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# OPTIMAL $H^1$ ERROR ANALYSIS OF A FRACTIONAL STEP FINITE ELEMENT SCHEME FOR A HYBRID MHD SYSTEM*

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**Abstract** This paper presents a fractional step finite element scheme for a hybrid MHD system coupled by the nonstationary Navier-Stokes equations and the steady Maxwell equations, which can be viewed that the magnetic phenomena reach their steady state "infinitely" faster than the fluid hydrodynamics phenomena. The proposed fractional step scheme has the following features: the first one is that the proposed scheme is a decoupled scheme, which means the magnetic field and velocity field can be solved independently at the same time discrete level. The second one is that the nonlinearity and the divergence-free of the Navier-Stokes equations are splitted by introducing an intermediate velocity field. We focus on a rigorous error analysis and obtain the optimal $\mathbf{H}^1$ convergence order $\mathcal{O}(\Delta t+h)$ for the magnetic and the velocity under the time step condition $\Delta t = \mathcal{O}(h)$, where $h$ is the mesh size. Finally, numerical results are shown to illustrate the theoretical convergence analysis.

**Keywords** Magnetohydrodynamics equations, fractional step method, finite element method, error analysis.

**MSC(2010)** 65M12, 76W05.

## 1. Introduction

The incompressible magnetohydrodynamics (MHD) equations are used to describe the flow of a viscous, incompressible and electrically conducting fluid, and consist of a coupling system by the incompressible Navier-Stokes equations of continuum fluid mechanics and the Maxwell equations of electromagnetism. Since it is difficult to find the analytical solution to the MHD equations in general domains, then that how to solve the numerical approximation solutions becomes more and more important. There have an amount of works devoted to the design and the analysis of numerical algorithms for the numerical simulations of the steady or nonstationary MHD equations. For example, for the steady MHD system, there have Galerkin finite element method [11], the stabilized mixed finite element method [6], the mixed finite element formulation based on $\mathbf{H}$(curl)-element for the approximations of the magnetic field [17]. For the nonstationary MHD system, there have the first-order semi-implicit scheme [5, 12, 22], the second-order Crank-Nicolson scheme [23], the

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projection methods [2,16], the viscosity-splitting and fractional-step scheme [3,19–21]. For a review of numerical methods for the MHD system, we refer the reader to [9].

Unlike the steady or nonstationary MHD equations, in this paper, we will consider the following hybrid MHD system which are coupled by the nonstationary Navier-Stokes equations and the steady Maxwell equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{R_e}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\widetilde{\mathbf{b}} \times \mathbf{curl}\ \widetilde{\mathbf{b}} = \mathbf{f} \qquad \text{in } \Omega\times(0,T], \tag{1.1}$$

$$\text{div } \mathbf{u} = 0 \qquad \text{in } \Omega\times(0,T], \tag{1.2}$$

$$\frac{1}{R_m}\mathbf{curl}\ (\mathbf{curl}\ \widetilde{\mathbf{b}}) - \mathbf{curl}\ (\mathbf{u}\times\widetilde{\mathbf{b}}) = 0 \qquad \text{in } \Omega\times(0,T], \tag{1.3}$$

$$\text{div } \widetilde{\mathbf{b}} = 0 \qquad \text{in } \Omega\times(0,T], \tag{1.4}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0 \qquad \text{in } \Omega, \tag{1.5}$$

$$\mathbf{u} = 0, \quad \widetilde{\mathbf{b}}\cdot\mathbf{n} = q, \quad \mathbf{curl}\ \widetilde{\mathbf{b}}\times\mathbf{n} = 0 \qquad \text{on } \partial\Omega\times[0,T], \tag{1.6}$$

with some $T > 0$, where the dependent variables are the fluid velocity $\mathbf{u}$, the pressure $p$ and the magnetic field $\mathbf{b}$. The domain $\Omega \subset \mathbf{R}^3$ is a bounded and convex domain with smooth boundary $\partial\Omega$. Three positive constants $R_e, R_m$ and $S$ are the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The $\mathbf{f}$ represents the body force applied to the fluid. The vector $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. The initial vector function $\mathbf{u}_0$ satisfies the compatibility condition div $\mathbf{u}_0 = 0$.

The hybrid MHD system (1.1)–(1.6) introduced in [7,8] is a simplified nonstationary MHD system, which are from the fact that the magnetic phenomena are known to reach their steady state "infinitely" faster than the hydrodynamics phenomena. From (1.3) we can see that the equation related to the magnetic field is an elliptic type equation. Moreover, the ellipticity of the equation heavily depends on the velocity field $\mathbf{u}$. If the velocity field becomes too large in some sense, the equation (1.3) may become ill-posed. Under some small assumptions of the initial data in some senses, Gerbeau & Bris in [7,8] proved that the hybrid MHD system (1.1)–(1.6) exists a unique local strong solution on a time interval $[0,T^\star]$ for some $T^\star < T$. Note that there has a nonhomogeneous boundary condition $\widetilde{\mathbf{b}}\cdot\mathbf{n} = q$ in (1.6). If $q = 0$, then $\widetilde{b} \equiv 0$ according to the existence and uniqueness of the local strong solution. In this case, the hybrid MHD system (1.1)–(1.6) will reduce to the incompressible Navier-Stokes equations. For the numerical methods of the hybrid MHD system (1.1)–(1.6), the first-order Euler semi-implicit scheme and the second-order Crank-Nicolson schemes based on the linear extrapolation were studied in [14] and [15], respectively.

In this paper, we will proposed a viscosity-splitting fraction step finite element scheme for the numerical simulations of (1.1)–(1.6) by using the MINI element and the piecewise linear element to approximate the velocity field, the pressure and the magnetic field, respectively. This type of viscosity-splitting fraction step algorithm was introduced by Blasco and Codina in [4] for solving the incompressible Navier-Stokes equations with the constant density numerically. Recently, it has been extended to the incompressible Navier-Stokes equations with variable density in [1]. Main feature of the fraction step algorithm is the decoupling of the nonlinearity and the incompressible condition of the velocity field. The proposed fractional step scheme for (1.1)–(1.6) is a three step scheme at a time discrete level in this paper. Firstly, we solve the magnetic equation by a semi-implicit scheme, and then solve an intermediate velocity by solving an linearized elliptic problem. Finally, we get the end-of-step velocity and pressure by solving a generalized Stokes problem. Thus, the proposed fractional step scheme is a fully decoupled scheme, which means that we can solve the magnetic and the velocity independently at a time discrete level. The rigorous error analysis are presented and we derive the optimal $\mathbf{H}^1$ temporal-spatial error estimate $\mathcal{O}(\Delta t + h)$ for the finite element approximations of the magnetic field and the velocity field.

The rest of this paper is organized as follows. In next section, we begin with some notation, and recall the existence and uniqueness of the local strong solution established in [7, 8]. The fractional step finite element fully scheme is proposed and main result is presented in Section 3. The optimal $\mathbf{H}^1$ error estimates for the magnetic and velocity are given in Section 4. In Section 5, numerical results are given to confirm the theoretical convergence analysis. Throughout this manuscript, we always use the symbol $C$ to denote the generic positive constant independent of the time step $\Delta t$ and the mesh size $h$.

## 2. Preliminaries

For the mathematical setting, we introduce some notations. For $k \in \mathbb{N}^+, 1 \leq p \leq +\infty$, let $W^{k,p}(\Omega)$ denote the standard Sobolev space. The norm in $W^{k,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{k,p}}$ defined by a classical way. Let $W_0^{k,p}(\Omega)$ be the subspace of $W^{k,p}(\Omega)$ of functions with zero trace on $\partial\Omega$. Especially, when $k = 0$, $W^{0,p}$ is the Lebesgue space $L^p(\Omega)$. When $p = 2$, $W^{k,2}(\Omega)$ is the Hilbert space which is simply denoted by $H^k(\Omega)$. The boldface notations $\mathbf{H}^k(\Omega), \mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector spaces $H^k(\Omega)^3, W^{k,p}(\Omega)^3$ and $L^p(\Omega)^3$, respectively. The $L^2$ or $L^2$ inner product is denoted by $(\cdot,\cdot)$. Let $X$ be a Banach space. For some $T > 0$, $L^p(0,T;X)$ is the space of measurable functions from the interval $[0,T]$ into $X$ such that

$$\int_0^T \|u(t)\|_X^p dt < +\infty, \quad \forall\ 1 \leq p < +\infty.$$

If $p = +\infty$, the functions in $L^\infty(0,T;X)$ are required to satisfy

$$\operatorname*{ess\ sup}_{t\in[0,T]} \|u(t)\|_X < +\infty.$$

Introduce the following function spaces:

$$\begin{aligned}
&\mathbf{H} = \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{V} = \mathbf{H}_0^1(\Omega), \quad \mathbf{V}_0 = \{\mathbf{u} \in \mathbf{V},\ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega\},\\
&\mathbf{W} = \{\mathbf{u} \in \mathbf{H}^1(\Omega),\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \mathbf{W}_0 = \{\mathbf{u} \in \mathbf{W},\ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega\},\\
&M = L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_\Omega q dx = 0\}.
\end{aligned}$$

For $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{W}$, the spaces $\mathbf{V}$ and $\mathbf{W}$ are equipped with norms

$$\|\mathbf{v}\|_V = \left(\int_\Omega |\nabla \mathbf{v}|^2 dx\right)^{1/2},$$

$$\|\mathbf{w}\|_W = \left( \int_\Omega (|\mathbf{curl}\ \mathbf{w}|^2 + |\mathrm{div}\ \mathbf{w}|^2) dx \right)^{1/2},$$

which are equivalent to the classical $\mathbf{H}^1$ norm. Define the trilinear term

$$\begin{aligned} b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} dx + \frac{1}{2} \int_\Omega (\mathrm{div}\ \mathbf{u})\mathbf{v} \cdot \mathbf{w} dx \\ &= \frac{1}{2} \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} dx - \frac{1}{2} \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mathbf{v} dx \quad \forall\ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}. \end{aligned}$$

It is clear that $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ satisfies

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \qquad \forall\ \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}. \tag{2.1}$$

Let us denote an orthogonal projection operator by $\mathbb{P}_\mathbf{H}$ from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. Introduce the Stokes operator $A$ which is defined by

$$A\mathbf{u} = -\mathbb{P}_\mathbf{H} \Delta \mathbf{u} \qquad \forall\ \mathbf{u} \in \mathbf{V}_0 \cap \mathbf{H}^2(\Omega) := \mathbf{D}(A).$$

As we know that $\|A\mathbf{u}\|_{L^2}$ is equivalent to the norm $\|\mathbf{u}\|_{H^2}$ (cf. [18]).

The following regularity results on the Stokes problem and the Maxwell problem are needed (cf. [9] and [18]).

**Lemma 2.1.** *Assume that the boundary of $\Omega$ is smooth such that for given $\boldsymbol{g}_1 \in \boldsymbol{L}^p(\Omega)$, $1 < p < +\infty$, the Stokes problem*

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{g}_1, \quad \mathit{div}\ \boldsymbol{v} &= 0 \qquad \mathit{in}\ \Omega, \\ \boldsymbol{v} &= 0 \qquad \mathit{on}\ \partial\Omega \end{aligned}$$

*admits a unique solution $(\boldsymbol{v}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{V} \times W^{1,p}(\Omega) \cap M$ such that*

$$\|\boldsymbol{v}\|_{W^{2,p}} + \|\pi\|_{W^{1,p}} \le C \|\boldsymbol{g}_1\|_{L^p}.$$

**Lemma 2.2.** *Assume that the boundary of $\Omega$ is smooth such that for given $\boldsymbol{g}_2 \in \boldsymbol{L}^p(\Omega)$, $1 < p < +\infty$, the Maxwell problem*

$$\begin{aligned} &\boldsymbol{curl}\ \boldsymbol{curl}\ \boldsymbol{v} = \boldsymbol{g}_2, \quad &&\mathit{div}\ \boldsymbol{v} = 0, \qquad &&\mathit{in}\ \Omega, \\ &\boldsymbol{v} \cdot \boldsymbol{n} = 0, &&\boldsymbol{curl}\ \boldsymbol{v} \times \boldsymbol{n} = 0, &&\mathit{on}\ \partial\Omega \end{aligned} \tag{2.2}$$

*admits a unique solution $\boldsymbol{v} \in \boldsymbol{W}^{2,p} \cap \boldsymbol{W}$ such that*

$$\|\boldsymbol{v}\|_{W^{2,p}} \le C \|\boldsymbol{g}_2\|_{L^p}.$$

In this paper, we always assume that

$$\mathbf{u}_0 \in \mathbf{D}(A), \quad q \in L^\infty(0, T; H^{3/2}(\partial\Omega)), \quad \mathbf{f} \in L^\infty(0, T; \mathbf{H}).$$

Notice that there has a nonhomogeneous boundary condition $\widetilde{\mathbf{b}} \cdot \mathbf{n}|_{\partial\Omega} = q$ in (1.6). Thus, we homogenize this boundary condition by the following lemma established in [7,8].

**Lemma 2.3.** *There exist* $\widetilde{\boldsymbol{B}} \in L^\infty(0,T;\boldsymbol{H}^2(\Omega)\cap \boldsymbol{W}_0)$ *satisfying*

$$\text{div}\ \widetilde{\boldsymbol{B}} = 0 \quad \text{and} \quad \textbf{curl}\ \widetilde{\boldsymbol{B}} = \boldsymbol{0} \quad \text{in } \Omega\times[0,T],$$

*and a constant* $C>0$ *such that* $\widetilde{\boldsymbol{B}}\cdot\boldsymbol{n} = q$ *on* $\partial\Omega\times[0,T]$, *and*

$$\|\widetilde{\boldsymbol{B}}(t)\|_{L^\infty(0,T;\boldsymbol{H}^2(\Omega))} \le C\|q\|_{L^\infty(0,T;H^{3/2}(\partial\Omega))}.$$

Let $\mathbf{b} = \widetilde{\mathbf{b}} - \widetilde{\mathbf{B}}$. Then the original system (1.1)–(1.6) can be rewritten as

$$\mathbf{u}_t - \frac{1}{Re}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times\textbf{curl}\ \mathbf{b} + S\widetilde{\mathbf{B}}\times\textbf{curl}\ \mathbf{b} = \mathbf{f}, \tag{2.3}$$

$$\text{div}\ \mathbf{u} = 0, \quad \text{div}\ \mathbf{b} = 0, \tag{2.4}$$

$$\frac{1}{Rm}\textbf{curl}\ (\textbf{curl}\ \mathbf{b}) - \textbf{curl}\ (\mathbf{u}\times\mathbf{b}) - \textbf{curl}\ (\mathbf{u}\times\widetilde{\mathbf{B}}) = 0, \tag{2.5}$$

with

$$\mathbf{u}(x,0) = \mathbf{u}_0, \qquad \text{in } \Omega, \tag{2.6}$$

$$\mathbf{u} = 0, \quad \mathbf{b}\cdot\mathbf{n} = 0, \quad \textbf{curl}\ \mathbf{b}\times\mathbf{n} = 0 \qquad \text{on } \partial\Omega\times[0,T]. \tag{2.7}$$

The existence and uniqueness of the local strong solution $\mathbf{b}$ to the new hybrid MHD system (2.3)–(2.7) were proved in [7,8].

**Theorem 2.1.** *If the initial data* $\boldsymbol{u}_0, Re, Rm, \boldsymbol{f}$ *and* $q$ *are "small enough" in an appropriate sense, then there exists a time* $T^\star < T$ *such that the coupled system (2.3)-(2.7) has a unique solution on* $[0,T^\star]$. *Moreover, the solution satisfies* $\boldsymbol{u}\in\mathcal{K}$, $p\in L^2(0,T^\star;H^1(\Omega)\cap M)$ *and* $\boldsymbol{b}\in L^\infty(0,T^\star;\boldsymbol{H}^2(\Omega)\cap\boldsymbol{W})$, *where*

$$\mathcal{K} = \{\|\boldsymbol{u}\|_{L^\infty(0,T^\star;\boldsymbol{V})} \le M,\ \|\boldsymbol{u}\|_{L^2(0,T^\star;\boldsymbol{D}(A))} \le M,\ \|\boldsymbol{u}_t\|_{L^2(0,T^\star;\boldsymbol{H})} \le M\} \tag{2.8}$$

*for some* $0 < M < \dfrac{1}{\kappa Rm}$, *where* $\kappa > 0$ *is from*

$$(\boldsymbol{u}\times\boldsymbol{v}, \textbf{curl}\ \boldsymbol{w}) \le \kappa\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_W\|\boldsymbol{w}\|_W.$$

# 3. Fractional step scheme and main result

In this section, we will propose a fractional step time-discrete scheme and the fully discrete finite element scheme for the numerical simulations of the new hybrid MHD system (2.3)–(2.7). Let $0 = t_0 < t_1 < \cdots < t_N = T^\star$ be a uniform partition of the time interval $[0,T^\star]$ with time step $\Delta t = T^\star/N$ and $t_n = n\Delta t, 0\le n\le N$, where $[0,T^\star]$ is the maximal time interval such that a unique local strong solution exists and satisfies the regularities mentioned in Theorem 2.1. For $1\le n\le N$, we denote $\mathbf{u}^n = \mathbf{u}(t_n), p^n = p(t_n)$, $\mathbf{b}^n = \mathbf{b}(t_n)$, $\mathbf{f}^n = \mathbf{f}(t_n)$ and $\widetilde{\mathbf{B}}^n = \widetilde{\mathbf{B}}(t_n)$. For any sequence $\{g^n\}_{n=0}^N$, we denote $D_t g^n = \frac{g^n - g^{n-1}}{\Delta t}$.

Let $\mathbf{U}^0 = \mathbf{u}_0$ be given. For $1\le n\le N$, we propose the following fractional step time-discrete scheme.

**Step I:** for given $\mathbf{U}^{n-1}$, we solve $\mathbf{B}^n$ from the following linearized Maxwell problem:

$$\frac{1}{Rm}\textbf{curl}\ (\textbf{curl}\ \mathbf{B}^n) - \textbf{curl}\ (\mathbf{U}^{n-1}\times\mathbf{B}^n) - \textbf{curl}\ (\mathbf{U}^{n-1}\times\widetilde{\mathbf{B}}^n) = 0,\ \text{div}\ \mathbf{B}^n = 0, \tag{3.1}$$

with the boundary condition $\mathbf{B}^n \cdot \mathbf{n} = 0$ and $\mathbf{curl}\ \mathbf{B}^n \times \mathbf{n} = 0$ on $\partial\Omega$.

**Step II:** for given $\mathbf{U}^{n-1}$ and $\mathbf{B}^n$, we solve an intermediate velocity $\widetilde{\mathbf{U}}^n$ from the following linearized elliptic problem:

$$\frac{\widetilde{\mathbf{U}}^n - \mathbf{U}^{n-1}}{\Delta t} - \frac{1}{Re}\Delta\widetilde{\mathbf{U}}^n + (\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n + S\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n + S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \mathbf{B}^n = \mathbf{f}^n \tag{3.2}$$

with the boundary condition $\widetilde{\mathbf{U}}^n = 0$ on $\partial\Omega$.

**Step III:** for given $\widetilde{\mathbf{U}}^n$, we solve $(\mathbf{U}^n, P^n)$ from the following generalized Stokes problem:

$$\frac{\mathbf{U}^n - \widetilde{\mathbf{U}}^n}{\Delta t} - \frac{1}{Re}\Delta(\mathbf{U}^n - \widetilde{\mathbf{U}}^n) + \nabla P^n = 0, \quad \text{div } \mathbf{U}^n = 0 \tag{3.3}$$

with the boundary condition $\mathbf{U}^n = 0$ on $\partial\Omega$.

Taking the sum of (3.2) and (3.3), we have

$$D_t\mathbf{U}^n - \frac{1}{Re}\Delta\mathbf{U}^n + \nabla P^n + (\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n + S\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n + S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \mathbf{B}^n = \mathbf{f}^n. \tag{3.4}$$

Next, we give the finite element fully discretization of (3.1)–(3.3). Let $\mathcal{T}_h$ be a quasi-uniform partition of $\Omega$ into triangle or tetrahedra of diameters by $h$ with $0 < h < 1$. We use the $P_1b - P_1$ element to approximate the velocity field and the pressure, and use the piecewise linear Lagrange element to approximate the magnetic field and the intermediate velocity field. The finite element spaces of $\mathbf{V}$, $M$ and $\mathbf{W}$ are denoted by $\mathbf{V}_h$, $M_h$ and $\mathbf{W}_h$, respectively. For this choice, the finite element spaces $\mathbf{V}_h$ and $M_h$ are required to satisfy the discrete inf-sup condition. In addition, with respect to the choice of finite element space to approximate the intermediate velocity $\widetilde{\mathbf{U}}^n$, we still use the finite element space $\mathbf{V}_h$ to get the optimal $H^1$ error estimate (3.19). However, this choice is not important to get the optimal error estimate for the end-of-step velocity if we can use the technique in [10].

Define the projection operators $(\mathbf{R}_h, Q_h) : \mathbf{V}\times M \longrightarrow \mathbf{V}_h \times M_h$ and $\Pi_h : \mathbf{W} \longrightarrow \mathbf{W}_h$ and $\mathbb{K}_h : \mathbf{V} \longrightarrow \mathbf{V}_h$ by

$$\begin{aligned}&\frac{1}{Re}(\nabla(\mathbf{R}_h\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}_h) - (\text{div } \mathbf{v}_h, Q_hp - p) = 0, \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\ &(\text{div } (\mathbf{R}_h\mathbf{u} - \mathbf{u}), q_h) = 0, \quad \forall\ q_h \in M_h,\end{aligned}$$

and

$$(\mathbf{curl}\ (\Pi_h\mathbf{b} - \mathbf{b}), \mathbf{curl}\ \mathbf{w}_h) + (\text{div } (\Pi_h\mathbf{b} - \mathbf{b}), \text{div } \mathbf{w}_h) = 0,\ \forall\ \mathbf{w}_h \in \mathbf{W}_h,$$

and

$$(\nabla(\mathbb{K}_h\mathbf{u} - \mathbf{u}), \nabla\mathbf{v}_h) = 0, \qquad \forall\ \mathbf{v}_h \in \mathbf{V}_h.$$

Then for any $(\mathbf{u}, p, \mathbf{b}) \in \mathbf{D}(A) \times H^1(\Omega) \times \mathbf{H}^2(\Omega)$, the following approximations and stabilities hold:

$$\|\mathbf{u} - \mathbf{R}_h\mathbf{u}\|_{L^2} + h\|\mathbf{u} - \mathbf{R}_h\mathbf{u}\|_V + h\|p - Q_hp\|_{L^2} \le Ch^2(\|A\mathbf{u}\|_{L^2} + \|p\|_{H^1}), \tag{3.5}$$

$$\|\mathbf{b} - \Pi_h\mathbf{b}\|_{L^2} + h\|\mathbf{b} - \Pi_h\mathbf{b}\|_W \le Ch^2\|\mathbf{b}\|_{H^2}, \tag{3.6}$$

$$\|\mathbf{u} - \mathbb{K}_h\mathbf{u}\|_{L^2} + h\|\mathbf{u} - \mathbb{K}_h\mathbf{u}\|_V \leq Ch^2\|\mathbf{u}\|_{H^2}, \tag{3.7}$$
$$\|\nabla\mathbf{R}_h\mathbf{u}\|_{L^6} + \|\nabla\mathbb{K}_h\mathbf{u}\|_{L^6} + \|\Pi_h\mathbf{b}\|_{W^{1,6}} + \|Q_hp\|_{L^6} \leq C. \tag{3.8}$$

Let $\mathbf{U}_h^0 = \rho_h\mathbf{U}^0 = \rho_h\mathbf{u}_0$, where $\rho_h$ is the $L^2$ projection operator from $\mathbf{V}$ to $\mathbf{V}_h$ and satisfies

$$\|\mathbf{u}_0 - \rho_h\mathbf{u}_0\|_{L^2} + h\|\nabla(\mathbf{u}_0 - \rho_h\mathbf{u}_0)\|_{L^2} \leq Ch^2\|A\mathbf{u}_0\|_{L^2}, \tag{3.9}$$

For $1 \leq n \leq N$, the finite element fully discrete scheme of (3.1)–(3.3) is described as follows.

**Step I:** for given $\mathbf{U}_h^{n-1} \in \mathbf{V}_h$, we find $\mathbf{B}_h^n \in \mathbf{W}_h$ such that

$$\begin{aligned}&\frac{1}{Rm}(\mathbf{curl}\ \mathbf{B}_h^n, \mathbf{curl}\ \mathbf{w}_h) + \frac{1}{Rm}(\text{div}\ \mathbf{B}_h^n, \text{div}\ \mathbf{w}_h) - (\mathbf{U}_h^{n-1} \times \mathbf{B}_h^n, \mathbf{curl}\ \mathbf{w}_h)\\ &- (\mathbf{U}_h^{n-1} \times \widetilde{\mathbf{B}}^n, \mathbf{curl}\ \mathbf{w}_h) = 0, \quad \forall\ \mathbf{w}_h \in \mathbf{W}_h.\end{aligned} \tag{3.10}$$

**Step II:** for given $\mathbf{U}_h^{n-1} \in \mathbf{V}_h$ and $\mathbf{B}_h^n \in \mathbf{W}_h$, we find $\widetilde{\mathbf{U}}_h^n \in \mathbf{V}_h$ such that

$$\begin{aligned}&\frac{1}{\Delta t}(\widetilde{\mathbf{U}}_h^n - \mathbf{U}_h^{n-1}, \mathbf{v}_h) + \frac{1}{Re}(\nabla\widetilde{\mathbf{U}}_h^n, \nabla\mathbf{v}_h) + b(\mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}_h^n, \mathbf{v}_h)\\ &+ S(\mathbf{B}_h^n \times \mathbf{curl}\ \mathbf{B}_h^n, \mathbf{v}_h) + S(\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \mathbf{B}_h^n, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h), \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h.\end{aligned} \tag{3.11}$$

**Step III:** for given $\widetilde{\mathbf{U}}_h^n \in \mathbf{V}_h$, we find $(\mathbf{U}_h^n, P_h^n) \in \mathbf{V}_h \times M_h$ such that

$$\frac{1}{\Delta t}(\mathbf{U}_h^n - \widetilde{\mathbf{U}}_h^n, \mathbf{v}_h) + \frac{1}{Re}\Big(\nabla(\mathbf{U}_h^n - \widetilde{\mathbf{U}}_h^n), \nabla\mathbf{v}_h\Big) - (\text{div}\ \mathbf{v}_h, P_h^n) + (\text{div}\ \mathbf{U}_h^n, q_h) = 0 \tag{3.12}$$

for any $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$.

Introduce the following temporal and spatial error splitting:

$$\|\mathbf{U}_h^n - \mathbf{u}^n\| \leq \|e^n\| + \|e_h^n\| + \|\mathbf{U}^n - \mathbf{R}_h\mathbf{U}^n\|, \tag{3.13}$$
$$\|\widetilde{\mathbf{U}}_h^n - \mathbf{u}^n\| \leq \|\widetilde{e}^n\| + \|\widetilde{e}_h^n\| + \|\widetilde{\mathbf{U}}^n - \mathbb{K}_h\widetilde{\mathbf{U}}^n\|, \tag{3.14}$$
$$\|\mathbf{B}_h^n - \mathbf{b}^n\| \leq \|\eta^n\| + \|\eta_h^n\| + \|\mathbf{B}^n - \Pi_h\mathbf{B}^n\|, \tag{3.15}$$
$$\|P_h^n - p^n\| \leq \|\theta^n\| + \|\theta_h^n\| + \|P^n - Q_hP^n\|, \tag{3.16}$$

for any norm $\|\cdot\|$, where

$$\begin{aligned}&e^n = \mathbf{u}^n - \mathbf{U}^n, \quad e_h^n = \mathbf{U}_h^n - \mathbf{R}_h\mathbf{U}^n, \quad \widetilde{e}^n = \mathbf{u}^n - \widetilde{\mathbf{U}}^n, \quad \widetilde{e}_h^n = \widetilde{\mathbf{U}}_h^n - \mathbb{K}_h\widetilde{\mathbf{U}}^n,\\ &\eta^n = \mathbf{b}^n - \mathbf{B}^n, \quad \eta_h^n = \mathbf{B}_h^n - \Pi_h\mathbf{B}^n, \quad \theta^n = p^n - P^n, \quad \theta_h^n = P_h^n - Q_hP^n.\end{aligned}$$

In this paper, we will prove the optimal first-order convergence order for the approximation of the velocity field and the magnetic field in $\mathbf{H}^1$ norm under the following regularity assumptions about the local strong solutions derived in Theorem 2.1:

$$\mathbf{b}_t \in L^2(0, T^\star; \mathbf{W}_0), \quad \mathbf{b}_{tt} \in L^2(0, T^\star; \mathbf{W}_0), \tag{3.17}$$
$$\mathbf{u} \in L^\infty(0, T^\star; \mathbf{W}^{2,4}(\Omega)), \quad \mathbf{u}_t \in L^2(0, T^\star; \mathbf{D}(A)), \quad \mathbf{u}_{tt} \in L^2(0, T^\star; \mathbf{H}), \tag{3.18}$$

where $T^\star$ is from Theorem 2.1.

**Theorem 3.1.** *Assume that the solution to (2.3)–(2.7) satisfies the regularities in Theorem 2.1 and (3.17)–(3.18). Then under the condition $\Delta t = \mathcal{O}(h)$, for the sufficiently small $h$ and $\Delta t$, there exists some $C > 0$ such that the following optimal $\boldsymbol{H}^1$ error estimate holds:*

$$\max_{1\le n\le N} (\|\boldsymbol{u}^n - \boldsymbol{U}_h^n\|_V + \|\boldsymbol{b}^n - \boldsymbol{B}_h^n\|_W) \le C\,(\Delta t + h)\,. \tag{3.19}$$

We will prove Theorem 3.1 in next section. Before beginning to the proof, we recall a discrete version of Gronwall's inequality established in [13] which is frequently used in the proof of Theorem 3.1.

**Lemma 3.1.** *Let $a_k, b_k, c_k$ and $\gamma_k$, for integers $k \ge 0$, be the nonnegative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \Delta t \sum_{k=0}^{n} \gamma_k a_k + \Delta t \sum_{k=0}^{n} c_k + B \qquad \text{for } n \ge 0. \tag{3.20}$$

*Suppose that $\Delta t \gamma_k < 1$, for all $k$, and set $\sigma_k = (1 - \Delta t \gamma_k)^{-1}$. Then*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \le \exp(\Delta t \sum_{k=0}^{n} \gamma_k \sigma_k)(\Delta t \sum_{k=0}^{n} c_k + B) \qquad \text{for } n \ge 0. \tag{3.21}$$

**Remark 3.1.** If the first sum on the right in (3.20) extends only up to $n-1$, then the estimate (3.21) holds for all $\Delta t > 0$ with $\sigma_k = 1$.

# 4. Error analysis

In this section, we will prove the optimal $\mathbf{H}^1$ convergence order $\mathcal{O}(\Delta t + h)$ for the approximation of the velocity field and the magnetic field. The temporal and spatial error analysis are presented in Subsection 4.1 and Subsection 4.2, respectively. Then the optimal error estimate (3.19) follows from the temporal and spatial error analysis and the error splitting (3.13) and (3.15). Please see Subsection 4.3.

## 4.1. Temporal error analysis

We first prove the first-order temporal convergence order $\mathcal{O}(\Delta t)$ for $(\mathbf{U}^n, \mathbf{B}^n)$ in $\mathbf{H}^1$-norm. For $1 \le n \le N$, we take $t = t_n$ in (2.3) and (2.5) to yield

$$\begin{aligned} &D_t \mathbf{u}^n - \frac{1}{Re}\Delta \mathbf{u}^n + (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n + \nabla p^n + S\mathbf{b}^n \times \mathbf{curl}\ \mathbf{b}^n \\ &+ S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \mathbf{b}^n = \mathbf{f}^n + \mathbf{R}_u^n, \qquad \text{div}\ \mathbf{u}^n = 0, \end{aligned} \tag{4.1}$$

and

$$\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b}^n) - \mathbf{curl}\ (\mathbf{u}^{n-1} \times \mathbf{b}^n) - \mathbf{curl}\ (\mathbf{u}^{n-1} \times \widetilde{\mathbf{B}}^n) = \mathbf{R}_b^n, \quad \text{div}\ \mathbf{b}^n = 0, \tag{4.2}$$

where the truncation errors $\mathbf{R}_u^n$ and $\mathbf{R}_b^n$ are given by

$$\mathbf{R}_u^n = D_t\mathbf{u}^n - \mathbf{u}_t^n + (\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n - (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,$$

$$\mathbf{R}_b^n = \mathbf{curl}\ ((\mathbf{u}^n - \mathbf{u}^{n-1}) \times \mathbf{b}^n) + \mathbf{curl}\ ((\mathbf{u}^n - \mathbf{u}^{n-1}) \times \widetilde{\mathbf{B}}^n).$$

It follows from (3.17)–(3.18) that

$$\Delta t \sum_{n=1}^{N} \|\mathbf{R}_u^n\|_{L^2}^2 + \|\mathbf{R}_b^n\|_{L^2}^2 \le C(\Delta t)^2 \tag{4.3}$$

by using Taylor formula with integral type.

For $1 \le n \le N$, subtracting (4.2) and (4.1) from (3.1) and (3.2), respectively, we get the following error equations:

$$\begin{aligned}&\frac{1}{Rm}\mathbf{curl}\ \mathbf{curl}\ \eta^n - \mathbf{curl}\ (\mathbf{u}^{n-1} \times \eta^n) - \mathbf{curl}\ (e^{n-1} \times \mathbf{B}^n)\\ &- \mathbf{curl}\ (e^{n-1} \times \widetilde{\mathbf{B}}^n) = \mathbf{R}_b^n, \quad \mathrm{div}\ \eta^n = 0,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\frac{\widetilde{e}^n - e^{n-1}}{\Delta t} - \frac{1}{Re}\Delta\widetilde{e}^n + \nabla p^n + (e^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n + (\mathbf{u}^{n-1}\cdot\nabla)\widetilde{e}^n + S\eta^n \times \mathbf{curl}\ \mathbf{b}^n\\ &+ S\mathbf{B}^n \times \mathbf{curl}\ \eta^n + S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \eta^n = \mathbf{R}_u^n.\end{aligned} \tag{4.5}$$

In addition, from (3.3), we have

$$\frac{e^n - \widetilde{e}^n}{\Delta t} - \frac{1}{Re}\Delta(e^n - \widetilde{e}^n) - \nabla P^n = 0, \quad \mathrm{div}\ e^n = 0. \tag{4.6}$$

Taking the sum of (4.5) and (4.6) leads to

$$\begin{aligned}&\frac{e^n - e^{n-1}}{\Delta t} - \frac{1}{Re}\Delta e^n + \nabla\theta^n + (e^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n + (\mathbf{u}^{n-1}\cdot\nabla)\widetilde{e}^n + S\eta^n \times \mathbf{curl}\ \mathbf{b}^n\\ &+ S\mathbf{B}^n \times \mathbf{curl}\ \eta^n + S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \eta^n = \mathbf{R}_u^n, \quad \mathrm{div}\ e^n = 0.\end{aligned} \tag{4.7}$$

First, we need to estimate $\mathbf{B}^n$ under some additional condition.

**Lemma 4.1.** *For $1 \le n \le N$, if $\|\boldsymbol{U}^{n-1}\|_V \le \widetilde{M}$, then the solution $\boldsymbol{B}^n$ to (3.1) belongs to $\boldsymbol{B}^n \in \boldsymbol{H}^2(\Omega)$, where $\widetilde{M} > 0$ satisfies $M < \widetilde{M} < \dfrac{1}{\kappa Rm}$.*

**Proof.** If $\|\mathbf{U}^{n-1}\|_V \le \widetilde{M}$, then testing (3.1) by $\mathbf{B}^n$ leads to

$$\|\mathbf{B}^n\|_W \le \frac{C\widetilde{M}Rm\|q\|_{L^\infty(0,T^\star;H^{3/2}(\partial\Omega))}}{1 - \kappa\widetilde{M}Rm}.$$

By using the following formula

$$\mathbf{curl}\ (\mathbf{u} \times \mathbf{v}) = (\mathrm{div}\ \mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathrm{div}\ \mathbf{u})\mathbf{v}, \tag{4.8}$$

an alternative of (3.1) is

$$\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{B}^n) = \mathbf{g}, \tag{4.9}$$

where

$$\mathbf{g} = (\mathbf{B}^n \cdot \nabla)\mathbf{U}^{n-1} - (\mathbf{U}^{n-1}\cdot\nabla)\mathbf{B}^n + (\widetilde{\mathbf{B}}^n\cdot\nabla)\mathbf{U}^{n-1} - (\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{B}}^n.$$

Then $\mathbf{g}$ belongs to $\mathbf{L}^{3/2}(\Omega)$ since

$$\begin{aligned}\|\mathbf{g}\|_{L^{3/2}} &\le (\|\mathbf{B}^n\|_{L^6} + \|\widetilde{\mathbf{B}}^n\|_{L^6})\|\mathbf{U}^{n-1}\|_V + (\|\nabla\mathbf{B}^n\|_{L^2} + \|\nabla\widetilde{\mathbf{B}}^n\|_{L^2})\|\mathbf{U}^{n-1}\|_{L^6}\\ &\le C(\|\mathbf{B}^n\|_W + \|\widetilde{\mathbf{B}}^n\|_W)\|\mathbf{U}^{n-1}\|_V \le C.\end{aligned}$$

By Lemma 2.2 and the Sobolev imbedding theorem, we have $\mathbf{B}^n \in \mathbf{W}^{2,3/2}(\Omega) \hookrightarrow \mathbf{W}^{1,3}(\Omega)$. By Lemma 2.2, again,

$$\begin{aligned}\|\mathbf{B}^n\|_{H^2} &\le \|(\mathbf{B}^n\cdot\nabla)\mathbf{U}^{n-1}\|_{L^2} + \|(\mathbf{U}^{n-1}\cdot\nabla)\mathbf{B}^n\|_{L^2}\\ &\quad + \|(\widetilde{\mathbf{B}}^n\cdot\nabla)\mathbf{U}^{n-1}\|_{L^2} + \|(\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{B}}^n\|_{L^2}\\ &\le C(\|\mathbf{B}^n\|_{L^\infty} + \|\widetilde{\mathbf{B}}^n\|_{L^\infty})\|\mathbf{U}^{n-1}\|_V + (\|\mathbf{B}^n\|_{W^{1,3}} + \|\widetilde{\mathbf{B}}^n\|_{W^{1,3}})\|\mathbf{U}^{n-1}\|_{L^6}\\ &\le C\|\mathbf{B}^n\|_W^{1/2}\|\mathbf{B}^n\|_{H^2}^{1/2}\|\mathbf{U}^{n-1}\|_V + C\|\widetilde{\mathbf{B}}^n\|_{H^2}\|\mathbf{U}^{n-1}\|_V\\ &\le \frac{1}{2Rm}\|\mathbf{B}^n\|_{H^2} + C,\end{aligned}$$

which complete the proof of Lemma 4.1. □

Since we will use the method of mathematical induction to prove the temporal convergence order, then we need to estimate $e^1$ and $\eta^1$ in $\mathbf{H}^1$-norm. Taking $n = 1$ in (4.4), testing it by $2\eta^1$ and using $e^0 = 0$, we get

$$\frac{1}{Rm}\|\mathbf{curl}\ \eta^1\|_{L^2}^2 \le \kappa M\|\mathbf{curl}\ \eta^1\|_{L^2}^2 + \|\mathbf{R}_b^1\|_{L^2}\|\eta^1\|_{L^2},$$

which implies that

$$\|\mathbf{curl}\ \eta^1\|_{L^2} \le C\|\mathbf{R}_b^1\|_{L^2} \le C\Delta t. \tag{4.10}$$

Testing (4.5) by $\Delta t\widetilde{e}^1$ and using $e^0 = 0$, we get

$$\begin{aligned}\|\widetilde{e}^1\|_{L^2}^2 + \frac{\Delta t}{Re}\|\widetilde{e}^1\|_V^2 &\le C\Delta t(\|\nabla p^1\|_{L^2} + \|\mathbf{curl}\ \eta^1\|_{L^2} + \|\mathbf{R}_u^1\|_{L^2})\|\widetilde{e}^1\|_{L^2}\\ &\le \frac{1}{2}\|\widetilde{e}^1\|_{L^2}^2 + C(\Delta t)^2,\end{aligned}$$

which means that

$$\|\widetilde{e}^1\|_{L^2}^2 + \frac{\Delta t}{Re}\|\widetilde{e}^1\|_V^2 \le C(\Delta t)^2. \tag{4.11}$$

Testing (4.7) by $\Delta t e^1$, and using $e^0 = 0$ and div $e^1 = 0$, we get

$$\begin{aligned}\|e^1\|_{L^2}^2 + \frac{\Delta t}{Re}\|e^1\|_V^2 &\le C\Delta t(\|\widetilde{e}^1\|_V + \|\mathbf{curl}\ \eta^1\|_{L^2} + \|\mathbf{R}_u^1\|_{L^2})\|e^1\|_{L^2}\\ &\le \frac{1}{2}\|e^1\|_{L^2}^2 + C(\Delta t)^3,\end{aligned}$$

which means that

$$\|e^1\|_{L^2}^2 + \frac{\Delta t}{Re}\|e^1\|_V^2 \le C(\Delta t)^3. \tag{4.12}$$

In addition, testing (4.7) by $\Delta t Ae^1$ leads to

$$\|e^1\|_V^2 + \frac{\Delta t}{Re}\|Ae^1\|_{L^2}^2 \le C\Delta t(\|\widetilde{e}^1\|_V + \|\mathbf{curl}\ \eta^1\|_{L^2} + \|\mathbf{R}_u^1\|_{L^2})\|Ae^1\|_{L^2}$$

$$\leq \frac{\Delta t}{2Re}\|Ae^1\|_{L^2}^2 + C(\Delta t)^3,$$

which means that

$$\|e^1\|_V^2 + \frac{\Delta t}{Re}\|Ae^1\|_{L^2}^2 \leq C(\Delta t)^3. \tag{4.13}$$

By (4.12), it is easy to check that $\|D_t\mathbf{U}^1\|_{L^2} \leq C$. Then from the regularity of Stokes problem and (3.4), we have

$$\|A\mathbf{U}^1\|_{L^2} + \|\nabla P^1\|_{L^2} \leq C.$$

Summing up the above estimates, we get

$$\Delta t\|\widetilde{e}^1\|_{L^2}^2 + (\Delta t)^2\|\widetilde{e}^1\|_V^2 + \|e^1\|_V^2 + \Delta t\|Ae^1\|_{L^2}^2 + \Delta t\|\eta^1\|_W^2 \leq C_1(\Delta t)^3 \tag{4.14}$$

for some $C_1 > 0$.

The main result in this subsection is the following theorem about the first-order temporal convergence order.

**Theorem 4.1.** *Assume that the solution to (2.3)–(2.7) satisfy regularities in Theorem 2.1 and (3.17)–(3.18). Then for the sufficiently small $\Delta t$, there exists some $C_0 > 0$ such that*

$$\max_{1\leq k\leq N}\left(\|\boldsymbol{u}^k - \boldsymbol{U}^k\|_V + \|\boldsymbol{b}^k - \boldsymbol{B}^k\|_W\right) \leq C_0\Delta t, \tag{4.15}$$

$$\max_{1\leq k\leq N}\|\boldsymbol{U}^k\|_V \leq \widetilde{M}, \tag{4.16}$$

$$\Delta t\sum_{n=1}^{N}\|D_t(A\boldsymbol{U}^n)\|_{L^2}^2 \leq C_0, \tag{4.17}$$

*where $\widetilde{M}$ is from Lemma 4.1.*

**Proof.** We firstly prove that (4.15) holds by the method of mathematical induction. Then (4.16) follows from (4.15) if we take sufficiently small $\Delta t$ such that $C_0\Delta t \leq \widetilde{M} - M$, then

$$\|\mathbf{U}^k\|_V \leq \|\mathbf{u}^k - \mathbf{U}^k\|_V + \|\mathbf{u}^k\|_V \leq M + C_0\Delta t \leq \widetilde{M}.$$

From (4.14), we can see that (4.15) is valid for $k = 1$ if we take sufficiently small $\Delta t$ such that $C_1\Delta t \leq C_0$. Now, we assume that (4.15) is valid for $k = n-1$ with $2 \leq n \leq N$. Under this assumption, we have $\|\mathbf{U}^{n-1}\|_V \leq \widetilde{M}$ and $\mathbf{B}^n \in \mathbf{H}^2(\Omega)$ by Lemma 4.1. To close the mathematical induction, we need to prove that (4.15) is valid for $k = n$.

Testing (4.4) by $\eta^n$, we have

$$\begin{aligned}&\frac{1}{Rm}\|\mathbf{curl}\ \eta^n\|_{L^2} - (\mathbf{u}^{n-1}\times\eta^n, \mathbf{curl}\ \eta^n) - (e^{n-1}\times\mathbf{B}^n, \mathbf{curl}\ \eta^n)\\ &- (e^{n-1}\times\widetilde{\mathbf{B}}^n, \mathbf{curl}\ \eta^n) = (\mathbf{R}_b^n, \eta^n).\end{aligned}$$

By the Hölder inequality, we get

$$\frac{1}{Rm}\|\mathbf{curl}\ \eta^n\|_{L^2}^2 \leq (\|\mathbf{B}^n\|_{L^\infty} + \|\widetilde{\mathbf{B}}^n\|_{L^\infty})\|e^{n-1}\|_{L^2}\|\mathbf{curl}\ \eta^n\|_{L^2}$$

$$
\begin{aligned}
&+\kappa\|\mathbf{u}^{n-1}\|_V\|\mathbf{curl}\ \eta^n\|_{L^2}^2+C\|\mathbf{R}_b^n\|_{L^2}\|\mathbf{curl}\ \eta^n\|_{L^2}\\
\le& C\left(\Delta t+\|e^{n-1}\|_{L^2}\right)\|\mathbf{curl}\ \eta^n\|_{L^2}+\kappa M\|\mathbf{curl}\ \eta^n\|_{L^2}^2,
\end{aligned}
$$

where we use $\|\mathbf{u}\|_V\le M$ with $0<M<\dfrac{1}{\kappa Rm}$. Then

$$
\|\eta^n\|_W=\|\mathbf{curl}\ \eta^n\|_{L^2}\le C\left(\|e^{n-1}\|_{L^2}+\Delta t\right) \tag{4.18}
$$

if we notice div $\eta^n=0$. Testing (4.5) by $2\Delta t\widetilde{e}^n$, we get

$$
\begin{aligned}
&\|\widetilde{e}^n\|_{L^2}^2+\|\widetilde{e}^n-e^{n-1}\|_{L^2}^2-\|e^{n-1}\|_{L^2}^2+\frac{2\Delta t}{Re}\|\widetilde{e}^n\|_V^2\\
\le& C\Delta t\left(\|e^{n-1}\|_{L^2}+\|\eta^n\|_W+\|\mathbf{R}_u^n\|_{L^2}\right)\|\widetilde{e}^n\|_V+2\Delta t\|\nabla p^n\|_{L^2}\|\widetilde{e}^n-e^{n-1}\|_{L^2}\\
\le& \frac{\Delta t}{Re}\|\widetilde{e}^n\|_V^2+\frac{1}{2}\|\widetilde{e}^n-e^{n-1}\|_{L^2}^2+C\Delta t\left(\|e^{n-1}\|_{L^2}^2+\|\eta^n\|_W^2+\|\mathbf{R}_u^n\|_{L^2}^2+\Delta t\right),
\end{aligned}
$$

where we use

$$
\begin{aligned}
&b(e^{n-1},\widetilde{\mathbf{U}}^n,\widetilde{e}^n)=b(e^{n-1},\mathbf{u}^n,\widetilde{e}^n)\le C\|A\mathbf{u}^n\|_{L^2}\|e^{n-1}\|_{L^2}\|\widetilde{e}^n\|_V,\\
&(\nabla p^n,\widetilde{e}^n)=(\nabla p^n,\widetilde{e}^n-e^{n-1})\le\|\nabla p^n\|_{L^2}\|\widetilde{e}^n-e^{n-1}\|_{L^2}.
\end{aligned}
$$

Then from (4.18), we get

$$
\begin{aligned}
&\|\widetilde{e}^n\|_{L^2}^2+\frac{1}{2}\|\widetilde{e}^n-e^{n-1}\|_{L^2}^2-\|e^{n-1}\|_{L^2}^2+\frac{\Delta t}{Re}\|\widetilde{e}^n\|_V^2\\
\le& C\Delta t\left(\|e^{n-1}\|_{L^2}^2+\|\mathbf{R}_u^n\|_{L^2}^2+\Delta t\right).
\end{aligned}
$$

Testing (4.6) by $2\Delta te^n$ leads to

$$
\|e^n\|_{L^2}^2+\|e^n-\widetilde{e}^n\|_{L^2}^2-\|\widetilde{e}^n\|_{L^2}^2+\frac{\Delta t}{Re}(\|e^n\|_V^2+\|e^n-\widetilde{e}^n\|_V^2-\|\widetilde{e}^n\|_V^2)=0.
$$

Taking the sum of the above two estimates leads to

$$
\begin{aligned}
&\|e^n\|_{L^2}^2+\frac{1}{2}\|\widetilde{e}^n-e^{n-1}\|_{L^2}^2+\|e^n-\widetilde{e}^n\|_{L^2}^2+\frac{\Delta t}{Re}\left(\|e^n\|_V^2+\|e^n-\widetilde{e}^n\|_V^2\right)\\
\le& \|e^{n-1}\|_{L^2}^2+C\Delta t\left(\|e^{n-1}\|_{L^2}^2+\|\mathbf{R}_u^n\|_{L^2}^2+\Delta t\right).
\end{aligned}
$$

From the discrete Gronwall's inequality, we get

$$
\|e^n\|_{L^2}^2+\sum_{k=1}^n\left(\|\widetilde{e}^k-e^{k-1}\|_{L^2}^2+\|e^k-\widetilde{e}^k\|_{L^2}^2+\frac{\Delta t}{Re}\|e^k\|_V^2+\frac{\Delta t}{Re}\|e^k-\widetilde{e}^k\|_V^2\right)\le C\Delta t \tag{4.19}
$$

for each $1\le n\le N$. The estimate (4.19) implies that $\mathbf{U}^n$ and $\widetilde{\mathbf{U}}^n$ in uniformly bounded in $\mathbf{V}$, i.e., there exists some $C>0$ such that for any $1\le n\le N$,

$$
\|\mathbf{U}^n\|_V+\|\widetilde{\mathbf{U}}^n\|_V\le C. \tag{4.20}
$$

Testing (4.7) by $2\Delta te^n$, we get

$$
\|e^n\|_{L^2}^2+\|e^n-e^{n-1}\|_{L^2}^2-\|e^{n-1}\|_{L^2}^2+\frac{2\Delta t}{Re}\|e^n\|_V^2
$$

$$\leq C\Delta t\left(\|\eta^n\|_W+\|\mathbf{R}_u^n\|_{L^2}+\|e^{n-1}\|_{L^2}^{1/2}\|e^{n-1}\|_V^{1/2}+\|\widetilde{e}^n-e^n\|_{L^2}\right)\|e^n\|_V$$
$$\leq \frac{\Delta t}{2Re}\|e^n\|_V^2+\frac{\Delta t}{2Re}\|e^{n-1}\|_V^2+C\Delta t\left(\|\eta^n\|_W^2+\|\mathbf{R}_u^n\|_{L^2}^2+\|e^{n-1}\|_{L^2}^2+\|\widetilde{e}^n-e^n\|_{L^2}^2\right),$$

where we use

$$b(\mathbf{u}^{n-1},\widetilde{e}^n,e^n)=-b(\mathbf{u}^{n-1},e^n,\widetilde{e}^n-e^n)\leq C\|A\mathbf{u}^{n-1}\|_{L^2}\|\widetilde{e}^n-e^n\|_{L^2}\|e^n\|_V,$$
$$b(e^{n-1},\widetilde{\mathbf{U}}^n,e^n)\leq C\|\widetilde{\mathbf{U}}^n\|_V\|e^{n-1}\|_{L^2}^{1/2}\|e^{n-1}\|_V^{1/2}\|e^n\|_V.$$

From the discrete Gronwall's inequality, we get

$$\|e^n\|_{L^2}^2+\sum_{k=1}^n\left(\|e^k-e^{k-1}\|_{L^2}^2+\frac{\Delta t}{Re}\|e^k\|_V^2\right)\leq C(\Delta t)^2 \tag{4.21}$$

for each $1\leq n\leq N$. Combining (4.21) with (4.18), the following temporal error estimate for magnetic field can be derived

$$\max_{1\leq n\leq N}\|\eta^n\|_W\leq C_2\Delta t \tag{4.22}$$

for some $C_2>0$ independent of $\Delta t$ and $C_0$.

Next, we estimate $\mathbf{U}^n$, $P^n$ and $\widetilde{\mathbf{U}}^n$. It follows from (4.21) that

$$\|D_t\mathbf{U}^n\|_{L^2}\leq\left\|\frac{e^n-e^{n-1}}{\Delta t}\right\|_{L^2}+\left\|\frac{\mathbf{u}^n-\mathbf{u}^{n-1}}{\Delta t}\right\|_{L^2}\leq C. \tag{4.23}$$

Rewritten (3.4) as

$$-\frac{1}{Re}\Delta\mathbf{U}^n+\nabla P^n=\mathbf{F}^n-(\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n, \tag{4.24}$$

where

$$\mathbf{F}^n=-D_t\mathbf{U}^n-S\mathbf{B}^n\times\mathbf{curl}\ \mathbf{B}^n-S\widetilde{\mathbf{B}}^n\times\mathbf{curl}\ \mathbf{B}^n+\mathbf{f}^n.$$

Since $\mathbf{B}^n,\widetilde{\mathbf{B}}^n\in\mathbf{H}^2(\Omega)$, then we have $\|\mathbf{F}^n\|_{L^2}\leq C$. In addition, from (4.20),

$$\|(\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n\|_{L^2}\leq\|\mathbf{U}^{n-1}\|_{L^\infty}\|\widetilde{\mathbf{U}}^n\|_V$$
$$\leq C\|\mathbf{U}^{n-1}\|_V^{1/2}\|A\mathbf{U}^{n-1}\|_{L^2}^{1/2}\|\widetilde{\mathbf{U}}^n\|_V\leq C\|A\mathbf{U}^{n-1}\|_{L^2}^{1/2}.$$

By Lemma 2.1, we have

$$\|A\mathbf{U}^n\|_{L^2}^2+\|P^n\|_{H^1}^2\leq C\|\mathbf{F}^n\|_{L^2}^2+C\|(\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n\|_{L^2}^2$$
$$\leq C+C\|A\mathbf{U}^{n-1}\|_{L^2}\leq\frac{1}{2}\|A\mathbf{U}^{n-1}\|_{L^2}^2+C.$$

Thus, we obtain

$$\Delta t\sum_{n=1}^N\left(\|A\mathbf{U}^n\|_{L^2}^2+\|P^n\|_{H^1}^2\right)\leq C. \tag{4.25}$$

By a similar method, we can prove that

$$\Delta t \sum_{n=1}^{N} \left\| \frac{\widetilde{\mathbf{U}}^n - \mathbf{U}^{n-1}}{\Delta t} \right\|_{L^2} \leq C \quad \text{and} \quad \Delta t \sum_{n=1}^{N} \|\widetilde{\mathbf{U}}^n\|_{H^2}^2 \leq C. \tag{4.26}$$

An alternative form of (4.7) is

$$\frac{e^n - e^{n-1}}{\Delta t} - \frac{1}{Re}\Delta e^n + \nabla\theta^n + (e^{n-1}\cdot\nabla)\mathbf{u}^n + (\mathbf{U}^{n-1}\cdot\nabla)\widetilde{e}^n + S\eta^n \times \mathbf{curl}\ \mathbf{b}^n$$
$$+ S\mathbf{B}^n \times \mathbf{curl}\ \eta^n + S\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ \eta^n = \mathbf{R}_u^n, \quad \operatorname{div} e^n = 0. \tag{4.27}$$

Testing (4.27) by $2\Delta t A e^n$, we obtain

$$\begin{aligned} &\frac{2\Delta t}{Re}\|Ae^n\|_{L^2}^2 + \|e^n\|_V^2 - \|e^{n-1}\|_V^2 + \|e^n - e^{n-1}\|_V^2 \\ \leq& C\Delta t\left(\|e^{n-1}\|_V + \|A\mathbf{U}^{n-1}\|_{L^2}\|\widetilde{e}^n\|_V + \|\eta^n\|_W + \|\mathbf{R}_u^n\|_{L^2}\right)\|Ae^n\|_{L^2} \\ \leq& \frac{\Delta t}{Re}\|Ae^n\|_{L^2}^2 + C\Delta t\left(\|e^{n-1}\|_V^2 + \|A\mathbf{U}^{n-1}\|_{L^2}^2\|\widetilde{e}^n\|_V^2 + \|\eta^n\|_W^2 + \|\mathbf{R}_u^n\|_{L^2}^2\right), \end{aligned}$$

which implies that

$$\max_{1\leq n\leq N}\|e^n\|_V^2 + \Delta t\sum_{n=1}^{N}\|Ae^n\|_{L^2}^2 \leq (C_3\Delta t)^2 \tag{4.28}$$

for some $C_3 > 0$ independent of $\Delta t$ and $C_0$. Thus, from (3.18), we get

$$\Delta t\sum_{n=1}^{N}\|D_t(A\mathbf{U}^n)\|_{L^2}^2 \leq \Delta t\sum_{n=1}^{N}\|D_t(Ae^n)\|_{L^2}^2 + \Delta t\sum_{n=1}^{N}\|D_t(A\mathbf{u}^n)\|_{L^2}^2 \leq C_4$$

for some $C_4 > 0$ independent of $\Delta t$ and $C_0$. Thus, we complete the proof of Theorem 4.1 if we take $C_0 \geq \max\{C_2, C_3, C_4\}$. □

From the estimate (4.28), we have

$$\|Ae^n\|_{L^2} \leq C_3\sqrt{\Delta t} \leq C,$$

which means that

$$\|A\mathbf{U}^n\|_{L^2} \leq \|Ae^n\|_{L^2} + \|A\mathbf{u}^n\|_{L^2} \leq C$$

for each $1 \leq n \leq N$. Moreover, it is easy to show

$$\|(\mathbf{U}^{n-1}\cdot\nabla)\widetilde{\mathbf{U}}^n\|_{L^2} \leq C\|A\mathbf{U}^{n-1}\|_{L^2}\|\widetilde{\mathbf{U}}^n\|_V \leq C.$$

Thus, by the regularity result of the Stokes problem, from (4.24), we get

$$\|A\mathbf{U}^n\|_{L^2}^2 + \|P^n\|_{H^1}^2 \leq C, \quad \forall\ 1 \leq n \leq N. \tag{4.29}$$

## 4.2. Spatial error analysis

In this subsection, we will prove the spatial error estimates of $(e_h^n, \eta_h^n)$ for $1 \leq n \leq N$. Testing (3.1) by $\mathbf{w}_h \in \mathbf{W}_h$ and subtracting the resulting equation from (3.10) yields

$$\begin{aligned} &\frac{1}{Rm}(\mathbf{curl}\ \eta_h^n, \mathbf{curl}\ \mathbf{w}_h) + \frac{1}{Rm}(\operatorname{div}\ \eta_h^n, \operatorname{div}\ \mathbf{w}_h) \\ =& ((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \times (\mathbf{B}_h^n - \mathbf{B}^n), \mathbf{curl}\ \mathbf{w}_h) + ((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \times \mathbf{B}^n, \mathbf{curl}\ \mathbf{w}_h) \\ &+ (\mathbf{U}^{n-1} \times (\mathbf{B}_h^n - \mathbf{B}^n), \mathbf{curl}\ \mathbf{w}_h) + ((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \times \widetilde{\mathbf{B}}^n, \mathbf{curl}\ \mathbf{w}_h). \end{aligned} \tag{4.30}$$

Testing (3.2) by $\mathbf{v}_h \in \mathbf{V}_h$ and subtracting the resulting equation from (3.11), we get

$$\begin{aligned}
&\frac{1}{\Delta t}(\widetilde{e}_h^n - e_h^{n-1}, \mathbf{v}_h) + \frac{1}{Re}(\nabla \widetilde{e}_h^n, \nabla \mathbf{v}_h) \\
&= \left( \frac{(\widetilde{\mathbf{U}}^n - \mathbb{K}_h \widetilde{\mathbf{U}}^n) - (\mathbf{U}^{n-1} - \mathbf{R}_h \mathbf{U}^{n-1})}{\Delta t}, \mathbf{v}_h \right) + b(\mathbf{U}^{n-1}, \widetilde{\mathbf{U}}^n, \mathbf{v}_h) - b(\mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}_h^n, \mathbf{v}_h) \\
&\quad + S(\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n, \mathbf{v}_h) - S(\mathbf{B}_h^n \times \mathbf{curl}\ \mathbf{B}_h^n, \mathbf{v}_h) + S(\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), \mathbf{v}_h).
\end{aligned} \tag{4.31}$$

Testing (3.3) by $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$ and subtracting the resulting equation from (3.12), we get

$$\begin{aligned}
&(\frac{e_h^n - \widetilde{e}_h^n}{\Delta t}, \mathbf{v}_h) + \frac{1}{Re}(\nabla e_h^n, \mathbf{v}_h) - \frac{1}{Re}(\nabla \widetilde{e}_h^n, \mathbf{v}_h) - d(\mathbf{v}_h, \theta_h^n) \\
&= \left( \frac{(\mathbf{U}^n - \mathbf{R}_h \mathbf{U}^n) - (\widetilde{\mathbf{U}}^n - \mathbb{K}_h \widetilde{\mathbf{U}}^n)}{\Delta t}, \mathbf{v}_h \right) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,
\end{aligned} \tag{4.32}$$

and

$$d(e_h^n, q_h) = 0, \quad \forall\ q_h \in M_h. \tag{4.33}$$

Taking the sum of (4.31) and (4.32), we have

$$\begin{aligned}
&(D_t e_h^n, \mathbf{v}_h) + \frac{1}{Re}(\nabla e_h^n, \nabla \mathbf{v}_h) - d(\mathbf{v}_h, \theta_h^n) \\
&= (D_t(\mathbf{U}^n - \mathbf{R}_h \mathbf{U}^n), \mathbf{v}_h) + b(\mathbf{U}^{n-1}, \widetilde{\mathbf{U}}^n, \mathbf{v}_h) - b(\mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}_h^n, \mathbf{v}_h) \\
&\quad + S(\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n, \mathbf{v}_h) - S(\mathbf{B}_h^n \times \mathbf{curl}\ \mathbf{B}_h^n, \mathbf{v}_h) \\
&\quad + S(\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), \mathbf{v}_h), \qquad \forall\ \mathbf{v}_h \in \mathbf{V}_h,
\end{aligned} \tag{4.34}$$

and

$$d(e_h^n, q_h) = 0, \quad \forall\ q_h \in M_h. \tag{4.35}$$

The main result in this subsection is the following optimal estimates of $e_h^n$ and $\eta_h^n$ in $\mathbf{H}^1$-norm.

**Theorem 4.2.** *Assume that the solution to (2.3)–(2.7) satisfies regularities in Theorem 2.1 and (3.17)–(3.18). Then for the sufficiently small $\Delta t$ and $h$, there exists $C_5 > 0$ and $C_6 > 0$ such that*

$$\max_{0 \le k \le N} \|e_h^k\|_V \le C_5 h, \tag{4.36}$$

$$\max_{1 \le k \le N} \|\eta_h^k\|_W \le C_6 h. \tag{4.37}$$

**Proof.** We will use the method of mathematical induction to prove (4.36). From (3.9), the error estimates (4.36) is valid for $k = 0$. For $1 \le n \le N$, we assume that (4.36) is valid for $k = n - 1$. Then

$$\|e_h^{n-1}\|_V \le C_5 h. \tag{4.38}$$

To complete the mathematical induction, we need to prove that (4.36) is valid for $k = n$. Setting $\mathbf{w}_h = \eta_h^n$ in (4.30) yields

$$\begin{aligned}\frac{1}{Rm}\|\eta_h^n\|_W^2 =& ((\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\times(\mathbf{B}_h^n-\mathbf{B}^n), \mathbf{curl}\ \eta_h^n)+((\mathbf{U}_h^{n-1}-\mathbf{U}^{n-1})\times\mathbf{B}^n, \mathbf{curl}\ \eta_h^n)\\ &+ ((\mathbf{U}_h^{n-1} - \mathbf{U}^{n-1}) \times \widetilde{\mathbf{B}}^n, \mathbf{curl}\ \eta_h^n) + (\mathbf{U}^{n-1} \times (\mathbf{B}_h^n - \mathbf{B}^n), \mathbf{curl}\ \eta_h^n)\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{4.39}$$

The right-hand side of (4.39) can be estimated as follow. For $I_1$, we have

$$\begin{aligned}I_1 =&(e_h^{n-1} \times \eta_h^n, \mathbf{curl}\ \eta_h^n) + ((\mathbf{R}_h\mathbf{U}^{n-1} - \mathbf{U}^{n-1}) \times \eta_h^n, \mathbf{curl}\ \eta_h^n)\\ &+(e_h^{n-1}\times(\Pi_h\mathbf{B}^n-\mathbf{B}^n), \mathbf{curl}\ \eta_h^n)+((\mathbf{R}_h\mathbf{U}^{n-1}-\mathbf{U}^{n-1})\times(\Pi_h\mathbf{B}^n-\mathbf{B}^n), \mathbf{curl}\ \eta_h^n)\\ \le& C\kappa h^{-1}\|e_h^{n-1}\|_{L^2}\|\eta_h^n\|_W^2 + \kappa\|\mathbf{R}_h\mathbf{U}^{n-1} - \mathbf{U}^{n-1}\|_V\|\eta_h^n\|_W^2 + C\kappa\|e_h^{n-1}\|_V\|\Pi_h\mathbf{B}^n\\ &-\mathbf{B}^n\|_W\|\eta_h^n\|_W + C\|\mathbf{R}_h\mathbf{U}^{n-1}-\mathbf{U}^{n-1}\|_V\|\Pi_h\mathbf{B}^n-\mathbf{B}^n\|_W\|\eta_h^n\|_W\\ \le& C(C_5h + h)\|\eta_h^n\|_W^2 + C(h + \|e_h^{n-1}\|_{L^2})\|\eta_h^n\|_W.\end{aligned}$$

For $I_2$ and $I_3$ we have

$$\begin{aligned}I_2 &= (e_h^{n-1} \times \mathbf{B}^n, \mathbf{curl}\ \eta_h^n) + ((\mathbf{R}_h\mathbf{U}^{n-1} - \mathbf{U}^{n-1}) \times \mathbf{B}^n, \mathbf{curl}\ \eta_h^n)\\ &\le C(\|e_h^{n-1}\|_{L^2} + h)\|\eta_h^n\|_W,\end{aligned}$$

and

$$I_3 \le C(\|e_h^{n-1}\|_{L^2} + h)\|\eta_h^n\|_W.$$

Finally, for $I_4$, we have

$$\begin{aligned}I_4 &= (\mathbf{U}^{n-1} \times \eta_h^n, \mathbf{curl}\ \eta_h^n) + (\mathbf{U}^{n-1} \times (\Pi_h\mathbf{B}^n - \mathbf{B}^n), \mathbf{curl}\ \eta_h^n)\\ &\le \kappa\widetilde{M}\|\eta_h^n\|_W^2 + Ch\|\eta_h^n\|_W.\end{aligned}$$

Substituting above estimates into (4.39) yields

$$\frac{1}{Rm}\|\eta_h^n\|_W^2 \le (CC_5h + Ch + \kappa\widetilde{M})\|\eta_h^n\|_W^2 + C(h + \|e_h^{n-1}\|_{L^2})\|\eta_h^n\|_W.$$

For sufficiently small $h$ such that $Rm(CC_0h + Ch + \kappa\widetilde{M}) < 1$, we get

$$\|\eta_h^n\|_W \le C(\|e_h^{n-1}\|_{L^2} + h). \tag{4.40}$$

Setting $\mathbf{v}_h = 2\Delta t\widetilde{e}_h^n$ in (4.31), we have

$$\begin{aligned}&\|\widetilde{e}_h^n\|_{L^2}^2 - \|e_h^{n-1}\|_{L^2}^2 + \|\widetilde{e}_h^n - e_h^{n-1}\|_{L^2}^2 + \frac{2\Delta t}{Re}\|\widetilde{e}_h^n\|_V^2\\ =&2\Delta t\left(\frac{(\widetilde{\mathbf{U}}^n - \mathbb{K}_h\widetilde{\mathbf{U}}^n) - (\mathbf{U}^{n-1} - \mathbf{R}_h\mathbf{U}^{n-1})}{\Delta t}, \widetilde{e}_h^n\right)\\ &+ 2\Delta t\left(b(\mathbf{U}^{n-1}, \widetilde{\mathbf{U}}^n, \widetilde{e}_h^n) - b(\mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}_h^n, \widetilde{e}_h^n)\right)\\ &+ 2S\Delta t\,((\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n, \widetilde{e}_h^n) - (\mathbf{B}_h^n \times \mathbf{curl}\ \mathbf{B}_h^n, \widetilde{e}_h^n))\\ &+ 2S\Delta t(\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), \widetilde{e}_h^n) = I_5 + I_6 + I_7 + I_8.\end{aligned} \tag{4.41}$$

The right-hand side of (4.41) can be estimated as follows. For $I_5$, we have

$$\begin{aligned}
I_5 =& 2\Delta t\left(\frac{(\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n)-(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1})}{\Delta t},\widetilde{e}_h^n-e_h^n\right)\\
&+2\Delta t\left(\frac{(\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n)-(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1})}{\Delta t},e_h^n\right)\\
\leq& 2\Delta t\left(\frac{(\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n)-(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1})}{\Delta t},e_h^n\right)\\
&+2\left(\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_{L^2}+\|\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1}\|_{L^2}\right)\|\widetilde{e}_h^n-e_h^n\|_{L^2}.
\end{aligned}$$

For $I_6$, we have

$$\begin{aligned}
I_6 =& 2\Delta t b(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1},\widetilde{\mathbf{U}}^n,\widetilde{e}_h^n)+2\Delta t b(e_h^{n-1},\widetilde{\mathbf{U}}^n,\widetilde{e}_h^n)\\
&+2\Delta t b(e_h^{n-1},\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n,\widetilde{e}_h^n)+2\Delta t b(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1},\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n,\widetilde{e}_h^n)\\
&+2\Delta t b(\mathbf{U}^{n-1},\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n,\widetilde{e}_h^n)\\
\leq& C\Delta t\Big(\|\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1}\|_{L^2}\|\widetilde{\mathbf{U}}^n\|_{H^2}+\|e_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{U}}^n\|_{H^2}+\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_V\Big)\|\widetilde{e}_h^n\|_V\\
&+C\Delta t\Big(h^{-1}\|e_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_V+\|\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1}\|_V\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_V\Big)\|\widetilde{e}_h^n\|_V\\
\leq& \frac{\Delta t}{4Re}\|\widetilde{e}_h^n\|_V^2+C\Delta t\left(\|e_h^{n-1}\|_{L^2}^2+h^2\right)\|\widetilde{\mathbf{U}}^n\|_{H^2}^2.
\end{aligned}$$

For $I_7$ and $I_8$, we can prove that

$$\begin{aligned}
I_7 =& 2S\Delta t(\mathbf{B}^n\times\mathbf{curl}\ (\mathbf{B}^n-\mathbf{B}_h^n),\widetilde{e}_h^n)+2S\Delta t((\mathbf{B}^n-\Pi_h\mathbf{B}^n)\times\mathbf{curl}\ (\mathbf{B}^n-\mathbf{B}_h^n),\widetilde{e}_h^n)\\
&+2S\Delta t(\eta_h^n\times\mathbf{curl}\ (\mathbf{B}^n-\mathbf{B}_h^n),\widetilde{e}_h^n)+2S\Delta t((\mathbf{B}^n-\Pi_h\mathbf{B}^n)\times\mathbf{curl}\ \mathbf{B}^n,\widetilde{e}_h^n)\\
&+2S\Delta t(\eta_h^n\times\mathbf{curl}\ \mathbf{B}^n,\widetilde{e}_h^n)\\
\leq& \frac{\Delta t}{4Re}\|\widetilde{e}_h^n\|_V^2+C\Delta t(\|\eta_h^n\|_W^2+h^2),
\end{aligned}$$

and

$$I_8\leq C\Delta t\|\mathbf{B}^n-\mathbf{B}_h^n\|_W\|\widetilde{e}_h^n\|_V\leq\frac{\Delta t}{4Re}\|\widetilde{e}_h^n\|_V^2+C\Delta t\left(\|\eta_h^n\|_W^2+h^2\right).$$

Substituting $I_5\cdots I_8$ into (4.41) we have

$$\begin{aligned}
&\|\widetilde{e}_h^n\|_{L^2}^2-\|e_h^{n-1}\|_{L^2}^2+\|\widetilde{e}_h^n-e_h^{n-1}\|_{L^2}^2+\frac{5\Delta t}{4Re}\|\widetilde{e}_h^n\|_V^2\\
\leq& 2\Delta t\left(\frac{(\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n)-(\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1})}{\Delta t},e_h^n\right)\\
&+2\left(\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_{L^2}+\|\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1}\|_{L^2}\right)\|\widetilde{e}_h^n-e_h^n\|_{L^2}\\
&+C\Delta t\left(\|e_h^{n-1}\|_{L^2}^2+h^2\right)\|\widetilde{\mathbf{U}}^n\|_{H^2}^2+C\Delta t\left(\|\eta_h^n\|_W^2+h^2\right).
\end{aligned}\tag{4.42}$$

Taking $(\mathbf{v}_h,q_h)=2\Delta t(e_h^n,\theta_h^n)$ in (4.32)–(4.33), we have

$$\|e_h^n\|_{L^2}^2-\|\widetilde{e}_h^n\|_{L^2}^2+\|e_h^n-\widetilde{e}_h^n\|_{L^2}^2+\frac{\Delta t}{Re}\left(\|e_h^n\|_V^2-\|\widetilde{e}_h^n\|_V^2+\|e_h^n-\widetilde{e}_h^n\|_V^2\right)$$

$$=2\Delta t\left(\frac{(\mathbf{U}^n-\mathbf{R}_h\mathbf{U}^n)-(\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n)}{\Delta t},e_h^n\right). \tag{4.43}$$

Taking the sum of (4.42) and (4.43), we have

$$\begin{aligned}
&\|e_h^n\|_{L^2}^2-\|e_h^{n-1}\|_{L^2}^2+\|e_h^n-\widetilde{e}_h^n\|_{L^2}^2+\|\widetilde{e}_h^n-e_h^{n-1}\|_{L^2}^2\\
&+\frac{\Delta t}{Re}\left(\|e_h^n\|_V^2+\frac{1}{4}\|\widetilde{e}_h^n\|_V^2+\|e_h^n-\widetilde{e}_h^n\|_V^2\right)\\
\le&2\Delta t\left(D_t(\mathbf{U}^n-\mathbf{R}_h\mathbf{U}^n),e_h^n\right)+C\Delta t\left(\|e_h^{n-1}\|_{L^2}^2+h^2\right)\|\widetilde{\mathbf{U}}^n\|_{H^2}^2+C\Delta t\left(\|\eta_h^n\|_W^2+h^2\right)\\
&+2\left(\|\widetilde{\mathbf{U}}^n-\mathbb{K}_h\widetilde{\mathbf{U}}^n\|_{L^2}+\|\mathbf{U}^{n-1}-\mathbf{R}_h\mathbf{U}^{n-1}\|_{L^2}\right)\|\widetilde{e}_h^n-e_h^n\|_{L^2}\\
\le&\frac{1}{2}\|\widetilde{e}_h^n-e_h^n\|_{L^2}^2+C\Delta t\left(\|e_h^{n-1}\|_{L^2}^2+\|e_h^n\|_{L^2}^2\right)+Ch^4\left(\|\widetilde{\mathbf{U}}^n\|_{H^2}^2+\|A\mathbf{U}^{n-1}\|_{L^2}^2\right)\\
&+C\Delta th^2\left(\|\widetilde{\mathbf{U}}^n\|_{H^2}^2+\|A\mathbf{U}^{n-1}\|_{L^2}^2+\|D_tA\mathbf{U}^n\|_{L^2}^2\right),
\end{aligned}$$

where we use (4.40). Under the condition $\Delta t=\mathcal{O}(h)$, it follows from the discrete Gronwall's inequality that

$$\begin{aligned}
&\|e_h^n\|_{L^2}^2+\sum_{k=1}^n\left(\|e_h^k-\widetilde{e}_h^k\|_{L^2}^2+\|\widetilde{e}_h^k-e_h^{k-1}\|_{L^2}^2\right)\\
&+\frac{\Delta t}{Re}\sum_{k=1}^n\left(\|e_h^k\|_V^2+\frac{1}{4}\|\widetilde{e}_h^k\|_V^2+\|e_h^k-\widetilde{e}_h^k\|_V^2\right)\le Ch^2,
\end{aligned} \tag{4.44}$$

which implies that

$$\sum_{k=1}^n\|e_h^k-e_h^{k-1}\|_{L^2}^2\le 2\sum_{k=1}^n\left(\|e_h^k-\widetilde{e}_h^k\|_{L^2}^2+\|\widetilde{e}_h^k-e_h^{k-1}\|_{L^2}^2\right)\le Ch^2, \tag{4.45}$$

$$\|\eta_h^n\|_W\le C_6h \tag{4.46}$$

for some $C_6>0$. From (4.44) and (4.46), we have

$$\|\eta_h^n\|_{W^{1,3}}\le Ch^{-1/2}\|\eta_h^n\|_W\le Ch^{1/2},\quad \|\mathbf{B}^n-\mathbf{B}_h^n\|_{W^{1,3}}\le C, \tag{4.47}$$

and

$$\|\widetilde{e}_h^n\|_{W^{1,3}}\le Ch^{-1/2}\|\widetilde{e}_h^n\|_V\le C,\quad \|\widetilde{\mathbf{U}}^n-\widetilde{\mathbf{U}}_h^n\|_{W^{1,3}}\le C \tag{4.48}$$

under the condition $\Delta t=\mathcal{O}(h)$.

Taking$(\mathbf{v}_h,q_h)=2\Delta t(D_te_h^n,\theta_h^n)$ in (4.34) and (4.35) yields

$$\begin{aligned}
&2\Delta t\|D_te_h^n\|_{L^2}^2+\frac{1}{Re}\left(\|e_h^n\|_V^2-\|e_h^{n-1}\|_V^2+\|e_h^n-e_h^{n-1}\|_V^2\right)\\
\le&2\Delta t(D_t(\mathbf{U}^n-\mathbf{R}_h\mathbf{U}^n),D_te_h^n)+2\Delta tb(\mathbf{U}^{n-1},\widetilde{\mathbf{U}}^n,D_te_h^n)\\
&-2\Delta tb(\mathbf{U}_h^{n-1},\widetilde{\mathbf{U}}_h^n,D_te_h^n)+2\Delta tS(\mathbf{B}^n\times\mathbf{curl}\ \mathbf{B}^n,D_te_h^n)\\
&-2\Delta tS(\mathbf{B}_h^n\times\mathbf{curl}\ \mathbf{B}_h^n,D_te_h^n)+2\Delta tS(\widetilde{\mathbf{B}}^n\times\mathbf{curl}\ (\mathbf{B}^n-\mathbf{B}_h^n),D_te_h^n).
\end{aligned}$$

Then

$$2\Delta t(D_t(\mathbf{U}^n-\mathbf{R}_h\mathbf{U}^n),D_te_h^n)\le C\Delta th^2\|D_te_h^n\|_{L^2}\|D_t(A\mathbf{U}^n)\|_{L^2}$$

$$\leq \frac{\Delta t}{4}\|D_t e_h^n\|_{L^2}^2 + C\Delta t h^4 \|D_t(A\mathbf{U}^n)\|_{L^2}^2,$$

and

$$\begin{aligned}
&2\Delta t b(\mathbf{U}^{n-1}, \widetilde{\mathbf{U}}^n, D_t e_h^n) - 2\Delta t b(\mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}_h^n, D_t e_h^n)\\
=&2\Delta t b(\mathbf{U}^{n-1} - \mathbf{U}_h^{n-1}, \widetilde{\mathbf{U}}^n, D_t e_h^n) + 2\Delta t b(e_h^{n-1}, \widetilde{\mathbf{U}}^n - \widetilde{\mathbf{U}}_h^n, D_t e_h^n)\\
&+ 2\Delta t b(\mathbf{R}_h \mathbf{U}^{n-1}, \widetilde{\mathbf{U}}^n - \widetilde{\mathbf{U}}_h^n, D_t e_h^n)\\
\leq& C\Delta t \left(\|\mathbf{U}^{n-1} - \mathbf{R}_h \mathbf{U}^{n-1}\|_V + \|e_h^{n-1}\|_V\right) \|\widetilde{\mathbf{U}}^n\|_{H^2} \|D_t e_h^n\|_{L^2}\\
&+ C\Delta t \|e_h^{n-1}\|_V \|\widetilde{\mathbf{U}}^n - \widetilde{\mathbf{U}}_h^n\|_{W^{1,3}} \|D_t e_h^n\|_{L^2}\\
&+ C\Delta t \|\mathbf{R}_h \mathbf{U}^{n-1}\|_{L^\infty} \left(\|\widetilde{\mathbf{U}}^n - \mathbb{K}_h \widetilde{\mathbf{U}}^n\|_V + \|\widetilde{e}_h^n\|_V\right) \|D_t e_h^n\|_{L^2}\\
\leq& \frac{\Delta t}{4}\|D_t e_h^n\|_{L^2}^2 + C\Delta t \left(\|e_h^{n-1}\|_V^2 + \|\widetilde{e}_h^n\|_V^2\right) + C\Delta t h^2 \left(\|A\mathbf{U}^{n-1}\|_{L^2}^2 + \|\widetilde{\mathbf{U}}^n\|_{H^2}^2\right)
\end{aligned}$$

and

$$\begin{aligned}
&2\Delta t S(\mathbf{B}^n \times \mathbf{curl}\ \mathbf{B}^n, D_t e_h^n) - 2\Delta t S(\mathbf{B}_h^n \times \mathbf{curl}\ \mathbf{B}_h^n, D_t e_h^n)\\
=&2S\Delta t(\mathbf{B}^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), D_t e_h^n) + 2S\Delta t((\mathbf{B}^n - \Pi_h \mathbf{B}^n)\\
&\times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), D_t e_h^n) + 2S\Delta t(\eta_h^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), D_t e_h^n)\\
&+ 2S\Delta t((\mathbf{B}^n - \Pi_h \mathbf{B}^n) \times \mathbf{curl}\ \mathbf{B}^n, D_t e_h^n) + 2S\Delta t(\eta_h^n \times \mathbf{curl}\ \mathbf{B}^n, D_t e_h^n)\\
\leq& C\Delta t \left(\|\mathbf{B}^n - \mathbf{B}_h^n\|_W + \|\mathbf{B}^n - \Pi_h \mathbf{B}^n\|_W + \|\eta_h^n\|_W\right) \|D_t e_h^n\|_{L^2}\\
\leq& \frac{\Delta t}{4}\|D_t e_h^n\|_{L^2}^2 + C\Delta t h^2,
\end{aligned}$$

and

$$2\Delta t S(\widetilde{\mathbf{B}}^n \times \mathbf{curl}\ (\mathbf{B}^n - \mathbf{B}_h^n), D_t e_h^n) \leq \frac{\Delta t}{4}\|D_t e_h^n\|_{L^2}^2 + C\Delta t h^2.$$

Thus, we get

$$\begin{aligned}
&\Delta t \|D_t e_h^n\|_{L^2}^2 + \frac{1}{Re}\left(\|e_h^n\|_V^2 - \|e_h^{n-1}\|_V^2 + \|e_h^n - e_h^{n-1}\|_V^2\right)\\
\leq& C\Delta t \left(\|e_h^{n-1}\|_V^2 + \|\widetilde{e}_h^n\|_V^2\right) + C\Delta t h^2 \left(\|A\mathbf{U}^{n-1}\|_{L^2}^2 + \|\widetilde{\mathbf{U}}^n\|_{H^2}^2 + \|D_t(A\mathbf{U}^n)\|_{L^2}^2 + 1\right).
\end{aligned}$$

By the discrete Gronwall's inequality, we obtain the following optimal error estimate in $H^1$-norm:

$$\max_{1\leq n\leq N}\left(\|e_h^n\|_V^2 + \Delta t \sum_{k=1}^{n} \|D_t e_h^n\|_{L^2}^2\right) \leq (C_5 h)^2$$

for some $C_5 > 0$. Thus, we complete the proof of Theorem 4.2. □

## 4.3. Proof of Theorem 3.1

The optimal $\mathbf{H}^1$ error estimate (3.19) follows from the error splitting (3.13), (3.15) and the temporal error estimate (4.15), and the spatial error estimate (4.36)-(4.37), where we use (3.5)-(3.6) and the regularity result (4.29).

# 5. Numerical results

In this section, we will give the numerical results to check the optimal $\mathbf{H}^1$ error estimate derived in Theorem 3.1. For the sake of simplicity, all numerical experiments are tested for a two-dimensional hybrid MHD system in the unit square domain $\Omega = (0,1) \times (0,1)$:

$$\mathbf{u}_t - \frac{1}{Re}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\mathbf{b}\times \mathrm{curl}\ \mathbf{b} = \mathbf{f}, \tag{5.1}$$

$$\mathrm{div}\ \mathbf{u} = 0, \tag{5.2}$$

$$\frac{1}{Rm}\mathbf{curl}\ (\mathrm{curl}\ \mathbf{b}) - \mathbf{curl}\ (\mathbf{u}\times\mathbf{b}) = \mathbf{g}, \tag{5.3}$$

$$\mathrm{div}\ \mathbf{b} = 0. \tag{5.4}$$

Here $\mathbf{u} = (u_1, u_2)$ and $\mathbf{b} = (b_1, b_2)$. In the two-dimensional case, the operator curl applied to a vector $\mathbf{v} = (v_1, v_2)$ is defined by curl $\mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$, while the operator **curl** applied to a scalar function $r$ is defined by **curl** $r = (\frac{\partial r}{\partial y}, -\frac{\partial r}{\partial x})$. In addition, the cross product of two vectors $\mathbf{u}$ and $\mathbf{b}$ is given by $\mathbf{u}\times\mathbf{b} = u_1b_2 - u_2b_1$. For a vector function $\mathbf{b}$ and a scalar function $r$, the cross product is given by $\mathbf{b}\times r = (rb_2, -rb_1)$.

In the above system, we take the appropriate $\mathbf{f}$ and $\mathbf{g}$ such that the analytical solutions $(\mathbf{b}, \mathbf{u}, p)$ are of the following forms:

$$\mathbf{b} = \left(x^2(x-1)^2y(y-1)(2y-1), \quad -y^2(y-1)^2x(x-1)(2x-1)\right)e^{-t},$$
$$\mathbf{u} = \left(y^2, x^2\right)e^{-t}, \quad p = (2x-1)(2y-1)e^{-t}.$$

The initial and boundary conditions are determined by the analytical solutions. In the numerical experiments, the coupling number $S = 1$ and the final time $T^\star = 1.0$.

To verify the optimal $\mathbf{H}^1$ error estimate derived in Theorem 3.1 under the condition $\Delta t = \mathcal{O}(h)$, we take gradually decreasing meshes $h = 1/2^i$, $i = 2, \cdots, 6$ with the time step size $\Delta t = h$. In this case, the optimal first-order error estimate holds from (3.19). The numerical results are displayed in Table 1 and Table 2 with different Reynolds numbers and magnetic Reynolds numbers $Re = Rm = 1$ and $Re = Rm = 10$, respectively. From these tables, we can see that the predicted optimal convergence order $\mathcal{O}(h)$ is obtained for $\mathbf{H}^1$ errors of the magnetic field and the velocity, which is in good agreement with our theoretical analysis.

**Table 1.** $\mathbf{H}^1$ numerical errors and convergence orders of $(\mathbf{b}, \mathbf{u})$ with $Re = Rm = 1$

| $h$ | $\Vert\mathbf{b}^N - \mathbf{B}_h^N\Vert_W$ | order | $\Vert\mathbf{u}^N - \mathbf{U}_h^N\Vert_V$ | order |
|---|---|---|---|---|
| 1/4 | 6.54585e-003 | | 7.64152e-002 | |
| 1/8 | 3.55364e-003 | 0.88 | 3.82926e-002 | 1.00 |
| 1/16 | 1.81775e-003 | 0.97 | 1.91165e-002 | 1.00 |
| 1/32 | 9.14238e-004 | 0.99 | 9.53074e-003 | 1.00 |
| 1/64 | 4.57799e-004 | 1.00 | 4.75054e-003 | 1.00 |
| 1/128 | 2.28985e-004 | 1.00 | 2.36890e-003 | 1.00 |

**Table 2.** $\mathbf{H}^1$ numerical errors and convergence orders of $(\mathbf{b}, \mathbf{u})$ with $Re = Rm = 10$

| $h$ | $\Vert\mathbf{b}^N - \mathbf{B}_h^N\Vert_W$ | order | $\Vert\mathbf{u}^N - \mathbf{U}_h^N\Vert_V$ | order |
|---|---|---|---|---|
| 1/4 | 6.55415e-003 | | 1.41420e-001 | |
| 1/8 | 3.55536e-003 | 0.88 | 6.78153e-002 | 1.06 |
| 1/16 | 1.81850e-003 | 0.97 | 3.14903e-002 | 1.10 |
| 1/32 | 9.14799e-004 | 0.99 | 1.46266e-002 | 1.10 |
| 1/64 | 4.58160e-004 | 1.00 | 6.83243e-003 | 1.10 |
| 1/128 | 2.29192e-004 | 1.00 | 3.21286e-003 | 1.09 |

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