]}(|u_n(s)|^{2p}) + C(|P_n g_*|^{2p}_{L^2(0,t;L_2^0)} + 1). \qquad (3.6)
\end{aligned}$$

Combining (3.4)–(3.6), we can easily obtain

$$\mathbb{E}\Big(\sup_{0\leq s\leq t}|u_n(s)|^{2p}\Big) + \mathbb{E}\int_0^t |u_n(s)|^{2(p-1)}\|u_n(s)\|^2 ds \leq K_1.$$

□

Proof of Lemma 3.2. By (3.3) for $p = 1$, we get

$$\begin{aligned}
4\nu^2\Big(\int_0^t \|u_n(s)\|^2 ds\Big)^2 \leq & \Big(|u_0|^2 + 2\int_0^t |\langle u_n(s), P_n f_*\rangle| ds + \int_0^t |P_n g_*|^2_{L_2^0}ds \\
& + 2\Big|\int_0^t \langle u_n, P_n g_* dW(s)\rangle\Big|\Big)^2 \\
\leq & 4(|u_0|^4 + I_1^2 + I_2^2 + I_3^2).
\end{aligned}$$

The Hölder's inequality and Young's inequality imply that

$$I_1^2 \leq \frac{\nu^2}{2}\Big(\int_0^t \|u_n(s)\|^2 ds\Big)^2 + \frac{8}{\nu^2}(|f_*|^2_{L^2(0,T;V^*)})^2.$$

Applying the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}|I_3^2|\Big) &\le C\mathbb{E}\Big(\int_0^t |u_n(s)|^2|P_*g|_{L_2^0}^2 ds\Big)\\ &\le C\mathbb{E}\Big[\Big(\sup_{0\le s\le t}|u_n(s)|^2\Big)\int_0^t |P_n g_*|_{L_2^0}^2 ds\Big]\\ &\le C\mathbb{E}\Big(\sup_{0\le s\le T}|u_n(s)|^4\Big)+C(|g_*|^2_{L^2(0,T;L_2^0)})^2.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{E}\Big(\int_0^t \|u_n(s)\|^2 ds\Big)^2 \le C\mathbb{E}\Big(&|u_0|^4+\sup_{0\le s\le t}|u_n(s)|^4+(|f_*|^2_{L^2(0,T;V^*)})^2\\ &+(|g_*|^2_{L^2(0,T;L_2^0)})^2\Big).\end{aligned}$$

By Lemma 3.1, the above formula means

$$\mathbb{E}\Big(\int_0^t \|u_n(s)\|^2 ds\Big)^2 \le C(|f_*|^2_{L^2(\Omega;L^2(0,T;V^*))},|g_*|^2_{L^2(\Omega;L^2(0,T;L_2^0))})\mathbb{E}|u_0|^4.$$

Then the proof of the lemma is complete. □

With the uniform estimates on the solutions of the Galerkin systems in hands, we proceed to identify a limit $u(t)$ and obtain the following proposition.

Proposition 3.1. *Let u_0 is an initial value of (3.2) with*

$$\mathbb{E}|u_0|^{2p}<\infty,\mathbb{E}|\partial_z u_0|^{2p}<\infty, p=1,2.$$

There exists a unique solution $u(t)$ to (3.2) in

$$u(t)\in L^4(\Omega;L^\infty(0,T;H)\cap L^2(\Omega\times[0,T];V)),$$

and it satisfies the following energy equality:

$$\begin{aligned}|u(t)|^2=|u_0|^2&+2\int_0^t\langle u(s),-\nu Au(s)-B(u(s))\rangle ds\\ &+2\int_0^t\langle u(s),f_*(s,\xi(s))\rangle ds+\int_0^t |g_*(s,\xi(s))|^2_{L_2^0}ds\\ &+2\int_0^t\langle u(s),g_*(s,\xi(s))dW(s)\rangle.\end{aligned}$$

Proof. By Lemma 3.1, we obtain that the subsequence $u_n(t)$ converges weakly to $u(t)$ in $L^4(\Omega;L^\infty(0,T;H)\cap L^2(\Omega\times[0,T];V))$. By an application of (2.3) and Lemmas 3.1-3.2,

$$\begin{aligned}&\mathbb{E}\int_0^T |P_nB(u_n(t))|^2_{V_*}\\ \le& C\mathbb{E}\Big[\sup_{t\in[0,T]}(|u_n(t)|^4+|\partial_z u_n(t)|^4)+\Big(\int_0^T\|u_n(t)\|^2ds\Big)^2\Big]<\infty.\end{aligned}$$

Furthermore since $\{Au_n\}, \{B(u_n)\}$ are uniformly bounded in $L^2(\Omega \times [0,T]; V^*)$, there exists a $\chi \in L^2(\Omega \times [0,T]; V^*)$ such that as $n \to \infty$,

$$-\nu Au_n - B(u_n) \rightharpoonup \chi \ \text{ weakly in } L^2(\Omega \times [0,T]; V^*).$$

We have that in V^*

$$u(t) = u_0 + \int_0^t \chi(s)ds + \int_0^t f_*(s, \xi(s))ds + \int_0^t g_*(s, \xi(s))dW(s).$$

According to Lemma 2.2, we use the Young inequality to get $\kappa > 0$ such that for any $u, v \in V$,

$$\langle E(u) - E(v), u - v\rangle \le \kappa(1 + |\partial_z v|^4 + \|v\|^2)|u - v|^2. \tag{3.7}$$

Given a function ω and let

$$\rho(t) = \int_0^t (1 + |\partial_z \omega|^4 + \|\omega\|^2)ds.$$

Then, we have that

$$\begin{aligned}\mathbb{E}e^{-2\kappa\rho(t)}|u(t)|^2 = \mathbb{E}|u_0|^2 - 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u(s)|^2ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle u(s), \chi(s)\rangle ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle u(s), f_*(s, \xi(s)))\rangle ds \\ + \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g_*(s, \xi(s)))|^2_{L_2^0}ds.\end{aligned}$$

By the Itô formula from (3.2) we also have

$$\begin{aligned}\mathbb{E}e^{-2\kappa\rho(t)}|u_n(t)|^2 = \mathbb{E}|u_n(0)|^2 - 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u_n(s)|^2ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle u_n(s), -\nu Au_n(s) - P_nB(u_n(s))\rangle ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle u_n(s), P_nf_*(s, \xi(s)))\rangle ds \\ + \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|P_ng_*(s, \xi(s))|^2_{L_2^0}ds.\end{aligned}$$

It is not difficult to get

$$\begin{aligned}&\mathbb{E}e^{-2\kappa\rho(t)}|u_n(t)|^2 - \mathbb{E}|u_n(0)|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle P_nf_*(s, \xi(s)), u_n(s)\rangle ds \\ &- \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|P_ng(s, \xi(s))|^2_{L_2^0}ds \\ &= -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u_n(s)|^2ds\end{aligned}$$

$$+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle -\nu A u_n(s) - P_n B(u_n(s)), u_n(s)\rangle ds.$$

According to Fatou lemma, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\Big(- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u_n(s)|^2 ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle -\nu A u_n(s) - B(u_n(s)), u_n(s)\rangle ds\Big) \\
&= \liminf_{n\to\infty}\Big(\mathbb{E}e^{-2\kappa\rho(t)}|u_n(t)|^2 - \mathbb{E}|u_n(0)|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle P_n f_*(s,\xi(s)), u_n(s)\rangle ds \\
&- \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|P_n g_*(s,\xi(s))|^2_{L_2^0} ds\Big) \\
&\geq \mathbb{E}e^{-2\kappa\rho(t)}|u(t)|^2 - \mathbb{E}|u_0|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f_*(s,\xi(s)), u_n(s)\rangle ds \\
&- \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g_*(s,\xi(s))|^2_{L_2^0} ds \\
&= -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)|^2 ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s), u(s)\rangle ds.
\end{aligned}$$

By (3.7)

$$\begin{aligned}
&2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s) - E(\omega(s)), u_n(s) - \omega(s)\rangle ds \\
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u_n(s)-\omega(s)|^2 ds \leq 0.
\end{aligned}$$

Namely

$$\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u_n(s)|^2 ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s)), u_n(s)\rangle ds \\
&\leq -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)(2u_n(s)-\omega(s), \omega(s)) ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s)), \omega(s)\rangle ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(z(s)), u_n(s) - \omega(s)\rangle ds.
\end{aligned}$$

Letting $n \to \infty$ and combining above formula, we get

$$- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)|^2 ds$$

$$
\begin{aligned}
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s), u(s)\rangle ds \\
&\leq -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)(2u(s)-\omega(s),\omega(s))ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s), \omega(s)\rangle ds \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(\omega(s)), u(s)-\omega(s)\rangle ds.
\end{aligned}
$$

Reorganizing the terms, we have

$$
\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)-\omega(s)|^2 \\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s)-E(\omega(s)), u(s)-\omega(s)\rangle ds \leq 0.
\end{aligned}
$$

Set $\varphi(s)$ is arbitrary. If we note $\omega(s) = u(s) - \mu\varphi(s), \mu > 0$, we have

$$
\begin{aligned}
&- \kappa\mu\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|\varphi(s)|^2 \\
&+ \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s)-E(u(s)-\mu\varphi(s)), \varphi(s)\rangle ds \leq 0.
\end{aligned}
$$

Letting $\mu \to 0$, we obtain

$$
\int_0^t \chi(s)ds = \int_0^t -\nu Au(s) - B(u(s))ds.
$$

Therefore it holds that in V^*

$$
u(t) = u_0 + \int_0^t -Au(s) - B(u(s))ds + \int_0^t f(s,u(s))ds + \int_0^t g(s,u(s))dW(s).
$$

Finally, the uniqueness of solution is similar as in Theorem4.1, we omit here. □

4. The proof of the main theorem

In this section, using the Proposition 3.1, we obtain the existence and uniqueness of the weak solution on local time to the non-Lipschitz 2D stochastic primitive equations as follows.

Theorem 4.1. *Let u_0 be a $\mathscr{F}_0$-random variable which is an initial value of (2.6). Suppose that Assumptions 1-2 are satisfied and let $\mathbb{E}|u_0|^{2p}, \mathbb{E}|\partial_z u_0|^{2p} < \infty (p = 1, 2)$. Then there exist a time $T_\epsilon > 0$ and a weak energy solution $u(t)$ to (2.6) satisfy*

$$
u(t) \in L^2(\Omega; L^2(0, T_\epsilon; V)) \cap L^2(\Omega; L^\infty([0, T_\epsilon]; H))
$$

and the solution $u(t)$ is unique in this space.

To prove the Theorem 4.1, we use iterative method, several moment estimations are required. Let $0 \le t \le 1$. Set $u^1(t) = u_0$ and let $E|u_0|^{2p}, E|\partial_z u_0|^{2p} < \infty$ for $p = 1, 2$. Assume that the process $u^n(s), n \ge 1$ satisfies

$$f(t, u^n(t)) \in L^4(\Omega \times [0,T]; V^*),$$
$$g(t, u^n(t)) \in L^4(\Omega \times [0,T]; L_2^0(K; H)).$$

Proposition 3.1 implies that for a given process $u^n(t)$, we have the unique energy weak solution $u^{n+1}(t)$ to the following stochastic equation

$$u^{n+1}(t) = u_0 + \int_0^t [-\nu A u^{n+1}(s) - B(u^{n+1}(s))]ds + \int_0^t f(s, u^n(s))ds + \int_0^t g(s, u^n(s))dW(s). \tag{4.1}$$

Then the sequence $\{u^n(t)\}$ is well defined.

Firstly, we give some moment estimates of the process $u^{n+1}(t)$.

Lemma 4.1. *If* (2.7) *with* $k = 1$ *is satisfied and* $\mathbb{E}|u_0|^2 < \infty$, *then there exists a time* $t_1 \in (0, 1]$ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le t} |u^{n+1}(s)|^2\Big) < 4\mathbb{E}|u_0|^2 \quad \text{and} \quad \int_0^t \mathbb{E}\|u^{n+1}(s)\|^2 ds < \frac{2}{\nu}\mathbb{E}|u_0|^2$$

for $t \in (0, t_1]$, *uniformly for all* $n \ge 1$.

Lemma 4.2. *If* (2.7) *with* $k = 2$ *is satisfied and* $\mathbb{E}|u_0|^4 < \infty$, *then there exists a time* $t_2 \in (0, 1]$ *and a positive constant* δ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le t} |u^{n+1}(s)|^4\Big) < 8\mathbb{E}|u_0|^4 \quad \text{and} \quad \int_0^t \mathbb{E}|u^{n+1}(s)|_4^4 ds < \delta^4\frac{2}{\nu}\mathbb{E}|u_0|^2$$

for $t \in (0, t_2]$, *uniformly for all* $n \ge 1$.

Lemma 4.3. *If* (2.8) *with* $k = 1$ *is satisfied and* $\mathbb{E}|\partial_z u_0|^2 < \infty$, *then there exists a time* $t_3 \in (0, 1]$ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le t} |\partial_z u^{n+1}(s)|^2\Big) < 4\mathbb{E}|\partial_z u_0|^2 \quad \text{and} \quad \int_0^t \mathbb{E}\|\partial_z u^{n+1}(s)\|^2 ds < \frac{2}{\nu}\mathbb{E}|\partial_z u_0|^2$$

for $t \in (0, t_3]$, *uniformly for all* $n \ge 1$.

Lemma 4.4. *If* (2.8) *with* $k = 2$ *is satisfied and* $\mathbb{E}|\partial_z u_0|^4 < \infty$, *then there exists a time* $t_4 \in (0, 1]$ *and a positive constant* δ *such that*

$$\mathbb{E}\Big(\sup_{0\le s\le t} |\partial_z u^{n+1}(s)|^4\Big) < 8\mathbb{E}|\partial_z u_0|^4 \quad \text{and} \quad \int_0^t \mathbb{E}|\partial_z u^{n+1}(s)|_4^4 ds < \delta^4\frac{2}{\nu}\mathbb{E}|\partial_z u_0|^2$$

for $t \in (0, t_4]$, *uniformly for all* $n \ge 1$.

For the sake of simplicity, we just give the proof of Lemma 4.4, the others are similar and more easier.

Proof of Lemma 4.4. It is obviously that the case where $n = 1$ holds. Assume that $\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^n(s)|^4\Big) < 8\mathbb{E}|\partial_z u_0|^4$ for any fixed $n \geq 1$. Applying the Itô formula to the function $|\partial_z u^{n+1}(t)|^4 = (|\partial_z u^{n+1}(t)|^2)^2$, we obtain

$$\begin{aligned}|\partial_z u^{n+1}(t)|^4 = |\partial_z u_0|^4 &\\ &+ 4\int_0^t |\partial_z u^{n+1}(s)|^2 \langle -\partial_{zz}u^{n+1}(s), -\nu A u^{n+1}(s) - B(u^{n+1}(s))\rangle ds\\ &+ 4\int_0^t |\partial_z u^{n+1}(s)|^2 \langle -\partial_{zz}u^{n+1}(s), f(s,u^n(s))\rangle ds\\ &+ 4\int_0^t |\partial_z u^{n+1}(s)|^2 \langle -\partial_{zz}u^{n+1}(s), g(s,u^n(s))dW(s)\rangle\\ &+ 6\int_0^t |\partial_z u^{n+1}(s)|^2 |\partial_z g(s,u^n(s))|^2_{L_2^0} ds.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^{n+1}(s)|^4\Big) + 4\nu\mathbb{E}\Big(\int_0^t |\partial_z u^{n+1}(s)|^2\|\partial_z u^{n+1}(s)\|^2 ds\Big)\\ &\le \mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u_0|^4\Big)\\ &\quad + 4\mathbb{E}\Big(\sup_{0\le\tau\le t}\int_0^\tau |\partial_z u^{n+1}(s)|^2|\partial_z u^{n+1}(s)|_V|\partial_z f(s,u^n(s))|_{V^*}ds\Big)\\ &\quad + 4\mathbb{E}\Big(\sup_{0\le\tau\le t}\Big|\int_0^\tau |\partial_z u^{n+1}(s)|^2\langle -\partial_{zz}u^{n+1}(s), g(s,u^n(s))\rangle dW(s)\Big|\Big)\\ &\quad + 6\mathbb{E}\Big(\sup_{0\le\tau\le t}\int_0^\tau |\partial_z u^{n+1}(s)|^2|\partial_z g(s,u^n(s))|^2_{L_2^0}ds\Big)\\ &= \mathbb{E}|\partial_z u_0|^4 + T_1 + T_2 + T_3.\end{aligned}$$

By Young inequality and Hölder inequality, we get

$$\begin{aligned}T_1 &\le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^{n+1}(s)|^4\Big) + \nu\mathbb{E}\int_0^t |\partial_z u^{n+1}(s)|^2\|\partial_z u^{n+1}(s)\|^2 ds\\ &\quad + (32 + \frac{32}{\nu^2})\int_0^t \mathbb{E}|\partial_z f(s,u^n(s))|^4_{V^*},\\ T_3 &\le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^{n+1}(s)|^4\Big) + 6\int_0^t \mathbb{E}|\partial_z g(s,u^n(s))|^4_{L_2^0}ds.\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, there exists a $k > 0$ such that

$$T_2 \le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^{n+1}(s)|^4\Big) + k\int_0^t \mathbb{E}|\partial_z g(s,u^n(s))|^4_{L_2^0}ds.$$

Thus we have

$$\mathbb{E}\Big(\sup_{0\le s\le t}|\partial_z u^{n+1}(s)|^4\Big) + 12\nu\mathbb{E}\Big(\int_0^t |\partial_z u^{n+1}(s)|^2\|\partial_z u^{n+1}(s)\|^2 ds\Big)$$

$$\leq 4\mathbb{E}\Big(\sup_{0\leq s\leq t}|\partial_z u_0|^4\Big)+4(\frac{32}{\nu^2}+38+k)\int_0^t H_2(s,\mathbb{E}|\partial_z u^n(s)|^4)ds.$$

Let $t_4>0$ be a time such that

$$4(\frac{32}{\nu^2}+38+k)\int_0^{t_4} H_2(s,\mathbb{E}|u^n(s)|^4)ds<4\mathbb{E}|\partial_z u_0|^4. \tag{4.2}$$

Then for $t\in(0,t_4]$

$$\mathbb{E}\Big(\sup_{0\leq s\leq t}|\partial_z u^{n+1}(s)|^4\Big)<8\mathbb{E}|\partial_z u_0|^4.$$

By the Gagliardo-Nirenberg inequality we have a $\delta>0$ such that

$$|\partial_z u^{n+1}(s)|_4\leq \delta|\partial_z u^{n+1}(s)|^{\frac{1}{2}}\|\partial_z u^{n+1}(s)\|^{\frac{1}{2}}\quad \text{for all}\;\; n\geq 0.$$

Furthermore, we have

$$\mathbb{E}\int_0^t|\partial_z u^{n+1}(s)|_4^4\leq \delta^4\mathbb{E}\Big(\int_0^t|\partial_z u^{n+1}(s)|^2\|\partial_z u^{n+1}(s)\|^2ds\Big)<\delta^4\frac{2}{\nu}\mathbb{E}|\partial_z u_0|^2.$$

Consequently, by the mathematical induction, the proof of the lemma is completed. □

Remark 4.1. In Lemma 4.1–Lemma 4.4, we obtain the moment estimates of the process $\{u^n(t)\}$ with some strict inequalities. These seem to be unusual and confusing. However, in order to obtain the uniform boundedness, these strict inequalities are not contradictory to the usual inequalities. Here, we choose strict inequality (as in [33]) for two reasons. Let's take Lemma 4.4 as an example. First, by Assumption 1, $H_k(t,u):R^+\times R^+\to R^+(k=1,2)$ are locally integrable in $t\geq 0$ for any fixed $u\geq 0$ and continuous, monotone nondecreasing in u for any fixed $t\geq 0$ with $H_k(t,0)=0$. therefore in (4.2), we can easily find small enough $t_4>0$ and satisfy

$$4(\frac{32}{\nu^2}+38+k)\int_0^{t_4} H_2(s,\mathbb{E}|u^n(s)|^4)ds<4\mathbb{E}|\partial_z u_0|^4,$$

there is no need for us to discuss whether there exists a t^* and satisfies

$$4(\frac{32}{\nu^2}+38+k)\int_0^{t^*} H_2(s,\mathbb{E}|u^n(s)|^4)ds=4\mathbb{E}|\partial_z u_0|^4.$$

Second, since $F_k(t,0)=0$, $H_k(t,0)=0$, if let initial value $u^0\equiv 0$, we obtain a constant solution $u\equiv 0$. It doesn't make any physical sense. To summarize, it is reasonable that we get the strict inequalities.

Lemma 4.5. *Let u_0 be an $\mathscr{F}_0$-random variable with $\mathbb{E}|u_0|^4<\infty$. Then there has a solution to (4.1) and satisfies*

$$\mathbb{E}\Big(\int_0^t\|u^n(s)\|^2ds\Big)^2\leq C(|f|^2_{L^2(\Omega;L^2(0,T;V^*))},|g|^2_{L^2(\Omega;L^2(0,T;L_2^0))})\mathbb{E}|u_0|^4.$$

Proof. We get

$$4\nu^2\Big(\int_0^T\|u^n(s)\|^2ds\Big)^2$$

$$\leq \Big(|u_0|^2 + 2\int_0^T |\langle u^n(s), f(s,u^{n-1}(s))\rangle| ds + \int_0^T |g(s,u^{n-1}(s))|^2_{L_2^0} ds$$
$$+ 2\Big|\int_0^T \langle u^n, g(s,u^{n-1}(s)) dW(s)\rangle\Big|\Big)^2$$
$$\leq 4(|u_0|^4 + I_1^2 + I_2^2 + I_3^2).$$

The Hölder's inequality and Young's inequality imply that

$$I_1^2 \leq \frac{\nu^2}{2}\Big(\int_0^T \|u^n(s)\|^2 ds\Big)^2 + \frac{8}{\nu^2}(|f(s,u^{n-1}(s))|^2_{L^2(0,T;V^*)})^2.$$

Applying the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq s\leq T}|I_3^2|\Big) &\leq C\mathbb{E}\Big(\int_0^T |u^n(s)|^2 |g(s,u^{n-1}(s))|^2_{L_2^0} ds\Big)\\ &\leq C\mathbb{E}\Big[\Big(\sup_{0\leq s\leq T}|u^n(s)|^2\Big)\int_0^T |g(s,u^{n-1}(s))|^2_{L_2^0} ds\Big]\\ &\leq C\mathbb{E}\Big(\sup_{0\leq s\leq T}|u^n(s)|^4\Big) + C(|g(s,u^{n-1}(s))|^2_{L^2(0,T;L_2^0)})^2.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{E}\Big(\int_0^T \|u^n(s)\|^2 ds\Big)^2 \leq C\mathbb{E}\Big(&|u_0|^4 + \sup_{0\leq s\leq T}|u^n(s)|^4 + (|f(s,u^{n-1}(s))|^2_{L^2(0,T;V^*)})^2\\ &+ (|g(s,u^{n-1}(s))|^2_{L^2(0,T;L_2^0)})^2\Big).\end{aligned}$$

Since $u^{n-1}(s)$ is well-defined, by Lemma 4.2, the above formula means

$$\mathbb{E}\Big(\int_0^T \|u^n(s)\|^2 ds\Big)^2 \leq C(|f|^2_{L^2(\Omega;L^2(0,T;V^*))}, |g|^2_{L^2(\Omega;L^2(0,T;L_2^0))})\mathbb{E}|u_0|^4.$$

Then the proof of the lemma is complete. □

Set $T_\epsilon = \min\{t_1, t_2, t_3, t_4\}$. By Assumption 1 and Lemma 4.2, it follows that

$$\begin{aligned}&f(t, u^{n+1}(t)) \in L^4(\Omega\times[0,T_\epsilon]; V^*),\\ &g(t, u^{n+1}(t)) \in L^4(\Omega\times[0,T_\epsilon]; L_2^0(K;H)).\end{aligned}$$

Then we consider the sequence $\{u^n(s)\}, t \in [0, T_\epsilon]$.

Lemma 4.6. *If Assumption 1 and 2 are satisfied and* $\mathbb{E}|u_0|^{2p}, \mathbb{E}|\partial_z u_0|^{2p} < \infty (p = 1, 2)$, *then the sequence* $\{u^n(t)\}, t \in [0, T_\epsilon]$, *which is defined by (4.1), is a Cauchy sequence in* $L^2(\Omega; L^\infty(0,T_\epsilon;H)) \cap L^2(\Omega\times[0,T_\epsilon];V)$, *moreover, it is also a Cauchy sequence in* $L^4(\Omega; L^\infty(0,T_\epsilon;H)) \cap L^4(\Omega\times[0,T_\epsilon];L^4(\mathcal{M}))$.

Proof. We only give the proof in $L^4(\Omega; L^\infty(0,T_\epsilon;H)) \cap L^4(\Omega\times[0,T_\epsilon];L^4(\mathcal{M}))$. According to Lemma 2.2, we use the Young inequality to get a $\lambda > 0$ such that for any $u, v \in V$,

$$\langle E(u) - E(v), u - v\rangle \leq -\nu\|u-v\|^2 + \lambda(1 + |\partial_z v|^4 + \|v\|^2)|u-v|^2. \tag{4.3}$$

Set

$$\zeta^n(t) = exp\Big(-4\lambda \int_0^t (1 + |\partial_z u^n|^4 + \|u^n\|^2) ds\Big), 0 \le t \le T_\epsilon.$$

By applying the energy equality to the function $\zeta^n(t)|u^{n+m}(t) - u^n(t)|^4$, we obtain

$$\begin{aligned}
&\zeta^n(t)|u^{n+m}(t) - u^n(t)|^4 \\
= &-4\lambda \int_0^t \zeta^n(s)(1 + |\partial_z u^n(s)|^4 + \|u^n\|^2)|u^{n+m}(s) - u^n(s)|^4 ds \\
&+ 4\int_0^t \zeta^n(s)|u^{n+m}(t) - u^n(t)|^2 \langle u^{n+m}(s) - n^n(s), E(u^{n+m}(s)) - E(u^n(s))\rangle ds \\
&+ 4\int_0^t \zeta^n(s)|u^{n+m}(t) - u^n(t)|^2 \langle u^{n+m}(s) - n^n(s), f(s, u^{n+m-1}) - f(s, u^{n-1})\rangle ds \\
&+ 6\int_0^t \zeta^n(s)|u^{n+m}(t) - u^n(t)|^2 |g(s, u^{n+m-1}) - g(s, u^{n-1})|^2_{L_2^0} ds \\
&+ 4\int_0^t \zeta^n(s)|u^{n+m}(t) - u^n(t)|^2 \\
&\times \langle u^{n+m}(s) - n^n(s), (g(s, u^{n+m-1}) - g(s, u^{n-1}))dW(s)\rangle.
\end{aligned}$$

By (4.3) it follows that

$$\begin{aligned}
&4\int_0^t \zeta^n(s)|u^{n+m}(t) - u^n(t)|^2 \langle u^{n+m}(s) - n^n(s), E(u^{n+m}(s)) - E(u^n(s))\rangle ds \\
\le &-4\nu \int_0^t \zeta^n(s)|u^{n+m}(s) - u^n(s)|^2 \|u^{n+m}(s) - u^n(s)\|^2 ds \\
&+ 4\lambda \int_0^t \zeta^n(s)(1 + |\partial_z u^n(s)|^4 + \|u^n\|^2)|u^{n+m}(s) - u^n(s)|^4 ds.
\end{aligned}$$

Define the stopping time as follows:

$$\tau_N^n = \inf \Big\{ t \le T_\epsilon : \int_0^t (1 + |\partial_z u^n(s)|^4 + \|u^n(s)\|^2) ds \ge N \Big\}.$$

Then we have $\exp(-4\lambda N) \le \zeta^n(t \wedge \tau_N^n) < 1$. It follows that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0 \le s \le t \wedge \tau_N^n} \zeta^n(s)|u^{n+m}(s) - u^n(s)|^4 \Big) \\
&+ 4\nu \mathbb{E}\Big(\int_0^{t \wedge \tau_N^n} \zeta^n(s)|u^{n+m}(s) - u^n(s)|^2 \|u^{n+m}(s) - u^n(s)\|^2 ds \Big) \\
\le &4\mathbb{E}\Big(\sup_{0 \le s \le \tau \wedge \tau_N^n} \Big| \int_0^\tau \zeta^n(s)|u^{n+m}(s) - u^n(s)|^2 \\
&\times \langle u^{n+m}(s) - u^n(s), f(s, u^{n+m-1}(s)) - f(s, u^{n-1}(s)\rangle ds \Big| \Big) \\
&+ 6\mathbb{E}\Big(\sup_{0 \le s \le \tau \wedge \tau_N^n} \Big| \int_0^\tau \zeta^n(s)|u^{n+m}(s) - u^n(s)|^2 \\
&\times |g(s, u^{n+m-1}(s)) - g(s, u^{n-1}(s)|^2_{L_2^0} ds \Big| \Big)
\end{aligned}$$

$$
\begin{aligned}
&+ 4\mathbb{E}\Big(\sup_{0\le s\le \tau\wedge\tau_N^n}\Big|\int_0^\tau \zeta^n(s)|u^{n+m}(s)-u^n(s)|^2\\
&\times \langle u^{n+m}(s)-u^n(s),(g(s,u^{n+m-1}(s))-g(s,u^{n-1}(s))dW(s)\rangle\Big|\Big)\\
&= N_1+N_2+N_3.
\end{aligned}
$$

And then this yields

$$
\begin{aligned}
N_1 &\le 4\mathbb{E}\int_0^{t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^2|u^{n+m}(s)-u^n(s)|_V\\
&\quad\times|f(s,u^{n+m-1}(s))-f(s,u^{n-1}(s)|_{V^*}ds\\
&\le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^4\Big)\\
&\quad+\nu\mathbb{E}\int_0^{t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^2\|u^{n+m}(s)-u^n(s)\|^2ds\\
&\quad+(\frac{4}{\nu}+16)\mathbb{E}\int_0^{t\wedge\tau_N^n}\zeta^n(s)|f(s,u^{n+m-1}(s))-f(s,u^{n-1}(s))|_{V^*}^4ds,\\
N_2 &\le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^4\Big)\\
&\quad+36\mathbb{E}\int_0^{t\wedge\tau_N^n}\zeta^n(s)|g(s,u^{n+m-1}(s))-g(s,u^{n-1}(s))|_{L_2^0}^4ds,
\end{aligned}
$$

and there exists a $k>0$ such that

$$
\begin{aligned}
N_3 &\le \frac{1}{4}\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^4\Big)\\
&\qquad+k\mathbb{E}\int_0^{t\wedge\tau_N^n}\zeta^n(s)|g(s,u^{n+m-1}(s))-g(s,u^{n-1}(s))|_{L_2^0}^4ds.
\end{aligned}
$$

Thus by Assumption 2 we have a $\beta>0$ such that for any fixed $n,m\ge 1$

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^4\Big)\\
&+12\nu\int_0^{t\wedge\tau_N^n}\zeta^n(s)|u^{n+m}(s)-u^n(s)|^2\|u^{n+m}(s)-u^n(s)\|^2ds\\
&\le\beta\int_0^{t\wedge\tau_N^n}G_2(s,\mathbb{E}|u^{n+m-1}(s)-u^{n-1}(s)|^4)ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t}|u^{n+m}(s\wedge\tau_N^n)-u^n(s\wedge\tau_N^n)|^4\Big)\\
&+12\nu\int_0^t|u^{n+m}(s\wedge\tau_N^n)-u^n(s\wedge\tau_N^n)|_4^4ds\\
&\le\beta e^{4\lambda N}\int_0^t G_2\Big(s\wedge\tau_N^n,\mathbb{E}\Big(\sup_{0\le\tau\le s}|u^{n+m-1}(\tau\wedge\tau_N^n)-u^{n-1}(\tau\wedge\tau_N^n)|^4\Big)\Big)ds. \quad (4.4)
\end{aligned}
$$

For any fixed $N > 0, 0 < t \leq T_\epsilon$, let

$$\begin{aligned}\jmath_N(t) = \limsup_{n,m\to\infty} \mathbb{E}\Big[\Big(\sup_{0\leq s\leq t\wedge\tau_N^n} |u^{n+m}(s) - u^n(s)|^4\Big) \\ + 12\nu\mathbb{E}\Big(\int_0^{t\wedge\tau_N^n} |u^{n+m}(s) - u^n(s)|_4^4 ds\Big)\Big],\end{aligned}$$

$$\jmath(t) = \limsup_{n,m\to\infty} \mathbb{E}\Big[\Big(\sup_{0\leq s\leq t} |u^{n+m}(s) - u^n(s)|^4\Big) + 12\nu\mathbb{E}\Big(\int_0^t |u^{n+m}(s) - u^n(s)|_4^4 ds\Big)\Big].$$

Since $G_2(t,u)$ is continuous, monotone nondecreasing in u, and we already have

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\leq s\leq t} |u^{n+m}(s\wedge\tau_N^n) - u^n(s\wedge\tau_N^n)|^2\Big) \\ &+ 2\nu\mathbb{E}\Big(\int_0^t \|u^{n+m}(s\wedge\tau_N^n) - u^n(s\wedge\tau_N^n)\|^2 ds\Big) \\ &\leq \alpha e^{2\lambda N}\int_0^t G_1\Big(s\wedge\tau_N^n, \mathbb{E}\Big(\sup_{0\leq\tau\leq s} |u^{n+m-1}(s\wedge\tau_N^n) - u^{n-1}(s\wedge\tau_N^n)|^2\Big)\Big)ds, \quad (4.5)\end{aligned}$$

by Fatou lemma, it follows that

$$\jmath_N(t) \leq \beta e^{4\lambda N}\int_0^t G_2(s, \jmath_N(s))ds,$$

which implies $\jmath_N(t) \equiv 0$. By Chebyshev inequality and Lemma 4.1, 4.4, we have

$$\begin{aligned}P(\tau_N^n \leq t) &= P\Big(\int_0^t (1 + |\partial_z u^n|^4 + \|u^n\|^2)ds \geq N\Big) \\ &\leq \frac{E\int_0^t (1 + |\partial_z u^n|^4 + \|u^n\|^2)ds}{N}.\end{aligned}$$

It follows that as $N \to \infty$, $P(\tau_N^n \leq t) \to 0$ and so $\jmath_N(t) \to \jmath(t)$. Consequently we obtain that $\jmath(t) \equiv 0$ for $0 < t \leq T_\epsilon$ and so $\{u^n(t)\}$ is a Cauchy sequence in $L^4(\Omega; L^\infty(0, T_\epsilon; H)) \cap L^4(\Omega \times [0, T_\epsilon]; L^4(\mathcal{M}))$. □

By Lemma 4.6, we say $\{u^n(t)\}$ is a Cauchy sequence, then we can find its limit $u(t)$ as $n \to \infty$. Finally, we proof the main Theorem 4.1 as follows.

Proof of Theorem 4.1. By lemma 4.6, we have an $u(t)$ such that as $n \to \infty$

$$u^n(t) \to u(t) \text{ strongly in } L^2(\Omega; L^2(0, T_\epsilon; V)) \cap L^4(\Omega; L^\infty([0, T_\epsilon]; H)).$$

By (2.9), as $n \to \infty$

$$\begin{aligned}f(t, u^n(t)) &\to f(t, u(t)) \text{ strongly in } L^2(\Omega \times [0, T_\epsilon]; V^*), \\ g(t, u^n(t)) &\to g(t, u(t)) \text{ strongly in } L^2(\Omega \times [0, T_\epsilon]; L_2^0(K; H)).\end{aligned}$$

Furthermore there exists a $\chi \in L^2(\Omega \times [0, T_\epsilon]; V^*)$ such that as $n \to \infty$

$$-\nu Au^n(t) - B(u^n(t)) \rightharpoonup \chi(t) \text{ weakly in } L^2(\Omega \times [0, T]; V^*).$$

Therefore letting $n \to \infty$, we have that in V^*

$$u(t) = u_0 + \int_0^t \chi(s)ds + \int_0^t f(s, u(s))ds + \int_0^t g(s, u(s))dW(s).$$

Define

$$\rho(t) = \int_0^t (1 + |\partial_z \omega|^4 + \|\omega\|^2) ds,$$

where $\omega \in L^2(\Omega \times [0, T_\epsilon]; V)$, and $\partial_z \omega \in L^4(\Omega \times [0, T_\epsilon]; H)$. Then, we have that

$$\begin{aligned}\mathbb{E}e^{-2\kappa\rho(t)}|u(t)|^2 = \mathbb{E}|u_0|^2 - 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u(s)|^2 ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle \chi(s), u(s)\rangle ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f(s, u(s)), u(s)\rangle ds \\ + \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g(s, u(s)))|^2_{L_2^0} ds.\end{aligned}$$

By the application of Itô formula to (4.1), we also have

$$\begin{aligned}\mathbb{E}e^{-2\kappa\rho(t)}|u^n(t)|^2 = \mathbb{E}|u_0|^2 - 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u^n(s)|^2 ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle -\nu A u^n(s) - B(u^n(s)), u^n(s)\rangle ds \\ + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f(s, u^{n-1}(s)), u^n(s)\rangle ds \\ + \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g(s, u^{n-1}(s))|^2_{L_2^0} ds.\end{aligned}$$

It is not difficult to get

$$\begin{aligned}&\mathbb{E}e^{-2\kappa\rho(t)}|u^n(t)|^2 - \mathbb{E}|u_0|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f(s, u^{n-1}(s)), u^n(s)\rangle ds \\ &- \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g(s, u^{n-1}(s))|^2_{L_2^0} ds \\ &= -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u^n(s)|^2 ds \\ &+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle -\nu A u^n(s) - B(u^n(s)), u^n(s)\rangle ds.\end{aligned}$$

According to Fatou lemma, we have

$$\begin{aligned}&\liminf_{n\to\infty}\Big(- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1 + |\partial_z\omega|^4 + \|\omega\|^2)|u^n(s)|^2 ds \\ &+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle -\nu A u^n(s) - B(u^n(s)), u^n(s)\rangle ds\Big) \\ &= \liminf_{n\to\infty}\Big(\mathbb{E}e^{-2\kappa\rho(t)}|u^n(t)|^2 - \mathbb{E}|u_0|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f(s, u^{n-1}(s)), u^n(s)\rangle ds \\ &- \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g(s, u^{n-1}(s))|^2_{L_2^0} ds\Big)\end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{E}e^{-2\kappa\rho(t)}|u(t)|^2 - \mathbb{E}|u_0|^2 - 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle f(s,u(s)),u(s)\rangle ds\\
&\quad - \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}|g(s,u(s))|^2_{L_2^0}ds\\
&= -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)|^2ds + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle\chi(s),u(s)\rangle ds.
\end{aligned}
$$

By (4.3)

$$
\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u_n(s)-\omega(s)|^2ds\\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s)-E(\omega(s)),u_n(s)-\omega(s)\rangle ds \leq 0,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u_n(s)|^2ds\\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s)),u_n(s)\rangle ds\\
&\leq -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)(2u(s)-\omega(s),\omega(s))ds\\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(u_n(s)),\omega(s)\rangle ds + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(\omega(s)),u_n(s)-\omega(s)\rangle ds.
\end{aligned}
$$

Letting $n\to\infty$ and combining above formulas, we get

$$
\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)|^2ds + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle\chi(s),u(s)\rangle ds\\
&\leq -2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)(2u(s)-\omega(s),\omega(s))ds\\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle\chi(s),\omega(s)\rangle ds + 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle E(\omega(s)),u(s)-\omega(s)\rangle ds.
\end{aligned}
$$

Reorganizing the terms, we have

$$
\begin{aligned}
&- 2\kappa\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|u(s)-\omega(s)|^2\\
&+ 2\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle\chi(s)-E(\omega(s)),u(s)-\omega(s)\rangle ds \leq 0.
\end{aligned}
$$

If we note $\omega(s)=u(s)-\mu\varphi(s),\mu>0$, we have

$$
\begin{aligned}
&- \kappa\mu\mathbb{E}\int_0^t e^{-2\kappa\rho(s)}(1+|\partial_z\omega|^4+\|\omega\|^2)|\varphi(s)|^2\\
&+ \mathbb{E}\int_0^t e^{-2\kappa\rho(s)}\langle\chi(s)-\mathbb{E}(u(s)-\mu\varphi(s)),\varphi(s)\rangle ds \leq 0.
\end{aligned}
$$

Letting $n \to \infty$, since $\varphi(s)$ is arbitrary, we obtain

$$\int_0^t \chi(s)ds = \int_0^t -\nu Au(s) - B(u(s))ds.$$

Therefore it holds that

$$u(t) = u_0 + \int_0^t -Au(s) - B(u(s))ds + \int_0^t f(s,u(s))ds + \int_0^t g(s,u(s))dW(s).$$

Thus the existence has been proved.

Finally we still need to prove the uniqueness of solution. Assume that $v(t)$ is another solution with the same initial value $v_0 = u_0$. Then we have $\mathbb{E}\int_0^t(1 + |\partial_z v|^4 + \|v\|)ds < \infty$. Set $\tau_N = \inf\Big\{t \le T_\epsilon : \int_0^t(1+|\partial_z v(s)|^4+\|v(s)\|^2)ds \ge N\Big\}$ and $\eta(t) = \exp\Big(-2\lambda\int_0^t(1+|\partial_z v(s)|^4 + \|v(s)\|)ds\Big)$, where λ is the same one as (4.3).

Applying the Itô formula to the function $\eta(t)|u(t)-v(t)|^2$, we have

$$\begin{aligned}
&\eta(t)|u(t)-v(t)|^2\\
&= -2\lambda\int_0^t \eta^n(s)(1+|\partial_z v(s)|^4 + \|v(s)\|^2)|u(s)-v(s)|^2ds\\
&\quad + 2\int_0^t \eta(s)\langle u(s)-v(s), E(u(s)) - E(u^n(s))\rangle ds\\
&\quad + 2\int_0^t \eta(s)\langle u(s)-v(s), f(s,u(s)) - f(s,v(s))\rangle ds\\
&\quad + 2\int_0^t \eta(s)\langle u(s)-v(s), (g(s,u(s)) - g(s,v(s)))dW(s)\rangle\\
&\quad + \int_0^t \eta(s)|g(s,u(s)) - g(s,v(s))|^2_{L_2^0}ds
\end{aligned}$$

According to (4.3), it follows that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t\wedge\tau_N}\eta(s)|u(s)-v(s)|^2\Big) + 2\nu\mathbb{E}\Big(\int_0^{t\wedge\tau_N}\eta(s)\|u(s)-v(s)\|^2ds\Big)\\
&\le 2\mathbb{E}\Big(\sup_{0\le s\le\tau\wedge\tau_N}\Big|\int_0^\tau \eta(s)\langle u(s)-v(s), f(s,u(s)) - f(s,v(s)\rangle ds\Big|\Big)\\
&\quad + 2\mathbb{E}\Big(\sup_{0\le s\le\tau\wedge\tau_N}\Big|\int_0^\tau \eta(s)\langle u^{n+m}(s)-u^n(s), (g(s,u(s)) - g(s,v(s))dW(s)\rangle\Big|\Big)\\
&\quad + \mathbb{E}\Big(\sup_{0\le s\le\tau\wedge\tau_N}\Big|\int_0^\tau \eta(s)|g(s,u(s)) - g(s,v(s)|^2_{L_2^0}ds\Big|\Big).
\end{aligned}$$

Therefore we deduce that there exists a $\gamma > 0$ such that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t}|u(s\wedge\tau_N) - v(s\wedge\tau_N)|^2\Big) + 2\nu\mathbb{E}\Big(\int_0^t\|u(s\wedge\tau_N) - v(s\wedge\tau_N)\|^2ds\Big)\\
&\le \gamma e^{2\lambda N}\int_0^t G_1\Big(s\wedge\tau_N, \mathbb{E}\Big(\sup_{0\le\tau\le s}|u(s\wedge\tau_N) - v(s\wedge\tau_N)|^2\Big)\Big)ds.
\end{aligned}$$

By Assumption 2, we know

$$\mathbb{E}\Big(\sup_{0\le s\le t}|u(s\wedge\tau_N)-v(s\wedge\tau_N)|^2\Big)+2\nu\mathbb{E}\Big(\int_0^t\|u(s\wedge\tau_N)-v(s\wedge\tau_N)\|^2ds\Big)=0.$$

Notice that $t\wedge\tau_N\to t$ as $N\to\infty$. Hence we have that $u(t)=v(t)$ in $L^2(\Omega;L^\infty([0,T_\epsilon];H))\cap L^2(\Omega\times[0,T_\epsilon];V)$. So the proof of theorem is completed. □

5. Global existence of solutions

In this section, we consider the global existence of solutions to (2.6) under following assumption.

Assumption 4. Let the function $F_1(t,X)$ be the same as given in Assumption 1. Then for any constant $C>0$ and any initial value X_0, the differential equations $\frac{dX(t)}{dt}=CF_1(t,X(t))$ have unique solutions $X(t)(0\le t\le T)$, respectively($T=\infty$ means $[0,T]=[0,\infty]$).

Theorem 5.1. *Suppose that Assumptions 1, 2 and 4 are satisfied and* $\mathbb{E}|u_0|^{2p}$, $\mathbb{E}|\partial_z u_0|^{2p}<\infty(p=1,2)$. *Then the solution* $u(t)$ *to (2.6) exists and satisfies* $u(t)\in L^2(\Omega;L^\infty([0,\,T];H))\cap L^2(\Omega\times[0,T];V)$.

Proof. From Theorem 4.8, there exists unique solution $u(t)$ in local time to (2.6) with the initial value u_0 under Assumptions 1 and 2. Using the same method as Lemma 4.1, we know there exists a real number $\varpi>0$ such that for any $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le s\le t}|u(s)|^2\Big)+2\nu\int_0^t\mathbb{E}\|u(s)\|^2ds\\&\le 2\mathbb{E}|u_0|^2+\varpi\int_0^t F_1\Big(s,\mathbb{E}\Big(\sup_{0\le\tau\le s}|u(\tau)|^2\Big)\Big)ds.\end{aligned}$$

Let $X_0>2\mathbb{E}|u_0|^2$. Then

$$\begin{aligned}&X(t)-\mathbb{E}\Big(\sup_{0\le s\le t}|u(s)|^2\Big)\\&>\varpi\int_0^t F_1\Big(s,X(s)\Big)ds-\varpi\int_0^t F_1\Big(s,\mathbb{E}\Big(\sup_{0\le\tau\le s}|u(\tau)|^2\Big)\Big)ds.\end{aligned}$$

Owing to the continuity and monotonicity of $F_1(t,X)$, we have

$$\begin{aligned}&X(t)>\mathbb{E}\Big(\sup_{0\le s\le t}|u(s)|^2\Big),\\&2\nu\int_0^t\mathbb{E}\|u(s)\|^2ds<2\mathbb{E}|u_0|^2+\varpi\int_0^t F_1(s,X(s))ds,\end{aligned}$$

for $0\le t\le T$. Since $X(t)$ is continuous on $[0,T]$, the proof of theorem is completed as in [33]. □

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