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THE NUMBER OF LIMIT CYCLES FROM A QUARTIC CENTER BY THE HIGHER-ORDER MELNIKOV FUNCTIONS*

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Abstract In this paper, by computing the higher-order Melnikov functions we study the number of limit cycles of the system $\dot{x} = -y(1+x)^3+\epsilon P(x,y)$, $\dot{y} = x(1+x)^3-\epsilon Q(x,y)$ where $P(x,y)$ and $Q(x,y)$ are arbitrary cubic polynomials. Our main results show that the first four Melnikov functions associated with the perturbed system yield at most five limit cycles.

Keywords Melnikov functions, bifurcation, limit cycles, generators.

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction

By using the recursive method developed by Françoise [6] and Iliev [9], the higher-order Melnikov functions have been computed of some perturbed systems to study the number of bifurcated limit cycles, see [1–3, 7, 10, 16–18]. Buica et al. [3] and Yu [18] studied the quadratic perturbations of the following system

$$\begin{cases} \dot{x} = y(1+x)^{\bar{m}} - \epsilon \sum\limits_{i+j=0}^{n} a_{ij}x^iy^j, \\ \dot{y} = -x(1+x)^{\bar{m}} + \epsilon \sum\limits_{i+j=0}^{n} b_{ij}x^iy^j, \end{cases}$$

where $\epsilon > 0$ is a small parameter, when $\bar{m} = 1$ or 2 and $n = 2$ they proved that the upper bound for the number of limit cycles is 3, respectively. For $\bar{m} = 1$ and $a_{00} = b_{00} = 0$, by using the averaging method, the authors in [11, 12] obtained up to first order in ϵ at most n limit cycles.

Based on their works, in this paper, we will consider the system

$$\begin{cases} \dot{x} = -y(1+x)^3 + \epsilon P(x,y), \\ \dot{y} = x(1+x)^3 - \epsilon Q(x,y), \end{cases} \tag{1.1}$$

where $P(x,y) = \sum\limits_{i+j=0}^{3} a_{ij}x^iy^j$, $Q(x,y) = \sum\limits_{i+j=0}^{3} b_{ij}x^iy^j$. For $\epsilon = 0$, there exists a family of periodic orbits $\Gamma_h : H(x,y) = \frac{1}{2}(x^2+y^2) = h$ $(0 < x^2+y^2 < 1)$ surrounding

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the origin. By fixing a transversal segment to the flow in (1.1) and using the energy level h to parameterize it, the corresponding displacement function is

$$d(h,\epsilon)=\epsilon M_1(h)+\epsilon^2 M_2(h)+\epsilon^3 M_3(h)+\cdots, h\in\left(0,\frac{1}{2}\right), \tag{1.2}$$

where $M_k(h)$ is called the k-th order Melnikov function of the system (1.1).

The first-order Melnikov function of the system (1.1) takes the form

$$M_1(h)=\oint\limits_{H=h}\frac{Q(x,y)}{(1+x)^3}dx+\frac{P(x,y)}{(1+x)^3}dy,$$

if $M_1(h)\not\equiv 0$, every simple zero $h_0\in\left(0,\frac{1}{2}\right)$ of $M_1(h)$ correspondings to a limit cycle of the perturbed system (1.1) near Γ_{h_0}.

If $M_1(h)\equiv 0$, then $M_2(h)$ will become very important to investigate the limit cycles of the perturbed system. Similarly, we may use $M_3(h)$, $M_4(h)$,$\cdots$ to study the limit cycles.

The organization of this paper is as follows: In Section 2, some preliminaries are listed. In Section 3, the higher-order Melnikov functions of system (1.1) are computed. In Section 4, a discussion is given.

2. Preliminaries

Let $\omega=\frac{Q(x,y)}{(1+x)^3}dx+\frac{P(x,y)}{(1+x)^3}dy$, then the Pfaffian form of the system (1.1) is $dH=\epsilon\omega$.

The following Lemmas and Remarks shall be used to prove the main Theorems. The algorithm of calculating $M_k(h)$ is shown by Françoise [6] and Iliev [9].

Lemma 2.1 ([9,18]). *Assume Γ_h is the period annulus defined by $H(x,y)=h$, the polynomial function H(x,y) and the 1-form ω satisfy $\oint\limits_{H=h}\omega=0$ if and only if there are two analytic function $q(x,y)$ and $S(x,y)$ in a neighborhood of Γ_h such that $\omega=qdH+dS$.*

Lemma 2.2 ([6,18]). *Let $\omega=\bar{q}_0dH+d\bar{Q}_0+N_0$, then $M_1(h)=\oint\limits_{H=h}\omega=\oint\limits_{H=h}N_0$;*

(1) *If $M_1(h)\equiv 0$, then $N_0=\tilde{q}_0dH+d\tilde{Q}_0$. Let $q_0=\bar{q}_0+\tilde{q}_0$, $Q_0=\bar{Q}_0+\tilde{Q}_0$, we have $\omega=q_0dH+dQ_0$ and $q_0\omega$ can be decomposed into $q_0\omega=\bar{q}_1dH+d\bar{Q}_1+N_1$, then $M_2(h)=\oint\limits_{H=h}q_0\omega=\oint\limits_{H=h}N_1$;*

(2) *If $M_2(h)\equiv 0$, then $N_1=\tilde{q}_1dH+d\tilde{Q}_1$. Let $q_1=\bar{q}_1+\tilde{q}_1$, $Q_1=\bar{Q}_1+\tilde{Q}_1$, we have $q_0\omega=q_1dH+dQ_1$ and $q_1\omega$ can be decomposed into $q_1\omega=\bar{q}_2dH+d\bar{Q}_2+N_2$, then $M_3(h)=\oint\limits_{H=h}q_1\omega=\oint\limits_{H=h}N_2$;*
By this way, if $M_1(h)=M_2(h)=\cdots=M_{i-1}(h)\equiv 0$, then $q_j\omega=q_{j+1}dH+dQ_{j+1}, j\le i-2$ and we assume $q_{-1}=1$. We have $M_i(h)=\oint\limits_{H=h}q_{i-2}\omega=\oint\limits_{H=h}N_{i-1}$, where $q_{i-2}\omega=\bar{q}_{i-1}dH+d\bar{Q}_{i-1}+N_{i-1}$.

Remark 2.1. The authors in [18] have noted that $q_i\omega=q_{i+1}dH+dQ_{i+1}$, dQ_i is not used in the subsequent calculative process when $i>0$. Thus, the specific form of dQ_i for $i\ge 1$ will not be shown in the following sections.

In the next lemma, an algorithm to decompose $q_i\omega$ is given to simplify the expression of $M_k(h)$.

Define

$$\omega_{ij}^k=\frac{x^iy^j}{(1+x)^k}dx,\quad \delta_{ij}^k=\frac{x^iy^j}{(1+x)^k}dy,\quad J_k(h)=\oint\limits_{H=h}\delta_{00}^k,\quad 0\le i+j\le k,\ k\ge 1,$$

where $J_k(h)$ are called the generators of $M_i(h)$. ω_{ij}^k, δ_{ij}^k are decomposed into the combination of δ_{00}^k, $h_{ij}^k(x,H)dH$ and $dQ_{ij}^k(x,H)$, and dQ_{ij}^k are the perfect differential functions in x and H. Therefore, ω can be rewritten as

$$\omega=\sum_{i+j=0}^{3}\left(a_{ij}\delta_{ij}^3+b_{ij}\omega_{ij}^3\right). \tag{2.1}$$

Lemma 2.3. *The 1-forms ω_{ij} and δ_{ij} are as follows:*

(i) For $k=3$, we have that

$$\begin{aligned}
\omega_{00}^3=&d\left(\frac{-1}{2(1+x)^2}\right),\ \omega_{10}^3=d\left(\frac{-(1+2x)}{2(1+x)^2}\right),\ \omega_{01}^3=d\left(\frac{-y}{2(1+x)^2}\right)+\frac{1}{2}\delta_{00}^2,\\
\omega_{20}^3=&d\left(\ln(1+x)+\frac{3+4x}{2(1+x)^2}\right),\\
\omega_{02}^3=&d\left(\frac{-H}{(1+x)^2}\right)+\frac{1}{(1+x)^2}dH-d\left(\ln(1+x)+\frac{3+4x}{2(1+x)^2}\right),\\
\omega_{30}^3=&d\left(x-\frac{5+6x}{2(1+x)^2}-3\ln(1+x)\right),\\
\omega_{12}^3=&d\left(\frac{-(1+2x)H}{(1+x)^2}\right)+\frac{1+2x}{(1+x)^2}dH-\omega_{30}^3,\\
\omega_{03}^3=&d\left(\frac{-y^3}{2(1+x)^2}\right)+\frac{3}{2}(2H\delta_{00}^2-\delta_{00}^2+2\delta_{00}^1-dy),\quad \delta_{10}^3=\delta_{00}^2-\delta_{00}^3,\\
\delta_{01}^3=&\frac{1}{(1+x)^3}dH+d\left(\frac{1+2x}{2(1+x)^2}\right),\quad \delta_{20}^3=\delta_{00}^1-2\delta_{00}^2+\delta_{00}^3,\\
\delta_{11}^3=&\frac{x}{(1+x)^3}dH-d\left(\frac{3+4x}{2(1+x)^2}+\ln(1+x)\right),\\
\delta_{02}^3=&2H\delta_{00}^3-\delta_{00}^1+2\delta_{00}^2-\delta_{00}^3,\quad \delta_{30}^3=dy-\delta_{00}^3-3\delta_{00}^1+3\delta_{00}^2,\\
\delta_{21}^3=&\frac{x^2}{(1+x)^3}dH-d\left(x-\frac{5+6x}{2(1+x)^2}-3\ln(1+x)\right),\\
\delta_{12}^3=&2H(\delta_{00}^2-\delta_{00}^3)-dy+\delta_{00}^3+3\delta_{00}^1-3\delta_{00}^2,\quad \delta_{03}^3=\frac{y^2}{(1+x)^3}dH-\omega_{12}^3,\\
\omega_{11}^3=&\frac{y}{(1+x)^3}dH-\delta_{02}^3,\quad \omega_{21}^3=\frac{xy}{(1+x)^3}dH-\delta_{12}^3.
\end{aligned}$$

(ii) For $k>3$, if j is even, then

$$\omega_{0j}^k=h_{0j}^k(x,H)dH+d(Q_{0j}^k(x,H)),\ \delta_{0j}^k=\sum_{r=0}^{j/2}(2H)^r\mathbb{L}(\delta_{00}^{k-j+2r},\cdots,\delta_{00}^k).$$

Concretely, we have

$$\begin{aligned}
\omega_{02}^k =&\frac{2}{(k-1)\,(1+x)^{k-1}}dH + dQ_{02}^k(x,H),\\
\omega_{04}^k =&\left[\frac{8H}{(k-1)(1+x)^{k-1}} + 4\int \frac{x^2}{(1+x)^k}dx\right] dH + dQ_{04}^k(x,H),\\
\omega_{06}^k =&\left[\frac{24H^2}{(k-1)(1+x)^{k-1}} + 24H\int \frac{x^2}{(1+x)^k}dx\right.\\
&\left. -6\int \frac{x^4}{(1+x)^k}dx\right] dH + dQ_{06}^k(x,H),\\
\omega_{08}^k =&\left(\frac{64H^3}{(k-1)(1+x)^{k-1}} + 96H^2\int \frac{x^2}{(1+x)^k}dx\right.\\
&\left. -48H\int \frac{x^4}{(1+x)^k}dx + 8\int \frac{x^6}{(1+x)^k}dx\right) dH + dQ_{08}^k(x,H),\\
\delta_{04}^k =&(2H-1)^2\,\delta_{00}^k + (8H-4)\,\delta_{00}^{k-1} + (6-4H)\,\delta_{00}^{k-2} - 4\delta_{00}^{k-3}\\
&+\delta_{00}^{k-4},\quad \delta_{02}^k = 2H\delta_{00}^k - \delta_{00}^k - \delta_{00}^{k-2} + 2\delta_{00}^{k-1},\\
\delta_{06}^k =&8H^3\delta_{00}^k - 12H^2\left(\delta_{00}^k - 2\delta_{00}^{k-1} + \delta_{00}^{k-2}\right) + 6H\left(\delta_{00}^k - 4\delta_{00}^{k-1}\right.\\
&\left.+6\delta_{00}^{k-2} - 4\delta_{00}^{k-3} + \delta_{00}^{k-4}\right) - \delta_{00}^k + 6\delta_{00}^{k-1} - 15\delta_{00}^{k-2}\\
&+20\delta_{00}^{k-3} - 15\delta_{00}^{k-4} + 6\delta_{00}^{k-5} - \delta_{00}^{k-6},
\end{aligned}\tag{2.2}$$

if j is odd, then

$$\begin{aligned}
\omega_{0j}^k =&d\left(\frac{-y^j}{(k-1)(1+x)^{k-1}}\right) + \frac{j}{k-1}\delta_{0,j-1}^{k-1} = dS_{0j}^k(x,y) + \frac{j}{k-1}\delta_{0,j-1}^{k-1},\\
\delta_{0j}^k =&\frac{y^{j-1}}{(1+x)^k}dH - \omega_{0,j-1}^{k-1} + \omega_{0,j-1}^k,
\end{aligned}\tag{2.3}$$

where $h_{0j}^k(x,H)$ and $Q_{0j}^k(x,H)$ are functions of x and H, $S_{0j}^k(x,y)$ are functions of x and y, $\mathbb{L}(\delta_{00}^1,\cdots,\delta_{00}^k)$ is a linear combination of δ_{00}^l $(l=1,2,\cdots,k)$ in $\mathbb{Z}$.

Proof.

(i) Here, we use partial integration and the Laurent expansion method to decompose ω_{ij}^3 and δ_{ij}^3 and some obvious decompositions are omitted.

$$\begin{aligned}
\omega_{11}^3 =&\frac{y}{2(1+x)^3}dx^2 = \frac{y}{(1+x)^3}dH - \frac{y^2}{(1+x)^3}dy = \frac{y}{(1+x)^3}dH - \delta_{02}^3,\\
\delta_{03}^3 =&\frac{y^2}{2(1+x)^3}dy^2 = \frac{y^2}{(1+x)^3}dH - \frac{xy^2}{(1+x)^3}dx,\\
\omega_{21}^3 =&\frac{xy}{2(1+x)^3}dx^2 = \frac{xy}{2(1+x)^3}d(2H-y^2) = \frac{xy}{(1+x)^3}dH - \delta_{12}^3.
\end{aligned}$$

(ii) For j even, we let $j=2m$. Then

$$\omega_{0j}^k =\frac{y^j}{(1+x)^k}dx = \frac{(2H-x^2)^m}{(1+x)^k}dx = \sum_{r=0}^{m} C_m^r(2H)^r\frac{(-x)^{2(m-r)}}{(1+x)^k}dx.$$

Let $Q^k_{0jr}=\int\limits_0^x \frac{(-s)^{2(m-r)}}{(1+s)^k}ds$, then

$$\omega^k_{0j}=\sum_{r=0}^m C^r_m(2H)^r dQ^k_{0jr}=\sum_{r=0}^m\left[d(C^r_m(2H)^r Q^k_{0jr})-C^r_m r(2H)^{r-1}2Q^k_{0jr}dH\right],$$

which can be rewritten as $\omega^k_{0j}=h^k_{0j}(x,H)dH+dQ^k_{0j}(x,H)$.
On the other hand, we have

$$\begin{aligned}\delta^k_{0j}=&\frac{y^{2m}}{(1+x)^k}dy=\frac{(2H-x^2)^m}{(1+x)^k}dy=\sum_{r=0}^m C^r_m(2H)^r\frac{(-1)^{m-r}x^{2m-2r}}{(1+x)^k}dy\\ =&\sum_{r=0}^m C^r_m(2H)^r\frac{(-1)^{m-r}(1+x-1)^{2m-2r}}{(1+x)^k}dy\\ =&\sum_{r=0}^m C^r_m(2H)^r(-1)^{m-r}\sum_{n=0}^{2m-2r}C^n_{2m-2r}\frac{(1+x)^n(-1)^{2m-2r-n}}{(1+x)^k}dy\\ =&\sum_{r=0}^m\sum_{n=0}^{2m-2r}(2H)^r a_{r,k-n}\delta^{k-n}_{00}=\sum_{r=0}^{j/2}(2H)^r\mathbb{L}(\delta^{k-j+2r}_{00},\cdots,\delta^k_{00}),\end{aligned}$$

where $a_{r,k-n}=C^r_mC^n_{2m-2r}(-1)^{3m-3r-n}$. The formula (2.2) is obtained.
For j odd, we have

$$\begin{aligned}\omega^k_{0j}=&\frac{y^j}{(1+x)^k}dx=y^j d\left(\frac{-1}{(k-1)(1+x)^{k-1}}\right)\\ =&d\left(\frac{-y^j}{(k-1)(1+x)^{k-1}}\right)+\frac{jy^{j-1}}{(k-1)(1+x)^{k-1}}dy\\ =&d\left(\frac{-y^j}{(k-1)(1+x)^{k-1}}\right)+\frac{j}{k-1}\delta^{k-1}_{0,j-1},\\ \delta^k_{0j}=&\frac{y^j}{(1+x)^k}dy=\frac{y^{j-1}}{2(1+x)^k}dy^2=\frac{y^{j-1}}{2(1+x)^k}d(2H-x^2)\\ =&\frac{y^{j-1}}{(1+x)^k}dH-\frac{xy^{j-1}}{(1+x)^k}dx=\frac{y^{j-1}}{(1+x)^k}dH-\frac{y^{j-1}}{(1+x)^{k-1}}dx+\frac{y^{j-1}}{(1+x)^k}dx\\ =&\frac{y^{j-1}}{(1+x)^k}dH-\omega^{k-1}_{0,j-1}+\omega^k_{0,j-1}.\end{aligned}$$

We get (2.3). This ends the proof. □

Lemma 2.4. *The generators $J_k(h)$ can be obtained through Maple software*

$$\begin{aligned}J_1(h)=&2\int_{-\sqrt{2h}}^{\sqrt{2h}}\frac{x}{(1+x)\sqrt{2h-x^2}}dx=2\pi\left(1-\frac{1}{\sqrt{1-2h}}\right),\\ J_2(h)=&-\frac{4h\pi}{(1-2h)^{3/2}},\quad J_3(h)=-\frac{6h\pi}{(1-2h)^{5/2}},\\ J_4(h)=&-\frac{4h\pi\,(h+2)}{(1-2h)^{7/2}},\quad J_5(h)=-\frac{5h\pi\,(3h+2)}{(1-2h)^{9/2}},\end{aligned}$$

$$J_6(h)=-\frac{6h\pi\left(h^2+6h+2\right)}{(1-2h)^{11/2}},\quad J_7(h)=-\frac{7h\pi\left(5h^2+10h+2\right)}{(1-2h)^{13/2}},$$
$$J_8(h)=-\frac{2h\pi\left(5h^3+60h^2+60h+8\right)}{(1-2h)^{15/2}},\ J_9(h)=-\frac{9h\pi\left(35h^3+140h^2+84h+8\right)}{4\,(1-2h)^{17/2}},$$
$$J_{10}(h)=-\frac{5\left(7h^4+140h^3+280h^2+112h+8\right)\pi h}{2\,(1-2h)^{19/2}},$$
$$J_{11}(h)=-\frac{11\left(63h^4+420h^3+504h^2+144h+8\right)\pi h}{4\,(1-2h)^{21/2}}.$$

The Chebyshev criterion will be used in the following, see [4,5,13–15] for example.

Let f_0, f_1,...,f_{n-1} be analytic functions on an open interval L of R. An ordered set $(f_0, f_1, ..., f_{n-1})$ is an extended complete Chebyshev system(in short, ECT-system) on Lif, for all $i = 1, 2, ..., n-1$, any nontrivial linear

$$\lambda_0 f_0(x)+\lambda_1 f_1(x)+...+\lambda_i f_i(x) \tag{2.4}$$

has at most $i-1$ isolated zeros on L counted with multiplicities.

Lemma 2.5 ([13]). *$(f_0, f_1, ..., f_{n-1})$ is an ECT-system on L if, and only if, for each $i = 1, 2, ..., n$, $W[f_0(x), f_1(x), \cdots, f_{i-1}(x)] \neq 0$ for all $x \in L$.*

Remark 2.2. If $(f_0, f_1, ..., f_{n-1})$ is an ECT-system on L, then for each $i = 1, 2, ..., n-1$, there exists a linear combination (2.4) with exactly $i-1$ simple zeros on L (see for instance Remark 3.7 in [8].

3. The calculation of $M_k(h)$

Note that δ_{0j}^k, $j = 2, 4, 6$ can be rewritten in the following forms that can be used to compute $\tilde{q}_i$ in Lemma 2.2.

$$\begin{aligned}
\delta_{04}^k =&\frac{y^3\,dH}{(1+x)^k}+d\left(\frac{y^3\,(xk-x+1)\,(1+x)}{(1+x)^k\,(k^2-3k+2)}\right)-\frac{3y\,(xk-x+1)\,(1+x)\,dH}{(1+x)^k\,(k^2-3k+2)}\\
&+\frac{3xy\,(xk-x+1)\,(1+x)\,dx}{(1+x)^k\,(k^2-3k+2)},\qquad \delta_{02}^k=\frac{y\,dH}{(1+x)^k}-\frac{xy\,dx}{(1+x)^k},\\
\delta_{06}^k =&\frac{y^5\,dH}{(1+x)^k}+d\left(\frac{y^5\,(xk-x+1)\,(1+x)}{(k^2-3k+2)\,(1+x)^k}\right)-\frac{5y^3\,(xk-x+1)\,(1+x)\,dH}{(k^2-3k+2)\,(1+x)^k}\\
&-5d\left(\frac{y^3\left(k^2x^2-4kx^2+3xk+3x^2-6x+3\right)(1+x)^2}{(k^2-3k+2)\,(1+x)^k\,(k^2-7k+12)}\right)\\
&+\frac{15y\left(k^2x^2-4kx^2+3xk+3x^2-6x+3\right)(1+x)^2\,dH}{(k^2-3k+2)\,(1+x)^k\,(k^2-7k+12)}\\
&-\frac{15yx\left(k^2x^2-4kx^2+3xk+3x^2-6x+3\right)(1+x)^2\,dx}{(k^2-3k+2)\,(1+x)^k\,(k^2-7k+12)}.
\end{aligned} \tag{3.1}$$

3.1. The Melnikov function $M_1(h)$

Lemma 3.1. *The 1-form ω of (2.1) can be expressed as $\omega = \bar{q}_0 dH + d\bar{Q}_0 + N_0$, where*

$$\begin{aligned}
\bar{q}_0 = \bar{q}_0(x,y) = & \frac{a_{03}y^2}{(1+x)^3} + \left(\frac{b_{11}}{(1+x)^3} + \frac{b_{21}x}{(1+x)^3}\right) y \\
& + \frac{a_{01}-a_{11}+a_{21}}{(1+x)^3} + \frac{a_{11}-2a_{21}+a_{03}+b_{02}-b_{12}}{(1+x)^2} + \frac{a_{21}-2a_{03}+2b_{12}}{1+x}, \\
d\bar{Q}_0 = d\bar{Q}_0(x,y,H) = & \frac{(2a_{03}-2b_{12})\,x + a_{03} - b_{02} - b_{12}}{(1+x)^2} dH \\
& - \frac{(a_{21}-b_{30})\,x^3 + (a_{11}-b_{20})\,x^2 + (a_{01}-b_{10})\,x - b_{00}}{(1+x)^3} dx \\
& + \left(-\frac{3b_{03}y^2}{2\,(1+x)^2} + a_{30} - a_{12} - \frac{b_{01}}{2\,(1+x)^2} + b_{21} - \frac{3}{2}b_{03}\right) dy \\
& + \left(\frac{b_{03}y^3}{(1+x)^3} - \frac{(a_{03}-b_{12})x - b_{02}}{(1+x)^3}y^2 + \frac{b_{01}y}{(1+x)^3}\right) dx, \\
N_0 = & \left(A_5\delta_{00}^3 + A_4\delta_{00}^2\right) H + A_3\delta_{00}^3 + A_2\delta_{00}^2 + A_1\delta_{00}^1,
\end{aligned}$$

with

$$\begin{aligned}
A_1 =& a_{20} - a_{02} - 3a_{30} + 3a_{12} + b_{11} - 3b_{21} + 3b_{03}, \\
A_2 =& a_{10} - 2a_{20} + 2a_{02} + 3a_{30} - 3a_{12} + \frac{1}{2}b_{01} - 2b_{11} + 3b_{21} - \frac{3}{2}b_{03}, \\
A_3 =& - a_{10} + a_{20} - a_{02} - a_{30} + a_{12} + b_{11} - b_{21} + a_{00}, \\
A_4 =& 2a_{12} - 2b_{21} + 3b_{03}, \quad A_5 = 2a_{02} - 2a_{12} - 2b_{11} + 2b_{21}.
\end{aligned}$$

Theorem 3.1. *$M_1(h)$ has at most three zeros, i.e., system (1.1) has at most three limit cycles by the first-order Melnikov function and the upper bound for the number of limit cycles is reached.*

Proof. Let $z = \sqrt{1-2h}$, $z \in (0,1)$, together with Lemmas 2.2 and 2.4

$$\begin{aligned}
& M_1(h) = \oint_{H=h} N_0 = \frac{(z-1)\pi}{2z^5}\left(k_0(z+1) + k_1 z^2(z+1) + k_2 z^4 + k_3 z^5\right), \\
& k_0 = 6A_3 + 3A_5 = 6(a_{00} - a_{10} + a_{20} - a_{30}), \\
& k_1 = 4A_2 + 2A_4 - 3a_5 = 4a_{10} - 8a_{20} + 2a_{02} + 12a_{30} - 2a_{12} + 2b_{01} - 2b_{11} + 2b_{21}, \\
& k_2 = 4A_1 - 2A_4 = 4a_{20} - 4a_{02} - 12a_{30} + 8a_{12} + 4b_{11} - 8b_{21} + 6b_{03}, \\
& k_3 = -2A_4 = 2a_{12} - 2b_{21} + 3b_{03},
\end{aligned}$$

and there exists $\frac{\partial(k_0,k_1,k_2,k_3)}{\partial(a_{00},a_{10},a_{12},a_{30})} = -288 \neq 0$. Since $\{z+1, z^2(z+1), z^4, z^5\}$ is an ECT system for $z \in (0,1)$ by Lemma 2.5 and $M_1(h)$ has precisely three zeros when taking appropriate coefficients by Remark 2.2. $\square$

3.2. The Melnikov function $M_2(h)$

If $A_1 = A_4 = 4A_2 - 3A_5 = 6A_3 + 3A_5 = 0$, then $M_1(h) \equiv 0$, i.e.,

$$\begin{aligned} a_{00} =& \frac{1}{3}a_{20} + \frac{1}{6}a_{02} - \frac{1}{2}b_{01} - \frac{1}{6}b_{11} + \frac{1}{4}b_{03},\ a_{12} = b_{21} - \frac{3}{2}b_{03}, \\ a_{10} =& a_{20} + \frac{1}{2}(a_{02} - b_{01} - b_{11}) + \frac{3}{4}b_{03},\ a_{30} = \frac{1}{3}(a_{20} - a_{02} + b_{11}) - \frac{1}{2}b_{03}. \end{aligned} \tag{3.2}$$

Let (3.2) hold. One can compute that

$$\begin{aligned} N_0 =& \bar{A}\left(4\delta_{300}H + 3\delta_{200} - 2\delta_{300}\right) = \frac{\bar{A}\left(2x^2 + 2y^2 + 3x + 1\right)dy}{(1+x)^3} \\ =& \bar{A}\left(\frac{2x+1}{(1+x)^2}dy + \frac{2y^2}{(1+x)^3}dy\right) = \bar{A}\left(\frac{2x+1}{(1+x)^2}dy + \frac{y}{(1+x)^3}dy^2\right) \\ =& \bar{A}\left(\frac{2x+1}{(1+x)^2}dy + \frac{y}{(1+x)^3}d(2H - x^2)\right) \\ =& \bar{A}\left(d\left(\frac{(2x+1)y}{(1+x)^2}\right) + \frac{2y}{(1+x)^3}dH\right), \quad \bar{A} = \frac{2a_{02} + 3b_{03} - 2b_{11}}{4}, \end{aligned}$$

together with Lemma 2.2, we obtain

$$\tilde{q}_0 = \frac{(2a_{02} + 3b_{03} - 2b_{11})\, y}{2\,(1+x)^3}, \quad d\tilde{Q}_0 = \frac{2a_{02} + 3b_{03} - 2b_{11}}{4}d\left(\frac{(2x+1)y}{(1+x)^2}\right).$$

Then

$$\begin{aligned} & q_0 = \bar{q}_0 + \tilde{q}_0 = f_0(x) + g_0(x,y), \\ & f_0(x) = \frac{a_{01} - a_{11} + a_{21}}{(1+x)^3} + \frac{a_{11} - 2a_{21} + a_{03} + b_{02} - b_{12}}{(1+x)^2} + \frac{a_{21} - 2a_{03} + 2b_{12}}{1+x}, \\ & g_0(x,y) = \frac{a_{03}y^2}{(1+x)^3} + \left(\frac{b_{11}}{(1+x)^3} + \frac{b_{21}x}{(1+x)^3}\right) y + \tilde{q}_0, \\ & dQ_0 = d\bar{Q}_0 + d\tilde{Q}_0 = \Omega_1(x) + \Omega_2(x,y) + h(x)dH, \\ & \Omega_1(x) = -\frac{\left((a_{21} - b_{30})\, x^3 + (a_{11} - b_{20})\, x^2 + (a_{01} - b_{10})\, x - b_{00}\right) dx}{(1+x)^3}, \\ & \Omega_2(x,y) = \left(-\frac{3b_{03}y^2}{2\,(1+x)^2} + \frac{1}{3}(a_{20} - a_{02} + b_{11}) - \frac{1}{2}b_{03} - \frac{b_{01}}{2\,(1+x)^2}\right) dy \\ & \qquad + \left(\frac{b_{03}y^3}{(1+x)^3} - \frac{((a_{03} - b_{12})\, x - b_{02})\, y^2}{(1+x)^3} + \frac{b_{01}y}{(1+x)^3}\right) dx + d\tilde{Q}_0, \\ & h(x) = \frac{(a_{03} - b_{12})\,(1+2x) - b_{02}}{(1+x)^2}. \quad \text{And} \quad \omega = q_0 dH + dQ_0. \end{aligned} \tag{3.3}$$

Lemma 3.2. *The 1-form $q_0\omega$ can be decomposed into*

$$q_0\omega = \bar{q}_1 dH + d\bar{Q}_1 + N_1, \tag{3.4}$$

where

$$\begin{aligned}\bar{q}_1 =& q_0h(x) + q_0^2 + \left(\frac{\beta_{03}^4}{(1+x)^4} + \frac{\beta_{03}^5}{(1+x)^5} + \frac{4\alpha_{04}^6}{5\,(1+x)^5} + \frac{\alpha_{04}^5}{(1+x)^4}\right) y^2\\
&+ \frac{\beta_{01}^3}{(1+x)^3} + \frac{\beta_{01}^2}{(1+x)^2} + \frac{\beta_{01}^4}{(1+x)^4} + \frac{\beta_{01}^5}{(1+x)^5} + \frac{2\alpha_{02}^6}{5\,(1+x)^5}\\
&+ \frac{\alpha_{02}^5}{2\,(1+x)^4} + \frac{2\alpha_{02}^4}{3\,(1+x)^3} + \frac{\alpha_{02}^3}{(1+x)^2} - \frac{1+3x}{3\,(1+x)^3}(\beta_{03}^4 + \alpha_{04}^5)\\
&- \frac{(1+4x)\left(4\alpha_{04}^6 + 5\beta_{03}^5\right)}{30\,(1+x)^4}, \quad N_1 = B_7\delta_{00}^5 H^2 + \sum_{i=3}^{5} B_{6i}\delta_{00}^i H + \sum_{l=1}^{5} B_l\delta_{00}^l,\end{aligned}$$

with

$$\begin{aligned}
&B_1 = -\beta_{02}^3 + \alpha_{01}^2 + \beta_{04}^5 + \alpha_{05}^6 - \alpha_{03}^4 + \beta_{00}^1,\\
&B_2 = 2\beta_{02}^3 - \beta_{02}^4 - 4\beta_{04}^5 - 4\alpha_{05}^6 - \frac{3}{4}\alpha_{03}^5 + 2\alpha_{03}^4 + \frac{1}{2}\alpha_{01}^3 + \beta_{00}^2,\\
&B_3 = -\beta_{02}^3 + 2\beta_{02}^4 + 6\beta_{04}^5 - \beta_{02}^5 + 6\,\alpha_{05}^6 - \frac{3}{5}\alpha_{03}^6 + \frac{3}{2}\alpha_{03}^5 - \alpha_{03}^4 + \frac{1}{3}\alpha_{01}^4 + \beta_{00}^3,\\
&B_4 = -\beta_{02}^4 - 4\beta_{04}^5 + 2\beta_{02}^5 - 4\alpha_{05}^6 + \frac{6}{5}\,\alpha_{03}^6 - \frac{3}{4}\alpha_{03}^5 + \frac{1}{4}\alpha_{01}^5 + \beta_{00}^4,\\
&B_5 = \beta_{04}^5 - \beta_{02}^5 + \alpha_{05}^6 - \frac{3}{5}\alpha_{03}^6 + \frac{1}{5}\alpha_{01}^6 + \beta_{00}^5,\\
&B_{63} = 2\beta_{02}^3 - 4\beta_{04}^5 - 4\alpha_{05}^6 + 2\alpha_{03}^4, \quad B_{64} = 2\beta_{02}^4 + 8\beta_{04}^5 + 8\alpha_{05}^6 + \frac{3}{2}\alpha_{03}^5,\\
&B_{65} = 2\beta_{02}^5 - 4\beta_{04}^5 - 4\alpha_{05}^6 + \frac{6}{5}\alpha_{03}^6, \quad B_7 = 4(\alpha_{05}^6 + \beta_{04}^5).
\end{aligned}$$

Proof. Following (3.3), we have

$$q_0\omega = q_0(q_0dH + dQ_0) = q_0^2dH + q_0dQ_0. \tag{3.5}$$

First, we decompose q_0dQ_0,

$$\begin{aligned}q_0dQ_0 =&(f_0(x) + g_0(x,y))(\Omega_1(x) + \Omega_2(x,y) + h(x)dH)\\
=&f_0(x)\Omega_1(x) + q_0h(x)dH + g_0(x,y)\Omega_1(x) + q_0\Omega_2(x,y)\\
=&f_0(x)\Omega_1(x) + q_0h(x)dH + \sum_{k=2}^{6}\sum_{j=1}^{k-1}\alpha_{0j}^k\omega_{0j}^k + \sum_{k=1}^{5}\sum_{j=0}^{k-1}\beta_{0j}^k\delta_{0j}^k,\end{aligned} \tag{3.6}$$

where α_{ij}^k and β_{ij}^k are the coefficients of ω_{ij}^k and δ_{ij}^k respectively. And the expressions of α_{ij}^k and β_{ij}^k are omitted for simplicity.

From (2.2) and (2.3), we can calculate the decomposed expressions of ω_{0j}^k and δ_{0j}^k. By (3.5), (3.6) and the expressions of ω_{0j}^k and δ_{0j}^k, (3.4) is given. The proof is completed. □

Theorem 3.2. *Let $M_1(h) \equiv 0$, $M_2(h)$ has at most four zeros, then system* (1.1) *has at most four limit cycles by the second-order Melnikov function, and the maximum number can be attained.*

Proof. Let $z=\sqrt{1-2h}$, $z\in(0,1)$. Together with Lemmas 2.2 and 2.4,

$$M_2(h)=\oint_{H=h} N_1=\frac{\pi(z-1)}{16z^9}(l_5z^8+l_4z^6(z+1)+l_3z^4(z+1)+l_2z^2(z+1)+l_1(z+1)),$$

where

$$\begin{aligned} l_1 =&140B_5+35B_7+70B_{65},\ l_2=80B_4-60B_5-85B_7+40B_{64}-100B_{65},\\ l_3 =&48B_3-16B_4+65B_7+24B_{63}-48B_{64}+30B_{65},\\ l_4 =&32B_2-15B_7-24B_{63}+8B_{64},\ \ l_5=32B_1. \end{aligned}$$

One can verify that $\{z+1, z^2(z+1), z^4(z+1), z^6(z+1), z^8\}$ is an ECT system for $z\in(0,1)$ by Lemma 2.5. Let $b_{11}=b_{03}=a_{02}=b_{01}=a_{01}=b_{00}=b_{02}=a_{21}=0$, $\frac{\partial(l_1,l_2,l_3,l_4,l_5)}{\partial(a_{20},a_{11},a_{03},b_{12},b_{21})}=-\frac{458752}{3}a_{11}a_{20}b_{21}^2\,(3\,b_{10}-b_{30})\neq 0$ if $b_{30}\neq 3b_{10}$, then $M_2(h)$ has exactly four zeros for $h\in(0,\frac{1}{2})$ by Remark 2.2. □

3.3. The higher-order Melnikov function

Let $M_2(h)\equiv 0$, i.e., $l_i=0$ for $i=1,2,\cdots,5$. By using the Maple software, there are 30 cases when $l_k=0$; some cases are short, however, some cases are very long, and we can not continue to compute the higher-order Melnikov functions. Since for other short cases, we obtain fewer limit cycles by calculating the higher-order Melnikov functions. Here, we only show three cases.

$$\begin{aligned} &Case\,(1)\quad a_{02}=a_{20}=b_{01}=b_{03}=b_{11}=b_{21}=0;\\ &Case\,(2)\quad a_{01}=\frac{1}{2}a_{21}-b_{02},\ a_{02}=a_{20}=b_{03}=b_{11}=b_{21}=0,\\ &\qquad\qquad a_{11}=\frac{3}{2}a_{21}-b_{02},\ b_{12}=\frac{9}{10}a_{03}-\frac{1}{2}a_{21};\\ &Case\,(3)\quad a_{02}=5b_{11},\ a_{03}=a_{21}-\frac{2}{3}\,a_{11}-\frac{2}{3}\,b_{02},\ a_{20}=2b_{11},\\ &\qquad\qquad b_{01}=b_{11},\ b_{03}=-\frac{10}{3}b_{11},\ b_{12}=-a_{11}+a_{21}-b_{02},\ b_{21}=0. \end{aligned} \tag{3.7}$$

3.3.1. Case (1):

Theorem 3.3. *Let* (3.2) *and Case (1) of* (3.7) *hold, then* $M_k(h)\equiv 0$ *for* $k\geq 3$, *i.e., the perturbation system* (1.1) *is integrable.*

Proof. Let (3.2) and Case (1) of (3.7) hold, then $M_1(h)=M_2(h)\equiv 0$, substituting Case (1) of (3.7) into (3.3) yields $N_0=0$ and $\omega=q_0dH+dQ_0$, where

$$\begin{aligned} q_0 =&\frac{a_{03}y^2}{(1+x)^3}+\frac{a_{01}-a_{11}+a_{21}}{(1+x)^3}+\frac{a_{11}-2a_{21}+a_{03}+b_{02}-b_{12}}{(1+x)^2}\\ &+\frac{-2a_{03}+a_{21}+2b_{12}}{1+x},\\ dQ_0 =&\frac{((2a_{03}-2b_{12})\,x+a_{03}-b_{02}-b_{12})\,dH}{(1+x)^2}-\frac{((a_{03}-b_{12})\,x-b_{02})\,y^2\,dx}{(1+x)^3}\\ &-\frac{\big((a_{21}-b_{30})\,x^3+(a_{11}-b_{20})\,x^2+(a_{01}-b_{10})\,x-b_{00}\big)\,dx}{(1+x)^3}. \end{aligned}$$

Substituting Case (1) of (3.7) into (3.4), one can obtain $q_0\omega = \bar{q}_1 dH + d\bar{Q}_1 + N_1$, where $N_1 = 0$ which implies $\tilde{q}_1 = d\tilde{Q}_1 = 0$ and hence

$$q_1 = \bar{q}_1 = f_1(x) + g_1(x,y),$$
$$q_1\omega = (q_1 q_0 + q_1 h(x))dH + f_1(x)\Omega_1(x) + g_1(x,y)\Omega_1(x) + q_1\Omega_2(x,y),$$

where

$$g_1(x,y) = \frac{a_{03}^2 y^4}{(1+x)^6} + \left(\frac{(a_{01} - a_{11} + a_{21})\, a_{03}}{2\,(1+x)^6} - \frac{a_{03}\,(-2a_{21} + 3a_{03} - 3b_{12})}{(1+x)^4} + \frac{a_{03}\,(10a_{11} - 20a_{21} + 9a_{03} + 9b_{02} - 9b_{12})}{5\,(1+x)^5}\right) y^2,$$
$$g_1(x,y)\Omega_1(x) + q_1\Omega_2(x,y) = m_{06}^9\omega_{06}^9 + \sum_{k=6}^{9} m_{04}^k\omega_{04}^k + \sum_{k=4}^{9} m_{02}^k\omega_{02}^k.$$

Note that since the expressions of $f_1(x)$ and m_{0j}^k are too long, we omit them here. By Lemma 2.3, we know that $\omega_{ij}^k = h_{ij}^k(x,H)dH + d(Q_{ij}^k(x,H))$ when j is even. Therefore, $q_1\omega = \bar{q}_2 dH + d\bar{Q}_2 + N_2$ where $N_2 = 0$, which implies $M_3(h) = \oint_{H=h} N_2 = 0$. Moreover, $N_2 = 0$ yields $\tilde{q}_2 = 0$, and $q_1\omega = \bar{q}_2 dH + d\bar{Q}_2 := q_2 dH + dQ_2$. By taking the similar step, we obtain $q_2\omega = \bar{q}_3 dH + d\bar{Q}_3$. In sum, $N_i = 0$, $i \geq 2$ and $M_k(h) = \oint_{H=h} N_{k-1} = 0$, $k \geq 3$. □

3.3.2. Case (2):

Lemma 3.3. *Let* (3.2) *and Case (2) of* (3.7) *hold. We have*

$$q_1\omega = \bar{q}_2 dH + d\bar{Q}_2 + N_2, \tag{3.8}$$

where

$$q_1 = \bar{q}_1 + \frac{b_{01}a_{03}y}{10(1+x)^5},$$
$$\bar{q}_2 = q_1 q_0 + q_1 h(x) + \left(\frac{3\xi_{06}^9}{4\,(1+x)^8} + \frac{6\xi_{06}^8}{7\,(1+x)^7}\right) y^4 + \left(\frac{\xi_{04}^9}{2\,(1+x)^8} + \frac{3\xi_{06}^9 + 4\xi_{04}^8}{7\,(1+x)^7} + \frac{\frac{4}{7}\xi_{06}^8 - \frac{1}{2}\xi_{06}^9 + \frac{2}{3}\xi_{04}^7}{(1+x)^6} + \frac{\frac{4\xi_{04}^6}{5} - \frac{24\xi_{06}^8}{35}}{(1+x)^5}\right) y^2$$
$$+ \frac{7\eta_{01}^7 + \xi_{04}^9 + 2\xi_{02}^8}{7\,(1+x)^7} + \frac{\frac{\xi_{06}^9}{7} - \frac{\xi_{04}^9}{6} + \frac{4\xi_{04}^8}{21} + \frac{\xi_{02}^7}{3}}{(1+x)^6} + \frac{\left(\frac{2\xi_{02}^4}{3} + \frac{16\xi_{06}^8}{35} - \frac{8\xi_{04}^6}{15}\right)}{(1+x)^3}$$
$$+ \frac{\frac{2\xi_{02}^6}{5} - \frac{13\xi_{06}^9}{35} + \frac{8\xi_{06}^8}{35} - \frac{8\xi_{04}^8}{35} + \frac{4\xi_{04}^7}{15}}{(1+x)^5} + \frac{\frac{\xi_{02}^5}{2} + \frac{\xi_{06}^9}{4} - \frac{22\xi_{06}^8}{35} - \frac{\xi_{04}^7}{3} + \frac{2\xi_{04}^6}{5}}{(1+x)^4},$$
$$N_2 = c_7\delta_{00}^8 H^2 + \left(c_{61}\delta_{00}^6 + c_{62}\delta_{00}^7 + c_{63}\delta_{00}^8\right) H + \sum_{i=1}^{5} c_i\delta_{00}^{i+3},$$

with

$$
\begin{aligned}
c_1 =& -\eta_{02}^6 + \eta_{04}^8 - \frac{1}{2}\xi_{03}^7 + \frac{5}{8}\xi_{05}^9 + \frac{1}{4}\xi_{01}^5 + \eta_{00}^4,\\
c_2 =& 2\eta_{02}^6 - 4\eta_{04}^8 - \eta_{02}^7 - \frac{3}{7}\xi_{03}^8 + \xi_{03}^7 - \frac{5}{2}\xi_{05}^9 + \frac{1}{5}\xi_{01}^6 + \eta_{00}^5,\\
c_3 =& -\eta_{02}^6 + 6\eta_{04}^8 + 2\eta_{02}^7 + \frac{6}{7}\xi_{03}^8 - \frac{1}{2}\xi_{03}^7 + \frac{15}{4}\xi_{05}^9 + \frac{1}{6}\xi_{01}^7 + \eta_{00}^6,\\
c_4 =& -4\eta_{04}^8 - \eta_{02}^7 - \frac{3}{7}\xi_{03}^8 - \frac{5}{2}\xi_{05}^9 + \frac{1}{7}\xi_{01}^8 + \eta_{00}^7,\\
c_5 =& \eta_{04}^8 + \frac{5}{8}\xi_{05}^9, \quad c_{61} = 2\eta_{02}^6 - 4\eta_{04}^8 + \xi_{03}^7 - \frac{5}{2}\xi_{05}^9,\\
c_{62} =& 8\eta_{04}^8 + 2\eta_{02}^7 + \frac{6}{7}\xi_{03}^8 + 5\xi_{05}^9,\ c_{63} = -4\eta_{04}^8 - \frac{5}{2}\xi_{05}^9,\ c_7 = 4\eta_{04}^8 + \frac{5}{2}\xi_{05}^9.
\end{aligned}
$$

Proof. Under (3.2) and Case (2) of (3.7), we have $M_1(h) = M_2(h) \equiv 0$, $N_0 = 0$ and

$$
\begin{aligned}
N_1 =& \frac{b_{01}a_{03}}{120}\left(12x^2\delta_{00}^5 + 12y^2\delta_{00}^5 - 8\delta_{00}^3 + 21\delta_{00}^4 - 12\delta_{00}^5\right)\\
=& \frac{b_{01}a_{03}}{120}\frac{\left(4x^2 + 12y^2 + 5x + 1\right)dy}{(1+x)^5} = \frac{b_{01}a_{03}}{120}\left(\frac{6y}{(1+x)^5}dy^2 + \frac{4x+1}{(1+x)^4}dy\right)\\
=& \frac{b_{01}a_{03}}{120}\left(\frac{12y}{(1+x)^5}dH - \frac{12xy}{(1+x)^5}dx + \frac{4x+1}{(1+x)^4}dy\right)\\
=& \frac{b_{01}a_{03}}{120}\left(\frac{12y}{(1+x)^5}dH + d\left(\frac{(4x+1)y}{(1+x)^4}\right)\right).
\end{aligned}
$$

Together with Lemma 2.2, N_1 can be rewritten as $N_1 = \tilde{q}_1 dH + d\tilde{Q}_1$ by $\oint_{H=h} N_1 = 0$. Therefore, $\tilde{q}_1 = \frac{b_{01}a_{03}y}{10(1+x)^5}$, and $q_1 = \bar{q}_1 + \tilde{q}_1 := f_1(x) + g_1(x,y)$. Then we have $q_1\omega = q_1 q_0 dH + q_1 dQ_0$, where

$$
\begin{aligned}
q_1 dQ_0 =& q_1(\Omega_1(x) + \Omega_1(x,y) + h(x)dH)\\
=& q_1 h(x)dH + f_1(x)\Omega_1(x) + g_1(x,y)\Omega_1(x) + q_1\Omega_2(x,y)\\
=& q_1 h(x)dH + f_1(x)\Omega_1(x) + \sum_{k=5}^{8}\xi_{01}^k\omega_{01}^k + \sum_{k=4}^{8}\xi_{02}^k\omega_{02}^k + \sum_{k=7}^{8}\xi_{03}^k\omega_{03}^k + \sum_{k=6}^{9}\xi_{04}^k\omega_{04}^k\\
&+ \xi_{05}^9\omega_{05}^9 + \xi_{06}^8\omega_{06}^8 + \xi_{06}^9\omega_{06}^9 + \sum_{k=4}^{7}\eta_{00}^k\delta_{00}^k + \sum_{k=6}^{7}\eta_{02}^k\delta_{02}^k + \eta_{01}^7\delta_{01}^7 + \eta_{04}^8\delta_{04}^8,
\end{aligned}
$$

here, the expressions of ξ_{0j}^k and η_{0j}^k are omitted for brevity. The decompositions of ω_{0j}^k and δ_{0j}^k can be obtained from (2.2) and (2.3). □

Theorem 3.4. *Let* (3.2) *and Case (2) of* (3.7) *hold.* $M_3(h)$ *has at most four zeros, moreover, system* (1.1) *has at most four limit cycles by the third order Melnikov function and the upper bound is reached.*

Proof. By Lemmas 2.2 and 2.4, we have

$$
M_3(h) = \oint_{H=h} N_2 = -\frac{3a_{03}b_{01}\pi h}{100\,(1-2h)^{13/2}}(k_4h^4 + k_3h^3 + k_2h^2 + k_1h + k_0),
$$

where

$$\begin{aligned} k_0 =& -60b_{00}, \quad k_1 = -75a_{21} - 300b_{00} + 60b_{02} + 240b_{10} - 30b_{20}, \\ k_2 =& -28a_{03} + 355a_{21} - 150b_{00} - 60b_{02} + 480b_{10} - 690b_{20} + 240b_{30}, \\ k_3 =& 49a_{03} - 340a_{21} - 120b_{02} + 60b_{10} - 480b_{20} + 1380b_{30}, \\ k_4 =& 14a_{03} - 140a_{21} + 240b_{30}. \end{aligned}$$

From the above formula, we can see that $M_3(h)$ has at most four zeros in $h \in (0, \frac{1}{2})$. And one can choose enough coefficients such that $M_3(h)$ has exactly four simple zeros. □

Then $M_3(h) \equiv 0$ is equivalent to

$$\begin{aligned} Case(2a) \quad a_{03} =& 8b_{02} + 28b_{20} - \frac{344b_{30}}{7}, \quad a_{21} = \frac{4}{5}b_{02} + \frac{14b_{20}}{5} - \frac{16b_{30}}{5}, \\ b_{00} =& 0, \quad b_{10} = b_{20} - b_{30}. \end{aligned}$$

By Case (2a) and (3.8), N_2 becomes

$$\begin{aligned} N_2 =& b_{01}\left(14b_{02} + 49b_{20} - 86b_{30}\right)\left(\frac{\left(28b_{02} + 98b_{20} - 172b_{30}\right)y^4}{49\left(1+x\right)^8}\right. \\ &+ \frac{\left(56b_{02} - 784b_{20} + 1616b_{30}\right)y^2}{8575\left(1+x\right)^7} \\ &- \frac{(126b_{02} + 441b_{20} - 684b_{30})y^2 + 154b_2 + 469b_{20} - 806b_{30}}{3675\left(1+x\right)^6} \\ &\left.+ \frac{\frac{4b_{02}}{35} + \frac{66b_{20}}{175} - \frac{114b_{30}}{175}}{\left(1+x\right)^5} + \frac{\left(-\frac{2}{25}b_{02} - \frac{7b_{20}}{25} + \frac{1189b_{30}}{2450}\right)}{\left(1+x\right)^4}\right) dy, \end{aligned}$$

and by (3.1), we have

$$\begin{aligned} N_2 =& b_{01}\left(14b_{02} + 49b_{20} - 86b_{30}\right) \\ &\times \left[n_1 dH + \left(\frac{172b_{30}}{49} - \frac{4}{7}b_{02} - 2b_{20}\right) d\left(\frac{\left(1+7x\right)y^3}{42\left(1+x\right)^7}\right) + m_1 dx + m_2 dy\right], \end{aligned}$$

where

$$\begin{aligned} n_1 =& -\frac{\left(28b_{02} + 98b_{20} - 172b_{30}\right)y^3}{49\left(1+x\right)^8} + \frac{\left(-\frac{44b_{02}}{175} - \frac{134b_{20}}{175} + \frac{1612b_{30}}{1225}\right)y}{\left(1+x\right)^7} \\ &+ \frac{\left(\frac{8b_{02}}{25} + \frac{28b_{20}}{25} - \frac{2378b_{30}}{1225}\right)y}{\left(1+x\right)^6}, \\ m_1 =& \frac{2xy\left(196b_{02}x + 686b_{20}x - 1189b_{30}x + 42b_{02} + 217b_{20} - 383b_{30}\right)}{1225\left(1+x\right)^7}, \\ m_2 =& \frac{\frac{806b_{30}}{3675} - \frac{22b_{02}}{525} - \frac{67b_{20}}{525}}{\left(1+x\right)^6} + \frac{\frac{4b_{02}}{35} + \frac{66b_{20}}{175} - \frac{114b_{30}}{175}}{\left(1+x\right)^5} + \frac{-\frac{2}{25}b_{02} - \frac{7b_{20}}{25} + \frac{1189b_{30}}{2450}}{\left(1+x\right)^4}, \end{aligned}$$

with $\frac{\partial m_1}{\partial y} = \frac{\partial m_2}{\partial x}$. Therefore, N_2 can be rewritten as $N_2 = \tilde{q}_2 dH + d\tilde{Q}_2(x,y)$, where

$$\tilde{q}_2 = b_{01}\left(14b_{02} + 49b_{20} - 86b_{30}\right) n_1.$$

Then, we have $q_2 = \bar{q}_2 + \tilde{q}_2 := f_2(x) + g_2(x,y)$, $\bar{q}_2$ is shown by (3.8). Applying Lemma 2.2 gives rise to

$$\begin{aligned}
M_4 &= \oint_{H=h} q_2\omega = \oint_{H=h} q_2(q_0 dH + dQ_0) = \oint_{H=h} q_2 dQ_0,\\
q_2 dQ_0 &= q_2 h(x) dH + f_2(x)\Omega_1(x) + q_2\Omega_2(x,y) + g_2(x,y)\Omega_1(x)\\
&= q_2 h(x) dH + f_2(x)\Omega_1(x) + \sum_{k=6}^{10} \lambda_{01}^k \omega_{01}^k + \sum_{k=5}^{10} \lambda_{02}^k \omega_{02}^k + \sum_{k=8}^{10} \lambda_{03}^k \omega_{03}^k + \sum_{k=7}^{11} \lambda_{04}^k \omega_{04}^k\\
&\quad + \sum_{k=10}^{11} \lambda_{05}^k \omega_{05}^k + \sum_{k=9}^{11} \lambda_{06}^k \omega_{06}^k + \lambda_{07}^{12}\omega_{07}^{12} + \sum_{k=11}^{12} \lambda_{08}^k \omega_{08}^k + \sum_{k=5}^{9} \mu_{00}^k \delta_{00}^k + \sum_{k=8}^{9} \mu_{01}^k \delta_{01}^k\\
&\quad + \sum_{k=7}^{9} \mu_{02}^k \delta_{02}^k + \mu_{03}^{10}\delta_{03}^{10} + \mu_{04}^{9}\delta_{04}^{9} + \mu_{04}^{10}\delta_{04}^{10} + \mu_{06}^{11}\delta_{06}^{11}.
\end{aligned}$$

Then, ω_{0j}^k and δ_{0j}^k can be deserved by (2.2) and (2.3), and their coefficients are omitted for simplicity.

$$M_4(h) = \frac{b_{01}\left(14b_{02} + 49b_{20} - 86b_{30}\right)\pi h}{343000\left(1-2h\right)^{17/2}}(d_5h^5 + d_4h^4 + d_3h^3 + d_2h^2 + d_1h + d_0),$$

where

$$\begin{aligned}
d_5 &= \left(\frac{1344368b_{30}}{6125} - \frac{128811b_{20}}{875}\right)b_{02} - \frac{19878b_{02}^2}{875} + \frac{4255758b_{20}b_{30}}{6125}\\
&\quad - \frac{29619b_{20}^2}{125} - \frac{20896452b_{30}^2}{42875},\\
d_4 &= \left(\frac{2066801b_{30}}{6125} - \frac{53547b_{20}}{250}\right)b_{02} - \frac{40821b_{02}^2}{875} + \frac{60819b_{20}b_{30}}{125}\\
&\quad - \frac{312537b_{20}^2}{1750} - \frac{13008264b_{30}^2}{42875},\\
d_3 &= \frac{92b_{02}^2}{7} + \left(\frac{3922b_{20}}{35} - \frac{79873b_{30}}{490}\right)b_{02} - \frac{33657b_{20}b_{30}}{49} + \frac{847041b_{30}^2}{1715} + \frac{16119b_{20}^2}{70} + \frac{3}{8}b_{01}^2,\\
d_0 &= \frac{3b_{01}^2}{35},\\
d_2 &= \frac{7036b_{02}^2}{875} + \left(\frac{14722b_{20}}{875} - \frac{197376b_{30}}{6125}\right)b_{02} + \frac{798624b_{20}b_{30}}{6125}\\
&\quad - \frac{4573446b_{30}^2}{42875} - \frac{34464b_{20}^2}{875} + \frac{3}{2}b_{01}^2,\\
d_1 &= \left(\frac{5918b_{30}}{6125} - \frac{396b_{20}}{875}\right)b_{02} - \frac{48b_{02}^2}{875} + \frac{13968b_{20}b_{30}}{6125} - \frac{10746b_{30}^2}{6125} - \frac{648b_{20}^2}{875} + \frac{9b_{01}^2}{10},
\end{aligned}$$

which implies the system (1.1) has at most five limit cycles.

3.3.3. Case (3):

The steps of computating the higher-order Melnikov functions are similar to Case (2). Here, some main elements are listed.

In this case, N_1 can be rewritten as

$$\begin{aligned}N_1 =& -\frac{5b_{11}\left(2a_{11}-3a_{21}+2b_{02}\right)y^4}{9\left(1+x\right)^5}dy-\frac{3b_{11}\left(a_{11}-a_{21}-a_{01}\right)y^2}{\left(1+x\right)^5}dy\\&-\frac{7b_{11}\left(-2a_{11}+3a_{21}-2b_{02}\right)y^2}{4\left(1+x\right)^4}-\frac{8b_{11}\left(2a_{11}-3a_{21}+2b_{02}\right)y^2}{9\left(1+x\right)^3}dy\\&-\frac{b_{11}}{36\left(1+x\right)^4}\left(\left(24a_{11}-36a_{21}+24b_{02}\right)x^3+\left(12b_{02}+12a_{11}-18a_{21}\right)x^2\right.\\&\left.+\left(-36a_{01}-16b_{02}+20a_{11}-12a_{21}\right)x-9a_{01}-4b_{02}+5a_{11}-3a_{21}\right)dy,\end{aligned}$$

and by the formula (3.1), one can obtain

$$\begin{aligned}\tilde{q}_1 =& -\frac{b_{11}y}{9\left(1+x\right)^5}\left(\left(10a_{11}-15a_{21}+10b_{02}\right)y^2+\left(6a_{11}-9a_{21}+6b_{02}\right)x^2\right.\\&\left.+\left(-12a_{11}+18a_{21}-12b_{02}\right)x-27a_{01}+9a_{11}-18b_{02}\right).\end{aligned}$$

Theorem 3.5. *Let* (3.2) *and Case (3) of* (3.7) *hold. $M_3(h)$ has at most five zeros, moreover, system* (1.1) *has at most five limit cycles by the third-order Melnikov function and the upper bound is reached.*

Proof. In this case,

$$M_3(h) = \frac{b_{11}h\pi}{18\left(1-2h\right)^{13/2}}\left(K_5h^5+K_4h^4+K_3h^3+K_2h^2+K_1h+K_0\right), \tag{3.9}$$

the expressions of K_i, $i=0,1,\cdots,5$ are omitted for brevity. Let $b_{02}=b_{30}=0$, and the Jacobian determinant

$$\begin{aligned}&\frac{\partial\left(K_0,K_1,K_2,K_3,K_4,K_5\right)}{\partial(a_{11},a_{01},a_{21},b_{10},b_{20},b_{00})}\\=&225270850387968\left(2a_{11}-3a_{21}\right)^2\Big[9\left(2a_{11}-3a_{21}\right)^2\left(a_{01}+a_{11}-2a_{21}\right)b_{00}\\&+\left(3a_{01}-a_{11}\right)\Big(\left(9a_{11}+18a_{21}\right)a_{01}^2+\Big(-22a_{11}^2+\left(-6b_{10}+48b_{20}\right)a_{11}+18a_{21}^2\\&+\left(9b_{10}-72b_{20}\right)a_{21}\Big)a_{01}+13a_{11}^3-\left(22a_{21}+30b_{10}+12b_{20}\right)a_{11}^2\\&+\Big(9a_{21}^2+\left(93b_{10}+12b_{20}\right)a_{21}\Big)a_{11}+9\left(b_{20}-8b_{10}\right)a_{21}^2\Big)\Big]\end{aligned}$$

still can be nonzero. And one can choose enough coefficients such that $M_3(h)$ has exactly five simple zeros in $h\in(0,\frac{1}{2})$. □

Furthermore, $M_3(h)\equiv 0$ is equivalent to

$$\begin{aligned}&\textit{Case}\,(3a) && a_{01}=-\frac{2}{3}b_{02}+\frac{1}{3}a_{11},\ a_{21}=\frac{2}{3}(b_{02}+a_{11});\\&\textit{Case}\,(3b) && a_{21}=\frac{2}{3}(b_{02}+a_{11}),\ b_{00}=0,\ b_{10}=\frac{1}{4}a_{01}+\frac{53a_{11}}{252}+\frac{11b_{02}}{63},\\& && b_{20}=\frac{25b_{02}}{63}+\frac{43a_{11}}{63},b_{30}=\frac{7a_{11}}{18}+\frac{7b_{02}}{18}.\end{aligned}$$

In Case (3a), we have $\tilde{q}_2=0$ and $q_2=0$, the perturbation system (1.1) is integrable. In Case (3b), we can compute

$$\tilde{q}_2 =\frac{b_{11}\left(3a_{01}-a_{11}+2b_{02}\right)y}{252\left(1+x\right)^8}\left(\left(98a_{11}+98b_{02}\right)x^2+\left(172a_{11}+100b_{02}\right)x+315a_{01}\right.$$

$$- 31a_{11} + 212b_{02})$$

and

$$M_4(h) = -\frac{\pi b_{11}\left(3a_{01} - a_{11} + 2b_{02}\right)h^2}{(1-2h)^{19/2}}\left(D_5h^5 + D_4h^4 + D_3h^3 + D_2h^2 + D_1h + D_0\right)$$

where

$$\begin{aligned}
D_0 =& \frac{9a_{01}^2}{16} - \left(\frac{11a_{11}}{56} + \frac{1}{28}b_{02}\right)a_{01} - \frac{281a_{11}b_{02}}{15876} + 2b_{11}^2 - \frac{88b_{02}^2}{3969} + \frac{689a_{11}^2}{63504},\\
D_1 =& \frac{63a_{01}^2}{16} + \left(\frac{89b_{02}}{84} - \frac{659a_{11}}{168}\right)a_{01} + \frac{68b_{11}^2}{9} + \frac{42271a_{11}^2}{63504} + \frac{353b_{02}^2}{7938} - \frac{991a_{11}b_{02}}{15876},\\
D_2 =& \frac{315a_{01}^2}{64} + \left(\frac{3547b_{02}}{336} - \frac{3571a_{11}}{672}\right)a_{01} - \frac{487b_{11}^2}{6} + \frac{647711a_{11}^2}{254016} + \frac{36047b_{02}^2}{63504}\\
&- \frac{287789a_{11}b_{02}}{63504},\ D_3 = \frac{63a_{01}^2}{64} + \left(\frac{4973b_{02}}{336} + \frac{3925a_{11}}{672}\right)a_{01} + \frac{392015b_{02}^2}{63504}\\
&+ \frac{437b_{11}^2}{3} - \frac{625901a_{11}b_{02}}{63504} - \frac{1181773a_{11}^2}{254016},\\
D_4 =& \left(\frac{149a_{11}}{56} + \frac{22b_{02}}{7}\right)a_{01} + \frac{45886b_{02}^2}{3969} - \frac{94b_{11}^2}{9} - \frac{15781a_{11}^2}{31752} + \frac{135599a_{11}b_{02}}{15876},\\
D_5 =& \frac{1145a_{11}^2}{567} + \frac{1640b_{02}^2}{567} - \frac{292b_{11}^2}{3} + \frac{2785a_{11}b_{02}}{567}.
\end{aligned}$$

From the expression of $M_4(h)$, we know that the system (1.1) has at most five limit cycles.

4. Discussion

When $M_1(h) = M_2(h) = 0$, there are many cases can be obtained, since some cases are very long and challenging to compute $M_3(h)$, and some cases can obtain less number limit cycles than Case (2) or Case (3), we do not list them in Section 3. From the expressions of $q_{i-1}dQ_0$, N_i can be shown, since $q_i = \bar{q}_i + \tilde{q}_i$, therefore, the critical problem is how to compute $\tilde{q}_i$, we solve it by giving the decomposition formulas (3.1). Some main formulas shown in Lemma 2.3, (2.2) and (2.3) can be used to consider higher-order degree perturbations problems.

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