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NITSCHE'S TYPE STABILIZATION FOR THE FULLY MIXED NAVIER-STOKES/DARCY PROBLEM*

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Abstract In this paper, we present and analyze a fully mixed finite element scheme for the Navier-Stokes/Darcy problem based on the Nitsche's type interface stabilizations, in the fluid region coupled with the porous media domain. The reasonable parameter $\delta > 0$, which is independent of mesh size h, will guarantee the stability and optimal convergence of our stabilized scheme. Moreover, we explicitly derive the dependence and requirement of the stabilization parameter δ for the optimal error estimates, while the numerical tests support the stability and efficiency of this stabilized mixed method.

Keywords Navier-Stokes/Darcy problem, mixed finite elements, Nitsche's method, Beavers-Joseph-Saffman.

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1. Introduction

The Navier-Stokes/Darcy or Stokes/Darcy fluid flow model coupling among surface flow and subsurface flow [7,14,18,20] is a very classical multi-domain, multi-physics model, which is popular in industrial processes and the groundwater fluid flow in the karst aquifer and so on. The coupled Navier-Stokes/Darcy model is composed of a nonlinear Navier-Stokes equations with Darcy law equation for fluid flow and porous media flow respectively, with specific interface conditions. Over the last few decades, lots of work are developed for the (Navier-)Stokes-Darcy model. In [20], Layton et al investigate a mixed variational formulation in both domains based on Beavers-Joseph-Saffman interface conditions and utilized a Lagrange multiplier, such ideas can also be found in [6, 12, 13, 19], other coupled finite element methods are given in [8,28]. There are also many decoupled schemes developed, such as two-grid or multi-grid methods [5, 10, 11, 17, 25, 27, 38–40] and domain decomposition methods [9, 15]. Lots of decouple schemes are developed for the time-dependent (Navier-)Stokes/Darcy problems, see [26,29,30,36]. More applications of the Navier-Stokes/Darcy model can be found in [4, 24, 31, 37].

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In this work, thanks to the conservation of mass on the interface, we will introduce two strongly consistent interface stabilization terms for the fully mixed formulation of the Navier-Stokes/Darcy model, to guarantee the stability. By the modified interface condition, we will overcome the technical difficulty caused by the nonlinear term. Here, the fully mixed stabilized method does not introduce any Lagrange multiplier and the computation is straightforward. Besides, the stability of the scheme only requires the reasonable mesh-independent stabilization parameter $\delta > 0$. This choice of the parameter is better than the condition derived in [34] (which is with Beavers-Joseph interface conditions), owing to the present Darcy-pressure consistent interface stabilization. Moreover, the dependence and requirement of stabilization parameter δ for the optimal error estimates are explicitly derived. The similar Nitsche's interface stabilized technique can be found in [3,23]. The former one mainly deals with the lowest finite element pairs for Stokes-Darcy problem to overcome the LBB condition, and also requires the pressure stabilization. In the later paper, the authors focus on the mixed-Stokes-dual-permeability fluid flow problems, only one interface stabilization norm is introduced, which will request the stabilized parameter large enough. Such stabilized techniques are also applied to deal with different interface problems, such as elliptic interface problems [33], steady mixed Stokes-Darcy model [32] and time-dependent Stokes-dual-permeability fluid flow problems [22] and so on.

The present paper is built up as follows. We briefly introduce the classical Navier-Stokes/Darcy fluid flow model with interface conditions and some preliminaries in section 2. The Nitsche's stabilized finite element method and its stability are discussed in section 3 while section 4 presents the error estimates of the stabilized finite element method. The paper ends with numerical experiments and conclusion.

2. The Navier-Stokes/Darcy model

Let two bounded connect domains $\Omega_f, \Omega_p \subset R^d (d = 2 \; or \; 3)$ with an interface Γ, i.e., $\Omega_f \bigcap \Omega_p = \emptyset$, and $\overline{\Omega}_f \bigcap \overline{\Omega}_p = \Gamma$, $\overline{\Omega} = \overline{\Omega}_f \bigcup \overline{\Omega}_p$. Let $\mathbf{n}_f$ and $\mathbf{n}_p$ denote the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively, and $\tau_i (i = 1, \cdots, d-1)$, the unit tangential vectors on the interface Γ. Besides, we denote $\Gamma_f = \partial\Omega_f \backslash \Gamma, \Gamma_p = \partial\Omega_p \backslash \Gamma$.

Given the external force $\mathbf{f}_f$ and fluid kinematic viscosity ν, the incompressible flow with the fluid velocity and pressure $\mathbf{u}_f$ and p satisfy in Ω_f:

$$-\nabla \cdot \mathbb{T} + \mathbf{u}_f \cdot \nabla \mathbf{u}_f = -2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_f) + \nabla p + \mathbf{u}_f \cdot \nabla \mathbf{u}_f = \mathbf{f}_f \quad \texttt{in } \Omega_f, \tag{2.1}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \texttt{in } \Omega_f, \tag{2.2}$$

with no slip conditions $\mathbf{u}_f = 0$ on Γ_f, and where $\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{u}_f)$ represents the stress tensor, and $\mathbb{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T)$ denotes the deformation tensor.

In the porous media domain Ω_p, the fluid velocity $\mathbf{u}_p$ and the piezometric head ϕ satisfy the following Darcy system:

$$\mathbf{u}_p = -\mathbf{K}\nabla\phi \quad \texttt{in } \Omega_p, \tag{2.3}$$

$$\nabla \cdot \mathbf{u}_p = f_p \quad \texttt{in } \Omega_p. \tag{2.4}$$

with $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p, on the exterior boundary. Here f_p is assumed to satisfy the solvability condition $\int_{\Omega_p} f_p dx = 0$, and $\mathbf{K}$ is the hydraulic conductivity tensor, and for the sake of simplicity, is assumed as the constant scalar matrix $\mathbf{K} = K\mathbf{I}$ with $K \leq 1$ in the porous medium.

The conservation of mass, balance of forces and a tangential condition on the fluid region for velocity on the interface on Γ are used:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p = 0, \tag{2.5}$$

$$p - 2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_f) \cdot \mathbf{n}_f + \frac{1}{2}(\mathbf{u}_f \cdot \mathbf{u}_f) = g\phi, \tag{2.6}$$

$$-2\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_f) \cdot \tau_i = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}} \mathbf{u}_f \cdot \tau_i, 1 \leq i \leq (d-1). \tag{2.7}$$

Here $\Pi = \frac{\mathbf{K}\nu}{g}$, g the gravitational acceleration.

Denote the L^2 norm and the inner product by $||\cdot||$ and $(\cdot,\cdot)$ for $L^2(\Omega_f)$ or $L^2(\Omega_p)$, and the L^2 norm by $||\cdot||_\Gamma$ for $L^2(\Gamma)$ (See Sobolev spaces and norms [1]), and $|\cdot|_1$, $||\cdot||_1$ mean the semi H^1-norm and H^1-norm, repectively. By setting the space

$$H(\text{div};\Omega_p) := \{\mathbf{v}_p \in L^2(\Omega_p)^d : \ \nabla \cdot \mathbf{v}_p \in L^2(\Omega_p)\},$$

we introduce the following modified spaces:

$$\mathbf{X}_f := \{\mathbf{v}_f \in H^1(\Omega_f)^d : \ \mathbf{v}_f = 0 \text{ on } \Gamma_f, \ \int_\Gamma \mathbf{v}_f \cdot \mathbf{n}_f = 0\}, \quad Q_f := L_0^2(\Omega_f),$$

$$\mathbf{X}_p := \{\mathbf{v}_p \in H(\text{div};\Omega_p) : \ \mathbf{v}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p, \ \int_\Gamma \mathbf{v}_p \cdot \mathbf{n}_p = 0\}, \quad Q_p := L_0^2(\Omega_p),$$

equipped with the norms for $\mathbf{X}_f$ and $\mathbf{X}_p$

$$||\mathbf{v}_f||_1 = \sqrt{||\mathbf{v}_f||^2 + |\mathbf{v}_f|_1^2}, \qquad ||\mathbf{v}_p||_{\text{div}} = \sqrt{||\mathbf{v}_p||^2 + ||\nabla \cdot \mathbf{v}_p||^2}.$$

Noting that the variational formulation related with (3.1) introduced in the next section under such spaces is equivalent to the classical formulation. The proof can follow Theorem 1 of [34]. Meanwhile, these modified spaces are much simple.

Then, we will introduce a family of regular triangulation T_h of Ω, consisting of T_h^f and T_h^p, with mesh size $h > 0$, and the interface Γ coincides the two meshes of T_h^f and T_h^p. Assuming the finite element spaces $\mathbf{X}_{fh} \subset \mathbf{X}_f, Q_{fh}^h \subset Q_f$ and $\mathbf{X}_{ph} \subset \mathbf{X}_p, Q_{ph} \subset Q_p$ which satisfy the classical inf-sup conditions [2], here, we only consider the P1b-P1/BDM1-P0 pairs.

3. The stabilized mixed finite element method and its stability

In this section, we present the stabilized finite element scheme for the Navier-Stokes/Darcy problem.

Find $(\mathbf{u}_f^h, p_f^h, \mathbf{u}_p^h, \phi_p^h) \in (\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph})$ satisfying that

$$\begin{aligned} &\mathscr{L}(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) + N(\mathbf{u}_f^h; \mathbf{u}_f^h, \mathbf{v}_f^h) \\ =&(\mathbf{f}_f, \mathbf{v}_f^h) + g(f_p, \nabla \cdot \mathbf{v}_p^h) + g(f_p, \psi^h) \\ &\forall (\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) \in (\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph}), \end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
&\mathscr{L}(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h;\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)\\
=&\mathscr{L}_f(\mathbf{u}_f^h,p^h;\mathbf{v}_f^h,q^h)+\mathscr{L}_p(\mathbf{u}_p^h,\phi^h;\mathbf{v}_p^h,\psi^h)+c_\Gamma(\mathbf{v}_f^h-\mathbf{v}_p^h,\phi^h)-c_\Gamma(\mathbf{u}_f^h-\mathbf{u}_p^h,\psi^h)\\
&+\frac{\delta}{h}\int_\Gamma((\mathbf{u}_f^h-\mathbf{u}_p^h)\cdot\mathbf{n}_f)((\mathbf{v}_f^h-\mathbf{v}_p^h)\cdot\mathbf{n}_f)d\Gamma,\\
&\mathscr{L}_f(\mathbf{u}_f,p;\mathbf{v}_f,q)=a_f(\mathbf{u}_f,\mathbf{v}_f)-b_f(\mathbf{v}_f,p)+b_f(\mathbf{u}_f,q),\\
&\mathscr{L}_p(\mathbf{u}_p,\phi;\mathbf{v}_p,\psi)=a_p(\mathbf{u}_p,\mathbf{v}_p)-b_p(\mathbf{v}_p,\phi)+b_p(\mathbf{u}_p,\psi),\\
&a_f(\mathbf{u}_f,\mathbf{v}_f)=2\nu(\mathbb{D}(\mathbf{u}_f),\mathbb{D}(\mathbf{v}_f))+\sum_{i=1}^{d-1}\frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\Pi)}}(\mathbf{u}_f\cdot\tau_i,\mathbf{v}_f\cdot\tau_i)_\Gamma,\\
&a_p(\mathbf{u}_p,\mathbf{v}_p)=g(\mathbf{K}^{-1}\mathbf{u}_p,\mathbf{v}_p)+g(\nabla\cdot\mathbf{u}_p,\nabla\cdot\mathbf{v}_p),\\
&b_f(\mathbf{v}_f,p)=(p,\nabla\cdot\mathbf{v}_f),\quad b_p(\mathbf{v}_p,\phi)=g(\phi,\nabla\cdot\mathbf{v}_p),\quad c_\Gamma(\mathbf{v}_f,\phi)=g(\phi,\mathbf{v}_f\cdot\mathbf{n}_f)_\Gamma.
\end{aligned}
$$

Thanks to the interface condition (2.6) and divergence free of the velocity, we can use the following nonlinear term form N as

$$
N(\mathbf{u};\mathbf{w},\mathbf{v})=\frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})-\frac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) \tag{3.2}
$$

and it satisfies the following property [21]:

$$
N(\mathbf{u};\mathbf{v},\mathbf{v})=0 \quad \forall\ \mathbf{u},\mathbf{v}\in\mathbf{X}_f \tag{3.3}
$$

$$
|N(\mathbf{u};\mathbf{w},\mathbf{v})|\le C_N||\mathbf{u}||_1||\mathbf{v}||_1||\mathbf{w}||_1 \quad \forall\ \mathbf{u},\mathbf{w},\mathbf{v}\in\mathbf{X}_f, \tag{3.4}
$$

with $C_N>0$ is a bounded constant.

By the inf-sup conditions, for arbitrarily given but fixed $p^h\in Q_{fh}$ and $\phi^h\in Q_{ph}$, there exist $\mathbf{w}_f^h\in\mathbf{X}_{fh}\bigcap H_0^1(\Omega_f)^d$ and $\mathbf{w}_p^h\in\mathbf{X}_{ph}\bigcap H_0^1(\Omega_f)^d$, and two constants $\beta_f,\beta_p>0$, independent of h, such that

$$
b_f(\mathbf{w}_f^h,p^h)\ge\beta_f||p^h||^2,\quad b_p(\mathbf{w}_p^h,\phi^h)\ge\beta_p||\phi^h||^2. \tag{3.5}
$$

Now, we begin to show the continuity and coercivity of the stabilized mixed method with the following norm:

$$
|||(\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)|||=||\mathbf{v}_f^h||_1+||q^h||+||\mathbf{v}_p^h||_{\mathtt{div}}+||\psi^h||+h^{-1/2}||(\mathbf{v}_f^h-\mathbf{v}_p^h)\cdot\mathbf{n}_f||_\Gamma.
$$

Theorem 3.1 (The continuity of $\mathscr{L}$)**.** *There exists a constant* $C_{max}=C\max\{\nu,gK^{-1},\delta\}$, *such that*

$$
\begin{aligned}
&\mathscr{L}(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h;\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)\\
\le&C_{max}|||(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h)|||\ |||(\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)|||.
\end{aligned} \tag{3.6}
$$

Proof. It is easy to get the result by applying the Schwarz inequality and inverse inequality. □

Theorem 3.2 (The coercivity of $\mathscr{L}$)**.** *There exists a constant* $\beta>0$ *such that the following inequality holds, for all* $(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h)\in(\mathbf{X}_{fh},Q_{fh},\mathbf{X}_{ph},Q_{ph})$,

$$
\begin{aligned}
&\sup_{(\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)\in(X_{fh},Q_{fh},X_{ph},Q_{ph})}\frac{\mathscr{L}(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h;\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)}{|||(\mathbf{v}_f^h,q^h,\mathbf{v}_p^h,\psi^h)|||}\\
&\ge\beta|||(\mathbf{u}_f^h,p^h,\mathbf{u}_p^h,\phi^h)|||.
\end{aligned} \tag{3.7}
$$

Proof. The proof is similar as that in [34], and can be improved according to the BJS condition and the Darcy-pressure consistent interface stabilization.

Firstly, for any $(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h) \in (\mathbf{X}_{fh}, Q_{fh}, \mathbf{X}_{ph}, Q_{ph})$, by choosing $(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) = (\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h)$, and we can derive

$$\begin{aligned}
&\mathscr{L}(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h; \mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h)\\
\geq& 2\nu||\mathbb{D}(\mathbf{u}_f^h)||^2 + g||\mathbf{K}^{-1/2}\mathbf{u}_p^h||^2 + g||\nabla\cdot\mathbf{u}_p^h||^2 + \frac{\delta}{h}||(\mathbf{u}_f^h - \mathbf{u}_p^h)\cdot\mathbf{n}_f||_\Gamma^2\\
\geq& C_\nu||\mathbf{u}_f^h||_1^2 + gK^{-1}||\mathbf{u}_p^h||^2 + g||\nabla\cdot\mathbf{u}_p^h||^2 + \frac{\delta}{h}||(\mathbf{u}_f^h - \mathbf{u}_p^h)\cdot\mathbf{n}_f||_\Gamma^2.
\end{aligned}$$

Secondly, selecting $(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) = (-\gamma\mathbf{w}_f^h, 0, -\gamma\mathbf{w}_p^h, 0)$, where $\mathbf{w}_f^h$, $\mathbf{w}_p^h$ satisfy (3.5), respectively, with $(\mathbf{w}_f^h - \mathbf{w}_p^h)\cdot\mathbf{n}_f = 0$ on Γ, and applying Young's inequalities, we arrive at

$$\begin{aligned}
&\widetilde{\mathscr{L}}(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h; -\gamma\mathbf{w}_f^h, 0, -\gamma\mathbf{w}_p^h, 0)\\
=& -\gamma a_f(\mathbf{u}_f^h, \mathbf{w}_f^h) - \gamma a_p(\mathbf{u}_p^h, \mathbf{w}_p^h) + \gamma b_f(\mathbf{w}_f^h, p^h) + \gamma b_p(\mathbf{w}_p^h, \phi^h)\\
\geq& -\gamma C_1||\mathbf{u}_f^h||_1||p^h|| - \gamma C_2||\mathbf{u}_p^h||_{\mathtt{div}}||\phi^h|| + \gamma\beta_f||p^h||^2 + \gamma\beta_p||\phi^h||^2\\
\geq& -\frac{\gamma C_1^2}{2\beta_f}||\mathbf{u}_f^h||_1^2 - \frac{\gamma C_2^2}{2\beta_p}||\mathbf{u}_p^h||_{\mathtt{div}}^2 + \frac{\gamma\beta_f}{2}||p^h||^2 + \frac{\gamma\beta_p}{2}||\phi^h||^2,
\end{aligned}$$

here, the continuity of a_f, a_p are used, namely:

$$a_f(\mathbf{u}_f, \mathbf{v}_f) \leq C_1||\mathbf{u}_f||_1||\mathbf{v}_f||_1,\ \ a_p(\mathbf{u}_p, \mathbf{v}_p) \leq C_2||\mathbf{u}_p||_{\mathtt{div}}||\mathbf{v}_p||_{\mathtt{div}}.$$

Then, choosing $(\widehat{\mathbf{v}_f^h}, \widehat{q^h}, \widehat{\mathbf{v}_p^h}, \widehat{\psi^h}) = (\mathbf{u}_f^h - \gamma\mathbf{w}_f^h, p^h, \mathbf{u}_p^h - \gamma\mathbf{w}_p^h, \phi^h)$, we obtain from above

$$\begin{aligned}
&\widetilde{\mathscr{L}}(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h; \mathbf{u}_f^h - \gamma\mathbf{w}_f^h, p^h, \mathbf{u}_p^h - \gamma\mathbf{w}_p^h, \phi^h)\\
\geq& (C_\nu - \frac{\gamma C_1^2}{2\beta_f})\|\mathbf{u}_f^h\|_1^2 + (g - \frac{\gamma C_2^2}{2\beta_p})\|\mathbf{u}_p^h\|_{\mathtt{div}}^2 + \frac{\gamma\beta_f}{2}\|p^h\|^2 + \frac{\gamma\beta_p}{2}\|\phi^h\|^2 + \frac{\delta}{h}||(\mathbf{u}_f^h - \mathbf{u}_p^h)\cdot\mathbf{n}_f||_\Gamma^2.
\end{aligned}$$

Then we enforce the following conditions on γ and δ:

$$C_\nu - \frac{\gamma C_1^2}{2\beta_f} \geq \frac{C_\nu}{2}, \quad g - \frac{\gamma C_2^2}{2\beta_p} \geq \frac{g}{2},\ \ \delta > 0.$$

Finally, letting $\gamma = \min\{\frac{\beta_f C_\nu}{C_1^2}, \frac{\beta_p g}{C_2^2}\}$ and $\delta > 0$, and then defining a positive constant $C_{min} = \min\{\frac{C_\nu}{2}, \frac{g}{2}, \frac{\gamma\beta_f}{2}, \frac{\gamma\beta_p}{2}, \delta\}$, which is independent of h, we can obtain that

$$\begin{aligned}
&\widetilde{\mathscr{L}}(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h; \mathbf{u}_f^h - \gamma\mathbf{w}_f^h, p^h, \mathbf{u}_p^h - \gamma\mathbf{w}_p^h, \phi^h)\\
\geq& \frac{C_\nu}{2}\|\mathbf{u}_f\|_1^2 + \frac{g}{2}\|\mathbf{u}_p^h\|_{\mathtt{div}}^2 + \frac{\gamma\beta_f}{2}\|p^h\|^2 + \frac{\gamma\beta_p}{2}\|\phi^h\|^2 + \frac{\delta}{h}||(\mathbf{u}_f^h - \mathbf{u}_p^h)\cdot\mathbf{n}_f||_\Gamma^2\\
\geq& C_{min}|||(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h|||^2\\
\geq& \tilde{C}C_{min}(|||(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h)|||\ |||(\mathbf{u}_f^h - \gamma\mathbf{w}_f^h, p^h, \mathbf{u}_p^h - \gamma\mathbf{w}_p^h, \phi^h)|||\\
=& \beta(|||(\mathbf{u}_f^h, p^h, \mathbf{u}_p^h, \phi^h)|||)(|||(\widehat{\mathbf{v}_f^h}, \widehat{q^h}, \widehat{\mathbf{v}_p^h}, \widehat{\psi^h})|||).
\end{aligned}$$

Therefore, we get the weak coercivity of $\mathscr{L}$. $\square$

4. Error estimates for the stabilized mixed method

In this section, we prove the error estimates for the stabilized mixed finite element method of the Navier-Stokes/Darcy problem.

First, in order to derive error estimates of the stabilized mixed finite element solution $(\mathbf{u}_f^h, p^h; \mathbf{u}_p^h, \phi^h)$, we need the special Galerkin projection as follows,

$$\mathscr{L}(\mathbf{u}_f, p, \mathbf{u}_p, \phi; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) = \mathscr{L}(P_h\mathbf{u}_f, P_h p, P_h\mathbf{u}_p, P_h\phi; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h). \quad (4.1)$$

By introducing $(\bar{\mathbf{u}}, \bar{p}, \bar{\mathbf{u}}_p, \bar{\phi})$ as the interpolation of $(\mathbf{u}_f, p, \mathbf{u}_p, \phi)$, the projection errors can be split as

$$\begin{aligned}
&\mathbf{u}_f - P_h\mathbf{u}_f = (\mathbf{u}_f - \bar{\mathbf{u}}_f) + (\bar{\mathbf{u}}_f - P_h\mathbf{u}_f) \triangleq \hat{\mathbf{e}}_f + \mathbf{e}_f^h,\\
&p - P_h p = (p - \bar{p}) + (\bar{p} - P_h p) \triangleq \hat{\eta} + \eta^h,\\
&\mathbf{u}_p - P_h\mathbf{u}_p = (\mathbf{u}_p - \bar{\mathbf{u}}_p) + (\bar{\mathbf{u}}_p - P_h\mathbf{u}_p) \triangleq \hat{\mathbf{e}}_p + \mathbf{e}_p^h,\\
&\phi - P_h\phi = (\phi - \bar{\phi}) + (\bar{\phi} - P_h\phi) \triangleq \hat{\theta} + \theta^h.
\end{aligned}$$

By the coercivity and the continuity of $\mathscr{L}$, and according to the interpolation error held by the chosen mixed finite element spaces, we obtain

$$\begin{aligned}
&\beta|||(\mathbf{e}_f^h, \eta^h, \mathbf{e}_p^h, \theta^h)|||\\
\le& \sup_{(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)\in(\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph})} \frac{\mathscr{L}(\mathbf{e}_f^h, \eta^h, \mathbf{e}_p^h, \theta^h; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)}{|||(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)|||}\\
=& \sup_{(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)\in(\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph})} \frac{-\mathscr{L}(\hat{\mathbf{e}}_f, \hat{\eta}, \hat{\mathbf{e}}_p, \hat{\theta}; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)}{|||(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)|||}\\
\le& C_{\mathtt{max}}|||(\hat{\mathbf{e}}_f, \hat{\eta}, \hat{\mathbf{e}}_p, \hat{\theta})|||\\
\le& C_{\mathtt{max}}(||\hat{\mathbf{e}}_f||_1 + ||\hat{\eta}|| + ||\hat{\mathbf{e}}_p||_{\mathtt{div}} + ||\hat{\theta}|| + h^{-\frac{1}{2}}(||\hat{\mathbf{e}}_f||_\Gamma + ||\hat{\mathbf{e}}_p||_\Gamma))\\
\le& CC_{\mathtt{max}}h.
\end{aligned}$$

Using the triangle inequality with the interpolation error, we derive the projection error estimate,

$$\begin{aligned}
&|||(\mathbf{u}_f - P_h\mathbf{u}_f, p - P_h p, \mathbf{u}_p - P_h\mathbf{u}_p, \phi - P_h\phi)|||\\
\le& |||(\mathbf{e}_f^h, \eta^h, \mathbf{e}_p^h, \theta^h)||| + |||(\hat{\mathbf{e}}_f, \hat{\eta}, \hat{\mathbf{e}}_p, \hat{\theta})||| \le \frac{CC_{\mathtt{max}}}{\beta}h.
\end{aligned}$$

Theorem 4.1. *Assume that $(\mathbf{u}_f, p; \mathbf{u}_p, \phi)$ is the exact solution of the Navier-Stokes/Darcy problem, and belongs to $(\mathbf{X}_f \bigcap H^2(\Omega_f)^d, Q_f \bigcap H^1(\Omega_f); \mathbf{X}_p \bigcap H^2(\Omega_p)^d, Q_p \bigcap H^1(\Omega_p))$, with the uniqueness condition $1 - \frac{C_N}{\nu^2}||\mathbf{f}_f|| > 0$. By choosing reasonable $\delta > 0$, $(\mathbf{u}_f^h, p^h; \mathbf{u}_p^h, \phi^h)$ is the stabilized mixed finite element solution, we can obtain*

$$||\mathbf{u}_f - \mathbf{u}_f^h||_1 + ||p - p^h|| + ||\mathbf{u}_p - \mathbf{u}_p^h||_{\mathrm{div}} + ||\phi - \phi^h|| \le \frac{CC_{max}}{\beta}h. \quad (4.2)$$

Proof. Thank to the interface conditions (2.5)–(2.6), $(\mathbf{u}_f - \mathbf{u}_p)\cdot\mathbf{n}_f = 0$, we rewrite the model (2.1)–(2.4) as follows, $\forall(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) \in (\mathbf{X}_{fh}, Q_{fh}, \mathbf{X}_{ph}, Q_{ph})$,

$$\mathscr{L}(\mathbf{u}_f, p, \mathbf{u}_p, \phi; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) + N(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}_f^h)$$

$$= (\mathbf{f}_f, \mathbf{v}_f^h) + g(f_p, \nabla \cdot \mathbf{v}_p^h) + g(f_p, \psi^h). \tag{4.3}$$

Subtracting (4.3) from (3.1) gives

$$\begin{aligned}
&\mathscr{L}(\mathbf{u}_f - \mathbf{u}_f^h, p - p^h, \mathbf{u}_p - \mathbf{u}_p^h, \phi - \phi^h; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) \\
&+ N(\mathbf{u}_f; \mathbf{u}_f, \mathbf{v}_f^h) - N(\mathbf{u}_f^h; \mathbf{u}_f^h, \mathbf{v}_f^h) = 0.
\end{aligned} \tag{4.4}$$

Thanks to the Galerkin projection (4.1), we get

$$\begin{aligned}
&\mathscr{L}(\mathbf{e}^f, \eta^f, \mathbf{e}^p, \theta^p; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) + N(\mathbf{u}_f - P_h\mathbf{u}_f + \mathbf{e}^f; \mathbf{u}_f, \mathbf{v}_f^h) \\
&+ N(\mathbf{u}_f^h; \mathbf{u}_f - P_h\mathbf{u}_f + \mathbf{e}^f, \mathbf{v}_f^h) = 0,
\end{aligned} \tag{4.5}$$

where $(\mathbf{e}^f, \eta^f, \mathbf{e}^p, \theta^p) = (P_h\mathbf{u}_f - \mathbf{u}_f^h, P_h p - p^h, P_h\mathbf{u}_p - \mathbf{u}_p^h, P_h\phi - \phi^h)$. By choosing $(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) = (\mathbf{e}^f, \eta^f, \mathbf{e}^p, \theta^p)$, similarly with the argument as in Theorem 3.2, it is easy to check that $\nu||\mathbf{u}_f||_1, \nu||\mathbf{u}_f^h||_1 \le ||\mathbf{f}_f||$, and choosing $\delta > 0$, we obtain

$$\begin{aligned}
&\nu(1 - \frac{N}{\nu^2}||\mathbf{f}_f||)||\mathbf{e}^f||_1^2 + gK^{-1}||\mathbf{e}^p||^2 + g||\nabla \cdot \mathbf{e}^p||^2 + \frac{\delta}{h}||(\mathbf{e}^f - \mathbf{e}^p) \cdot \mathbf{n}_f||_\Gamma^2 \\
\le& \max\{C_\nu, \nu\}||\mathbf{e}^f||_1^2 + gK^{-1}||\mathbf{e}^p||^2 + g||\nabla \cdot \mathbf{e}^p||^2 + \frac{\delta}{h}||(\mathbf{e}^f - \mathbf{e}^p) \cdot \mathbf{n}_f||_\Gamma^2 + N(\mathbf{e}^f; \mathbf{u}_f, \mathbf{e}^f) \\
\le& \max\{1, \frac{\nu}{C_\nu}\}|N(\mathbf{u}_f - P_h\mathbf{u}_f; \mathbf{u}_f, \mathbf{e}^f) + N(\mathbf{u}_f^h; \mathbf{u}_f - P_h\mathbf{u}_f, \mathbf{e}^f)| \\
\le& \max\{1, \frac{\nu}{C_\nu}\}C_N||\mathbf{u}_f - P_h\mathbf{u}_f||_1(||\mathbf{u}_f||_1 + ||\mathbf{u}_f^h|_1)||\mathbf{e}^f||_1 \\
\le& Ch||\mathbf{e}^f||_1.
\end{aligned}$$

Thus, $||\mathbf{e}^f||_1 \le Ch$. In view of the triangle inequality, we can obtain $||\mathbf{u}_f - \mathbf{u}_f^h||_1 \le Ch$.

Using the coercivity of $\mathscr{L}$ and the trilinear inequality again,

$$\begin{aligned}
&\beta|||(\mathbf{e}^f, \eta^f, \mathbf{e}^p, \theta^p)||| \\
\le& \sup_{(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) \in (\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph})} \frac{\mathscr{L}(\mathbf{e}^f, \eta^f, \mathbf{e}^p, \theta^p; \mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)}{|||(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)|||} \\
=& \sup_{(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h) \in (\mathbf{X}_{fh}, Q_{fh}; \mathbf{X}_{ph}, Q_{ph})} \frac{-N(\mathbf{u}_f - \mathbf{u}_f^h; \mathbf{u}_f, \mathbf{v}_f^h) + N(\mathbf{u}_f^h; \mathbf{u}_f - \mathbf{u}_f^h, \mathbf{v}_f^h)}{|||(\mathbf{v}_f^h, q^h, \mathbf{v}_p^h, \psi^h)|||} \\
\le& C_N||\mathbf{u}_f - \mathbf{u}_f^h||_1(||\mathbf{u}_f||_1 + ||\mathbf{u}_f^h|_1) \le Ch.
\end{aligned}$$

Finally, we get (4.2) by using the triangle inequality. □

Remark 4.1. According to Theorem 3.1, 3.2, if the stabilization parameter δ is very small, then $C_{min} = \delta$ and $\beta = C\delta$, and C_{max} is independent of δ, from Theorem 4.1, the error will be large. Otherwise, if δ is very large, then $C_{max} = C\delta$, and β is independent of δ, from Theorem 4.1, the error will be large too. So to keep optimal convergence of the scheme, the reasonable parameter δ is required.

5. Numerical tests

In this section, we present two numerical tests to illustrate the stability, efficiency of the stabilized fully mixed finite element method. The first experiment is a model test, mainly shows the stability of this stabilized technique, and the second experiment mainly demonstrates the convergence orders which supports the theoretical analysis and also the impact of the stabilization parameter. The finite element spaces are constructed by well-known MINI elements $P1b - P1$ for Navier-Stokes problem and to capture the fully mixed technique in the porous medium region, piecewise constant finite element $P0$ used for hydraulic(piezometric) head, the $BDM1$ space for Darcy velocity. For the nonlinear term, the Newton iterative technique is used. For computational convenience, in both examples, all the physical parameters $\nu, g, \mathbf{K}, \alpha$ are simply taken as 1.0 and the stabilization parameteris is always chosen $\delta = 0.1$. All the numerical tests executed by a specialized free domain software FreeFEM++ [16] and figures are drawn by Tecplot.

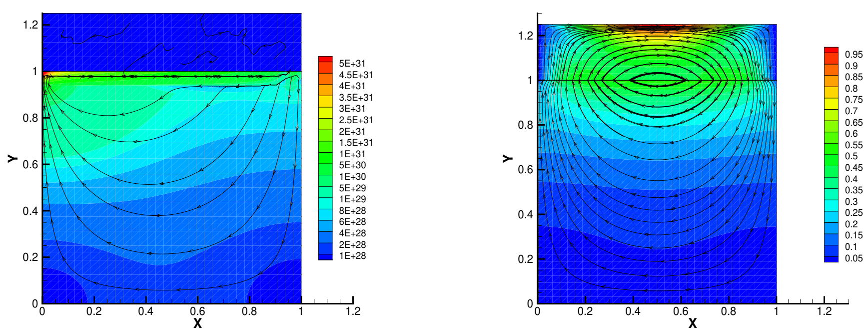

Figure 1. The streamlines Left, SFEM without stabilization; Right, Stabilized mixed method.

5.1. Model test

In the first example, we consider the domain $\Omega_f = (0, 1) \times (1, 1.25)$ and $\Omega_p = (0, 1) \times (0, 1)$ with interface $\Gamma = (0, 1) \times \{1\}$. A modified driven cavity flow with the Dirichlet boundary conditions with $\mathbf{f}_f = 0$ for the flow region is used as follows; See also [35],

$$u = (\sin(\pi x), 0), \ \texttt{on} \ (0, 1) \times \{1.25\}$$

$$u = (0, 0), \ \texttt{on} \ \{0, 1\} \times (1, 1.25),$$

and $\mathbf{u}_p \cdot n_p = 0$ on Γ_p with $f_p = 0$ are used for the porous medium domain.

In this test, we test both the standard finite element method (SFEM) without the stabilization and our stabilized mixed finite element method. The streamline and contour plots of the pressures p, ϕ obtained by both methods with the finite element pairs $P1b - P1/BDM1 - P0$, with a uniform mesh of $h = 1/32$, are depicted in Figures 1–2, respectively. From Figure 1, we find that SFEM without stabilization can not capture the correct flow behavior for the mixed finite element pairs, while the Streamline shows stable by our stabilized mixed method with $P1b-P1/BDM1-P0$. From Figure 2, SFEM without stabilization shows the pressure oscillation, especially the pressure in the free flow region Ω_f, while the stabilized mixed method w captures the smooth numerical pressure in the fluid region Ω_f and the porous domain Ω_p.

Therefore, the interface stabilization terms are necessary and efficient for the mixed finite element method.

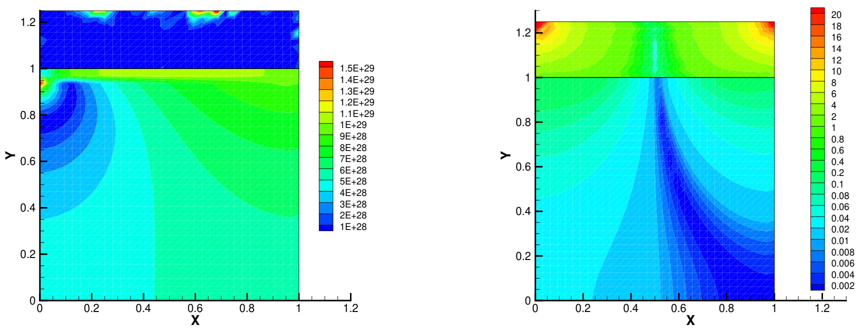

Figure 2. The pressure contours: Left, SFEM without stabilization; Right, Stabilized mixed method.

5.2. Convergence test

In the second test, the global domain Ω consists of two subdomains with free medium fluid flow region $\Omega_f = [0,1] \times [1,2]$ and porous medium domain $\Omega_p = [0,1] \times [0,1]$. The interface of the current computational domain is $\Gamma = [0,1] \times \{1\}$. The exact solutions for this test satisfy the Beavers-Joseph-Saffman interface condition is given by:

$$\begin{aligned}
\mathbf{u}_f &= [x^2(y-1)^2 + y, -\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)],\\
p &= [2 - \pi\sin(\pi x)]\sin(0.5\pi y),\\
\phi &= [2 - \pi\sin(\pi x)][1 - y - \cos(\pi y)],\\
\mathbf{u}_p &= -K\nabla\phi.
\end{aligned}$$

The boundary conditions and source terms of the model problem are chosen such that the above-listed functions are the exact solutions of the model problem.

To demonstrate the convergence order of the stabilized mixed method, Table 1 demonstrates the relative errors between the computed solution and exact solution by our stabilized mixed method with different mesh size $h = 1/8, 1/16, 1/32, 1/64$. From Table 1, we can observe that, all the optimal convergence orders obtained with the stabilized mixed method, which supports the theoretical analysis, and the pressure shows a little better than 1th order accuracy.

In all, our stabilized mixed finite element method is stable and efficient.

Finally, to investigate the impact of the stabilization parameter δ on the stabilized mixed method as discussed in Remark 4.1, we test with the different value of $\delta = 0, 0.0001, 0.01, 0.1, 10$, we use Tables 2, 3 to demonstrate the order of the convergence of $\frac{\|\mathbf{u}_p^h - \mathbf{u}_p\|_{0,\Omega_p}}{\|\mathbf{u}_p\|_{0,\Omega_p}}$ and $\frac{\|\mathbf{u}_p^h - \mathbf{u}_p\|_{\mathtt{div}}}{\|\mathbf{u}_p\|_{\mathtt{div}}}$, respectively. This study admits that too large or too small values ($\delta = 10,\ 0.0001$) of the stabilization parameter δ will affect significantly the convergence orders, which agree with our theorem and remark. While $\delta = 0$, without the stabilization, the accuracy of SFEM is also not ok.

Therefore, the reasonable stabilization parameter δ is required, such as $\delta = 0.01,\ 0.1$ in this test.

Table 1. Errors by Stabilized mixed method

h	$\frac{\|\mathbf{u}_f^h-\mathbf{u}_f\|_{0,\Omega_f}}{\|\mathbf{u}_f\|_{0,\Omega_f}}$	$\frac{\|\mathbf{u}_f^h-\mathbf{u}_f\|_{1,\Omega_f}}{\|\mathbf{u}_f\|_{1,\Omega_f}}$	$\frac{\|p^h-p\|_{0,\Omega_f}}{\|p\|_{0,\Omega_f}}$	$\frac{\|\phi^h-\phi\|_{0,\Omega_p}}{\|\phi\|_{0,\Omega_p}}$	$\frac{\|\mathbf{u}_p^h-\mathbf{u}_p\|_{0,\Omega_p}}{\|\mathbf{u}_p\|_{0,\Omega_p}}$	$\frac{\|\mathbf{u}_p^h-\mathbf{u}_p\|_{\mathtt{div}}}{\|\mathbf{u}_p\|_{\mathtt{div}}}$
1/8	0.0168591	0.102318	0.304966	0.223899	0.0323872	0.165125
1/16	0.0042179	0.051035	0.099039	0.111179	0.0072510	0.081861
1/32	0.0010541	0.025479	0.033554	0.055468	0.0016684	0.040633
1/64	0.0002629	0.012728	0.011621	0.027729	0.0004451	0.020230
order	2.00274	1.00113	1.52970	1.00026	1.90624	1.00498

Table 2. $\frac{\|\mathbf{u}_p^h-\mathbf{u}_p\|_{0,\Omega_p}}{\|\mathbf{u}_p\|_{0,\Omega_p}}$ by Stabilized mixed method with different δ

$h\backslash\delta$	0	0.0001	0.01	0.1	10
1/8	0.28362	0.278862	0.112717	0.0323872	0.0235662
1/16	0.20641	0.192281	0.027536	0.0072510	0.0059825
1/32	0.14909	0.114327	0.005366	0.0016684	0.0024373
1/64	0.10696	0.047898	0.000996	0.0004451	0.0166355

Table 3. $\frac{\|\mathbf{u}_p^h-\mathbf{u}_p\|_{\mathtt{div}}}{\|\mathbf{u}_p\|_{\mathtt{div}}}$ by Stabilized mixed method with different δ

$h\backslash\delta$	0	0.0001	0.01	0.1	10
1/8	0.97102	0.94802	0.327945	0.165125	0.160148
1/16	1.44487	1.31252	0.159125	0.081861	0.080452
1/32	2.10751	1.49782	0.066386	0.040633	0.040417
1/64	3.04086	1.15836	0.027699	0.020230	0.031035

6. Conclusions

In this contribution, we investigated a new fully mixed finite element method to solve the Navier-Stokes/Darcy model without introducing any Lagrange multiplier. The interface stabilization terms were introduced to ensure the well-posedness of the mixed finite element scheme. The stability and convergence analysis including optimal error estimates were derived for the proposed method. In the further study, we can extend the present method to the time-dependent Navier-Stokes/Darcy equations and related coupled problems and so on.

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