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GROUND STATE SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS WITH ASYMPTOTICALLY PERIODIC POTENTIALS*

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Abstract In this paper, by using a concentration-compactness argument, we study the existence of ground state solutions for nonlinear Schrödinger equations with asymptotically periodic potentials. In particular, when the coefficients are "competing", some sufficient conditions are given to guarantee the existence of ground state solutions, which improve and generalize some previous results in the literature.

Keywords Nonlinear Schrödinger equation, asymptotically periodic, ground state solution

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1. Introduction

In this paper, we are interested in the following problem:

$$\begin{cases} -\Delta u + V(x)u = K(x)|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \\ u(x) > 0, \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$ is a integer, $p \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$. $V(x)$ and $K(x)$ are continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$ to be specified later.

During the past years, starting from the pioneering papers [4], there has been a considerable interest in problems like (1.1) due essentially to two reasons: on one hand such problems arise naturally in various branches of Mathematical Physics, and on the other hand they present specific mathematical difficulties that make them

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challenging to the researchers. Indeed, in spite of its variational nature, a lack of compactness, due to the invariance of $\mathbb{R}^N$ under the action of the noncompact group of translations, prevents a straight application of the usual variational methods.

The earliest results were obtained in radially symmetric situations, taking advantage of the compact embedding in $L^p(\mathbb{R}^N), p \in (2, 2N/(N-2))$ of the subspace of $H^1(\mathbb{R}^N)$ consisting of radial functions. When the coefficients do not enjoy symmetry, many different devices have been exploited to obtain the desired solutions. We refer readers to some survey papers [5] and references therein.

When $V(x) \to V_\infty$ from below and $K(x) \to K_\infty$ from above, as $|x| \to +\infty$, the existence of a positive ground state solution to (1.1) can be shown by using a minimization method together concentration-compactness type arguments [8, 13]. Conversely, if $V(x) \to V_\infty$ from above and $K(x) \to K_\infty$ from below, (1.1) may not have a least energy solution. This is the case, for instance, when $V(x) = V_\infty$, $K(x) \leq K_\infty$, and $K(x) \neq K_\infty$ on a positive measure set. Nevertheless, it is well-known that also these situations can be successfully handled (see [2, 3]). We point out that in the above results, the coefficients $V(x)$ and $K(x)$ act on (1.1) in a "cooperative" way. Very recently, G. Cerami and D. Passaseo [6, 7] study the existence of positive solutions of (1.1) to describe some phenomena that can occur when the coefficients are "competing", that is the case when $V(x) \to V_\infty$ from above and $K(x) \to K_\infty$ from above. We remark that this type results have been generalized to Kirchhoff problem by Hu and Lu [10], to Choquard equation by Wang, Qu and Xiao [19]. For more related results, we refer the readers to [9, 14] and references therein.

Many authors have obtained the existence of ground state solutions and nontrivial solutions for periodic nonlinear Schrödinger equations

$$-\Delta u + V(x)u = f(x, u), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

under variant conditions on the nonlinearities, see [20] and references therein. However, to the best of our knowledge, there are only a few works concerning the existence of solutions of nonlinear Schrödingger equations with asymptotically periodic potentials and the nonlinearity $f(x, u)$ having subcritical or critical growth and being asymptotically periodic at infinity [1, 12, 15]. Some results are also been obtained concerning the existence of positive ground state solutions for asymptotically periodic quasilinear Schrödinger equation [21, 22]. We emphasize that in all these previous works, among other assumptions, the authors always assume that $V(x) \leq V_0(x)$ with $V_0(x) \in C(\mathbb{R}^N, \mathbb{R})$ being 1-periodic in $x_i, i = 1, 2, \cdots, N$, see [11, 15, 18, 21, 22] and references therein.

In the present paper, we assume that the potentials $V(x)$ and $K(x)$ are asymptotically periodic functions. The goal of this paper is studying the existence of ground state solutions for (1.1) when the coefficients are "competing" by using a concentration-compactness argument [13]. The remainder of this paper is organized as follows. In Section 2, we state the main results in our paper. In Section 3, we formulate the variational setting and introduce some preliminaries. We complete the proof of main result in Sections 4. Finally, we present a global compactness result in the appendix.

2. Main Result

To state our main result, we make the following assumptions:

(**H1**) There exist $V_0(x), K_0(x)$ which are continuous and $\mathbb{Z}^N$-periodic in x, and satisfy

$$\inf_{\mathbb{R}^N} V_0(x) > 0, \qquad \inf_{\mathbb{R}^N} K_0(x) > 0$$

such that

$$\lim_{|x|\to\infty} (V(x) - V_0(x)) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} (K(x) - K_0(x)) = 0.$$

(**H2**) $V_0(x) \geq V(x) \geq 0$, and there exists $\xi \in C(\mathbb{R}^N, \mathbb{R})$ such that

$$K_0(x) - K(x) \leq (V_0(x) - V(x))\xi(x), \quad \text{for all} \;\; x \in \mathbb{R}^N,$$

and for some open subset U of $S^{N-1} = \{\sigma \in \mathbb{R}^N \mid |\sigma| = 1\}$, there holds

$$\lim_{r\to+\infty} \xi(r\sigma) = 0 \quad \text{for } \sigma \in U, \qquad \lim_{|x|\to\infty} \xi(x)e^{-\alpha|x|} = 0 \quad \text{for all } \alpha > 0.$$

(**H3**) $K(x) \geq K_0(x), V - V_0 \in L^{N/2}(\mathbb{R}^N)$, and there exist $\eta \in C(\mathbb{R}^N, \mathbb{R})$ and $R_0 > 0$ such that

$$0 \leq V(x) - V_0(x) \leq (K(x) - K_0(x))\eta(x), \quad \text{for all } |x| \geq R_0, \; x \in \mathbb{R}^N,$$

and for some $\tau \in (0, 1)$, there holds

$$\lim_{|x|\to\infty} \eta(x)|x|^{\frac{(p-2)(N-1)}{2}} e^{\frac{(p-2)\tau}{1-\tau}\sqrt{V_+}|x|} = 0, \qquad \lim_{|x|\to\infty} (K(x)-K_0(x))e^{\frac{2\tau}{1+\tau}\sqrt{V_-}|x|} = +\infty,$$

where $V_+ = \max_{\mathbb{R}^N} V_0(x)$ and $V_- = \min_{\mathbb{R}^N} V_0(x)$.

Our main result is the following theorem:

Theorem 2.1. *Assume (H1) and either (H2) or (H3) hold. Then (1.1) admits a positive ground state solution.*

Remark 2.2. It is easy to show that if $V(x) \leq V_0(x)$ and $K(x) \geq K_0(x)$ for all $x \in \mathbb{R}^N$, then (1.1) admits a ground state solution. It is worth observing that most results in the literature concern these cases, in which the coefficients $V(x)$ and $K(x)$ act on (1.1) in a ”cooperative” way. On the other hand, it is also easy to see that if $V(x) \geq V_0(x)$, $K(x) \leq K_0(x)$ for all $x \in \mathbb{R}^N$ and either $V(x) \not\equiv V_0(x)$ or $K(x) \not\equiv K_0(x)$, then (1.1) have no ground state solution. In the present paper, we describe some phenomena that can occur when the coefficients are ”competing”. We emphasize that in (H2) it is allowed that $V(x) \leq V_0(x)$ and $K(x) \leq K_0(x)$ on the whole space $\mathbb{R}^N$. Also, in (H3) it is allowed that $V(x) \geq V_0(x)$ and $K(x) \geq K_0(x)$ on the whole space $\mathbb{R}^N$.

Remark 2.3. Clearly, (H2) is valid if $V_0(x) \geq V(x) \geq 0$, and there exists $\xi \in C(\mathbb{R}^N, \mathbb{R})$ satisfying $\lim_{|x|\to\infty} \xi(x) = 0$ such that

$$0 \leq K_0(x) - K(x) \leq (V_0(x) - V(x))\xi(x), \quad \text{for all } |x| \geq R_0.$$

This shows that when $V(x) \to V_\infty > 0$ from below and $K(x) \to K_\infty > 0$ from below, as $|x| \to +\infty$, the problem (1.1) may have a positive ground state solution if the decay rate of $K_\infty - K(x)$ is faster than that of $V_\infty - V(x)$.

Remark 2.4. Theorem 2.1 is even new in the case where $V(x) \to V_\infty > 0$ and $K(x) \to K_\infty > 0$ as $|x| \to \infty$. In this case, (H3) reads as follows

(H3′) $K(x) \geq K_\infty, V - V_\infty \in L^{N/2}(\mathbb{R}^N)$, and there exist $\eta \in C(\mathbb{R}^N, \mathbb{R})$ and $R_0 > 0$ such that

$$0 \leq V(x) - V_\infty \leq (K(x) - K_\infty)\eta(x), \quad \text{for all } |x| \geq R_0,\ x \in \mathbb{R}^N,$$

and for some $\tau \in (0,1)$ there holds

$$\lim_{|x|\to\infty} \eta(x)|x|^{\frac{(p-2)(N-1)}{2}} e^{\frac{(p-2)\tau}{1-\tau}\sqrt{V_\infty}|x|} = 0, \quad \lim_{|x|\to\infty}(K(x) - K_\infty)e^{\frac{2\tau}{1+\tau}\sqrt{V_\infty}|x|} = +\infty.$$

G. Cerami and A.Pomponio [7] prove that (1.1) has a ground state solution under the following conditions:
(C1) $\lim_{|x|\to\infty} V(x) = V_\infty > 0, \lim_{|x|\to\infty} K(x) = K_\infty > 0$,
(C2) $V(x) \geq V_\infty$ and $K(x) \geq K_\infty$ for all $x \in \mathbb{R}^N$,
(C3) For some $\tau \in (0,1)$ there holds

$$\lim_{|x|\to\infty}(V(x) - V_\infty)|x|^{\frac{p(N-1)}{2}} e^{\frac{p\tau}{1-\tau}\sqrt{V_\infty}|x|} = 0, \quad \lim_{|x|\to\infty}(K(x) - K_\infty)e^{\frac{2\tau}{1+\tau}\sqrt{V_\infty}|x|} = +\infty.$$

This result shows that when $V(x) \to V_\infty > 0$ from above and $K(x) \to K_\infty > 0$ from above, as $|x| \to +\infty$, the problem (1.1) may have a positive ground state solution if the decay rate of $V(x) - V_\infty$ is faster than that of $K(x) - K_\infty$.

The following corollary is a sharp improvement of the above result.

Corollary 2.5. *The problem (1.1) admits a positive ground state solution if (C1), (C2) and the following condition hold*
(C3′) For some $\tau \in (0,1)$ *there holds*

$$\limsup_{|x|\to\infty}(V(x) - V_\infty)|x|^{(\frac{p}{2}-1)(N-1)} e^{(\frac{p\tau}{1-\tau} - \frac{4\tau^2}{1-\tau^2})\sqrt{V_\infty}|x|} < +\infty,$$
$$\lim_{|x|\to\infty}(K(x) - K_\infty)e^{\frac{2\tau}{1+\tau}\sqrt{V_\infty}|x|} = +\infty.$$

Proof. Clearly, it is suffices to check (H3′). Since $K(x) - K_\infty > 0$ for large $|x|$, there is $\eta \in C(\mathbb{R}^N, \mathbb{R})$ and $R_0 > 0$ such that

$$V(x) - V_\infty = (K(x) - K_\infty)\eta(x), \qquad \text{for all } |x| \geq R_0.$$

Notice that for $|x| \geq R_0$, there holds

$$\begin{aligned}&(V(x) - V_\infty)|x|^{\frac{p(N-1)}{2}} e^{\frac{p\tau}{1-\tau}\sqrt{V_\infty}|x|}\\ &= (K(x) - K_\infty)e^{\frac{2\tau}{1+\tau}\sqrt{V_\infty}|x|}\eta(x)|x|^{\frac{(p-2)(N-1)}{2}} e^{\frac{(p-2)\tau}{1-\tau}\sqrt{V_\infty}|x|} \cdot |x|^{N-1}e^{\frac{4\tau^2}{1-\tau^2}\sqrt{V_\infty}|x|}.\end{aligned}$$

Then (H3′) follows from (C3′). Moreover, since $p > 2$, it follows that

$$\frac{p\tau}{1-\tau} - \frac{4\tau^2}{1-\tau^2} > 0 \qquad \text{for all } \tau \in (0,1),$$

and hence $V - V_\infty \in L^{N/2}(\mathbb{R}^N)$. Thus (H1), (H2) and (H3′) hold and the proof is complete. □

3. Preliminaries

In this section we present some preliminaries for the proofs of our main theorem. Throughout the paper, we assume that the potential V satisfies (H1) and $V(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then the norm and inner product in $H^1(\mathbb{R}^N)$ may be defined by

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx\right)^{1/2}, \qquad u \in H^1(\mathbb{R}^N)$$

and

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + V(x)uv dx, \qquad u, v \in H^1(\mathbb{R}^N),$$

respectively. We denote by $B_R(y)$ the open ball in $\mathbb{R}^N$ of radius $R > 0$ and centre at y, B_R denotes the ball of radius R centered at 0. For $1 \leq p \leq \infty$, $\|\cdot\|_p$ denotes the norm in $L_p(\mathbb{R}^N)$. Finally, we use C and c denote positive constants which may vary from line to line.

When $V(x) = V_0(x)$ and $K(x) = K_0(x)$ for all $x \in \mathbb{R}^N$, (1.1) reduces to a periodic nonlinear Schrödinger equation:

$$-\Delta u + V_0(x)u = K_0(x)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N). \tag{3.1}$$

The associated functional is given by

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V_0(x)u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} K_0(x)|u|^p dx,$$

and its Nehari manifold $\mathcal{N}_0 := \left\{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle I_0'(u), u\rangle = 0\right\}$. It is well-known that

$$\varpi_0 := \inf_{u\in\mathcal{N}_0} I_0(u) > 0. \tag{3.2}$$

For the problem (1.1), the associated energy functional is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|u|^p dx.$$

As usual, the critical points of I correspond to the nontrivial solutions of (1.1).

In order to prove our main result, we are going to minimize the functional I restricted to its Nehari manifold

$$\mathcal{N} := \left\{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle I'(u), u\rangle = 0\right\}.$$

Lemma 3.1. *The following statements hold true:*

(1) $\mathcal{N}$ is non-empty and it is a C^1-manifold;

(2) for any $u \in \mathcal{N}$, we have $I(u) = \max_{t\geq 0} I(tu)$;

(3) we have $\varpi := \inf_{u\in\mathcal{N}} I(u) > 0$.

Proof. (1) For $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ and fixed $\mathbf{k} \in \mathbb{Z}^N$, we define $u^{\mathbf{k}} = u(\cdot + \mathbf{k})$. Then by (H1) it follows from the Lebesgue's dominated convergence theorem that

$$\int_{\mathbb{R}^N} K(x)|u^{\mathbf{k}}|^p dx = \int_{\mathbb{R}^N} K(x - \mathbf{k})|u|^p dx \to \int_{\mathbb{R}^N} K_0(x)|u|^p dx > 0, \quad \text{as} \quad |\mathbf{k}| \to \infty.$$

Therefore, without loss of generality, we assume that $\int_{\mathbb{R}^N} K(x)|u|^p dx > 0$. Define $g(t) := I(tu),\ t \geq 0$, then

$$g'(t) = \langle I'(tu), u\rangle = t\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx - t^{p-1}\int_{\mathbb{R}^N} K(x)|u|^p dx = 0$$

has a unique solution $t_u > 0$, and $t_u u \in \mathcal{N}$. Therefore, $\mathcal{N} \neq \emptyset$.

For any $u \in \mathcal{N}$, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx = \int_{\mathbb{R}^N} K(x)|u|^p dx \leq C\|u\|_p^p. \tag{3.3}$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2^*]$, we have $\|u\| \geq c > 0$ for all $u \in \mathcal{N}$.

Define $J(u) := \langle I'(u), u\rangle$. Then for any $u \in \mathcal{N}$, by (2.2), we have

$$\begin{aligned}\langle J'(u), u\rangle &= 2\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx - p\int_{\mathbb{R}^N} K(x)|u|^p dx \\ &\leq (2-p)\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx \\ &\leq -c < 0.\end{aligned}$$

Hence $\mathcal{N}$ is a C^1-manifold.

(2) For any $u \in \mathcal{N}$, let $g(t) = I(tu)$, then

$$g'(t) = \langle I'(tu), u\rangle = t(1 - t^{p-2})\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx.$$

Thus $g'(t) > 0$ for $t \in (0, 1)$ and $g'(t) < 0$ for $t > 1$. Therefore, $I(u) > I(tu)$ for all $t \in (0, 1) \cup (1, +\infty)$.

(3) For any $u \in \mathcal{N}$, we have

$$I(u) = I(u) - \frac{1}{p}\langle I'(u), u\rangle \geq (\frac{1}{2} - \frac{1}{p})\|u\|^2 \geq c > 0,$$

which implies that $\varpi = \inf_{\mathcal{N}} I > 0$. The proof is complete. □

We say that (u_k), $u_k \in \mathcal{N}$, is a $(PS)_d$ sequence if $I(u_k) \to d$ and $I'|_{\mathcal{N}}(u_k) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

Lemma 3.2. *Let (u_k) be a $(PS)_d$ sequence. Then (u_k) is relatively compact for all $d \in (0, \varpi_0)$.*

Proof. Let us consider a $(PS)_d$ sequence (u_k) with $d \in (0, \varpi_0)$. Noting that $I_0(v^j) \geq \varpi_0$ for all j, we get $m = 0$ in Lemma A.2 and, then, $u_k \to u$ in $H^1(\mathbb{R}^N)$. □

The following result is proved in [16]. See also [17].

Lemma 3.3. *Let $\rho > 0$ and $W \in C^1((\rho, \infty), \mathbb{R})$. If*

$$\lim_{s\to\infty} W(s) > 0$$

and for some $\beta > 0$

$$\lim_{s\to\infty} W'(s)s^{1+\beta} = 0,$$

then there exists a nonnegative radial function $w : \mathbb{R}^N \setminus B_\rho \to \mathbb{R}$ such that

$$-\Delta w + Ww = 0$$

in $\mathbb{R}^N \setminus B_\rho$ *and some* $\rho_0 \in (\rho, \infty)$,

$$\lim_{|x|\to\infty} w(x)|x|^{\frac{N-1}{2}} \exp \int_{\rho_0}^{|x|} \sqrt{W(s)}ds = 1.$$

Lemma 3.4. *Let* $v \in H^1(\mathbb{R}^N)$ *be a positive ground state solution of (3.1), then there exist constants* $R_1 > 0, c_0 > 0, C_0 > 0$ *such that*

$$c_0|x|^{-\frac{N-1}{2}} e^{-\sqrt{V_+}|x|} \leq v(x) \leq C_0|x|^{-\frac{N-1}{2}} e^{-\sqrt{V_-}|x|}, \qquad \text{for all } |x| \geq R_1.$$

Proof. Since $v \in H^1(\mathbb{R}^N)$ is a positive ground state slution of (3.1) and $p > 2$, we have

$$\lim_{|x|\to\infty} K_0(x)v^{p-2}(x) = 0. \tag{3.4}$$

Therefore, for any fixed $\epsilon \in (0, V_-)$, there exists $\rho > 0$ such that

$$-\Delta v + (V_- - \epsilon)v \leq -\Delta v + V_0(x)v - K_0(x)v^{p-1} = 0, \qquad \text{in } \mathbb{R}^N \setminus B_\rho.$$

Let $w \in C^2(\mathbb{R}^N \setminus B_\rho, \mathbb{R})$ be such that

$$\begin{cases} -\Delta w + (V_- - \epsilon)w = 0, & \text{if } \ x \in \mathbb{R}^N \setminus B_\rho, \\ w(x) = \max\limits_{x\in\partial B_\rho} v(x), & \text{if } \ x \in \partial B_\rho, \\ \lim\limits_{|x|\to\infty} w(x) = 0. \end{cases}$$

By Lemma 3.3 with $W = V_- - \epsilon$, there exists $C_\epsilon > 0$ such that for all $x \in \mathbb{R}^N \setminus B_\rho$,

$$w(x) \leq C_\epsilon |x|^{-\frac{N-1}{2}} e^{-\sqrt{V_- - \epsilon}|x|}.$$

Hence, by the comparison principle, for all $x \in \mathbb{R}^N \setminus B_\rho$, we have

$$v(x) \leq w(x) \leq C_\epsilon |x|^{-\frac{N-1}{2}} e^{-\sqrt{V_- - \epsilon}|x|}.$$

Therefore, there exists some $\mu \in (0, \sqrt{V_-}/2)$ and $R_1 > \rho$ such that

$$K_0(x)v(x)^{p-2} \leq \mu e^{-(p-2)\sqrt{V_- - \epsilon}|x|}, \qquad \text{for all } x \in \mathbb{R}^N \setminus B_{R_1} \tag{3.5}$$

and

$$-\Delta v + (V_- - \mu e^{-(p-2)\sqrt{V_- - \epsilon}|x|})v \leq 0, \qquad \text{in } \mathbb{R}^N \setminus B_{R_1}.$$

Let $\bar{w} \in C^2(\mathbb{R}^N \setminus B_{R_1}, \mathbb{R})$ be such that

$$\begin{cases} -\Delta \bar{w} + (V_- - \epsilon)\bar{w} = 0, & \text{if } \ x \in \mathbb{R}^N \setminus B_{R_1}, \\ \bar{w}(x) = \max\limits_{x\in\partial B_{R_1}} v(x), & \text{if } \ x \in \partial B_{R_1}, \\ \lim\limits_{|x|\to\infty} \bar{w}(x) = 0. \end{cases}$$

Then by Lemma 3.3 with $W = V_- - \mu e^{-(p-2)\sqrt{V_- - \epsilon}|x|}$, it follows that

$$\limsup_{|x|\to\infty} \bar{w}(x)|x|^{\frac{N-1}{2}} e^{\int_{R_1}^{|x|} \sqrt{W(s)}ds} < +\infty.$$

Note that

$$\int_{R_1}^{|x|} \sqrt{W(s)}ds \geq \sqrt{V_-}\int_{R_1}^{|x|} W(s)/V_- ds \geq \sqrt{V_-}|x| - \sqrt{V_-}R_1 - \frac{\mu}{(p-2)\sqrt{V_- - \epsilon}}.$$

Therefore,

$$\exp\left(\sqrt{V_-}|x| - \int_{R_1}^{|x|}\sqrt{W(s)}ds\right) \leq \exp\left(\sqrt{V_-}R_1 - \frac{\mu}{(p-2)\sqrt{V_- - \epsilon}}\right).$$

Then the comparison principle implies that

$$\limsup_{|x|\to\infty} v(x)|x|^{\frac{N-1}{2}} e^{\sqrt{V_-}|x|} \leq \lim_{|x|\to\infty} \bar{w}(x)|x|^{\frac{N-1}{2}} e^{\sqrt{V_-}|x|} < +\infty.$$

Thus, $v(x) \leq C_0|x|^{-\frac{N-1}{2}} e^{-\sqrt{V_-}|x|}$ for all $|x| \geq R_1$ and some $C_0 > 0$.

Since

$$-\Delta v + V_+ v \geq -\Delta v + V_0(x)v - K_0(x)v^{p-1} = 0,$$

a similar argument implies that $v(x) \geq c_0|x|^{-\frac{N-1}{2}} e^{-\sqrt{V_+}|x|}$ for all $|x| \geq R_1$ and some $c_0 > 0$. The proof is complete. □

4. Proof of Main Result

In this section, we give the proof of our main result. To this end, we need the following results.

Lemma 4.1. *Assume that (H1) and (H2) hold. Then we have*

$$\varpi \leq \varpi_0.$$

The equality $\varpi = \varpi_0$ holds only if $V(x) = V_0(x)$ and $K(x) = K_0(x)$ for a.e. $x \in \mathbb{R}^N$.

Proof. Let $v \in H^1(\mathbb{R}^N)$ be a positive ground state solution of (3.1), that is, $I_0(v) = \varpi_0$. Without loss of generality, we assume $U = \{\sigma \in S^{N-1} \mid |\sigma - e_1| < \delta\}$ and $\lim_{r\to+\infty}\xi(r\sigma) = 0$ uniformly for $\sigma \in U$. Define $v^k = v(\cdot - ke_1)$, then by the translation invariance, it follows that $v^k \in H^1(\mathbb{R}^N)$ is also a positive ground state of (4.1) and

$$\lim_{k\to\infty} v^k(x) = 0, \quad \text{for all } x \in \mathbb{R}^N. \tag{4.1}$$

It is easy to see that $g(t) := I(tv^k)$ achieves its global maximum in some $t_k > 0$ such that $\langle I'(t_kv^k), t_kv^k\rangle = 0$, and hence $t_kv^k \in \mathcal{N}$. Moreover, we have

$$t_k^{p-2} = \frac{\int_{\mathbb{R}^N} |\nabla v|^2 + V(x+ke_1)v^2dx}{\int_{\mathbb{R}^N} K(x+ke_1)|v|^pdx} \to \frac{\int_{\mathbb{R}^N} |\nabla v|^2 + V_0(x)v^2dx}{\int_{\mathbb{R}^N} K_0(x)|v|^pdx} > 0,$$

as $k \to \infty$. By the translation invariance and Lemma 3.1, we have that

$$I_0(t_kv^k) = I_0(t_kv) \leq I_0(v) = \varpi_0. \tag{4.2}$$

Clearly, if $V(x) \equiv V_0(x)$ and $K(x) \equiv K_0(x)$ for all $x \in \mathbb{R}^N$, then $\varpi = \varpi_0$. In what follows, we assume that $V(x) \not\equiv V_0(x)$ or $K(x) \not\equiv K_0(x)$. By Lemma 3.4, we can find $\tilde{C} \geq C_0$ such that

$$|v(x)| \leq \tilde{C} e^{-\sqrt{V_-}|x|}, \quad \text{for all } x \in \mathbb{R}^N.$$

Hence, we obtain

$$|\xi(x)| \cdot |v^k(x)|^{p-2} \leq \tilde{C}^{p-2} |\xi(x)| e^{-(p-2)\sqrt{V_-}|x-ke_1|}, \quad \text{for all } x \in \mathbb{R}^N.$$

By (H2) and a compact argument, it follows that $\lim_{k\to\infty} |\xi(x)| \cdot |v^k(x)|^{p-2} = 0$ uniformly in $x \in \mathbb{R}^N$. If $V_0(x) - V(x) \not\equiv 0$, taking into account that $0 < c \leq t_k \leq C < \infty$, we get

$$\begin{aligned}
\varpi &\leq I(t_k v^k) \\
&= I_0(t_k v^k) - \tfrac{1}{2}\int_{\mathbb{R}^N} (V_0(x) - V(x)) t_k^2 v^k(x)^2 dx + \tfrac{1}{p}\int_{\mathbb{R}^N} (K_0(x) - K(x)) t_k^p |v^k|^p dx \\
&\leq \varpi_0 - t_k^2 \int_{\mathbb{R}^N} [V_0(x) - V(x)] v^k(x)^2 \left[\tfrac{1}{2} - \tfrac{1}{p}|\xi(x)| t_k^{p-2} \tilde{C}^{p-2} e^{-(p-2)\sqrt{V_-}|x-ke_1|}\right] dx \\
&< \varpi_0
\end{aligned}$$

for large k. Thus, we obtain $\varpi < \varpi_0$. If $V(x) \equiv V_0(x)$ and $K(x) \not\equiv K_0(x)$, then $K_0(x) \leq K(x)$ and it is also easy to see that $\varpi < \varpi_0$. The proof is complete. □

Lemma 4.2. *Assume that (H1) and (H3) hold. Then we have*

$$\varpi \leq \varpi_0.$$

The equality $\varpi = \varpi_0$ holds only if $V(x) = V_0(x)$ and $K(x) = K_0(x)$ for a.e. $x \in \mathbb{R}^N$.

Proof. Let $v \in H^1(\mathbb{R}^N)$ be a positive ground state solution of (3.1), that is, $I_0(v) = \varpi_0$. Let $v^k = v(\cdot - ke_1)$ and $t_k > 0$ be such that $t_k v^k \in \mathcal{N}$. Then we have

$$\varpi \leq I(t_k v^k) = I_0(t_k v^k) + J \leq \varpi_0 + J,$$

where

$$\begin{aligned}
J &= \frac{1}{2}\int_{\mathbb{R}^N} (V(x) - V_0(x)) t_k^2 |v^k|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} (K(x) - K_0(x)) t_k^p |v^k|^p dx \\
&= \int_{\mathbb{R}^N} \frac{1}{2}(V(x+ke_1) - V_0(x+ke_1)) t_k^2 |v|^2 - \frac{1}{p}(K(x+ke_1) - K_0(x+ke_1)) t_k^p |v|^p dx.
\end{aligned} \tag{4.3}$$

It is suffices to show that $J < 0$ for some large k. To this end, we adopt an argument used in [6]. Since $V - V_0 \in L^{N/2}(\mathbb{R}^N)$, by Lemma 3.4, it is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^N \setminus B_{\tau k}} \frac{1}{2}(V(x+ke_1) - V_0(x+ke_1)) t_k^2 |v(x)|^2 dx \\
&\quad - \int_{\mathbb{R}^N \setminus B_{\tau k}} \frac{1}{p}(K(x+ke_1) - K_0(x+ke_1)) t_k^p |v(x)|^p dx \\
&\leq \int_{\mathbb{R}^N \setminus B_{\tau k}} \frac{1}{2}(V(x+ke_1) - V_0(x+ke_1)) t_k^2 |v(x)|^2 dx
\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^N\setminus B_{\tau k}} |V(x+ke_1)-V_0(x+ke_1)|^{\frac{N}{2}}\right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N\setminus B_{\tau k}} |v(x)|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}$$
$$\leq Ce^{-2\tau k\sqrt{V_-}}. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}
&-\int_{B_{\tau k}} \frac{1}{2}(V(x+ke_1)-V_0(x+ke_1))t_k^2|v(x)|^2dx \\
&-\int_{B_{\tau k}} \frac{1}{p}(K(x+ke_1)-K_0(x+ke_1))t_k^p|v(x)|^p dx \\
\geq &\int_{B_{\tau k}} (K(x+ke_1)-K_0(x+ke_1))t_k^2|v(x)|^2 \left[-\frac{1}{2}\eta(x+ke_1)+\frac{1}{p}t_k^{p-2}|v(x)|^{p-2}\right]dx.
\end{aligned} \tag{4.5}$$

By (H3), we have $\limsup_{|x|\to\infty}\eta(x)=0$, and then it follows from the fact that $v(x)>0$ for all $x\in\mathbb{R}^N$ that

$$\eta(x+ke_1)\leq \varepsilon v^{p-2}(x),$$

for all $x\in B_{R_1}$ and large k. For any $x\in B_{\tau k}\setminus B_{R_1}$, by Lemma 3.4, we have

$$\begin{aligned}
\eta(x+ke_1) &\leq \varepsilon|x+ke_1|^{-\frac{(p-2)(N-1)}{2}} e^{-\frac{(p-2)\tau}{1-\tau}\sqrt{V_+}|x+ke_1|} \\
&\leq \varepsilon(1-\tau)^{-\frac{(p-2)(N-1)}{2}}\tau^{\frac{(p-2)(N-1)}{2}}|\tau k|^{-\frac{(p-2)(N-1)}{2}} e^{-(p-2)\tau k\sqrt{V_+}} \\
&\leq \varepsilon(1-\tau)^{-\frac{(p-2)(N-1)}{2}}\tau^{\frac{(p-2)(N-1)}{2}}|x|^{-\frac{(p-2)(N-1)}{2}} e^{-(p-2)\sqrt{V_+}|x|} \\
&\leq \varepsilon v^{p-2}(x).
\end{aligned}$$

Therefore, for all $x\in B_{\tau k}$ with large k, we have

$$-\frac{1}{2}\eta(x+ke_1)+\frac{1}{p}t_k^{p-2}|v(x)|^{p-2}\geq cv^{p-2}(x).$$

Hence, by (4.5), we have

$$\begin{aligned}
&-\int_{B_{\tau k}} \frac{1}{2}(V(x+ke_1)-V_0(x+ke_1))t_k^2|v(x)|^2dx \\
&-\int_{B_{\tau k}} \frac{1}{p}(K(x+ke_1)-K_0(x+ke_1))t_k^p|v(x)|^p dx \\
\geq &c\int_{B_{\tau k}} (K(x+ke_1)-K_0(x+ke_1))t_k^2|v(x)|^p dx \\
\geq &cMe^{-2\tau k\sqrt{V_-}}\int_{B_{R_1}} |v(x)|^p dx \\
\geq &cMe^{-2\tau k\sqrt{V_-}}.
\end{aligned} \tag{4.6}$$

Hence, by (4.3), (4.4), (4.6) and the arbitrariness of M, we conclude that $J<0$. The proof is complete. □

Now, we are in a position to prove our main result.

Proof of Theorem 1.1. By the Ekeland Variational Principle, we obtain a sequence $\{u_k\} \subset \mathcal{N}$ such that $I(u_k) \to \varpi$ and $I'|_{\mathcal{N}}(u_k) \to 0$ as $k \to \infty$. Then we obtain

$$\varpi + o(1) = I(u_k) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx,$$

which implies that (u_k) is bounded in $H^1(\mathbb{R}^N)$. Moreover, there exists a $\lambda_k \in \mathbb{R}$ such that

$$o(1) = I'|_{\mathcal{N}}(u_k) = I'(u_k) - \lambda_k J'(u_k). \tag{4.7}$$

Taking the scalar product with u_k in the above equality, we obtain

$$o(1) = \langle I'|_{\mathcal{N}}(u_k), u_k\rangle = \langle I'(u_k), u_k\rangle - \lambda_k \langle J'(u_k), u_k\rangle.$$

Since $\langle I'(u_k), u_k\rangle = 0$ and $\langle J'(u_k), u_k\rangle \leq -c < 0$. It follows that $\lambda_k \to 0$ as $k \to +\infty$. Moreover, by the boundedness of (u_k), $J'(u_k)$ is bounded and this implies that $\lambda_k J'(u_k) \to 0$. Therefore, $I'(u_k) \to 0$ as $k \to \infty$. By the construction, we have $I(u_k) \to \varpi > 0$. Therefore, (u_k) is a Palais-Smale sequence of I at level ϖ.

If $V(x) \equiv V_0(x)$ and $K(x) \equiv K_0(x)$ for all $x \in \mathbb{R}^N$, then (1.1) reduces to the periodic equation (3.1), and it is well-known that there is a ground sate solution for such an equation. So we assume that either $V(x) \not\equiv V_0(x)$ or $K(x) \not\equiv K_0(x)$. Therefore, by Lemma 4.1 and Lemma 4.2, we have $\varpi \in (0, \varpi_0)$.

By Lemma 3.2, (u_k) is relatively compact. Therefore, up to a subsequence, $u_k \to u$. Moreover, we have $I'(u) = 0$ and $I(u) = \varpi > 0$. Thus, $u \neq 0$ and u is a ground state solution. Lastly, since $|u|$ is also a ground state solution, the Maximum Principle implies that $u > 0$ on $\mathbb{R}^N$ or $u < 0$ on $\mathbb{R}^N$. The proof is complete. □

Appendix

In this appendix, for the sake of the completeness and for the reader's convenience, we prove a global compactness lemma by using a standard argument.

In what follows, the following well known Brezis-Lieb type lemma is needed [20].

Lemma A.1. *If $(u_k)_{k\in\mathbb{N}} \subset H^1(\mathbb{R}^N)$ is a bounded sequence and $u_k \to u$ almost everywhere on $\mathbb{R}^N$. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} K(x)|u_k|^p - K(x)|u_k - u|^p dx = \int_{\mathbb{R}^N} K(x)|u|^p dx.$$

Lemma A.2. *Let (u_k) be a (PS) sequence of I constrained on $\mathcal{N}$, i.e. $u_k \in \mathcal{N}$ satisfies*

$$(a) \quad I(u_k) \text{ is bounded}; \quad (b) \quad I'|_{\mathcal{N}}(u_k) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

Then replacing (u_k), if necessary, with a subsequence, there exist a solution u of (1.1), a number $m \in \mathbb{N} \cup \{0\}$, m functions $v^1, \cdots, v^m$ of $H^1(\mathbb{R}^N)$ and m sequences of points $(x_k^j) \subset \mathbb{R}^N$, $1 \leq j \leq m$, such that

(i) $|x_k^j| \to +\infty, \quad |x_k^j - x_k^i| \to +\infty$ if $i \neq j$, $k \to +\infty$;

(ii) $u_k = u + \sum_{j=1}^m v^j(\cdot - x_k^j) + o(1)$ in $H^1(\mathbb{R}^N)$;

(iii) $I(u_k) = I(u) + \sum_{j=1}^m I_0(v^j) + o(1)$;

(iv) v^j are nontrivial weak solutions of (3.1).

Moreover, we agree that in the case $m = 0$ the above holds without v^j.

Proof. Since $(u_k) \subset \mathcal{N}$ is a (PS) sequence, we have

$$+\infty > C \geq I(u_k) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_k\|^2,$$

and hence (u_k) is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exists $u \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, $u_k \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Argued as in the proof of Theorem 1.1, we can show that $I'(u_k) \to 0$ as $k \to \infty$. Then the weakly sequentially continuity of I' implies that $I'(u) = 0$.

If $u_k \to u$ in $H^1(\mathbb{R}^N)$, we are done. So we can assume that (u_k) does not converge strongly to u in $H^1(\mathbb{R}^N)$. Set

$$z_k^1 = u_k - u.$$

Then $z_k^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, but not strongly. Obviously, we have

$$\|u_k\|^2 = \|u\|^2 + \|z_k^1\|^2 + o(1). \tag{A.1}$$

By Lemma A.1, we also have

$$\int_{\mathbb{R}^N} K(x)|u_k|^p = \int_{\mathbb{R}^N} K(x)|u|^p + \int_{\mathbb{R}^N} K(x)|z_k^1|^p + o(1), \tag{A.2}$$

and for any $h \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} K(x)|u_k|^{p-2}u_k h = \int_{\mathbb{R}^N} K(x)|u|^{p-2}uh + \int_{\mathbb{R}^N} K(x)|z_k^1|^{p-2}z_k^1 h + o(1). \tag{A.3}$$

Therefore, we obtain

$$I(u_k) = \frac{1}{2}\|u_k\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|u_k|^p = I(u) + I_0(z_k^1) + o(1),$$

and for all $h \in H^1(\mathbb{R}^N)$,

$$o(1) = \langle I'(u_k), h\rangle = \langle I_0'(z_k^1), h\rangle + o(1).$$

Let

$$\delta := \limsup_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |z_k^1|^2 dx.$$

Then $\delta > 0$. Otherwise, for any $p \in (2, 2^*)$, $z_k^1 \to 0$ in $L^p(\mathbb{R}^N)$. Therefore, we have

$$o(1) = \langle I'(u_k) - I'(u), z_k^1\rangle = \|z_k^1\|^2 - \int_{\mathbb{R}^N} K(x)\left[|u_k|^{p-2}u_k - |u|^{p-2}u\right] z_k^1 dx. \tag{A.4}$$

By the Hölder inequality, we know

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)\left[|u_k|^{p-2}u_k - |u|^{p-2}u\right] z_k^1 dx \\
\leq & C\left(\int_{\mathbb{R}^N} K(x)\left||u_k|^{p-2}u_k - |u|^{p-2}u\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |z_k^1|^p\right)^{\frac{1}{p}} \\
\leq & C\left(\int_{\mathbb{R}^N} (|u_k|^p + |u|^p)\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |z_k^1|^p\right)^{\frac{1}{p}} \\
\leq & C\|z_k^1\|_p \to 0,
\end{aligned}$$

which together with (A.4) yields $z_k^1 \to 0$ in $H^1(\mathbb{R}^N)$. This contradicts to the fact that (u_k) does not converge strongly to u in $H^1(\mathbb{R}^N)$.

Let $x_k^1 \in \mathbb{Z}^N$ be such that for some suitable $r_0 \geq 1$, we have

$$\int_{B(0,r_0)} |z_k^1(x + x_k^1)|^2 dx \geq \frac{1}{2}\delta.$$

Then $z_k^1(\cdot + x_k^1)$ is bounded in $H^1(\mathbb{R}^N)$ and we may assume that

$$z_k^1(\cdot + x_k^1) \rightharpoonup u^1 \quad \text{in } H^1(\mathbb{R}^N).$$

Then $u^1 \neq 0$. But, since $z_k^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, (x_k^1) must be unbounded and, up to a subsequence, we can assume that $|x_k^1| \to +\infty$.

Furthermore, $I_0'(z_k^1) = o(1)$ in $[H^1(\mathbb{R}^N)]^*$ implies $I_0'(u^1) = 0$.

Finally, let us set

$$z_k^2 = z_k^1 - u^1(\cdot - x_k^1).$$

Then by the translation invariance of the functional I_0, a similar argument shows that

$$\begin{aligned} I(u_k) &= I(u) + I_0(z_k^1) + o(1) \\ &= I(u) + I_0(u^1) + I_0(z_k^2) + o(1) \end{aligned}$$

and

$$I'(z_k^2) = o(1).$$

Now, if $z_k^2 \to 0$ in $H^1(\mathbb{R}^N)$, we are done. Otherwise $z_k^2 \rightharpoonup 0$ and not strongly and we repeat the argument. By iterating this procedure we obtain sequences of integers $x_k^j \in \mathbb{Z}^N$ such that

$$|x_k^j| \to +\infty, \quad |x_k^j - x_k^i| \to +\infty \;\text{ if } j \neq i,$$

as $k \to +\infty$ and a sequence of functions

$$z_k^j = z_k^{j-1} - u^{j-1}(\cdot - x_k^{j-1})$$

with $j \geq 2$ such that

$$z_k^j(\cdot + x_k^j) \rightharpoonup u^j \quad \text{in } H^1(\mathbb{R}^N).$$

Moreover, we have

$$I(u_k) = I(u) + \sum_{j=1}^{m} I_0(u^j) + I(z_k^m) + o(1),$$

and

$$I_0'(u^j) = 0.$$

Then, since $I_0(u^j) \geq \varpi_0 > 0$ for all j and $I(u_k)$ is bounded, the iteration must stop at some finite index m and $z_k^m \to 0$ in $H^1(\mathbb{R}^N)$. Thus $I(u_k) = I(u) + \sum_{j=1}^m I_0(u^j) + o(1)$ and the proof is complete. □

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