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LEAST ENERGY SIGN-CHANGING SOLUTIONS FOR SUPER-QUADRATIC SCHRÖDINGER-POISSON SYSTEMS IN $\mathbb{R}^3$

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Abstract In this paper, we study the following Schrödinger-Poisson systems

$$\begin{cases} -\Delta u + Vu + \lambda\phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where V, $\lambda > 0$ and $f \in C(\mathbb{R}, \mathbb{R})$. Under some relaxed assumptions on f, using variational methods in combination with the Pohozăev identity, we prove that the above system possesses a least energy sign-changing solution and a ground state solution provided that λ is sufficiently small. Moreover, we prove that the energy of a sign-changing solution is strictly larger than that of the ground state solution. Our results generalize and extend some recent results in the literature.

Keywords Schrödinger-Poisson system, sing-changing solution, ground state solution.

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1. Introduction and main results

Consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + Vu + \lambda\phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V > 0$ is constant, $\lambda > 0$ is a parameter. Moreover, we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following assumptions:

(f_1) $\lim\limits_{t\to 0} \frac{f(t)}{t} = 0$.

(f_2) $\lim\limits_{t\to \infty} \frac{f(t)}{t^5} = 0$.

(f_3) $\lim\limits_{t\to +\infty} \frac{F(t)}{t^2} = +\infty$, where $F(t) = \int_0^t f(s)ds$.

(f_4) $\frac{f(t)}{|t|}$ is a non-decreasing function of $\mathbb{R} \setminus \{0\}$.

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System (1.1) is also called Schrödinger-Maxwell system, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations. For more information on the physical relevance of the Schrödinger-Poisson system, we refer the readers to the papers [4,8,20] and the references therein.

Recently, Schrödinger-Poisson systems setting on the the whole space $\mathbb{R}^3$ have attracted a lot of attention. Many solvability conditions on the nonlinearity have been given to obtain the existence and multiplicity of solutions for Schrödinger-Poisson systems in $\mathbb{R}^3$, we refer the readers to [2,3,6,11,15,17–19,22–24,27–30] and references therein. Besides, the existence of sign-changing solutions for problem (1.1) was studied in [1,7,16,21,25].

In [25], Wang and Zhou considered the following system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where, $p \in (3,5)$ and $V \in C(\mathbb{R}^3, \mathbb{R})$. Using variational methods in combination with the Brouwer degree theory, the authors proved that system (1.2) has a sign-changing solution. Furthermore, they also proved that the energy of any sign-changing solution of (1.2) is strictly larger than twice the least energy, that is, the"energy doubling" property of sign-changing solutions of (1.2).

Alves et al. [1] studied the following class of Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $(f_1) - (f_2)$ and

(f_3') $\lim\limits_{t\to+\infty} \frac{F(t)}{t^4} = +\infty$,

(f_4') $\frac{f(t)}{|t|^3}$ is a non-decreasing function of $\mathbb{R} \setminus \{0\}$.

Under more assumptions on the potential V, the authors proved that system (1.3) has a least energy sing-changing solution by means of variational methods combined with the deformation lemma and Miranda's theorem.

Shuai and Wang [21] investigated the existence of sign-changing solution for the system (1.1) with a non-constant potential $V(x)$ instead of V and $f \in C^1(\mathbb{R})$ satisfies $(f_1) - (f_2)$ and $(f_3') - (f_4')$. Based on variational methods in association with the deformation lemma and the implicit function theorem, they obtained the existence of a least energy sign-changing solution. Moreover, the"energy doubling" property and the asymptotic behavior of the sign-changing solution was discussed there.

Chen and Tang [7] have improved the results obtained in [16,21,25] by relaxed the condition (f_3') to the following one:

(F_3) there exists $\theta_0 \in (0,1)$ such that for all $t > 0$ and $\tau \in \mathbb{R} \setminus \{0\}$

$$\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right] sign(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0.$$

Inspired by the above facts, in the present paper, we investigate the existence of least energy sign-changing solution for problem (1.1) with the nonlinearity f is only continuous and satisfies a relaxed assumptions (f_3) and (f_4) instead of (f_3') and (f_4') respectively.

Before stating our main results, we introduce the following notations. As usual, for $1 \leq p < +\infty$, we let

$$\|u\|_p := \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}, \quad u \in L^p(\mathbb{R}^3).$$

Let

$$H^1(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) \,:\, \nabla u \in L^2(\mathbb{R}^3)\right\}.$$

with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V uv)\, dx, \quad \|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V u^2\right) dx\right)^{\frac{1}{2}}.$$

It is well known that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p\left(\mathbb{R}^3\right)$ is continuous for $p \in [2, 6]$, that is, there exist $\tau_p > 0$ such that

$$\|u\|_p \leq \tau_p \|u\|, \quad \forall u \in H^1(\mathbb{R}^3), \quad p \in [2, 6]. \tag{1.4}$$

Define our working space

$$H := H^1_r(\mathbb{R}^3) = \left\{u \in H^1(\mathbb{R}^3) \,:\, u(x) = u(|x|)\right\}.$$

Therefore, the embedding $H \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $p \in (2, 6)$ (see [26]).

Let $\mathcal{D}^{1,2}(\mathbb{R}^3) := \left\{u \in L^6(\mathbb{R}^3) \,:\, \nabla u \in L^2(\mathbb{R}^3)\right\}$ be the Sobolev space equipped with the norm

$$\|u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Then, the embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is continuous (see [26]), and the following Sobolev inequality holds

$$S := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{1/3}}. \tag{1.5}$$

For every $u \in H^1(\mathbb{R}^3)$, by the Lax-Milgram theorem, we know that there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = u^2, \quad \text{in } \mathbb{R}^3.$$

Moreover, ϕ_u has the following properties (for a proof, see [6, 19])

Lemma 1.1. *For $u \in H^1(\mathbb{R}^3)$ we have*

(i) $\phi_u \geq 0$, $\forall u \in H^1(\mathbb{R}^3)$;

(ii) $\phi_t u = t^2 \phi_u$, $\forall t > 0$, $\forall u \in H^1(\mathbb{R}^3)$;

(iii) If $u_n \rightharpoonup u$ *in* H*, then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx;$$

(*iv*) *There exists a constant* $C_0 >$ *such that*

$$\|\phi_u\|^2_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^3} \phi_u u^2 dx \le C_0 \|u\|^4, \quad \forall u \in H.$$

(*v*) *If* u *is a radial function (i.e.,* $u(x) = u(|x|)$*), the* ϕ_u *is radial.*

Now, we define the energy functional $J_\lambda : H \to \mathbb{R}$ associated with problem (1.1) by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V|u|^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx. \tag{1.6}$$

Therefore, under the assumptions $(f_1) - (f_2)$, J_λ is well defined and $J \in C^1(H, \mathbb{R})$ with

$$\langle J'_\lambda(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + Vuv)\, dx + \lambda \int_{\mathbb{R}^3} \phi_u uv dx - \int_{\mathbb{R}^3} f(u) v dx, \quad \forall v \in E. \tag{1.7}$$

Note that $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of J_λ and $\phi = \phi_u$. Furthermore, by the principle of symmetric criticality, the critical points of J_λ on H are the critical points of J_λ on $H^1(\mathbb{R}^3)$, see [8]. Consequently, find a weak solution to problem (1.1) is equivalent to finding a critical point of the functional J_λ.

Throughout this paper, we denote

$$u^+ = \max\{u(x), 0\} \quad \text{and} \quad u^- = \min\{u(x), 0\},$$

then $u = u^+ + u^-$.

As usual, for problem (1.1), we define the associated Nehari manifold by

$$\mathcal{N}_\lambda = \left\{u \in H \setminus \{0\},\ \langle J_\lambda^{'}(u), u\rangle = 0\right\}, \tag{1.8}$$

and the nodal-Nehari manifold by

$$\mathcal{M}_\lambda = \left\{u \in H,\ u^\pm \ne 0 \text{ and } \langle J'_\lambda(u), u^\pm\rangle = 0\right\}. \tag{1.9}$$

Moreover, we denote

$$c_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) \quad \text{and} \quad m_\lambda := \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u). \tag{1.10}$$

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Assume that* $(f_1) - (f_4)$ *hold. Then there exists* $\tilde{\lambda} > 0$ *such that for all* $\lambda \in (0, \tilde{\lambda})$ *the problem (1.1) has a least energy sign-changing solution* $w_\lambda \in \mathcal{M}_\lambda$ *which has exactly two nodal domains. If in addition* $f \in C^1(\mathbb{R})$*, then for all* $\lambda \in (0, \tilde{\lambda})$ *the problem (1.1) has a ground state solution* $u_\lambda \in \mathcal{N}_\lambda$ *which is constant sign. Furthermore, it holds that*

$$m_\lambda = J_\lambda(w_\lambda) > J_\lambda(u_\lambda) = c_\lambda.$$

Remark 1.1. Compared with the results obtained in [1, 16, 21], we only need $f \in C(\mathbb{R}, \mathbb{R})$ for establishing the existence of sign-changing solutions, also, the weaker conditions (f_3) and (f_4) are employed to replace (f_3') and (f_4') respectively. Thus, our results extend and generalize the existing results to more general nonlinearity. Further, there are many functions satisfying $(f_3)-(f_4)$, but not $(f_3')-(f_4')$. For example, let

$$f(t)=t|t|^{p-2}, \quad \forall t\in\mathbb{R},$$

where $p\in(2,4]$. Then, by a simple calculation, we have $F(t)=\frac{1}{p}|t|^p$. Therefore, it is easy to check that f satisfies $(f_1)-(f_4)$ and does not satisfy $(f_3')-(f_4')$. Consequently, our results extend the results of [1, 7, 16, 21, 25].

The paper is organized as follows. In Section 2, we provide some lemmas, which are crucial to prove the main results of this paper. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In the sequel, C_i or C will denote different positive constants and $\to$ $(\rightharpoonup)$ denotes the strong (weak) convergence.

Motivated by [29], we shall use a truncated technique which is due to Jeanjean and Le Coz [13] (see also [3, 14]). Therefore, we define the cut-off function $\xi\in C^1(\mathbb{R}_+,\mathbb{R})$ satisfying $0\le\xi(t)\le 1$, $\|\xi'\|_\infty\le 2$,

$$\xi(t)=\begin{cases}1, & t\in[0,1],\\ 0, & t\in[2,\infty),\end{cases}$$

and ξ is decreasing on $[1,2]$. Similar to [29], we consider the truncated functional $J_{\lambda,T}:H\to\mathbb{R}$ defined by

$$J_{\lambda,T}(u)=\frac{1}{2}\|u\|^2+\frac{\lambda}{4}B_T(u)\int_{\mathbb{R}^3}\phi_u u^2dx-\int_{\mathbb{R}^3}F(u)dx, \tag{2.1}$$

where $B_T(u)=\xi\left(\frac{\|u\|^2}{T^2}\right)$. Under assumptions $(f_1)-(f_2)$, it is easy to check that $J_\lambda^T\in C^1(H,\mathbb{R})$ and

$$\begin{aligned}\langle J_{\lambda,T}^{'}(u),v\rangle =&\langle u,v\rangle+\lambda B_T(u)\int_{\mathbb{R}^3}\phi_u uvdx+\frac{\lambda}{2T^2}\xi'\left(\frac{\|u\|^2}{T^2}\right)\langle u,v\rangle\int_{\mathbb{R}^3}\phi_u u^2dx\\ &-\int_{\mathbb{R}^3}f(u)vdx.\end{aligned} \tag{2.2}$$

In what follows, we try to find a critical point w_λ of $J_{\lambda,T}$ on H for small $\lambda>0$. Then we will show that w_λ also solves the original problem (1.1) By showing that $\|w_\lambda\|\le T$. Define the Nehari manifold of $J_{\lambda,T}$ as

$$\mathcal{N}_{\lambda,T}=\left\{u\in H\setminus\{0\},\ \langle J_{\lambda,T}^{'}(u),u\rangle=0\right\} \tag{2.3}$$

and the nodal-Nehari manifold

$$\mathcal{M}_{\lambda,T}=\left\{u\in H,\ u^{\pm}\neq 0 \text{ and } \langle J_{\lambda,T}'(u),u^{\pm}\rangle=0\right\}. \tag{2.4}$$

Furthermore, we denote

$$c_{\lambda,T} := \inf_{u\in\mathcal{N}_{\lambda,T}} J_{\lambda,T}(u) \quad \text{and} \quad m_{\lambda,T} := \inf_{u\in\mathcal{M}_{\lambda,T}} J_{\lambda,T}(u). \tag{2.5}$$

We have the following results.

Theorem 2.1. *Assume that $(f_1)-(f_4)$ hold. Then there exists $\lambda^*>0$ such that for all $\lambda\in(0,\lambda^*)$ the functional $J_{\lambda,T}$ has a least energy sign-changing critical point $w_\lambda\in\mathcal{M}_{\lambda,T}$. If in addition $f\in C^1(\mathbb{R})$, then for all $\lambda\in(0,\lambda^*)$ the functional $J_{\lambda,T}$ has a critical point $u_\lambda\in\mathcal{N}_{\lambda,T}$ which is constant sign. Furthermore, it holds that*

$$c_{\lambda,T} = J_{\lambda,T}(u_\lambda) < J_{\lambda,T}(w_\lambda) = m_{\lambda,T}.$$

Lemma 2.1. *For each $u\in H$ with $u^\pm\neq 0$, there exists a pair $(t_u,s_u)\in\mathbb{R}\times\mathbb{R}$ with $t_u,s_u>0$ such that $t_uu^++s_uu^-\in\mathcal{M}_{\lambda,T}$, moreover*

$$J_{\lambda,T}(t_uu^++s_uu^-) = \max_{t,s\geq 0} J_{\lambda,T}(tu^++su^-).$$

Proof. For any $u\in H$ with $u^\pm\neq 0$, we define the function $G:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ by

$$G(t,s) = J_{\lambda,T}(tu^++su^-).$$

Obviously, G is well defined on $\mathbb{R}_+\times\mathbb{R}_+$ and $G\in C^1(\mathbb{R}_+\times\mathbb{R}_+)$ due to $J_{\lambda,T}\in C^1$. For $(t,s)\in\mathbb{R}_+\times\mathbb{R}_+$, by a simple computation we obtain

$$\begin{aligned}\nabla G(t,s) &= \big(\langle J'_{\lambda,T}(tu^++su^-),u^+\rangle, \langle J'_{\lambda,T}(tu^++su^-),u^-\rangle\big)\\ &= \left(\frac{1}{t}\langle J'_{\lambda,T}(tu^++su^-),tu^+\rangle, \frac{1}{s}\langle J'_{\lambda,T}(tu^++su^-),su^-\rangle\right).\end{aligned}$$

Therefore, $tu^++su^-\in\mathcal{M}_{\lambda,T}$ if and only if the pair (t,s) is a critical point of G with $t,s>0$. By (f_1) and (f_2), for any $\varepsilon>0$ and $p\in(2,6)$, there exist $C_\varepsilon>0$ such that

$$|f(u)|\leq\varepsilon|u|+C_\varepsilon|u|^5 \quad \text{and} \quad |F(u)|\leq\varepsilon\frac{1}{2}|u|^2+\frac{C_\varepsilon}{6}|u|^6, \quad \forall u\in\mathbb{R}. \tag{2.6}$$

Hence, using (1.4), (2.6) and the conclusion (i) of Lemma 1.1, we get

$$\begin{aligned}G(t,s) &= J_{\lambda,T}(tu^++su^-)\\ &\geq \frac{t^2}{2}\|u^+\|^2+\frac{s^2}{2}\|u^-\|^2-\int_{\mathbb{R}^3}\big(F(tu^+)+F(su^-)\big)\,dx\\ &\geq \frac{t^2}{2}\|u^+\|^2+\frac{s^2}{2}\|u^-\|^2-\varepsilon\frac{t^2}{2}\|u^+\|_2^2-\varepsilon\frac{s^2}{2}\|u^-\|_2^2-C_\varepsilon\frac{t^6}{6}\|u^+\|_6^6-C_\varepsilon\frac{s^6}{6}\|u^-\|_6^6\\ &\geq \frac{t^2}{2}\|u^+\|^2\left(1-\varepsilon\tau_2^2\right)+\frac{s^2}{2}\|u^-\|^2\left(1-\varepsilon\tau_2^2\right)-C_1\tau_6^6\frac{t^6}{6}\|u^+\|^6-C_2\tau_6^6\frac{s^6}{6}\|u^-\|^6,\end{aligned}$$

for some positive constant C_i $(i=1,2)$. Thus, $G(t,s)>0$ for (t,s) small. On the other hand, for $t>0$ sufficiently large, it follows from (f_3) that there exist a large $M>0$ such that

$$f(t)\geq M|t| \quad \text{and} \quad F(t)\geq M|t|^2. \tag{2.7}$$

Consequently, for (t,s) large, it has

$$\begin{aligned}
G(t,s) =& J_{\lambda,T}(tu^+ + su^-)\\
=&\frac{1}{2}\|tu^+ + su^-\|^2 + \frac{\lambda}{4}B_T(tu^+ + su^-)\int_{\mathbb{R}^3}\phi_{tu^+ + su^-}(tu^+ + su^-)^2dx\\
&-\int_{\mathbb{R}^3}F(tu^+ + su^-)dx\\
=&\frac{t^2}{2}\|u^+\|^2 + \frac{s^2}{2}\|u^-\|^2 - \int_{\mathbb{R}^3}\left(F(tu^+) + F(su^-)\right)dx\\
\le&\frac{t^2}{2}\|u^+\|^2 + \frac{s^2}{2}\|u^-\|^2 - Mt^2\int_{\mathbb{R}^3}|u^+|^2dx - Ms^2\int_{\mathbb{R}^3}|u^-|^2dx.
\end{aligned}$$

This implies that $G(t,s) \to -\infty$ as $|(t,s)| \to +\infty$. We conclude that there exists a pair $(t_u, s_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$G(t_u, s_u) = \max_{t,s\ge 0} G(t,s).$$

Next, we show that $t_u, s_u > 0$. Without loss of generality, we may assume that $(t_u, 0)$ is the maximum point of $G(t,s)$. Then, we have

$$\begin{aligned}
\frac{\partial G(t_u,s)}{\partial s} =& \langle J'_{\lambda,T}(t_uu^+ + su^-), u^-\rangle\\
=& s\|u^-\|^2 + \lambda s B_T(t_uu^+ + su^-)\int_{\mathbb{R}^3}\phi_{t_uu^+ + su^-}(u^-)^2dx\\
&+\frac{\lambda s}{2T^2}\xi'\left(\frac{\|t_uu^+ + su^-\|^2}{T^2}\right)\|u^-\|^2\int_{\mathbb{R}^3}\phi_{t_uu^+ + su^-}(t_uu^+ + su^-)^2dx\\
&-\int_{\mathbb{R}^3}f(su^-)u^-dx\\
\ge& s\|u^-\|^2 - \int_{\mathbb{R}^3}f(su^-)u^-dx - \frac{\lambda s}{T^2}\|u^-\|^2\int_{\mathbb{R}^3}\phi_{t_uu^+ + su^-}(t_uu^+ + su^-)^2dx\\
\ge& s\|u^-\|^2 - \int_{\mathbb{R}^3}f(su^-)u^-dx - \frac{\lambda s}{T^2}C_0\|u^-\|^2\|t_uu^+ + su^-\|^4,
\end{aligned}$$

by virtue of (f_1), for λ, s sufficiently small, we see that $\frac{\partial G(t_u,s)}{\partial s} > 0$, which implies that $G(t_u, s)$ is increasing for s small. This is a contradiction with the fact that $(t_u, 0)$ is the maximum point of $G(t,s)$. We conclude that $t_u, s_u > 0$.

Finally, since (t_u, s_u) is a positive maximum point of $G(t,s)$ it follows that

$$\frac{\partial G(t,s)}{\partial t}|_{(t_u,s_u)} = \frac{\partial G(t,s)}{\partial s}|_{(t_u,s_u)} = 0,$$

and then

$$\langle J'_{\lambda,T}(t_uu^+ + s_uu^-), u^+\rangle = \langle J'_{\lambda,T}(t_uu^+ + s_uu^-), u^-\rangle = 0,$$

which implies that $t_uu^+ + s_uu^- \in \mathcal{M}_{\lambda,T}$, since $t_u, s_u > 0$. This completes the proof. □

Corollary 2.1. *For each $u \in H \setminus \{0\}$, there exists a $t_u \in \mathbb{R}$ with $t_u > 0$ such that $t_uu^+ \in \mathcal{N}_{\lambda,T}$, moreover*

$$J_{\lambda,T}(t_uu^+) = \max_{t\ge 0} J_{\lambda,T}(tu^+).$$

Lemma 2.2. *Assume that $(f_1)-(f_4)$ hold. Then, for any $u \in H$ with $\|u\|^2 > 2T^2$, it has*

$$J_{\lambda,T}(u) \geq J_{\lambda,T}(tu) + \frac{1-t^2}{2}\langle J'_{\lambda,T}(u), u\rangle, \quad \forall t \geq 0.$$

Proof. It follows from (f_4) that for any $t > 0$ and $\tau \neq 0$

$$\left[\frac{f(\tau)}{\tau} - \frac{f(t\tau)}{t\tau}\right] sign(1-t) \geq 0,$$

which implies that

$$\frac{1-t^2}{2} f(\tau)\tau + F(t\tau) - F(\tau) = \int_t^1 \left[\frac{f(\tau)}{\tau} - \frac{f(s\tau)}{s\tau}\right] s\tau^2 ds \geq 0, \ \forall t \geq 0, \ \tau \in \mathbb{R} \setminus \{0\}. \tag{2.8}$$

On the other hand, from the definition of $B_T(u)$ we have

$$\begin{aligned} J_{\lambda,T}(u) &= \frac{1}{2}\|u\|^2 + \frac{\lambda}{4} B_T(u) \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(u)dx, \end{aligned} \tag{2.9}$$

and

$$\langle J'_{\lambda,T}(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}^3} f(u)udx. \tag{2.10}$$

Hence, by (2.8), (2.9) and (2.10), we have

$$\begin{aligned} J_{\lambda,T}(u) - J_{\lambda,T}(tu) &= \frac{1-t^2}{2}\|u\|^2 + \int_{\mathbb{R}^3} (F(tu) - F(u))dx \\ &= \frac{1-t^2}{2}\langle J'_{\lambda,T}(u), u\rangle + \int_{\mathbb{R}^3} \left[\frac{1-t^2}{2} f(u)u + F(tu) - F(u)\right] dx \\ &\geq \frac{1-t^2}{2}\langle J'_{\lambda,T}(u), u\rangle, \quad \forall t \geq 0. \end{aligned}$$

Thus, the proof is completed. □

Lemma 2.3. *Let $\{u_n\} \subset \mathcal{N}_{\lambda,T}$ be a minimizing sequence of $c_{\lambda,T}$, then $\{u_n\}$ is bounded in H.*

Proof. Let $\{u_n\} \subset \mathcal{N}_{\lambda,T}$ be a minimizing sequence of $c_{\lambda,T}$, that is,

$$J_{\lambda,T}(u_n) \to c_{\lambda,T}, \quad \text{as} \quad n \to \infty.$$

We claim that $\{u_n\}$ is bounded in H. To this end, arguing by contradiction, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $v_n := u_n/\|u_n\|$, then $\|v_n\| = 1$. Going if necessary to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \text{ in } H; \\ &v_n \to v \text{ in } L^p(\mathbb{R}^3), \quad 2 < p < 6; \\ &v_n \to v \text{ a.e. in } \mathbb{R}^3. \end{aligned}$$

So, we have two cases need to be considered: $v = 0$ or $v \not\equiv 0$.

If $v = 0$, then $v_n \to 0$ strongly in $L^p(\mathbb{R}^3)$, for $p \in (2,6)$. Let $L = \sqrt{2(c_{\lambda,T} + T^2)}$. Then, it follows from $(f_1) - (f_2)$, for any $\varepsilon > 0$ and $p \in (2,6)$, there exists $C_\varepsilon > 0$ such that

$$|f(u)| \le \varepsilon(|u| + |u|^5) + C_\varepsilon |u|^{p-1} \text{ and } |F(u)| \le \varepsilon(\frac{1}{2}|u|^2 + \frac{1}{6}|u|^6) + \frac{C_\varepsilon}{p}|u|^p, \ \forall u \in \mathbb{R}. \tag{2.11}$$

Combining (2.11) with Lemma 2.2, for n large enough so that $L^2/\|u_n\|^2 \le 1$, one has

$$\begin{aligned} c_{\lambda,T} &= J_{\lambda,T}(u_n) + o(1) \\ &\ge J_{\lambda,T}(Lv_n) + \left(\frac{1}{2} - \frac{L^2}{2\|u_n\|^2}\right)\langle J'_{\lambda,T}(u_n), u_n\rangle + o(1) \\ &\ge J_{\lambda,T}(Lv_n) + \left(\frac{1}{2} - \frac{L^2}{2\|u_n\|^2}\right)\langle J'_{\lambda,T}(u_n), u_n\rangle + o(1) \\ &= \frac{L^2}{2} - \int_{\mathbb{R}^3} F(Lv_n)dx + o(1) \\ &\ge \frac{L^2}{2} - \int_{\mathbb{R}^3} |F(Lv_n)|\, dx + o(1) \\ &\ge \frac{1}{2}L^2 - \varepsilon\left(L^2\|v_n\|_2^2 + L^6\|v_n\|_6^6\right) - C_\varepsilon L^p \|v_n\|_p^p + o(1) \\ &= c_{\lambda,T} + T^2 - \varepsilon\left(L^2\|v_n\|_2^2 + L^6\|v_n\|_6^6\right) - C_\varepsilon L^p\|v_n\|_p^p + o(1) \\ &\ge c_{\lambda,T} + T^2 - \varepsilon C_3 + o(1). \end{aligned}$$

This is an obvious contradiction in view of the arbitrariness of ε.

Now, we consider the case $v \not\equiv 0$. Set $A = \{x \in \mathbb{R}^3 \ : \ v(x) \neq 0\}$. Then, for $x \in A$ we have $\lim\limits_{n\to\infty} |u_n(x)| = \infty$. By (f_3) and Fatou's Lemma, we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{J_{\lambda,T}(u_n)}{\|u_n\|^2} \\ &= \lim_{n\to\infty}\left[\frac{1}{2} - \int_{\mathbb{R}^3} \frac{F(u_n)}{u_n^2} v_n^2 dx\right] \\ &\le \frac{1}{2} - \liminf_{n\to\infty} \int_A \frac{F(u_n)}{u_n^2} v_n^2 dx \\ &\le \frac{1}{2} - \int_A \liminf_{n\to\infty} \frac{F(u_n)}{u_n^2} v_n^2 dx \\ &= -\infty. \end{aligned}$$

This is a contradiction. Hence $\{u_n\} \subset H$ is bounded. □

Corollary 2.2. *Let $\{u_n\} \subset \mathcal{M}_{\lambda,T}$ be a minimizing sequence of $m_{\lambda,T}$, then $\{u_n\}$ is bounded in H.*

Lemma 2.4. *Assume that $(f_1) - (f_4)$ hold. Then, there exists $\overline{\lambda} > 0$ such that for all $\lambda \in (0, \overline{\lambda})$, $m_{\lambda,T}$ is achieved by some $w_\lambda \in \mathcal{M}_{\lambda,T}$.*

Proof. Let $\{u_n\} \subset \mathcal{M}_{\lambda,T}$ be such that

$$J_{\lambda,T}(u_n) \to m_{\lambda,T}, \quad \text{as} \quad n \to \infty.$$

Then, $\{u_n\}$ is bounded in H in view of Corollary 2.2, that is, there exists a constant $d > 0$ such that

$$\|u_n\| \leq d, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

On the other hand, since $u_n \in \mathcal{M}_{\lambda,T}$, we have

$$\begin{aligned}&\|u_n^{\pm}\|^2 + \lambda B_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}(u_n^{\pm})^2dx + \frac{\lambda}{2T^2}\xi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n^{\pm}\|^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\\ =&\int_{\mathbb{R}^N} f(u_n^{\pm})u_n^{\pm}dx\end{aligned} \tag{2.13}$$

Then, it follows from (1.4), Lemma 1.1(iv), (2.6), (2.12) and (2.13) that

$$\begin{aligned}\left(1-\frac{\lambda d^4}{T^2}C_0\right)\|u_n^{\pm}\|^2 \leq&\|u_n^{\pm}\|^2 - \frac{\lambda}{T^2}C_0\|u_n\|^4\|u_n^{\pm}\|^2\\ \leq&\|u_n^{\pm}\|^2 + \lambda B_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}(u_n^{\pm})^2dx\\ &+\frac{\lambda}{2T^2}\xi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n^{\pm}\|^2\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\\ =&\int_{\mathbb{R}^N} f(u_n^{\pm})u_n^{\pm}dx\\ \leq&\varepsilon\int_{\mathbb{R}^3}|u_n^{\pm}|^2dx + C_\varepsilon\int_{\mathbb{R}^3}|u_n^{\pm}|^6dx\\ \leq&C_4\|u_n^{\pm}\|^2 + C_5\|u_n^{\pm}\|^6.\end{aligned}$$

Therefore, there exists a constant $\rho > 0$ such that

$$\|u_n^{\pm}\|^2 \geq \rho, \quad \forall \lambda \in \left(0, \frac{T^2}{d^4 C_0}\right). \tag{2.14}$$

By Ekeland's variational principle (see [9]), $\{u_n\}$ is a $(PS)_{m_{\lambda,T}}$ sequence for $J_{\lambda,T}\big|_{\mathcal{M}_{\lambda,T}}$, that is,

$$J_{\lambda,T}(u_n) \to m_{\lambda,T}, \quad J'_{\lambda,T}(u_n) \to 0 \text{ in } H^*. \tag{2.15}$$

Since $\{u_n\}$ is bounded in H, going if necessary to a subsequence, there exists a $u \in H$ so that

$$\begin{aligned}&u_n \rightharpoonup u \text{ in } H;\\ &u_n \to u \text{ in } L^p(\mathbb{R}^3), \quad 2 < p < 6;\\ &u_n \to u \text{ a.e. in } \mathbb{R}^3.\end{aligned} \tag{2.16}$$

It follows from (1.4), (2.11), (2.12), (2.16) and the Hölder inequality

$$\int_{\mathbb{R}^3} f(u_n)(u_n - u)dx \to 0 \text{ as } n \to \infty. \tag{2.17}$$

On the other hand, by (1.5), Lemma 1.1-(iv), (2.16) and the Hölder inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)dx \leq& \|\phi_{u_n}\|_6\|u_n\|_2\|u_n-u\|_3\\ \leq& S^{-\frac{1}{2}}\|\phi_{u_n}\|_{\mathcal{D}^{1,2}}\|u_n\|_2\|u_n-u\|_3\\ \leq& C\|u_n\|^2\|u_n\|_2\|u_n-u\|_3 \longrightarrow 0 \text{ as } n \to \infty.\end{aligned} \tag{2.18}$$

Then combining (2.15), (2.17) and (2.18) with (2.2), for a large n, we infer that

$$\begin{aligned}
o(1) =&\langle J'_{\lambda,T}(u_n), u_n - u\rangle \\
=&\langle u_n, u_n - u\rangle + \lambda B_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n-u)dx \\
&+ \frac{\lambda}{2T^2}\xi'\left(\frac{\|u_n\|^2}{T^2}\right)\langle u_n, u_n-u\rangle\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx - \int_{\mathbb{R}^3}f(u_n)(u_n-u)dx \\
=&\left(1+\frac{\lambda}{2T^2}\xi'\left(\frac{\|u_n\|^2}{T^2}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\right)\langle u_n, u_n-u\rangle + o(1),
\end{aligned}$$

which implies that $\|u_n\| \to \|u\|$. Since $u_n \rightharpoonup u$ in H which is a uniformly convex Banach space, we deduce that

$$u_n \to u, \qquad u_n^+ \to u^+, \qquad u_n^- \to u^- \quad \text{in } H \ \text{ as } \ n \to \infty. \tag{2.19}$$

Moreover, from (2.14) we have $\|u^\pm\|^2 \geq \rho > 0$, thus $u^\pm \neq 0$.

Using (2.6), (2.19) and the compactness lemma of Strauss (see [5, Theorem A1]), we derive

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^3}f(u_n^\pm)u_n^\pm dx &= \int_{\mathbb{R}^3}f(u^\pm)u^\pm dx, \\
\lim_{n\to\infty}\int_{\mathbb{R}^3}F(u_n^\pm)dx &= \int_{\mathbb{R}^3}F(u^\pm)dx.
\end{aligned} \tag{2.20}$$

Note that from $u_n \in \mathcal{M}_{\lambda,T}$, we have

$$\langle J'_{\lambda,T}(u_n), u_n^\pm\rangle = 0,$$

by (2.20), Lemma 1.1-(iii) and passing to the limit, we deduce that

$$\langle J'_{\lambda,T}(u), u^\pm\rangle = 0,$$

which implies that $u \in \mathcal{M}_{\lambda,T}$ and $J_{\lambda,T}(u) = m_{\lambda,T}$ for all $\lambda \in (0,\overline{\lambda})$ where

$$\overline{\lambda} = \frac{T^2}{d^4C_0}. \tag{2.21}$$

Thus, $J_{\lambda,T}|_{\mathcal{M}_{\lambda,T}}$ attains its minimum $m_{\lambda,T}$ at $w_\lambda := u \in \mathcal{M}_{\lambda,T}$ for all $\lambda \in (0,\overline{\lambda})$. The proof is completed. □

Corollary 2.3. *Assume that $(f_1)-(f_4)$ hold. Then for all $\lambda \in (0,\overline{\lambda})$, $c_{\lambda,T}$ is achieved by some $u_\lambda \in \mathcal{N}_{\lambda,T}$, where $\overline{\lambda}$ is given by (2.21).*

Proof of Theorem 2.1. To complete the proof of Theorem 2.1, we consider the following four steps.

Step 1. We first show that the minimizer w_λ obtained in Lemma 2.4 for the minimization problem (2.5) is a sign-changing critical point of $J_{\lambda,T}$, that is $J'_{\lambda,T}(w_\lambda) = 0$. To this end, using Lemma 2.1 to replace Lemmas 2.1 and 2.3 in [21], the rest proof can be concluded by some slightly modifications of the proof of Theorem 1.1 in [21]. Noting that it only needs $f(u) \in C(\mathbb{R})$ throughout the proof.

Step 2. Next, we prove that u_λ obtained in Corollary 2.3 is a critical point of $J_{\lambda,T}$ in H. By corollary 2.3, we know that u_λ is a critical point of $J_{\lambda,T}$ in $\mathcal{N}_{\lambda,T}$.

If $f \in C^1(\mathbb{R})$, then $\mathcal{N}_{\lambda,T}$ is manifold of C^1 and the critical points of the functional $J_{\lambda,T}$ on $\mathcal{N}_{\lambda,T}$ are critical points of $J_{\lambda,T}$ on H in view of Corollary 2.9 in [10], thus u_λ is a critical point of $J_{\lambda,T}$ in H with $J_{\lambda,T}(u_\lambda) = c_{\lambda,T}$.

Step 3. Finally, we show that the energy of the sign-changing solution is strictly greater than the least energy, i.e.,

$$m_{\lambda,T} > c_{\lambda,T}.$$

Let w_λ be the sign-changing critical point of $J_{\lambda,T}$ obtained in Lemma 2.4. For w_λ^+, by Corollary 2.1, there exists a $t = t_{w_\lambda^+} > 0$ such that $t_{w_\lambda^+} w_\lambda^+ \in \mathcal{N}_{\lambda,T}$. Therefore, it follows from Lemma 2.1, Corollary 2.1 and Corollary 2.3 that

$$\begin{aligned}0 < c_{\lambda,T} = J_{\lambda,T}(u_\lambda) \leq & J_{\lambda,T}(t_{w_\lambda^+} w_\lambda^+) \\ = & J_{\lambda,T}(t_{w_\lambda^+} w_\lambda^+ + 0 w_\lambda^-) < J_{\lambda,T}(w_\lambda^+ + w_\lambda^-) = J_{\lambda,T}(w_\lambda) = m_{\lambda,T}.\end{aligned}$$

Step 4. Finally, we prove that u_λ is constant sign. Suppose to the contrary that u_λ is sing-changing, then $u_\lambda \in \mathcal{M}_{\lambda,T}$ and

$$c_{\lambda,T} = J_{\lambda,T}(u_\lambda) \geq J_{\lambda,T}(w_\lambda) = m_{\lambda,T},$$

which is absurd. The proof is completed. □

3. Proof of the main results

To establish the proof of Theorem 1.1, we shall make use of the following Pohozăev identity (see [29]).

Lemma 3.1. *If $u \in H$ is a critical point of $J_{\lambda,T}$, then for $\lambda > 0$ small, u satisfies*

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} V|u|^2 dx + \frac{5\lambda}{4} B_T(u) \int_{\mathbb{R}^3} \phi_u u^2 dx \\ &+ \frac{3\lambda}{T^2}\xi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^2 \int_{\mathbb{R}^3} \phi_u u^2 dx \\ =& 3\int_{\mathbb{R}^3} F(u) dx.\end{aligned}$$

Lemma 3.2. *For w_λ and u_λ obtained in Theorem 1.1, if $T > 0$ large enough and $\lambda > 0$ small enough, then we have $\|w_\lambda\|, \|u_\lambda\| \leq T$.*

Proof. Part of the proof is similar to that of [29], Lemma 2.3. For the reader's convenience, we sketch the proof here briefly. Since $J'_{\lambda,T}(w_\lambda) = 0$, by Lemma 3.1 and Lemma 1.1(iv) we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla w_\lambda|^2 dx =& 3J_{\lambda,T}(w_\lambda) + \frac{\lambda}{2} B_T(w_\lambda) \int_{\mathbb{R}^3} \phi_{w_\lambda} w_\lambda^2 dx \\ &+ \frac{3\lambda}{T^2}\xi'\left(\frac{\|w_\lambda\|^2}{T^2}\right)\|w_\lambda\|^2 \int_{\mathbb{R}^3} \phi_{w_\lambda} w_\lambda^2 dx \\ \leq& 3m_{\lambda,T} + \frac{\lambda}{2} B_T(w_\lambda) C_0 \|w_\lambda\|^4 + \frac{3\lambda}{T^2} C_0 \left|\xi'\left(\frac{\|w_\lambda\|^2}{T^2}\right)\right| \|w_\lambda\|^6.\end{aligned} \tag{3.1}$$

If $\|w_\lambda\|^2 > 2T^2$, we have $B_T(w_\lambda) = 0$. Therefore, it follows from the last inequality that

$$\int_{\mathbb{R}^3} |\nabla w_\lambda|^2 dx \le C_8 + \lambda C_9 T^4. \tag{3.2}$$

Furthermore, by (2.6) and $\langle J'_{\lambda,T}(w_\lambda), w_\lambda\rangle = 0$, for $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\|w_\lambda\|^2 + \left(\lambda B_T(w_\lambda) + \frac{\lambda}{2T^2}\xi'\left(\frac{\|w_\lambda\|^2}{T^2}\right)\|w_\lambda\|^2\right)\int_{\mathbb{R}^3}\phi_{w_\lambda} w_\lambda^2 dx\\ = \int_{\mathbb{R}^3} f(w_\lambda)w_\lambda dx \le \varepsilon\int_{\mathbb{R}^3}|w_\lambda|^2 dx + C_\varepsilon\int_{\mathbb{R}^3}|w_\lambda|^6 dx.\end{aligned}$$

By $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we have

$$\begin{aligned}(1-\frac{\varepsilon}{V})\|w_\lambda\|^2 &\le C_\varepsilon\int_{\mathbb{R}^3}|w_\lambda|^6 dx - \frac{\lambda}{2T^2}\xi'\left(\frac{\|w_\lambda\|^2}{T^2}\right)\|w_\lambda\|^2\int_{\mathbb{R}^3}\phi_{w_\lambda}w_\lambda^2 dx\\ &\le C_{10}\left(\int_{\mathbb{R}^3}|\nabla w_\lambda|^2 dx\right)^3 + C_{11}\lambda T^4.\end{aligned}$$

Hence, for $\varepsilon \le \frac{V}{2}$, using (3.2) we then get

$$\|w_\lambda\|^2 \le C_{12}\left(C_8 + \lambda C_9 T^4\right)^3 + C_{13}\lambda T^4. \tag{3.3}$$

Arguing by contradiction suppose that $\|w_\lambda\| > T$, then, by (3.3) one has

$$\begin{aligned}T^2 \le \|w_\lambda\|^2 &\le C_{12}\left(C_8 + \lambda C_9 T^4\right)^3 + C_{13}\lambda T^4\\ &\le C_{14}\left(1 + \lambda T^4 + \lambda^2 T^8 + \lambda^3 T^{12}\right).\end{aligned}$$

Choosing $T^2 > \max\{1, 4C_{14}\}$ and $\lambda < \frac{1}{T^4}$, the last inequality yields

$$T^2 \le C_{14}\left(1 + \lambda T^4 + \lambda^2 T^8 + \lambda^3 T^{12}\right) < 4C_{14},$$

which is impossible. Thus $\|w_\lambda\| \le T$, similarly, we obtain $\|u_\lambda\| \le T$. This completes the proof. □

Proof of Theorem 1.1. Let T be large enough and λ small. We know from Theorem 2.1 that $J_{\lambda,T}$ has a least energy critical u_λ at level $c_{\lambda,T}$ and a least energy sign-changing critical point w_λ at level $m_{\lambda,T}$, and by Lemma 3.2 we have that $\|u_\lambda\|, \|w_\lambda\| \le T$, therefore $J_{\lambda,T} = J_\lambda$ and u_λ, w_λ are critical points of J_λ with $J_\lambda(u_\lambda) = c_\lambda$ and $J_\lambda(w_\lambda) = m_\lambda$. Hence, system (1.1) has a least energy sign-changing solution w_λ and a ground state solution u_λ which is constant sign. Moreover, since $J_{\lambda,T} = J_\lambda$, it follows from Lemma 2.1 that

$$0 < c_\lambda = J_\lambda(u_\lambda) < J_\lambda(w_\lambda) = m_\lambda.$$

The proof is completed. □

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References

[1] C. O. Alves, M. A. S. Souto and S. H. M. Soares, *A sign-changing solution for the Schrödinger-Poisson equation*, Rocky Mountain J. Math., 2017, 47(1), 1–25.

[2] A. Ambrosetti and D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*. Commun. Contemp. Math., 2008, 10, 391–404.

[3] A. Azzollini, P. d'Avenia and A. Pomponio, *On the Schrödinger-Maxwell equations under the effect of a general nonlinear term*, Ann. Inst. H. Poincaré Anal. Non Linéair, 2010, 27, 779–791.

[4] V. Benci and F. Donato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal., 1998, 11, 283–293.

[5] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations I, existence of a ground state*, Arch. Ration. Mech. Anal., 1983, 82, 313–346.

[6] G. Cerami and G. Vaira, *Positive solutions for some non-autonomous Schrödinger-Poisson systems*, J. Diff. Eqs., 2010, 248, 521–543.

[7] S. Chen and X. Tang, *Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in* $\mathbb{R}^3$, Z. Angew. Math. Phys., 2016, 67(4), 102–120.

[8] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, 2004, 134, 893–906.

[9] D. G. De Figueiredo, *Lectures on the Ekeland variational principle with applications and detours*. Vol. 81. Berlin: Springer, 1989.

[10] X. He and W. Zou, *Existence and concentration behavior of positive solutions of Kirchhoff type equation in* $\mathbb{R}^3$, J. Diff. Eqs., 2012, 252(2), 1813–1834.

[11] W. Huang and X. Tang, *The existence of infinitely many solutions for the nonlinear Schrödinger-Maxwell equations*, Results Math., 2014, 65, 223–234.

[12] W. Huang and X. Tang, *Ground-state solutions for asymptotically cubic Schrödinger-Maxwell equations*, Mediterr. J. Math., 2016, 13, 3469–3481.

[13] L. Jeanjean and S. Le Coz, *An existence and stability result for standing waves of nonlinear Schrödinger equations*, Adv. Diff. Eqs., 2006, 11, 813–840.

[14] H. Kikuchi, *Existence and stability of standing waves for SchrÖdinger-Poisson-Slater equation*, Adv. Nonlinear Stud., 2007, 7, 403–437.

[15] S. Khoutir and H. Chen, *Multiple nontrivial solutions for a nonhomogeneous Schrödinger-Poisson system in* $\mathbb{R}^3$, Electron. J. Qual. Theo. Diff. Eqs., 2017, 28, 1–17.

[16] Z. Liang, J. Xu and X. Zhu, *Revisit to sign-changing solutions for the nonlinear Schrödinger-Poisson system in* $\mathbb{R}^3$, J. Math. Anal. Appl., 2016, 435, 783–799.

[17] H. Liu, H. Chen and G. Wang, *Multiplicity for a 4-sublinear Schrödinger-Poisson system with sign-changing potential via Morse theory*, Compt. Rendus Math., 2016, 354, 75–80.

[18] H. Liu and H. Chen, *Multiple solutions for a nonlinear Schrödinger-Poisson system with sign-changing potential*, Comput. Math. Appl., 2016, 71(7), 1405–1416.

[19] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal., 2006, 237, 655–674.

[20] O. Sánchez and J. Soler, *Long-time dynamics of the Schrödinger-Poisson-Slater system*, J. Stat. Phys., 2004, 114, 179–204.

[21] W. Shuai and Q. Wang, *Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger-Poisson system in* $\mathbb{R}^3$, Z. Angew. Math. Phys., 2015, 66(6), 3267–3282.

[22] J. Sun, H. Chen and L. Yang, *Positive solutions of asymptotically linear Schrödinger-Poisson systems with a radial potential vanishing at infinity*, Nonlinear Anal., 2011, 74, 413–423.

[23] J. Sun, *Infinitely many solutions for a class of sublinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl., 2012, 390, 514–522.

[24] J. Sun, T. Wu and Z. Feng, *Multiplicity of positive solutions for a nonlinear Schrödinger-Poisson system*, J. Diff. Eqs., 2016, 260, 586–627.

[25] Z. Wang and H. Zhou, *Sign-changing solutions for the nonlinear Schrödinger-Poisson system in* $\mathbb{R}^3$, Calc. Var. Partial Differential Equations, 2015, 52, 927–943.

[26] M. Willem, *Minimax Theorems*, Birkhäuser Boston Inc, Boston, 1996.

[27] L. Xu and H. Chen, *Multiplicity of small negative-energy solutions for a class of nonlinear Schrödinger-Poisson systems*, Appl. Math. Comput., 2014, 243, 817–824.

[28] M. Yang and Z. Han, *Existence and multiplicity results for the nonlinear Schrödinger-Poisson systems*, Nonlinear Anal. RWA, 2012, 13, 1093–1101.

[29] J. Zhang, *On the Schrödinger-Poisson equations with a general nonlinearity in the critical growth*, Nonlinear Anal., 2012, 75, 6391–6401.

[30] J. Zhang, *On ground state and nodal solutions of Schrödinger-Poisson equations with critical growth*, J. Math. Anal. Appl., 2015, 428, 387–404.