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# NEW EXISTENCE, UNIQUENESS RESULTS FOR MULTI-DIMENSIONAL MULTI-TERM CAPUTO TIME-FRACTIONAL MIXED SUB-DIFFUSION AND DIFFUSION-WAVE EQUATION ON CONVEX DOMAINS

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**Abstract** In this paper, we investigate an efficient analytical method known as two step Adomian decomposition method (TSADM). This method does not require approximation/discretization, lengthy calculations and due to involvement of fractional operators and provides an exact solution. In this study, we generalize the multi-term time-fractional mixed sub-diffusion and diffusion-wave equation into multi dimensions with Caputo derivative for time fractional operators and obtain the exact solution. Furthermore, we establish the new results of existence and uniqueness of the solution using fixed point theory. To demonstrate the effectiveness of the proposed method, several generalized examples on the convex domain are considered.

**Keywords** Fixed point theorems, Caputo's fractional operators, diffusion-wave equation, two-step Adomian decomposition method.

**MSC(2010)** 47H10, 35R11.

## 1. Introduction

Fractional differential equations (FDEs) are essential tools for biology, physics, classical mechanics, quantum mechanics, nuclear physics, astrophysics, hadron spectroscopy, engineering, and in various areas of science. Recently, the FDEs have been widely using in the applied for the discussion of the nonlinear phenomena (such as wave propagation problems, diffusion processes, and solid mechanics), and as a consequence, the find for exact solutions of the FDEs is becoming an emerging field of current research. Several approximations that is numerical methods have been used to solve the FDEs for examples, the Adomian decomposition method (ADM), fractional splines, Chebyshev collocation, backward differentiation formulas, spectral method, Pseudo-spectral method and spline collocation methods. These provide numerical solutions for the FDEs using approximations, and many authors discussed the stability and convergence theorems for these methods [1, 8, 11, 12].

Fractional partial differential equations (FPDEs) is one of the class of the FDEs. The multi-term time-fractional diffusion equation, which mainly has three types: the multi-term time-fractional sub-diffusion equation for fractional-order $(0, 1)$, the

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multi-term time-fractional diffusion-wave equation for fractional-order $(1,2)$ and multi-term time-fractional diffusion equations for fractional-order $(0,a), a \geq 2$. The analytical solutions for the two and three-dimension are available but more than three dimension multi-term time-space fractional diffusion equations have not been obtained in the literature. Whereas, some analytical solutions of the multi-term time-fractional diffusion equation have been provided generally in the form of multinomial Mittag-Leffler functions, which increase the complexity and difficulty for calculations. Therefore, numerical methods are mostly preferred to solve these equations [2–4, 9].

The authors [2] developed an effective numerical method for the fractional sub-diffusion equation and obtained solution with Neumann boundary conditions for two/three-dimensional. Here, the time-fractional derivative has been used to approximated with the help of the $L1$ scheme on graded meshes, and the compact finite difference methods have been used with the spatial discretization. The fully discrete alternating direction implicit ADI method has been discussed with some corrected terms. Also, the convergence of the scheme has been obtained under some assumptions of the weak singularity of the solutions. In [9], the initial/boundary value two-dimensional multi-term time-fractional mixed diffusion and diffusion-wave equations are considered and obtained the solution using the alternating direction implicit (ADI) spectral method is developed based on Legendre spectral approximation for space and finite difference discretization for time. The stability and convergence of the schemes are shown with proof and find the optimal error. They added some correction terms for the non-smooth solution case. The numerical results confirm that the techniques can be applied to model diffusion and transport of viscoelastic non-Newtonian fluids.

A novel two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on convex domains has been solved using finite element method in [4]. They utilized the mixed $L$ schemes to approximate the time-fractional sub-diffusion term, space derivative, and the coupled time- diffusion-wave term. Authors investigated the variational formulation and used the finite element method to discretize the equation. Then they have used linear polynomial basis functions on triangular elements to derive the matrix form for the numerical scheme. Additionally, they established the stability and convergence analysis of the numerical scheme and showed the effectiveness of the scheme. Some examples are considered in the two-dimension multi-term time-fractional mixed diffusion equation on a convex domain for analyzing the results. In [3], the authors used time-space spectral collocation method to solve the two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion-wave equation and increase the accuracy of the obtained solution and used less number of Legendre polynomials.

Xing-Guo Luo [10] presented a very effective method named as the two-step Adomian decomposition method (TSADM) to obtain the exact solution of linear and non-linear ordinary/partial differential equations. Here, we extend this method for solving the multi-term time-fractional mixed sub-diffusion and diffusion-wave equation for multi-dimensions. In most of the cases, an exact solution for multi-dimensions problems is not possible via numerical methods, but we obtain an exact solution successfully using the TSADM and derive an algorithm for extended problem. The effectiveness and applicability of the proposed method is tested on five generalized examples and compared to the results with the popular and efficient numerical methods available in the literature. The proposed method is easy to implement on the considered problems without any approximation or tools used in the numerical methods.

The authors [13] proved the efficiency and application of the TSADM in which they considered such a problem with two different types of fractional operators involve without dealing with these operators, the authors provided an exact solution in just one iteration. Additionally, the new conditions for the existence and uniqueness of a solution have been discussed. In [13], the authors have obtained results for the multi-dimensional time-space tempered fractional diffusion-wave equation. They mentioned that the solution of the considered problem is not possible by using other existing numerical methods because of the high dimensions of the problem.

In this article, we shall discuss the new existence and uniqueness results for the considered problem. We give two such findings; the first one is based on the Banach fixed point theorem, and another is based on the Schaefer's fixed point theorem. The multi-term time-fractional mixed sub-diffusion and diffusion-wave equation is extended into multi-dimension using the Caputo's time-fractional derivatives and find the exact solution without using such tools involved in the numerical methods and reduce the computation effort. Moreover, we obtain an exact solution of the considered problem using the TSADM. The TSADM provides the exact solution without approximation/discretization, and the solution does not involve multinomial Mittag-Leffler functions. The only requirement of the TSADM is that the first term of the series contains the verifying term, which satisfies the equation as well as the associated the initial/boundary conditions. Overall, we observe that the adopted method is more suitable for the multi-dimensional multi-term Caput's time-fractional mixed sub-diffusion and diffusion-wave equation, while other existing methods are not applicable for solving the considered problems via numerically or analytically.

This whole article is arranged as follows. In section 2, we mention basic definitions of fractional calculus and describe some essential theorems and lemmas. Section 3, we prove the main results for the existence and uniqueness conditions of the solution. In section 4, we present the algorithm of the TSADM. In section 5, we consider five examples to show the effectiveness of the TSADM. Finally, we summarize our findings in section 6.

## 2. Basic Concepts

Here, we describe elementary concepts of the Caputo's fractional operators (integrals/derivatives) and their properties. The Caputo's definition of fractional derivatives, which is very famous in applied mathematics. Furthermore, we present some theorems and lemmas that will be helpful to prove our main theorems for the existence and uniqueness of a considered problem [1, 13–17, 19].

**Definition 2.1.** The Gamma function is an extension of the fractional function to real numbers, and defined by

$$\Gamma(\theta) = \int_0^\infty \tau^{\theta-1} \exp(-\tau) d\tau, \theta > 0, \tag{2.1}$$

and

$$\Gamma(\theta+1) = \theta\Gamma(\theta), \tag{2.2}$$

where $\Gamma(\cdot)$ is Gamma function.

**Definition 2.2.** A real function $w(\vartheta)$, $\vartheta > 0$, is said to be in the space $C_\theta$, if $\theta \in \mathbb{R}$, there exist a real number $v(> \theta)$, such that $w(\vartheta) = \vartheta^v w_1(\vartheta)$, where $w_1(\vartheta) \in C[0, \infty)$ and it is said to be in the space $C_\theta^m$ if $w^m \in C_\theta$, $m \in \mathbb{N} \cup \{0\}$.

**Definition 2.3.** The Riemann-Liouville fractional integral operator of order $\sigma \geq 0$, of a function $w \in C_\theta$, $\theta \geq -1$, is defined as

$$\begin{aligned} J^\sigma{}_\vartheta w(\vartheta) &= \frac{1}{\Gamma(\sigma)} \int_0^\vartheta (\vartheta - \xi)^{\sigma-1} w(\xi) d\xi, \quad \sigma > 0, \;\; \vartheta > 0. \\ J^0{}_\vartheta w(\vartheta) &= w(\vartheta). \end{aligned} \tag{2.3}$$

The following properties of the operator $J^\sigma{}_\vartheta$ can be found in [1, 13]. For $w \in C_\theta$, $\theta \geq -1$, $\sigma, \omega \geq 0$ and $\eta > -1$,

$(i)$ $J_\vartheta^\sigma J_\vartheta^\omega w(\vartheta) = J_\vartheta^{\sigma+\omega} w(\vartheta)$. (2.4)

$(ii)$ $J_\vartheta^\sigma J_\vartheta^\omega w(\vartheta) = J_\vartheta^\omega J_\vartheta^\sigma w(\vartheta)$. (2.5)

$(iii)$ $J_\vartheta^\sigma \vartheta^\eta = \dfrac{\Gamma(\eta+1)}{\Gamma(\sigma+\eta+1)} \vartheta^{\sigma+\eta}$. (2.6)

The modified version of the Riemann-Liouville derivative is known as Caputo derivative.

**Definition 2.4.** The fractional derivative of $w(\vartheta)$ in the Caputo sense is defined as

$$D_\vartheta^\sigma w(\vartheta) = J_\vartheta^{m-\sigma} D_\vartheta^m w(\vartheta) = \frac{1}{\Gamma(m-\sigma)} \int_0^\vartheta (\vartheta - \xi)^{m-\sigma-1} w^m(\xi) d\xi, \tag{2.7}$$

for $m - 1 < \sigma \leq m, \;\; m \in \mathbb{N}, \;\; \vartheta > 0, \;\; w \in C_{-1}^m$, and

$$D_\vartheta^\sigma A_1 = 0. \tag{2.8}$$

Due to linearity of Caputo's fractional derivatives, we can write

$$D_\vartheta^\sigma(A_1 h_1(\vartheta) + A_2 h_2(\vartheta)) = A_1 D_\vartheta^\sigma h_1(\vartheta) + A_2 D_\vartheta^\sigma h_2(\vartheta), \tag{2.9}$$

where $A_1$ and $A_2$ are constants.

**Definition 2.5.** For every smallest integer $m$, which exceeds $\sigma$, the Caputo time-fractional derivative operator of order $\sigma > 0$ can be defined as

$$D_\xi{}^\sigma w(\vartheta, \xi) = \frac{\partial^\sigma w(\vartheta, \xi)}{\partial \xi^\sigma} = \begin{cases} \frac{1}{\Gamma(m-\sigma)} \int_0^\xi (\xi - \tau)^{m-\sigma-1} \frac{\partial^m w(\vartheta, \tau)}{\partial \tau^m} d\tau \; ; & m - 1 < \sigma < m, \\ \frac{\partial^m w(\vartheta, \xi)}{\partial \xi^m}; & \sigma = m \in \mathbb{N}. \end{cases} \tag{2.10}$$

**Remark.** If $\sigma = m \in \mathbb{N}$, then the Riemann-Liouville and the Caputo derivatives become a conventional $m$th order derivative of the function $w(\vartheta)$.

The following lemma help us for the solution of considered problems.

**Lemma 2.1.** *For $m - 1 < \sigma \leq m, \;\; m \in \mathbb{N}$ and $w \in C_\theta^m$, $\theta \geq -1$, then*

$$D_\vartheta{}^\sigma J_\vartheta^\sigma w(\vartheta) = w(\vartheta), \tag{2.11}$$

$$J_\vartheta^\sigma D_\vartheta{}^\sigma w(\vartheta) = w(\vartheta) - \sum_{q=0}^{m-1} g^q(0^+) \frac{\vartheta^q}{q!} \;, \;\; y > 0. \tag{2.12}$$

The detail analysis of this study is based on the following results.

**Lemma 2.2.** *Let $\sigma > 0$,*

$$D_{0+}^{\sigma}\psi(\vartheta) = 0, \tag{2.13}$$

*then a general solution of the equation (2.13) to the homogeneous equation is given by*

$$\psi(\vartheta)=a_0+a_1\vartheta+a_2\vartheta^2+a_3\vartheta^3+\cdots+a_{m-1}\vartheta^{m-1}, a_k \in \mathbb{R}, k=1,2,\cdots,m-1(m=[\sigma]+1). \tag{2.14}$$

**Lemma 2.3.** *Let $\sigma > 0$, we have*

$$\begin{aligned} &J_{0+}^{\sigma}D_{0+}^{\sigma}\psi(\vartheta) = \psi(\vartheta) + a_0 + a_1\vartheta + a_2\vartheta^2 + a_3\vartheta^3 + \cdots + a_{m-1}\vartheta^{m-1}, \\ &a_k \in \mathbb{R}, k = 1,2,\cdots,m-1(m = [\sigma]+1). \end{aligned} \tag{2.15}$$

**Theorem 2.1** ( [5], Banach fixed point theorem)**.** *Let $(P,d)$ be a nonempty complete metric space. Let $V : P \to P$ be a map such that for every $p_1, p_2 \in P$, then the inequity*

$$d(Vp_1, Vp_2) = ad(p_1, p_2), a \in [0,1)$$

*holds. Then the operator $V$ has a unique fixed point $p^* \in P$.*

**Theorem 2.2** ( [6], Schaefer's fixed point theorems)**.** *Let $H : T \to T$ is completely continuous operator. If the set $S(H) = \{t \in T : t = c^*H(t) \text{ for some } c^* \in [0,1]\}$ is bounded, then, $H$ has fixed points in $H$.*

**Theorem 2.3** ( [18], Arzelà-Ascoli theorem)**.** *Let $T$ be a compact metric space. Let $C(T,\mathbb{R})$ be given the sup norm metric. Then a set $H \subset C(T)$ is compact iff $H$ is bounded, closed and equicontinous.*

## 3. Existence and uniqueness results

In this section, we develop the existence and uniqueness conditions of the solution for the considered problem by applying some standard fixed point theorems.

We denote $C(\eta,\mathbb{R})$ as the Banach space of all continous functions from $\eta = \Lambda \times I$ into $\mathbb{R}$ with the norm $\|\cdot\|_\infty$, defined by $\|\zeta\|_\infty := \sup\{|\zeta(\Phi,t)|; (\Phi,t) \in \eta\}$ (see [5–7, 18, 19]).

We consider following multi-dimensional multi-term Caputo's time-fractional mixed sub-diffusion and diffusion-wave equation on convex domain with Neumann boundary conditions, describe as

$$\begin{aligned} &\sum_{\nu_1=1}^{r_1} d_{1,\nu_1} D_t^{\sigma_{\nu_1}}(\zeta(\Phi,t)) + d_2\frac{\partial\zeta(\Phi,t)}{\partial t} + \sum_{\nu_2=1}^{r_1} d_{3,\nu_2} D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t)) + d_4\zeta(\Phi,t) \\ =&d_5\nabla\zeta(\Phi,t) + d_6 D_t^{\mu}(\nabla\zeta(\Phi,t)) + \Psi(\Phi,t), \\ &(\Phi,t) = (\Phi_1,\Phi_2,\cdots,\Phi_m,t) \in \Lambda \times I, \end{aligned} \tag{3.1}$$

with the initial and Neumann boundary conditions

$$\begin{aligned} &\zeta(\Phi,0) = \phi_1(\Phi), \zeta_t(\Phi,0) = \phi_2(\Phi), \Phi \in \bar{\Lambda}, \\ &\zeta(\Phi,t) = 0, \Phi \in \partial\Lambda, t \in \bar{I}, \end{aligned} \tag{3.2}$$

where $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_m) \in \Lambda$, $\Lambda = \prod_{i=0}^{m}(-a, a), a > 0$, $\bar{\Lambda} = \prod_{i=0}^{m}[-a, a]$, $\partial\Lambda = \bar{\Lambda}/\Lambda$, $t \in (0, T]$, $d_{1,\nu_1} > 0$, $d_{3,\nu_2} \geq 0$, $d_j \geq 0$, in which $d_j$'s are not zero simultaneously, $r_1$, $r_2$ are integers, $\Psi(\Phi, t)$, $\phi_1(\Phi)$ and $\phi_2(\Phi)$ are known smooth functions, $\zeta(\Phi, t)$ is the unknown function, $\nabla\zeta(\Phi, t) = \sum_{i=1}^{m} \frac{\partial}{\partial\Phi_i^2}(\zeta(\Phi, t))$, $D_t^{\sigma_{\nu_1}}$, $D_t^{\varrho_{\nu_2}}$, $D_t^{\mu}$ $(n - 1 < \sigma_{\nu_1}, \varrho_{\nu_2}, \mu \leq n)$ are the Caputo time-fractional derivatives.

The Caputo time-fractional derivative $D_t^{\sigma}\zeta(\Phi, t)$ $(n - 1 \leq \sigma < n)$ is given by

$$D_t^{\sigma}\zeta(\Phi, t) = \frac{1}{\Gamma(n - \sigma)} \int_0^t (t - s)^{n-\sigma-1}\zeta^n(\Phi, s)ds. \tag{3.3}$$

**Definition 3.1.** A function $\zeta \in C(\eta, \mathbb{R})$ with its $\sigma_{\nu_1}$ derivative existing on $\eta$ is said to be a solution of the considered problem (3.1)–(3.2), if $\zeta$ satisfies the equation (3.1) on $\eta$, and the associated initial/boundary conditions in the equation (3.2).

**Lemma 3.1.** *The functions $\Psi(\Phi, t) : C(\eta) \to C(\eta)$ and $\zeta(\Phi, t) : C(\eta, \mathbb{R}) \to C(\eta, \mathbb{R})$ are continuous. A function $\zeta \in C(\eta, \mathbb{R})$ is a solution of the fractional integral equation*

$$\begin{aligned}\zeta(\Phi, t) =& \phi_1 + t\phi_2 + J_t^{\sigma_{\nu_1}}\Bigg(\bigg(\delta_4\nabla\zeta(\Phi, t) + \delta_5 D_t^{\mu}(\nabla\zeta(\Phi, t)) + \delta_6\Psi(\Phi, t)\bigg) \\ &- \bigg(\delta_1\frac{\partial\zeta(\Phi, t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta(\Phi, t)) + \delta_3\zeta(\Phi, t)\bigg)\Bigg),\end{aligned} \tag{3.4}$$

*if only if $\zeta$ is a solution of the problem* (3.1)–(3.2).

To transform the considered problem (3.1)–(3.2) to a fixed point problem, we define the operator $\Delta : C(\eta, \mathbb{R}) \to C(\eta, \mathbb{R})$ such that

$$\begin{aligned}\Delta\zeta(\Phi, t) =& \phi_1 + t\phi_2 + J_t^{\sigma_{\nu_1}}\Bigg(\bigg(\delta_4\nabla\zeta(\Phi, t) + \delta_5 D_t^{\mu}(\nabla\zeta(\Phi, t)) + \delta_6\Psi(\Phi, t)\bigg) \\ &- \bigg(\delta_1\frac{\partial\zeta(\Phi, t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta(\Phi, t)) + \delta_3\zeta(\Phi, t)\bigg)\Bigg),\end{aligned} \tag{3.5}$$

where

$$\delta_1 = \frac{d_2}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}}, \quad \delta_2 = \frac{\sum_{\nu_2=1}^{r_1} d_{3,\nu_2}}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}}, \quad \delta_3 = \frac{d_4}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}},$$
$$\delta_4 = \frac{d_5}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}}, \quad \delta_5 = \frac{d_6}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}}, \quad \delta_6 = \frac{1}{\sum_{\nu_1=1}^{r_1} d_{1,\nu_1}}.$$

Where the fixed points of the operator $\Delta$ are the solutions of the considered problem (3.1)–(3.2).

Before going to prove main results, we assume the following hypotheses:

$(E_1)$ For any $(\Phi, t) \in C(\eta)$, there exist $\chi_1, \chi_2, \chi_3, \chi_4, \chi' > 0$ such that

$$\begin{aligned}&|\nabla\zeta_1(\Phi, t) - \nabla\zeta_2(\Phi, t)| \leq \chi_1|\zeta_1 - \zeta_2|, \\ &|D_t^{\mu}(\nabla\zeta_1(\Phi, t)) - D_t^{\mu}(\nabla\zeta_2(\Phi, t))| \leq \chi_2|\nabla\zeta_1 - \nabla\zeta_2| = \chi_1\chi_2|\zeta_1 - \zeta_2| = \chi'|\zeta_1 - \zeta_2|, \\ &|D_t^{\varrho_{\nu_2}}(\zeta_1(\Phi, t)) - D_t^{\varrho_{\nu_2}}(\zeta_2(\Phi, t))| \leq \chi_3|\zeta_1 - \zeta_2|,\end{aligned}$$

and

$$\left|\frac{\partial\zeta_1(\Phi, t)}{\partial t} - \frac{\partial\zeta_2(\Phi, t)}{\partial t}\right| \leq \chi_4|\zeta_1 - \zeta_2|.$$

$(E_2)$ For all $\Phi \in \Lambda$ there exist $M_i, M > 0$, $i = 1, 2$ such that

$$|\phi_i(\Phi)| \leq M_i,$$

and

$$|\Psi(\Phi, t)| \leq M.$$

$(E_3)$ There exist constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda' > 0$ such that, for all $(\Phi, t) \in C(\eta)$

$$\begin{aligned}
&|\nabla\zeta(\Phi, t)| \leq \lambda_1 |\zeta(\Phi, t)|,\\
&|D_t^{\mu}(\nabla\zeta(\Phi, t))| \leq \lambda_1\lambda_2 |\zeta(\Phi, t)| = \lambda' |\zeta(\Phi, t)|,\\
&|D_t^{\varrho_{\nu_2}}(\zeta(\Phi, t))| \leq \lambda_3 |\zeta(\Phi, t)|,
\end{aligned}$$

and

$$\left|\frac{\partial\zeta(\Phi, t)}{\partial t}\right| \leq \lambda_4 |\zeta(\Phi, t)|.$$

$(E_4)$ There exist constants $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau', \tau'' > 0$ such that

$$\begin{aligned}
&|\nabla\zeta(\Phi', t') - \nabla\zeta(\Phi'', t'')| \leq \tau_1 |\Phi' - \Phi''| + \tau_2 |t' - t''|,\\
&|D_{t'}^{\mu}\nabla\zeta(\Phi', t') - D_{t''}^{\mu}\nabla\zeta(\Phi'', t'')| \leq \tau' |\Phi' - \Phi''| + \tau'' |t' - t''|,\\
&|D_{t'}^{\varrho_{\nu_2}}\zeta(\Phi', t') - D_{t''}^{\varrho_{\nu_2}}\zeta(\Phi'', t'')| \leq \tau_3 |\Phi' - \Phi''| + \tau_4 |t' - t''|,\\
&\left|\frac{\partial\zeta(\Phi', t')}{\partial t'} - \frac{\partial\zeta(\Phi'', t'')}{\partial t''}\right| \leq \tau_5 |\Phi' - \Phi''| + \tau_6 |t' - t''|,
\end{aligned}$$

and

$$|\zeta(\Phi', t') - \zeta(\Phi'', t'')| \leq \tau_7 |\Phi' - \Phi''| + \tau_8 |t' - t''|.$$

**Theorem 3.1.** *If assumptions $(E_1)$ hold, then the problem* (3.1)–(3.2) *has at least one solution.*

Here, with the help of the Schaefer fixed point theorem. We need to show that the operator $\Delta$ has at least one fixed point.

**Proof.** This proof consists of a number of steps:

**Step 1:** We will show that the operator $\Delta$ is continuous.

Consider a sequence $\zeta_n \to \zeta \in C(\eta, \mathbb{R})$. For any $(\Phi, t) \in \eta$, we have

$$\begin{aligned}
&|\Delta\zeta_n(\Phi, t) - \Delta\zeta(\Phi, t)|\\
=&\Bigg|J_t^{\sigma_{\nu_1}}\Bigg(\Big(\delta_4\nabla\zeta_n(\Phi, t) + \delta_5 D_t^{\mu}(\nabla\zeta_n(\Phi, t))\Big)\\
&-\left(\delta_1\frac{\partial\zeta_n(\Phi, t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta_n(\Phi, t)) + \delta_3\zeta_n(\Phi, t)\right)\Bigg)\\
&-J_t^{\sigma_{\nu_1}}\Bigg(\Big(\delta_4\nabla\zeta(\Phi, t) + \delta_5 D_t^{\mu}(\nabla\zeta(\Phi, t))\Big)\\
&-\left(\delta_1\frac{\partial\zeta(\Phi, t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta(\Phi, t)) + \delta_3\zeta(\Phi, t)\right)\Bigg)\Bigg|\\
\leq&J_t^{\sigma_{\nu_1}}\Bigg(|\delta_4|\Big|\nabla\zeta_n(\Phi, t) - \nabla\zeta(\Phi, t)\Big| + |\delta_5|\Big|D_t^{\mu}(\nabla\zeta_n(\Phi, t)) - D_t^{\mu}(\nabla\zeta(\Phi, t))\Big|\\
&+|\delta_1|\left|\frac{\partial\zeta_n(\Phi, t)}{\partial t} - \frac{\partial\zeta(\Phi, t)}{\partial t}\right| + |\delta_2|\Big|D_t^{\varrho_{\nu_2}}(\zeta_n(\Phi, t)) - D_t^{\varrho_{\nu_2}}(\zeta(\Phi, t))\Big|
\end{aligned}$$

$$
\begin{aligned}
&+|\delta_3|\Big|\zeta_n(\Phi,t)-\zeta(\Phi,t)\Big|\Big) \\
\leq & J_t^{\sigma_{\nu_1}}\Big(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Big)|\zeta_n(\Phi,t)-\zeta(\Phi,t)| \\
\leq & \frac{t^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Big)|\zeta_n(\Phi,t)-\zeta(\Phi,t)| \\
\leq & \frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Big)\|\zeta_n(\Phi,t)-\zeta(\Phi,t)\|_\infty. \quad (3.6)
\end{aligned}
$$

Since $\zeta$ is continuous. Hence, we obtain

$$\|\Delta\zeta_n(\Phi,t)-\Delta\zeta(\Phi,t)\|_\infty\to 0 \quad \text{as} \quad n\to\infty.$$

Therefore, the operator $\Delta$ is continuous.
**Step 2:** We show that $\Delta$ maps bounded sets into bounded sets.

Indeed, we need to show that, for any $\beta>0$, there exists a constant $\xi>0$ such that, for every $\zeta\in B_\beta=\{\zeta\in C(\eta,\mathbb{R}):\|\zeta\|_\infty\leq\beta\}$, one has $\|\Delta\zeta\|_\infty\leq\xi$. For every $t\in I$, we obtain

$$
\begin{aligned}
&|\Delta\zeta(\Phi,t)| \\
=&\Bigg|\phi_1+t\phi_2+J_t^{\sigma_{\nu_1}}\Bigg(\Big(\delta_4\nabla\zeta(\Phi,t)+\delta_5D_t^{\mu}(\nabla\zeta(\Phi,t))+\delta_6\Psi(\Phi,t)\Big) \\
&-\Big(\delta_1\frac{\partial\zeta(\Phi,t)}{\partial t}+\delta_2D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t))+\delta_3\zeta(\Phi,t)\Big)\Bigg)\Bigg| \qquad (3.7)\\
\leq&|\phi_1|+T|\phi_2|+J_t^{\sigma_{\nu_1}}\Bigg(\Big(|\delta_4||\nabla\zeta(\Phi,t)|+|\delta_5||D_t^{\mu}(\nabla\zeta(\Phi,t))|+|\delta_6||\Psi(\Phi,t)|\Big) \\
&+\Big(|\delta_1|\Big|\frac{\partial\zeta(\Phi,t)}{\partial t}\Big|+|\delta_2||D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t))|+|\delta_3||\zeta(\Phi,t)|\Big) \\
\leq&M_1+TM_2+J_t^{\sigma_{\nu_1}}\Big(\lambda_1|\delta_4|+\lambda'|\delta_5|+\lambda_4|\delta_1|+\lambda_3|\delta_2|+|\delta_3|\Big)|\zeta(\Phi,t)| \\
&+J_t^{\sigma_{\nu_1}}\Big(|\delta_6||\Psi(\Phi,t)|\Big) \\
\leq&M_1+TM_2+\frac{t^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(\lambda_1|\delta_4|+\lambda'|\delta_5|+\lambda_4|\delta_1|+\lambda_3|\delta_2|+|\delta_3|\Big)|\zeta(\Phi,t)| \\
&+\frac{t^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}M|\delta_6| \\
\leq&M_1+TM_2+\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(\lambda_1|\delta_4|+\lambda'|\delta_5|+\lambda_4|\delta_1|+\lambda_3|\delta_2|+|\delta_3|\Big)\|\zeta(\Phi,t)\|_\infty \\
&+\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}M|\delta_6|, \qquad (3.8)
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|\Delta\zeta(\Phi,t)\|_\infty\leq&M_1+TM_2+\frac{\beta T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(\lambda_1|\delta_4|+\lambda'|\delta_5|+\lambda_4|\delta_1|+\lambda_3|\delta_2|+|\delta_3|\Big) \\
&+\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}M|\delta_6|:=\xi. \qquad (3.9)
\end{aligned}
$$

This shows that bounded sets are mapped into bounded sets under $\Delta$.
**Step 3:** We will show that the operator $\Delta$ is equi-continuous.

The set $B_\beta$ be a bounded set of $C(\eta,\mathbb{R})$ as in the step 2, and let $B_\beta \subset C(\eta,\mathbb{R})$, then, for $\zeta \in B_\beta$ and $\Phi',\Phi'',t',t'' \in \eta$ with $\Phi' < \Phi'', t' < t''$, we get

$$\begin{aligned}
&|\Delta\zeta(\Phi',t') - \Delta\zeta(\Phi'',t'')| \\
=&\Bigg| J_{t'}^{\sigma_{\nu_1}}\Bigg(\Big(\delta_4\nabla\zeta(\Phi',t') + \delta_5 D_{t'}^{\mu}(\nabla\zeta(\Phi',t'))\Big) \\
&-\Big(\delta_1\frac{\partial\zeta(\Phi',t')}{\partial t'} + \delta_2 D_{t'}^{\varrho_{\nu_2}}(\zeta(\Phi',t')) + \delta_3\zeta(\Phi',t')\Big)\Bigg) \\
&-J_{t''}^{\sigma_{\nu_1}}\Bigg(\Big(\delta_4\nabla\zeta(\Phi'',t'') + \delta_5 D_{t''}^{\mu}(\nabla\zeta(\Phi'',t''))\Big) \\
&-\Big(\delta_1\frac{\partial\zeta(\Phi'',t'')}{\partial t''} + \delta_2 D_{t''}^{\varrho_{\nu_2}}(\zeta(\Phi'',t'')) + \delta_3\zeta(\Phi'',t'')\Big)\Bigg)\Bigg| \\
\leq& J_t^{\sigma_{\nu_1}}\Bigg(|\delta_4||\nabla\zeta(\Phi',t') - \nabla\zeta(\Phi'',t'')| \\
&+|\delta_5||D_{t'}^{\mu}(\nabla\zeta(\Phi',t')) - D_{t''}^{\mu}(\nabla\zeta(\Phi'',t''))| + |\delta_1|\left|\frac{\partial\zeta(\Phi'',t'')}{\partial t''} - \frac{\partial\zeta(\Phi'',t'')}{\partial t''}\right| \\
&+|\delta_2||D_{t'}^{\varrho_{\nu_2}}(\zeta(\Phi',t')) - D_{t''}^{\varrho_{\nu_2}}(\zeta(\Phi'',t''))| + |\delta_3||\zeta(\Phi',t') - \zeta(\Phi'',t'')|\Bigg) \\
\leq& J_t^{\sigma_{\nu_1}}\Bigg(|\delta_4|\Big(\tau_1|\Phi'-\Phi''| + \tau_2|t'-t''|\Big) \\
&+|\delta_5|\Big(\tau'|\Phi'-\Phi''| + \tau''|t'-t''|\Big) + |\delta_1|\Big(\tau_5|\Phi'-\Phi''| + \tau_6|t'-t''|\Big) \\
&+|\delta_2|\Big(\tau_3|\Phi'-\Phi''| + \tau_4|t'-t''|\Big) + |\delta_3|\Big(\tau_7|\Phi'-\Phi''| + \tau_8|t'-t''|\Big)\Bigg) \\
\leq&\frac{t^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Bigg(\Big(\tau_1|\delta_4| + \tau'|\delta_5| + \tau_5|\delta_1| + \tau_3|\delta_2| + \tau_7|\delta_3|\Big)|\Phi'-\Phi''| \\
&+\Big(\tau_2|\delta_4| + \tau''|\delta_5| + \tau_6|\delta_1| + \tau_4|\delta_2| + \tau_8|\delta_3|\Big)|t'-t''|\Bigg).
\end{aligned} \tag{3.10}$$

Finally, we obtain

$$\begin{aligned}
&\|\Delta\zeta(\Phi',t') - \Delta\zeta(\Phi'',t'')\|_\infty \\
\leq&\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Bigg(\Big(\tau_1|\delta_4| + \tau'|\delta_5| + \tau_5|\delta_1| + \tau_3|\delta_2| + \tau_7|\delta_3|\Big)\|\Phi'-\Phi''\|_\infty \\
&+\Big(\tau_2|\delta_4| + \tau''|\delta_5| + \tau_6|\delta_1| + \tau_4|\delta_2| + \tau_8|\delta_3|\Big)\|t'-t''\|_\infty\Bigg).
\end{aligned} \tag{3.11}$$

The above equation is independent of $\zeta$. Hence

$$\|\Delta\zeta(\Phi',t') - \Delta\zeta(\Phi'',t'')\|_\infty \to 0 \quad \text{as} \quad \Phi' \to \Phi'', t' \to t''.$$

Therefore by the Arzelà-Ascoli Theorem [18] the operator $\Delta : C(\eta,\mathbb{R}) \to C(\eta,\mathbb{R})$ is completely continuous, and consequently, the operator $\Delta$ is completely continuous.
**Step 4:** A priori bound.

Define the set $\omega = \{\zeta \in C(\eta, \mathbb{R}) : \zeta = \epsilon\Delta\zeta, 0 < \epsilon < 1\}$. We will show that $\omega$ is bounded. If $\zeta \in \omega$, then by definition $\zeta = \epsilon\Delta\zeta$ with $0 < \epsilon < 1$. Thus for any $t \in I$, we can write

$$\begin{aligned}|\zeta| &= |\epsilon\Delta\zeta| \\ &= \Bigg|\epsilon \times \Bigg(\phi_1 + t\phi_2 + J_t^{\sigma_{\nu_1}}\Bigg(\Bigg(\delta_4\nabla\zeta(\Phi,t) + \delta_5 D_t^{\mu}(\nabla\zeta(\Phi,t)) + \delta_6\Psi(\Phi,t)\Bigg) \\ &\quad - \Bigg(\delta_1\frac{\partial\zeta(\Phi,t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t)) + \delta_3\zeta(\Phi,t)\Bigg)\Bigg)\Bigg)\Bigg|. \end{aligned} \tag{3.12}$$

By using inequality in the equation (3.9), we obtain

$$\begin{aligned}|\zeta| \leq \epsilon \times \Bigg(M_1 + TM_2 + \frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(\lambda_1|\delta_4| + \lambda'|\delta_5| + \lambda_4|\delta_1| + \lambda_3|\delta_2| + |\delta_3|\Big) \\ \times \|\zeta(\Phi,t)\|_\infty + \frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}M|\delta_6|\Bigg).\end{aligned} \tag{3.13}$$

This further gives

$$\begin{aligned}\|\zeta\|_\infty \leq \epsilon \times \Bigg(M_1 + TM_2 + \frac{\beta T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(\lambda_1|\delta_4| + \lambda'|\delta_5| + \lambda_4|\delta_1| + \lambda_3|\delta_2| + |\delta_3|\Big) \\ + \frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}M|\delta_6|\Bigg) := C.\end{aligned} \tag{3.14}$$

Which shows that the set $\omega$ is bounded.

Hence, by the Schaefer fixed point Theorem, $\Delta$ has at least one fixed point. Consequently, the considered problem (3.1)–(3.2) has at least one solution. ☐

**Theorem 3.2.** *The problem* (3.1)–(3.2) *has a unique solution under the hypotheses ($E_1$) if the following inequality holds*

$$\begin{aligned}&\Omega_{(T,\sigma_{\nu_1},\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\chi_1,\chi',\chi_3,\chi_4,)} \\ &= \Bigg(\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Big(|\delta_4|\chi_1 + |\delta_5|\chi' + |\delta_1|\chi_3 + |\delta_2|\chi_4 + |\delta_3|\Big)\Bigg) < 1.\end{aligned} \tag{3.15}$$

**Proof.** Let $\zeta_1, \zeta_2 \in C(\eta, \mathbb{R})$, then, for each $t \in I$, we have

$$\begin{aligned}&|\Delta\zeta_1(\Phi,t) - \Delta\zeta_2(\Phi,t)| \\ &= \Bigg|J_t^{\sigma_{\nu_1}}\Bigg(\Bigg(\delta_4\nabla\zeta_1(\Phi,t) + \delta_5 D_t^{\mu}(\nabla\zeta_1(\Phi,t))\Bigg) \\ &\quad - \Bigg(\delta_1\frac{\partial\zeta_1(\Phi,t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta_1(\Phi,t)) + \delta_3\zeta_1(\Phi,t)\Bigg)\Bigg) \\ &\quad - J_t^{\sigma_{\nu_1}}\Bigg(\Bigg(\delta_4\nabla\zeta_2(\Phi,t) + \delta_5 D_t^{\mu}(\nabla\zeta_2(\Phi,t))\Bigg) \\ &\quad - \Bigg(\delta_1\frac{\partial\zeta_2(\Phi,t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta_2(\Phi,t)) + \delta_3\zeta_2(\Phi,t)\Bigg)\Bigg)\Bigg| \\ &\leq J_t^{\sigma_{\nu_1}}\Bigg(|\delta_4|\Big|\nabla\zeta_1(\Phi,t) - \nabla\zeta_2(\Phi,t)\Big|\end{aligned}$$

$$
\begin{aligned}
&+|\delta_5|\left|D_t^{\mu}(\nabla\zeta_1(\Phi,t))-D_t^{\mu}(\nabla\zeta_2(\Phi,t))\right|+|\delta_1|\left|\frac{\partial\zeta_1(\Phi,t)}{\partial t}-\frac{\partial\zeta_2(\Phi,t)}{\partial t}\right|\\
&+|\delta_2|\left|D_t^{\varrho_{\nu_2}}(\zeta_1(\Phi,t))-D_t^{\varrho_{\nu_2}}(\zeta_2(\Phi,t))\right|+|\delta_3|\left|\zeta_1(\Phi,t)-\zeta_2(\Phi,t)\right|\Bigg)\\
\leq & J_t^{\sigma_{\nu_1}}\Bigg(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Bigg)|\zeta_1(\Phi,t)-\zeta_2(\Phi,t)|\\
\leq & \frac{t^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\times\Bigg(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Bigg)|\zeta_1(\Phi,t)-\zeta_2(\Phi,t)|\\
\leq & \frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\times\Bigg(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Bigg)\|\zeta_1(\Phi,t)-\zeta_2(\Phi,t)\|_{\infty}.
\end{aligned}
\tag{3.16}
$$

Therefore, we get

$$
\|\Delta\zeta_1(\Phi,t)-\Delta\zeta_2(\Phi,t)\|_{\infty}\leq\Omega_{(T,\sigma_{\nu_1},\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\chi_1,\chi',\chi_3,\chi_4,)}\|\zeta_1(\Phi,t)-\zeta_2(\Phi,t)\|_{\infty},
\tag{3.17}
$$

where

$$
\Omega_{(T,\sigma_{\nu_1},\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\chi_1,\chi',\chi_3,\chi_4)}=\frac{T^{\sigma_{\nu_1}}}{\Gamma(1+\sigma_{\nu_1})}\Bigg(|\delta_4|\chi_1+|\delta_5|\chi'+|\delta_1|\chi_3+|\delta_2|\chi_4+|\delta_3|\Bigg).
$$

Hence $\Delta$ is a contraction, and therefore, by the Banach fixed point theorem, $\Delta$ has a unique fixed point. □

# 4. TSADM Algorithm

In this section, we present the description of TSADM in the several steps and also includes the advantages and limitation of the method.

Consider the problem (3.1)–(3.2), we describe the TSADM as follows

$$
\begin{aligned}
&\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}D_t^{\sigma_{\nu_1}}(\zeta(\Phi,t))+d_2\frac{\partial\zeta(\Phi,t)}{\partial t}+\sum_{\nu_2=1}^{r_1}d_{3,\nu_2}D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t))+d_4\zeta(\Phi,t)\\
&=d_5\nabla\zeta(\Phi,t)+d_6D_t^{\mu}(\nabla\zeta(\Phi,t))+\Psi(\Phi,t),\\
&(\Phi,t)=(\Phi_1,\Phi_2,\cdots,\Phi_m,t)\in\Lambda\times I.
\end{aligned}
\tag{4.1}
$$

Multiplying the equation (4.1) by $1/\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}$ into both sides, we have

$$
\begin{aligned}
&D_t^{\sigma_{\nu_1}}(\zeta(\Phi,t))+\delta_1\frac{\partial\zeta(\Phi,t)}{\partial t}+\delta_2D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t))+\delta_3\zeta(\Phi,t)\\
&=\delta_4\nabla\zeta(\Phi,t)+\delta_5D_t^{\mu}(\nabla\zeta(\Phi,t))+\delta_6\Psi(\Phi,t),
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\delta_1&=\frac{d_2}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}}, & \delta_2&=\frac{\sum_{\nu_2=1}^{r_1}d_{3,\nu_2}}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}}, & \delta_3&=\frac{d_4}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}},\\
\delta_4&=\frac{d_5}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}}, & \delta_5&=\frac{d_6}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}}, & \delta_6&=\frac{1}{\sum_{\nu_1=1}^{r_1}d_{1,\nu_1}}.
\end{aligned}
$$

The algorithm consists five steps:
**Step 1.** On applying the inverse operator $J_t^{\sigma_{\nu_1}}$ of $D_t^{\sigma_{\nu_1}}$ into both sides of the equation (4.2), we obtain

$$\begin{aligned}\zeta(\Phi,t) = \phi_1 + t\phi_2 + J_t^{\sigma_{\nu_1}}\Bigg(\Bigg(&\delta_4\nabla\zeta(\Phi,t) + \delta_5 D_t^{\mu}(\nabla\zeta(\Phi,t)) \\ &+ \delta_6\Psi(\Phi,t)\Bigg) - \left(\delta_1\frac{\partial\zeta(\Phi,t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta(\Phi,t)) + \delta_3\zeta(\Phi,t)\right)\Bigg).\end{aligned} \tag{4.3}$$

**Step 2.** Write the recursion formula for the TSADM from the equation (4.3) in step 1 as

$$\zeta_0(\Phi,t) = \phi_1 + t\phi_2 + J_t^{\sigma_{\nu_1}}\Big(\delta_6\Psi(\Phi,t)\Big), \tag{4.4}$$

and

$$\begin{aligned}\zeta_q(\Phi,t) = J_t^{\sigma_{\nu_1}}\Bigg(&\Big(\delta_4\nabla\zeta_q(\Phi,t) + \delta_5 D_t^{\mu}(\nabla\zeta_q(\Phi,t))\Big) \\ &- \left(\delta_1\frac{\partial\zeta_q(\Phi,t)}{\partial t} + \delta_2 D_t^{\varrho_{\nu_2}}(\zeta_q(\Phi,t)) + \delta_3\zeta_q(\Phi,t)\right)\Bigg),\end{aligned} \tag{4.5}$$

where $q = 1, 2, \cdots$.
**Step 3.** The first iteration (zeroth term) in the equation (4.4) can be split into several components as

$$\zeta_0(\Phi,t) = \psi_1 + \psi_2 + \psi_3 + \cdots + \psi_M = \psi, \tag{4.6}$$

where $\psi_1$, $\psi_2$, $\psi_3$, $\cdots$, $\psi_M$ are the terms obtained from integrating the source term $\Psi(\Phi,t)$ and from the associted intial/boundary conditions.
**Step 4.** The first component of $\zeta_0$, that is $\psi_1$ and verify that the component satisfies the equation (3.1) and given the initial condition, if so $\psi_1$ is the exact solution of the equation (3.1). If the first component satisfy neither the equation (3.1) nor the initial condition then we go to the next component and proceed with the same process. If any component in $\zeta_0$ satisfies the equation (3.1) and the initial condition both then that component becomes our exact solution of the equation (3.1) with the initial conditions. If all the components involve in $\zeta_0$ do not satisfy the equation (3.1) or the initial conditions (3.2) then we go to the next step.
**Step 5.** On applying the ADM to obtain the solution by choosing $\zeta_0(\Phi,t) = \psi$ and iterates the solution by using equation (4.6) in step 2.
**Advantages.** The advantages of the TSADM for the considered problem are listed as follows:

1. If the TSADM is applicable on the problem then the obtained solution is an exact solution.
2. The TSADM gives the exact solution in just one iteration and reduce the computation effort.
3. The TSADM is powerful and efficient method for such types of problems in comparison to other existing methods without linearization, discrtization, Adomain polynomial and reduce the memory space with low cost. Thus, the TSADM is superior than ADM, MADM, and other numerical methods.

**Limitations.** Proposed method will fail under the condition stated as:
The first term/zeroth term of the obtained series solution involves verifying term, which satisfies the considered equation and the related initial/boundary conditions. If there is no such term involves in the zeroth term of the series then we will obtained semi-analytical solution.

# 5. Applications

**Example 5.1.** Consider the multi-dimension multi-term time-fractional mixed diffusion equation on a rectangular domain, as

$$\begin{aligned} &d_1 D_t^{\sigma}(\zeta(\Phi,t)) + d_2 \frac{\partial \zeta(\Phi,t)}{\partial t} + d_3 D_t^{\varrho}(\zeta(\Phi,t)) + d_4 \zeta(\Phi,t) \\ &= d_5 \nabla \zeta(\Phi,t) + d_6 D_t^{\mu}(\nabla \zeta(\Phi,t)) + \Psi(\Phi,t), \\ &(\Phi,t) = (\Phi_1,\Phi_2,\cdots,\Phi_m,t) \in \Lambda \times I, \end{aligned} \tag{5.1}$$

with the initial and Neumann boundary conditions

$$\begin{aligned} &\zeta(\Phi,0) = \prod_{i=1}^{m} \sin(\pi\Phi_i), \zeta_t(\Phi,0) = 0, \Phi \in \bar{\Lambda}, \\ &\zeta(\Phi,t) = 0, \Phi \in \partial\Lambda, t \in \bar{I}, \end{aligned} \tag{5.2}$$

where $1 < \sigma < 2$, $0 < \varrho, \mu < 1$, $\Lambda = \prod_{i=1}^{m}(0,1)$, $I = (0,1]$,

$$\begin{aligned} \Psi(\Phi,t) = \prod_{i=1}^{m} \sin(\pi\Phi_i) \bigg[ & d_1 \frac{\Gamma(n+1)t^{n-\sigma}}{\Gamma(n+1-\sigma)} + n d_2 t^{n-1} + d_3 \frac{\Gamma(n+1)t^{n-\varrho}}{\Gamma(n+1-\varrho)} \\ & + (d_4 + m d_5 \pi^2)(t^n + 1) + m d_6 \pi^2 \frac{\Gamma(n+1)t^{n-\mu}}{\Gamma(n+1-\mu)} \bigg]. \end{aligned} \tag{5.3}$$

The corresponding exact solution is $\zeta(\Phi,t) = (t^n + 1)\prod_{i=1}^{m} \sin(\pi\Phi_i)$.

On multiplying the term $\frac{1}{d_1}$ into both sides of the equation (5.1), we obtain

$$\begin{aligned} &D_t^{\sigma}(\zeta(\Phi,t)) + \frac{d_2}{d_1}\frac{\partial \zeta(\Phi,t)}{\partial t} + \frac{d_3}{d_1} D_t^{\varrho}(\zeta(\Phi,t)) + \frac{d_4}{d_1}\zeta(\Phi,t) \\ =&\frac{d_5}{d_1}\nabla\zeta(\Phi,t) + \frac{d_6}{d_1} D_t^{\mu}(\nabla\zeta(\Phi,t)) + \frac{1}{d_1}\Psi(\Phi,t). \end{aligned} \tag{5.4}$$

On applying $J_t^{\sigma}$ into the equation (5.4), we have

$$\begin{aligned} \zeta(\Phi,t) = \zeta(\Phi,0) + J_t^{\sigma}\bigg( & \frac{d_5}{d_1}\nabla\zeta(\Phi,t) + \frac{d_6}{d_1} D_t^{\mu}(\nabla\zeta(\Phi,t)) + \frac{1}{d_1}\Psi(\Phi,t) \\ & - \frac{d_2}{d_1}\frac{\partial \zeta(\Phi,t)}{\partial t} + \frac{d_3}{d_1} D_t^{\varrho}(\zeta(\Phi,t)) + \frac{d_4}{d_1}\zeta(\Phi,t)\bigg), \end{aligned} \tag{5.5}$$

where $J_t^{\sigma}$ is the inverse operator of the operator $D_t^{\sigma}$.

The recursion formula of the solution from the equation (5.5) works as follows

$$\zeta_0(\Phi,t) = \zeta(\Phi,0) + J_t^{\sigma}\left(\frac{1}{d_1}\Psi(\Phi,t)\right), \tag{5.6}$$

and

$$\zeta_{k+1}(\Phi,t) = J_t^{\sigma}\bigg(\frac{d_5}{d_1}\nabla\zeta_k(\Phi,t) + \frac{d_6}{d_1}D_t^{\mu}(\nabla\zeta_k(\Phi,t)) - \frac{d_2}{d_1}\frac{\partial\zeta_k(\Phi,t)}{\partial t} + \frac{d_3}{d_1}D_t^{\varrho}(\zeta_k(\Phi,t)) + \frac{d_4}{d_1}\zeta_k(\Phi,t)\bigg), \tag{5.7}$$

where $k = 1, 2, \cdots$.

By solving the equation (5.6), we obtain

$$\begin{aligned}\zeta_0(\Phi,t) = \prod_{i=1}^{m}\sin(\pi\Phi_i) + \prod_{i=1}^{m}\sin(\pi\Phi_i)\bigg[t^n + \frac{d_2\Gamma(n+1)}{d_1\Gamma(n+\sigma)}t^{n+\sigma-1} \\ + \frac{d_3\Gamma(n+1)}{d_1\Gamma(n+1-\varrho+\sigma)}t^{n-\varrho+\sigma} + \bigg(\frac{d_4+md_5\pi^2}{d_1}\bigg)\bigg(\frac{\Gamma(n+1)}{\Gamma(n+\sigma+1)}t^{n+\sigma} \\ + \frac{1}{\Gamma(1+\sigma)}t^{\sigma}\bigg) + \bigg(\frac{md_6\pi^2}{d_1}\bigg)\bigg(\frac{\Gamma(n+1)t^{n-\mu+\sigma}}{\Gamma(n+1-\mu+\sigma)}\bigg)\bigg].\end{aligned} \tag{5.8}$$

From the above equation (5.8), the first iteration of TSADM can be split into five terms as

$$\zeta_0(\Phi,t) = X_0 + X_1 + X_2 + X_3 + X_4, \tag{5.9}$$

where

$$X_0 = \prod_{i=1}^{m}\sin(\pi\Phi_i)\bigg(1+t^n\bigg), \tag{5.10}$$

$$X_1 = \frac{d_2\Gamma(n+1)}{d_1\Gamma(n+\sigma)}t^{n+\sigma-1}, \tag{5.11}$$

$$X_2 = \frac{d_3\Gamma(n+1)}{d_1\Gamma(n+1-\varrho+\sigma)}t^{n-\varrho+\sigma}, \tag{5.12}$$

$$X_3 = \bigg(\frac{d_4+md_5\pi^2}{d_1}\bigg)\bigg(\frac{\Gamma(n+1)}{\Gamma(n+\sigma+1)}t^{n+\sigma} + \frac{1}{\Gamma(1+\sigma)}t^{\sigma}\bigg), \tag{5.13}$$

$$X_4 = \bigg(\frac{md_6\pi^2}{d_1}\bigg)\bigg(\frac{\Gamma(n+1)t^{n-\mu+\sigma}}{\Gamma(n+1-\mu+\sigma)}\bigg). \tag{5.14}$$

Here we generalized our problem and obtained a general solution for the equation (5.1).

According to the TSADM process, we select the first iteration as the term involved in the equation (5.9) and the term satisfies the problem and the given condition, so we terminate the process and obtain the solution of the problem (5.31)–(5.32).

Let us consider $\zeta_0 = X_0$ and check that the chosen term as $\zeta_0$ is satisfying the equation (5.1) and also the given conditions. If this choice of $\zeta_0$ is approved then this implies that the chosen term is a solution to the problem.

Let us consider $\zeta_0 = X_0$ as a solution of the equation (5.1), so that it satisfies the equation (5.1).

To show that $\zeta = X_0$ is an exact solution of the equation (5.1), we substitute $\zeta = X_0$ in the left-hand side (LHS) of the equation (5.1),

$$d_1 D_t^{\sigma} X_0 + d_2 \frac{\partial X_0}{\partial t} + d_3 D_t^{\varrho} X_0 + d_4 X_0 = \prod_{i=1}^{m} \sin(\pi \Phi_i) \bigg[ d_1 \frac{\Gamma(n+1) t^{n-\sigma}}{\Gamma(n+1-\sigma)} + n d_2 t^{n-1} + d_3 \frac{\Gamma(n+1) t^{n-\varrho}}{\Gamma(n+1-\varrho)} + d_4 (t^n + 1) \bigg]. \tag{5.15}$$

Now, we will calculate the terms on the right-hand side (RHS) of the equation (5.1) for $X_0$, as follows,

$$d_5 \nabla X_0 + d_6 D_t^{\mu} (\nabla X_0) + \Psi(\Phi, t) = \prod_{i=1}^{m} \sin(\pi \Phi_i) \bigg[ d_1 \frac{\Gamma(n+1) t^{n-\sigma}}{\Gamma(n+1-\sigma)} + n d_2 t^{n-1} + d_3 \frac{\Gamma(n+1) t^{n-\varrho}}{\Gamma(n+1-\varrho)} + d_4 (t^n + 1) \bigg]. \tag{5.16}$$

As we see that the equations (5.15) and (5.16) the same results. This implies that the LHS of the equation (5.15) is equal to the RHS of the equation (5.16). This proves that $\zeta_0 = X_0$ satisfies the equation (5.1) and the related conditions. Thus, the investigated solution is the exact solution of the problem (5.1)–(5.2) using the TSADM.

**Example 5.2.** Consider the multi-dimension multi-term time-fractional mixed diffusion equation on circular domain, described as

$$\begin{aligned} &D_t^{\sigma}(\zeta(\Phi,t)) + \frac{\partial \zeta(\Phi,t)}{\partial t} + D_t^{\varrho}(\zeta(\Phi,t)) + \zeta(\Phi,t) = \nabla \zeta(\Phi,t) + D_t^{\mu}(\nabla \zeta(\Phi,t)) + \Psi(\Phi,t), \\ &(\Phi,t) = (\Phi_1, \Phi_2, \cdots, \Phi_m, t) \in \Lambda \times I, \end{aligned} \tag{5.17}$$

with the initial and Neumann boundary conditions

$$\begin{aligned} &\zeta(\Phi,t) = \bigg(1 - \sum_{i=1}^{m} \Phi_i^2\bigg), \zeta_t(\Phi,t) = 0, \Phi \in \bar{\Lambda}, \\ &\zeta(\Phi,t) = 0, \Phi \in \partial\Lambda, t \in \bar{I}, \end{aligned} \tag{5.18}$$

where $1 < \sigma < 2$, $0 < \varrho, \mu < 1$, $\Lambda = \{\Phi = (\Phi_1, \Phi_2, \cdots, m) | \sum_{i=1}^{m} \Phi_i^2 < 1\}$, $I = (0,1]$,

$$\Psi(\Phi,t) = (1 - \sum_{i=1}^{m} \Phi_i^2) \bigg[ \frac{\Gamma(n+1) t^{n-\sigma}}{\Gamma(n+1-\sigma)} + n t^{n-1} + \frac{\Gamma(n+1) t^{n-\varrho}}{\Gamma(n+1-\varrho)} + (t^n + 1) + \bigg( 2m(t^n+1) + \frac{2m\Gamma(n+1) t^{n-\mu}}{\Gamma(n+1-\mu)} \bigg) \bigg]. \tag{5.19}$$

The corresponding solution is $\zeta(\Phi,t) = (t^n + 1)\bigg(1 - \sum_{i=1}^{m} \Phi_i^2\bigg)$.

On applying $J_t^{\sigma}$ into the equation (5.17), we have

$$\zeta(\Phi,t) = \bigg(1 - \sum_{i=1}^{m} \Phi_i^2\bigg) + J_t^{\sigma} \bigg( \nabla \zeta(\Phi,t) + D_t^{\mu} \zeta(\Phi,t) + \Psi(\Phi,t) - \bigg( \frac{\partial \zeta(\Phi,t)}{\partial t} + D_t^{\varrho}(\zeta(\Phi,t)) + \zeta(\Phi,t) \bigg) \bigg), \tag{5.20}$$

where $J_t^\sigma$ is the inverse operator of the operator $D_t^\sigma$.

The recursion formula of the solution from the equation (5.20) works as follows

$$\zeta_0(\Phi,t)=\left(1-\sum_{i=1}^{m}\Phi_i^2\right)+J_t^\sigma\left(\Psi(\Phi,t)\right), \tag{5.21}$$

$$\zeta_k(\Phi,t)=J_t^\sigma\left(\nabla\zeta_k(\Phi,t)+D_t^\mu(\nabla\zeta_k(\Phi,t))-\left(\frac{\partial\zeta_k(\Phi,t)}{\partial t}+D_t^\varrho(\zeta_k(\Phi,t))+\zeta_k(\Phi,t)\right)\right), \tag{5.22}$$

where $k=1,2,\cdots$.

By solving the equation (5.21), we obtain

$$\begin{aligned}\zeta_0(\Phi,t)=\left(1-\sum_{i=1}^{m}\Phi_i^2\right)(1+t^n)&+\left[\frac{\Gamma(n+1)}{\Gamma(n+\sigma)}t^{n+\sigma-1}+\frac{\Gamma(n+1)}{\Gamma(n+1-\varrho+\sigma)}t^{n-\varrho+\sigma}\right.\\&+\frac{\Gamma(n+1)}{\Gamma(n+\sigma+1)}t^{n+\sigma}+\frac{1}{\Gamma(\sigma+1)}t^{\sigma}+\left(2m\left(\frac{\Gamma(n+1)}{\Gamma(n+\sigma+1)}t^{n+\sigma}\right.\right.\\&\left.\left.\left.+\frac{1}{\Gamma(\sigma+1)}t^{\sigma}\right)+\frac{2m\Gamma(n+1)}{\Gamma(n+1-\mu+\sigma)}t^{n-\mu+\sigma}\right)\right].\end{aligned} \tag{5.23}$$

From the above equation (5.23), the first iteration of TSADM can be split into five terms as

$$\zeta_0(\Phi,t)=X_0+X_1+X_2+X_3+X_4+X_5, \tag{5.24}$$

where

$$X_0=\left(1-\sum_{i=1}^{m}\Phi_i^2\right)(1+t^n), \tag{5.25}$$

$$X_1=\frac{\Gamma(n+1)}{\Gamma(n+\sigma)}t^{n+\sigma-1}, \tag{5.26}$$

$$X_2=\frac{\Gamma(n+1)}{\Gamma(n+1-\varrho+\sigma)}t^{n-\varrho+\sigma}, \tag{5.27}$$

$$X_3=\frac{1}{\Gamma(\sigma+1)}t^{\sigma}, \tag{5.28}$$

$$X_4=\left(2m\left(\frac{\Gamma(n+1)}{\Gamma(n+\sigma+1)}t^{n+\sigma}+\frac{1}{\Gamma(\sigma+1)}t^{\sigma}\right)\right), \tag{5.29}$$

$$X_5=\frac{2m\Gamma(n+1)}{\Gamma(n+1-\mu+\sigma)}t^{n-\mu+\sigma}. \tag{5.30}$$

Here we generalized our problem and obtained a general solution for the equation (5.17).

According to the TSADM process, we select the first iteration as the term involved in the equation (5.24) and the term satisfies the problem and the given condition, so we terminate the process and obtain the solution of the problem.

Let us consider $\zeta_0=X_0$ and check that the chosen term $\zeta_0$ is satisfying the equation (5.17) and also the related conditions. If this choice of $\zeta_0$ is approved then this implies that the chosen term is a solution to the problem.

Let us consider $\zeta_0 = X_0$ as a solution of the equation (5.17), so that it satisfies the equation (5.17).

To prove $\zeta = X_0$ is the exact solution of the equation (5.17), we substitute $\zeta = X_0$ in the left-hand side of the equation (5.17), and hence we obtain

$$D_t^{\sigma} X_0 + \frac{\partial X_0}{\partial t} + D_t^{\varrho} X_0 + X_0 = \left[\frac{\Gamma(n+1)t^{n-\sigma}}{\Gamma(n+1-\sigma)} + nt^{n-1} + \frac{\Gamma(n+1)t^{n-\varrho}}{\Gamma(n+1-\varrho)} + (t^n+1)\right]. \tag{5.31}$$

Now, we will calculate the terms on the right-hand side of the equation (5.17) for $X_0$, as follows,

$$d_5 \nabla X_0 + D_t^{\mu}(\nabla X_0) + \Psi(\Phi, t) = \left[\frac{\Gamma(n+1)t^{n-\sigma}}{\Gamma(n+1-\sigma)} + nt^{n-1} + \frac{\Gamma(n+1)t^{n-\varrho}}{\Gamma(n+1-\varrho)} + (t^n+1)\right]. \tag{5.32}$$

As we see that the equations (5.31) and (5.32) gives the same results. This implies that the LHS of the equation (5.31) is equal to the RHS of the equation (5.32). This proves that $\zeta_0 = X_0$ satisfies the equation (5.17) and the related conditions. Thus, the discovered solution is the exact solution of the problem (5.31)–(5.32) using the TSADM.

**Example 5.3.** Consider the multi-dimension multi-term time-fractional mixed diffusion equation on a rectangular domain, as

$$\begin{aligned} &D_t^{\sigma}(\zeta(\Phi,t)) + d_2 \frac{\partial \zeta(\Phi,t)}{\partial t} + d_3 D_t^{\varrho}(\zeta(\Phi,t)) + d_4 \zeta(\Phi,t) \\ &= d_5 \nabla \zeta(\Phi,t) + d_6 D_t^{\mu}(\nabla \zeta(\Phi,t)) + \Psi(\Phi,t), \\ &(\Phi,t) = (\Phi_1, \Phi_2, \cdots, \Phi_m, t) \in \Lambda \times I, \end{aligned} \tag{5.33}$$

with the initial and Neumann boundary conditions

$$\begin{aligned} &\zeta(\Phi,0) = 0, \zeta_t(\Phi,0) = 0, \Phi \in \bar{\Lambda}, \\ &\zeta(\Phi,t) = 0, \Phi \in \partial\Lambda, t \in \bar{I}, \end{aligned} \tag{5.34}$$

where $1 < \sigma < 2$, $0 < \varrho, \mu < 1$, $\Lambda = \prod_{i=1}^{m}(0,1)$, $I = (0,1]$,

$$\begin{aligned} \Psi(\Phi,t) = \prod_{i=1}^{m} \cos(\pi \Phi_i) \Bigg[ &\left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\sigma+1)} t^{k+\alpha-\sigma} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\sigma+1)} t^{\alpha-\sigma} \right) \\ &+ d_2 \left( (k+\alpha) t^{k+\alpha-1} + \alpha t^{\alpha-1} \right) + d_3 \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\varrho+1)} t^{k+\alpha-\varrho} \right. \\ &\left. + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\varrho+1)} t^{\alpha-\varrho} \right) + (d_4 + d_5 m\pi^2)(t^{k+\alpha} + t^{\alpha}) \\ &+ m d_6 \pi^2 \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\mu+1)} t^{k+\alpha-\mu} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} t^{\alpha-\mu} \right) \Bigg]. \end{aligned} \tag{5.35}$$

The corresponding exact solution is $\zeta(\Phi,t) = \prod_{i=1}^{m} \cos(\pi\Phi_i)(t^{k+\alpha} + t^{\alpha})$.

On applying $J_t^{\sigma}$ into the equation (5.33), we have

$$\begin{aligned} \zeta(\Phi,t) = \zeta(\Phi,0) + J_t^{\sigma} \Bigg( &d_5 \nabla \zeta(\Phi,t) + d_6 D_t^{\mu}(\nabla \zeta_k(\Phi,t)) + \Psi(\Phi,t) \\ &- \left( d_2 \frac{\partial \zeta(\Phi,t)}{\partial t} + d_3 D_t^{\varrho}(\zeta(\Phi,t)) + d_4 \zeta(\Phi,t) \right) \Bigg), \end{aligned} \tag{5.36}$$

where $J_t^\sigma$ is the inverse operator of the operator $D_t^\sigma$.

The recursion formula of the solution from the equation (5.36) works as follows

$$\zeta_0(\Phi,t) = \zeta(\Phi,0) + J_t^\sigma\Big(\Psi(\Phi,t)\Big), \tag{5.37}$$

and

$$\begin{aligned}\zeta_{k+1}(\Phi,t) = J_t^\sigma\Bigg(& d_5\nabla\zeta_k(\Phi,t) + d_6 D_t^\mu(\nabla\zeta_k(\Phi,t)) \\ & - \left(d_2\frac{\partial\zeta_k(\Phi,t)}{\partial t} + d_3 D_t^\varrho(\zeta_k(\Phi,t)) + d_4\zeta_k(\Phi,t)\right)\Bigg),\end{aligned} \tag{5.38}$$

where $k = 1, 2, \cdots$.

By solving the equation (5.37), we obtain

$$\begin{aligned}\zeta_0(\Phi,t) = \prod_{i=1}^m \cos(\pi\Phi_i)\Bigg[&(t^{k+\alpha}+t^\alpha) + d_2\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma)}t^{k+\alpha+\sigma-1} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma)}t^{\alpha+\sigma-1}\right) \\ & + d_3\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma-\varrho+1)}t^{k+\alpha+\sigma-\varrho} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma-\varrho+1)}t^{\alpha+\sigma-\varrho}\right) \\ & + (d_4 + d_5 m\pi^2)\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma+1)}t^{k+\alpha+\sigma} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1)}t^{\alpha+\sigma}\right) \\ & + m d_6\pi^2\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma-\mu+1)}t^{k+\alpha+\sigma-\mu} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma-\mu+1)}t^{\alpha+\sigma-\mu}\right)\Bigg].\end{aligned} \tag{5.39}$$

From the above equation (5.39), the first iteration of TSADM can be split into five terms as

$$\zeta_0(\Phi,t) = X_0 + X_1 + X_2 + X_3 + X_4, \tag{5.40}$$

where

$$X_0 = \prod_{i=1}^m \cos(\pi\Phi_i)(t^{k+\alpha} + t^\alpha), \tag{5.41}$$

$$X_1 = \prod_{i=1}^m \cos(\pi\Phi_i)\left(d_2\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma)}t^{k+\alpha+\sigma-1}\right)\right), \tag{5.42}$$

$$X_2 = \prod_{i=1}^m \cos(\pi\Phi_i)\left(d_3\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma-\varrho+1)}t^{k+\alpha+\sigma-\varrho} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma-\varrho+1)}t^{\alpha+\sigma-\varrho}\right)\right), \tag{5.43}$$

$$X_3 = \prod_{i=1}^m \cos(\pi\Phi_i)\left((d_4 + d_5 m\pi^2)\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma+1)}t^{k+\alpha+\sigma} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1)}t^{\alpha+\sigma}\right)\right), \tag{5.44}$$

$$X_4 = \prod_{i=1}^m \cos(\pi\Phi_i)\left(m d_6\pi^2\left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+\sigma-\mu+1)}t^{k+\alpha+\sigma-\mu} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma-\mu+1)}t^{\alpha+\sigma-\mu}\right)\right). \tag{5.45}$$

Here, we generalized our problem and obtained a general solution for the equation (5.33).

According to the TSADM process, we select the first iteration as the term involved in the equation (5.40) and the term satisfies the equation (5.33) and the given conditions in the equation (5.34), so we terminate the process and obtain the solution of the the equation (5.34).

Let us consider $\zeta_0 = X_0$ and check that the chosen term as $\zeta_0$ is satisfying the equation (5.33) and also the given conditions. If this choice of $\zeta_0$ is approved then this implies that the chosen term is a solution to the problem (5.33)–(5.34).

Let us consider $\zeta_0 = X_0$ as a solution of the equation (5.33), so that it satisfies the equation (5.33) with conditions given in the equation (5.34).

To show that $\zeta = X_0$ is an exact solution of the equation (5.33), we substitute $\zeta = X_0$ in the left-hand side (LHS) of the equation (5.33), and we obtain

$$\begin{aligned}
&D_t^\sigma X_0 + d_2 \frac{\partial X_0}{\partial t} + d_3 D_t^\varrho X_0 + d_4 X_0 \\
&= \prod_{i=1}^{m} \cos(\pi\Phi_i) \Bigg[ \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\sigma+1)} t^{k+\alpha-\sigma} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\sigma+1)} t^{\alpha-\sigma} \right) \\
&\quad + d_2 \left( (k+\alpha) t^{k+\alpha-1} + \alpha t^{\alpha-1} \right) + d_3 \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\varrho+1)} t^{k+\alpha-\varrho} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\varrho+1)} t^{\alpha-\varrho} \right) \\
&\quad + d_4 (t^{k+\alpha} + t^\alpha).
\end{aligned} \tag{5.46}$$

Now, we will calculate the terms on the right-hand side (RHS) of the equation (5.33) for $X_0$, as follows,

$$\begin{aligned}
&d_5 \nabla X_0 + d_6 D_t^\mu X_0 + \Psi(\Phi, t) \\
&= \prod_{i=1}^{m} \cos(\pi\Phi_i) \times \Bigg[ \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\sigma+1)} t^{k+\alpha-\sigma} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\sigma+1)} t^{\alpha-\sigma} \right) \\
&\quad + d_2 \left( (k+\alpha) t^{k+\alpha-1} + \alpha t^{\alpha-1} \right) \\
&\quad + d_3 \left( \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\varrho+1)} t^{k+\alpha-\varrho} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\varrho+1)} t^{\alpha-\varrho} \right) \\
&\quad + d_4 (t^{k+\alpha} + t^\alpha).
\end{aligned} \tag{5.47}$$

As we see that the equations (5.46) and (5.47) give the same results. This implies that the LHS of the equation (5.46) is equal to the RHS of the equation (5.47). This proves that $\zeta_0 = X_0$ satisfies the equation (5.33) and the related conditions. Thus, the obtained solution is the exact solution of the problem (5.33)–(5.34) using the TSADM.

**Example 5.4.** Consider the multi-dimension multi-term time-fractional mixed diffusion equation on a rectangular domain, as

$$\begin{aligned}
&D_t^\sigma(\zeta(\Phi,t)) + d_2 \frac{\partial \zeta(\Phi,t)}{\partial t} + d_3 D_t^\varrho(\zeta(\Phi,t)) + d_4 \zeta(\Phi,t) \\
&= d_5 \nabla\zeta(\Phi,t) + d_6 D_t^\mu(\nabla\zeta(\Phi,t)) + \Psi(\Phi,t), \\
&(\Phi,t) = (\Phi_1, \Phi_2, \cdots, \Phi_m, t) \in \Lambda \times I,
\end{aligned} \tag{5.48}$$

with the initial and Neumann boundary conditions

$$\begin{aligned}&\zeta(\Phi,0)=\prod_{i=1}^{m}\Phi_i^n(\Phi_i-a)^n,\zeta_t(\Phi,0)=0,\Phi\in\bar{\Lambda},\\&\zeta(\Phi,t)=0,\Phi\in\partial\Lambda,t\in\bar{I},\end{aligned}\tag{5.49}$$

where $1<\sigma<2$, $0<\varrho,\mu<1$, $\Lambda=\prod_{i=1}^{m}(0,a)$, $I=(0,T]$,

$$\begin{aligned}\Psi(\Phi,t)=\prod_{i=1}^{m}\Phi_i^n(\Phi_i-a)^n\bigg(&\frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha-\sigma)}t^{\alpha-\sigma}+d_2\alpha t^{\alpha-1}\\&+d_3\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\varrho)}t^{\alpha-\varrho}+d_4(1+t^{\alpha})\bigg)\\&-d_5n(n-1)\prod_{\substack{j=1\\i\neq j}}^{k}\Phi_j^n(\Phi_j-a)^n\bigg(\sum_{i=1}^{m}(\Phi_i^{n-2}+(\Phi_i-a)^{n-2})\bigg)(1+t^{\alpha})\\&-d_6n(n-1)\prod_{\substack{j=1\\i\neq j}}^{k}\Phi_j^n(\Phi_j-a)^n\bigg(\sum_{i=1}^{m}(\Phi_i^{n-2}+(\Phi_i-a)^{n-2})\bigg)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\mu)}t^{\alpha-\mu}.\end{aligned}\tag{5.50}$$

The corresponding exact solution is $\prod_{i=1}^{m}\Phi_i^n(\Phi_i-a)^n(1+t^{\alpha})$.

On applying $J_t^{\sigma}$ into the equation (5.48), we have

$$\begin{aligned}\zeta(\Phi,t)=\zeta(\Phi,0)+J_t^{\sigma}\bigg(&d_5\nabla\zeta(\Phi,t)+d_6D_t^{\mu}(\nabla\zeta(\Phi,t))+\Psi(\Phi,t)\\&-\bigg(d_2\frac{\partial\zeta(\Phi,t)}{\partial t}+d_3D_t^{\varrho}(\zeta(\Phi,t))+d_4\zeta(\Phi,t)\bigg)\bigg),\end{aligned}\tag{5.51}$$

where $J_t^{\sigma}$ is the inverse operator of the operator $D_t^{\sigma}$.

The recursion formula of the solution from the equation (5.51) works as follows

$$\zeta_0(\Phi,t)=\zeta(\Phi,0)+J_t^{\sigma}\bigg(\Psi(\Phi,t)\bigg),\tag{5.52}$$

and

$$\begin{aligned}\zeta_{k+1}(\Phi,t)=\zeta(\Phi,0)+J_t^{\sigma}\bigg(&d_5\nabla\zeta_k(\Phi,t)+d_6D_t^{\mu}(\nabla\zeta_k(\Phi,t))\\&-\bigg(d_2\frac{\partial\zeta_k(\Phi,t)}{\partial t}+d_3D_t^{\varrho}(\zeta_k(\Phi,t))+d_4\zeta_k(\Phi,t)\bigg)\bigg),\end{aligned}\tag{5.53}$$

where $k=1,2,\cdots$.

By solving the equation (5.52), we obtain

$$\begin{aligned}\zeta_0(\Phi,t)=&\prod_{i=1}^{m}\Phi_i^n(\Phi_i-a)^n(1+t^{\alpha})+\prod_{i=1}^{m}\Phi_i^n(\Phi_i-a)^n\bigg(d_2\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma)}t^{\alpha+\sigma-1}\\&+d_3\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1-\varrho)}t^{\alpha+\sigma-\varrho}+d_4\bigg(\frac{1}{\Gamma(1+\sigma)}t^{\alpha}+\frac{1+\alpha}{\Gamma(1+\sigma+\alpha)}t^{\alpha+\sigma}\bigg)\bigg)\end{aligned}$$

$$
\begin{aligned}
&- d_5 n(n-1) \prod_{\substack{j=1\\ i\neq j}}^{k} \Phi_j^n (\Phi_j - a)^n \Bigg( \sum_{i=1}^{m} (\Phi_i^{n-2} + (\Phi_i - a)^{n-2}) \Bigg) \\
&\times \left( \frac{1}{\Gamma(1+\sigma)} t^{\alpha} + \frac{1+\alpha}{\Gamma(1+\sigma+\alpha)} t^{\alpha+\sigma} \right) - d_6 n(n-1) \prod_{\substack{j=1\\ i\neq j}}^{k} \Phi_j^n (\Phi_j - a)^n \\
&\times \Bigg( \sum_{i=1}^{m} (\Phi_i^{n-2} + (\Phi_i - a)^{n-2}) \Bigg) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1-\mu)} t^{\alpha+\sigma-\mu}. \qquad (5.54)
\end{aligned}
$$

From the above equation (5.54), the first iteration of TSADM can be split into five terms as

$$\zeta_0(\Phi, t) = X_0 + X_1 + X_2 + X_3 + X_4, \qquad (5.55)$$

where

$$X_0 = \prod_{i=1}^{m} \Phi_i^n (\Phi_i - a)^n (1 + t^{\alpha}), \qquad (5.56)$$

$$X_1 = \prod_{i=1}^{m} \Phi_i^n (\Phi_i - a)^n \Bigg( d_2 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma)} t^{\alpha+\sigma-1} + d_3 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1-\varrho)} t^{\alpha+\sigma-\varrho} \Bigg), \qquad (5.57)$$

$$X_2 = \prod_{i=1}^{m} \Phi_i^n (\Phi_i - a)^n \Bigg( d_4 \Bigg( \frac{1}{\Gamma(1+\sigma)} t^{\alpha} + \frac{1+\alpha}{\Gamma(1+\sigma+\alpha)} t^{\alpha+\sigma} \Bigg), \qquad (5.58)$$

$$
\begin{aligned}
X_3 = &- d_5 n(n-1) \prod_{\substack{j=1\\ i\neq j}}^{k} \Phi_j^n (\Phi_j - a)^n \Bigg( \sum_{i=1}^{m} (\Phi_i^{n-2} + (\Phi_i - a)^{n-2}) \Bigg) \\
&\times \left( \frac{1}{\Gamma(1+\sigma)} t^{\alpha} + \frac{1+\alpha}{\Gamma(1+\sigma+\alpha)} t^{\alpha+\sigma} \right),
\end{aligned} \qquad (5.59)
$$

$$
\begin{aligned}
X_4 = &- d_6 n(n-1) \prod_{\substack{j=1\\ i\neq j}}^{k} \Phi_j^n (\Phi_j - a)^n \Bigg( \sum_{i=1}^{m} (\Phi_i^{n-2} + (\Phi_i - a)^{n-2}) \Bigg) \\
&\times \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1-\mu)} t^{\alpha+\sigma-\mu}.
\end{aligned} \qquad (5.60)
$$

Here we generalized our problem and obtained a general solution for the equation (5.48).

According to the TSADM process, we select the first iteration as the term involved in the equation (5.55) and the term satisfies the problem (5.48)–(5.49) , so we terminate the process and obtain the solution of the problem.

Let us consider $\zeta_0 = X_0$ and check that the chosen term as $\zeta_0$ is satisfying the equation (5.48) and also the given conditions. If this choice of $\zeta_0$ is approved then this implies that the chosen term is a solution to the problem (5.48)–(5.49).

Let us consider $\zeta_0 = X_0$ as a solution of the equation (5.48), so that it satisfies the equation (5.48).

To prove $\zeta = X_0$ is the exact solution of the equation (5.48), we substitute

$\zeta = X_0$ in the left-hand side (LHS) of the equation (5.48), and obtain

$$D_t^\sigma X_0 + d_2 \frac{\partial X_0}{\partial t} + d_3 D_t^\varrho X_0 + d_4 X_0$$
$$= \prod_{i=1}^{m} \Phi_i^n (\Phi_i - a)^n \left( \frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha-\sigma)} t^{\alpha-\sigma} + d_2 \alpha t^{\alpha-1} + d_3 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\varrho)} t^{\alpha-\varrho} + d_4(1+t^\alpha) \right). \tag{5.61}$$

Now, we will calculate the terms on the right-hand side (RHS) of the equation (5.48) for $X_0$, as follows,

$$d_5 \nabla X_0 + d_6 D_t^\mu X_0 + \Psi(\Phi, t) = \prod_{i=1}^{m} \Phi_i^n (\Phi_i - a)^n \left( \frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha-\sigma)} t^{\alpha-\sigma} + d_2 \alpha t^{\alpha-1} + d_3 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\varrho)} t^{\alpha-\varrho} + d_4(1+t^\alpha) \right). \tag{5.62}$$

As we see that the equations (5.61) and (5.62) give the same results. This implies that the LHS of the equation (5.61) is equal to the RHS of the equation (5.62). This proves that $\zeta_0 = X_0$ satisfies the equation (5.48) and the related conditions. Thus, the discovered solution is the exact solution of the problem (5.48)–(5.49) using the TSADM.

**Example 5.5.** We consider the problem (3.1), the domain $\Lambda = (?\,2, 1) \times (?\,1, 2)$, and the parameters in the case $r_1 = d_{1,\nu_1} = 1$, $\sigma_{\nu_1} = 1.5$, $d_2 = 0$, $r_2 = 2$, $d_{3,1} = d_{3,2} = 1$, $\varrho_{\nu_1} = 1$, $\varrho_{\nu_2} = 0.4$, $d_5 = 2, d_4 = d_6 = 0$, described as

$$D_t^{1.5} \zeta(\Phi, t) + D_t \zeta(\Phi, t) + D_t^{0.4} \zeta(\Phi, t) = 2\nabla \zeta(\Phi, t) + \Psi(\Phi, t), \tag{5.63}$$

with the initial and boundary conditions

$$\zeta(\Phi, t) = \zeta_t(\Phi, t) = 0, \Phi \in \bar{\Lambda}, \zeta(\Phi, t) = 0, \Phi \in \partial\Lambda, t \in \bar{I}, \tag{5.64}$$

and

$$\Psi(\Phi, t) = \left( \frac{\Gamma(n+1)}{\Gamma(n-0.5)} t^{n-1.5} + \frac{\Gamma(n+1)}{\Gamma(n-0.6)} t^{n-0.4} + n t^{n-1} + n(n-1) m t^n \right) \exp\left( -\sum_{i=1}^{m} \Phi_i^n \right), n \in \mathbb{N}, \tag{5.65}$$

with an exact solution $\zeta(\Phi, t) = t^n \exp\left( -\sum_{i=1}^{m} \Phi_i^n \right)$.

On applying $J_t^{1.1}$ into the equation (5.63), we have

$$\zeta(\Phi, t) = J_t^{1.1} \left( 2\nabla \zeta(\Phi, t) + \Psi(\Phi, t) - \left( D_t \zeta(\Phi, t) + D_t^{0.4} \zeta(\Phi, t) \right) \right) \tag{5.66}$$

where $J_t^\sigma$ is the inverse operator of the operator $D_t^{1.1}$.

The recursion formula for the solution from the equation (5.66) is

$$\zeta_0(\Phi, t) = J_t^{1.1} \left( \Psi(\Phi, t) \right), \tag{5.67}$$

$$\zeta_{k+1}(\Phi,t) = J_t^{1.1}\left(2\nabla\zeta(\Phi,t) - \left(D_t\zeta(\Phi,t) + D_t^{0.4}\zeta(\Phi,t)\right)\right), \tag{5.68}$$

where $k = 1, 2, \cdots$.

By solving the equation (5.66), we obtain

$$\begin{aligned}\zeta_0(\Phi,t) = \Bigg(t^n + \frac{\Gamma(n+1)}{\Gamma(n+2.1)}t^{n+1.1} + \frac{\Gamma(n+1)}{\Gamma(n+1.5)}t^{n+00.5} \\ + (n-1)mt^{n+1}\Bigg)\exp\left(-\sum_{i=1}^{m}\Phi_i^n\right).\end{aligned} \tag{5.69}$$

From the above equation (5.69), the first iteration of TSADM can be split into five terms as

$$\zeta_0(\Phi,t) = X_0 + X_1 + X_2 + X_3, \tag{5.70}$$

where

$$X_0 = t^n \exp\left(-\sum_{i=1}^{m}\Phi_i^n\right), \tag{5.71}$$

$$X_1 = \frac{\Gamma(n+1)}{\Gamma(n+2.1)}t^{n+1.1}\exp\left(-\sum_{i=1}^{m}\Phi_i^n\right), \tag{5.72}$$

$$X_2 = \frac{\Gamma(n+1)}{\Gamma(n+1.5)}t^{n+00.5}\exp\left(-\sum_{i=1}^{m}\Phi_i^n\right), \tag{5.73}$$

$$X_3 = (n-1)mt^{n+1}\exp\left(-\sum_{i=1}^{m}\Phi_i^n\right). \tag{5.74}$$

Here we generalized our problem and obtained a general solution for the equation (5.63).

According to the TSADM process, we select the first iteration as the term involved in the equation (5.70) and the term satisfies the problem (5.63)–(5.64), so we terminate the process and obtain the solution of the consider problem.

Let us consider $\zeta_0 = X_0$ and check that the chosen term as of $\zeta_0$ is satisfying the equation (5.63) and also the given conditions. If this choice of $\zeta_0$ is approved then this implies that the chosen term is a solution to the problem.

Let us consider $\zeta_0 = X_0$ as a solution of the equation (5.63), so that it satisfies the equation (5.63).

To prove $\zeta = X_0$ is the exact solution of the equation (5.63), we substitute $\zeta = X_0$ in the left-hand side (LHS) of the equation (5.63), and obtain

$$\begin{aligned}D_t^{1.5}X_0 + D_tX_0 + D_t^{0.4}X_0 = \exp\left(-\sum_{i=1}^{m}\Phi_i^n\right)\Bigg(\frac{\Gamma(n+1)}{\Gamma(n-0.5)}t^{n-1.5} \\ + \frac{\Gamma(n+1)}{\Gamma(n-0.6)}t^{n-0.4} + nt^{n-1}\Bigg).\end{aligned} \tag{5.75}$$

Now, we will calculate the terms on the right-hand side (RHL) of the equation (5.63) for $X_0$, as follows,

$$2\nabla X_0 + \Psi(\Phi,t) = \exp\left(-\sum_{i=1}^{m}\Phi_i^n\right)\left(\frac{\Gamma(n+1)}{\Gamma(n-0.5)}t^{n-1.5} + \frac{\Gamma(n+1)}{\Gamma(n-0.6)}t^{n-0.4} + nt^{n-1}\right). \tag{5.76}$$

As we see that the equations (5.75) and (5.76) give the same results. This implies that the LHS of the equation (5.75) is equal to the RHS of the equation (5.76). This proves that $\zeta_0 = X_0$ satisfies the equation (5.63) and the related conditions. Thus, the discovered solution is the exact solution of the problem (5.63)–(5.64) using the TSADM.

**Remark.** The considered equation has also been solved using various existing numerical methods [3, 4] for two or three dimensions only. The considered problem has importance in many real life phenomena due to this fact, we are interested to discuss the solution of these types of equations. The analytical/exact solution of the equation does not discussed in the literature for more than three dimensions. The numerical methods are generally used to solve this problem as the standard tools to get solutions. These methods either discretization or iteration methods. In this article present study overcomes the following deficiencies to give the efficiency of the proposed method as compare to numerical methods for solving the considered problem:
1. Numerical methods for solving such problems, involve discretization. It is well known fact that if we discretize the problem domain, there are so many difficulties occur, and the results depend on the number of messing points considered to solve the problem and the chosen methods, whether it is suitable or not for the particular equation or its domain. The obtained solutions are always approximate using numerical methods.
2. If we do not use the discretization techniques, we have another option to solve the problem using approximations. There are so many numerical techniques exist which give the approximations to the solution. However, it is not possible to always obtain solutions using numerical methods because of the convergence rate of the methods. The numerical methods do not give the guarantee of the existence of solutions. Also, accuracy of the solution depends on number of iterations that may be very large in most of the cases.

From above, we observe that the numerical methods provide approximate solutions via multiple methods and steps. Also, it is not possible to obtain the solution for multi-dimension via these methods. In [3,4], they used several Methods to obtain the approximate solutions with complicated polynomials. Finally, we analyze that the proposed method [10, 13–17] is the most suitable method and straight forward to obtain an analytical solution in just one iteration without discretization.

# 6. Conclusion

The main objectives of this work is to find the exact solution of the considered problems (3.1)–(3.2) without meshing via discretization and approximation of the fractional operators. We also generalized the problem and obtained the general solution for the multi-dimensional multi-term Caputo time-fractional mixed sub-diffusion and diffusion-wave equation. The proposed method neither requires meshing of the time-fractional operators nor the correction process of the solution for obtaining high accuracy. The TSADM has straightforward steps and easy to implement on the problems and provides an exact solution with one iteration and also reduces the computational cost/effort for the process as compared to other existing numerical methods. The obtained solutions are smooth on the convex domain.

Moreover, we have established the new conditions for the existence and uniqueness of the solution using fixed point theory. We have considered five examples, out of which the first four have arbitrary order, and the rest Example 5 has fixed parameter values. These examples show the applicability and efficiency of the proposed method, whereas the numerical method gives approximate solutions and hard to apply on the multi-dimensions. Generally, the numerical methods have been applied up to four dimensional problems which requires high level programming skills, and the cost/time of these methods is also very high. Thus, the proposed method is more useful compared to other existing numerical methods.

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