Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 1678–1690
Website:http://www.jaac-online.com
DOI:10.11948/20190146

ASYMPTOTIC FLOCKING VELOCITY AND POSITION FORMULAS FOR THE DELAYED CUCKER-SMALE MODEL*

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Abstract The Cucker-Smale (short for C-S) model was modified by introduced multiple time delays, and the flocking characteristics of the processing delay C-S model of systems was obtained. Based on the fixed point theorem, we present the existence and uniqueness of the flocking solution for our delayed C-S model when the influence function has the property of Lipschitz and the initial value satisfies certain conditions. At last, we present the asymptotic flocking velocity and the final relative position between agents of the unique flocking solution for such system.

Keywords C-S model, flocking solution, delay adaptation, flocking velocity formula, fixed point.

MSC(2010) 93C15, 35D30.

1. Introduction

Self-organized systems arise very naturally in physical, biological, artificial intelligence, and social sciences. For such systems, it is important to understand how the flow of information from distinct and independent components can be adaptively regulated to achieve a prescribed performance, and in particular to understand how systems develop their emerging behaviours such as flocking, herding and schooling, in which self-propelled individuals using only limited environmental information and simple rules, organize into an ordered motion. Examples of flocking phenomena include fish swimming in schools [20], birds flying in flocks for the purpose of enhancing the foraging success [3], and the flight guidance in honeybee swarms [12].

Cucker and Smale [7] presented a mathematical model to investigate the emergent behaviors of flocks. Such a model could be used to explain self-organized behaviours in various complex systems aforementioned, for example those from macroscopic world (bird flying and honeybee swarming) to microscopic phenomenon (Kinetic models and mean field models [2]). Recent developments about the C-S model extend the pioneering work, to include asymmetric influence functions [18], Kinetic version of C-S models [14, 22], and multi-agent systems with hierarchical leadership [8–10, 19, 21], pattern formation [15, 16], collision avoidance [1, 4, 6, 16]. Also, the time lag is introduced in recent work [5, 11, 17, 19, 23], but the flocking phenomenon described in the most previous works is considered with the delay between

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*The authors were supported by National Natural Science Foundation of China (11671011).

different agents is equal, there are little work about the flocking performance of the multiple time delays C-S model. In fact, due to the differences between agents, the time delay between agents may not equal.

As we known, for the emerging behaviors, the main issue for engineers and scientists is to evaluate the final position distance for each pair agents and the stability of the whole multi-agent systems. To this aim, we try to find a new insight for modelling the flocking or swarming behaviors. Comparing with other works, we present a new method via fixed point theorem to analysis the asymptotic flocking velocity and position formulas.

In this work, we focus on the existence, uniqueness and asymptotic flocking velocity of conditional flocking solutions for the C-S model incorporating a multiple time delayed influence function which is assumed to be Lipschitz continuous. We show that there is a unique conditional flocking solution for the multiple time delayed C-S models, and we deduce an asymptotic flocking for the self-organized group for general influence functions, and then derive an asymptotic flocking velocity formula and a position formula.

2. Model formulation

We consider the motion of a self-organized group with N agents, with each agent i being characterized by two quantities: the position $\mathbf{x}_i(t) \in \mathbb{R}^d$, and velocity $\mathbf{v}_i(t) \in \mathbb{R}^d$, where $d \geq 1$ is an integer. Thus, $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1,2,\cdots,N}$ describes the agent system at any give time t, and the C-S model [7] is given by

$$\begin{cases} \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{v}_i, i = 1, 2, \cdots, N, \\ \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \frac{\alpha}{N} \sum\limits_{j=1, j\neq i}^{N} a_{ij}(\mathbf{v}_j - \mathbf{v}_i). \end{cases}$$

Here α measures the interaction strength, and $a_{ij} = \psi(|\mathbf{x}_j - \mathbf{x}_i|)$ in the original C-S model quantifies the pairwise influence of agent j on the alignment of agent i, as a function of the distance, the so-called influence function, $\psi(\cdot)$ is a strictly positive decreasing function. A popular example is $\psi(r) = (1 + r^2)^{-\beta}$ for $r \geq 0$ and $\beta \geq 0$ is a constant. In the recently modified C-S model [18], the non-symmetric pairwise influence

$$a_{ij} = \frac{\psi\left(\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|\right)}{\sum\limits_{k=1}^{N} \psi\left(\|\mathbf{x}_k(t) - \mathbf{x}_i(t)\|\right)}. \tag{2.1}$$

was also used.

Here, we extend the above model by considering the delay in the pairwise influence due to the finite speed in processing the influence. In general, the influence of agent j on agent i is realized in various fashions including smell, sound and vision. For examples, The influence between honey bees is transferred mainly by a certain chemical material [13], the influence between geese is mainly made through vision [18]. The influence of an agent on another is naturally transferred with finite speed. In biological and artificial neural networks, time delays arise naturally due to the inter-neural distances and finite axonal conduction. We will focus in this study on the case of processing the information about the location and velocity of neighbouring agents, resulting in the following modified self-organized C-S model with delay

$$\begin{cases} \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, i = 1, 2, \cdots, N, \\ \frac{d\mathbf{v}_i}{dt} = \alpha \sum\limits_{j=1, j\neq i}^{N} a_{ij}(\mathbf{x}(t-\tau_{ij}))(\mathbf{v}_j(t-\tau_{ij}) - \mathbf{v}_i(t)). \end{cases} \tag{2.2}$$

where τ_{ij} denotes the communication time between agents i and j. In general, the time delay τ_{ij} is non-symmetric so that $\tau_{ij} \neq \tau_{ji}$. In what follows, we assume also $\tau_{ii} = 0$ for all i. In the above model, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N) \in \mathbb{R}^{dN}, \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N) \in \mathbb{R}^{dN}$, and $a_{ij}(\mathbf{x}(t-\tau_{ij}))$ quantifies the pairwise influence of agent j on the alignment of agent i (specific forms of a_{ij} will be given in the next subsection). This self-organized system (2.2) is subjected to the following initial conditions

$$\mathbf{x}_i(\theta) = \mathbf{f}_i(\theta), \quad \mathbf{v}_i(\theta) = \mathbf{g}_i(\theta), i = 1, 2, \cdots, N, \tag{2.3}$$

where $\mathbf{f}$ and $\mathbf{g}$ are given continuous vector-value functions, $\tau = max_{i,j}\{\tau_{ij}\}$. Note that an agent may receive influences from multiple agents in a group and, an agent i may also receive influence from another agent j via the agent l. For examples, in the bee swarm, the small minority of informed bees manage to provide guidance to the rest, and the entire swarm is able to fly to the new nest intact [12]. It is similar for the network system in power engineering and intelligent engineering. Therefore, the pairwise influence a_{ij} may take the following form

$$a_{ij}(\mathbf{x}(t)) = \sum_{k=0}^{N-1} \delta_{ij}^k(\mathbf{x}(t)), \tag{2.4}$$

where $\delta_{ij}^k(\mathbf{x}(t))$ is defined inductively as follows:

$$\begin{aligned} &\delta_{ij}^0(\mathbf{x}(t)) = \psi_{ij}(\mathbf{x}(t)), \\ &\delta_{ij}^k(\mathbf{x}(t)) = \sum_{l\neq i,j} \max\{\delta_{il}^{k-1}(\mathbf{x}(t)) - \delta_{lj}^{k-1}(\mathbf{x}(t)), 0\}, k = 1, \cdots, N-1. \end{aligned}$$

the matrix $(\psi_{ij})_{N\times N}$ sketches the pairwise influence of each agent, formulated by (2.1). The term $\delta_{ij}^0(\mathbf{x}(t))$ represents the direct impact from j to i, $\delta_{ij}^1(\mathbf{x}(t))$ represents the impact from i to j by dint of one middle agent. Similarly, $\delta_{ij}^k(\mathbf{x}(t))$ denotes the impact from i to j by dint of k middle agents. If the long range cohesion is ignored and delay is neglected, then $a_{ij}(\mathbf{x}(t)) = \delta_{ij}^0(\mathbf{x}(t)) = (\psi_{ij})(\mathbf{x}(t))$, and our model (2.2) deduces to the Motsch's modified version of the C-S system [18]. Similarly, if we consider the case of the first order approximation of formula (2.4) then the pairwise influence a_{ij} is given by

$$\begin{aligned} a_{ij}(\mathbf{x}(t)) &= \delta_{ij}^0(\mathbf{x}(t)) + \delta_{ij}^1(\mathbf{x}(t)) \\ &= \psi_{ij}(\mathbf{x}(t)) + \sum_{l\neq i,j} \max\{\psi_{il}(\mathbf{x}(t)) - \psi_{lj}(\mathbf{x}(t)), 0\}. \end{aligned}$$

The influence function (2.4) permits us to highlight the influences of the wholeness instead of individual agents, so we may be able to for more subtle performances of the self-organized system, such as flocking behaviours and asymptotic flocking velocities. Let d_X and d_V denote the diameters in position and velocity phase spaces, namely

$$d_X = \max_{i,j}\{|\mathbf{x}_i - \mathbf{x}_j|\}, d_V = \max_{i,j}\{|\mathbf{v}_i - \mathbf{v}_j|\}.$$

A solution $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^N$of system (2.2) - (2.3) is called a flocking solution if it converges to a flock in the sense that

$$\sup_{t\geq 0} d_X(t) < +\infty, \lim_{t\to+\infty} d_V(t) = 0.$$

We now simplify the model by using similar arguments used in [18]. By rescaling α if necessary, without loss of generality, we may assume that the a_{ij} are normalized such that

$$\sum_{j\neq i} a_{ij}(\mathbf{x}(t-\tau_{ij})) < 1.$$

Letting $a_{ii}(\mathbf{x}(t)) = 1 - \sum_{j\neq i} a_{ij}(\mathbf{x}(t-\tau_{ij}))$, then we can rewrite the system (2.2) in the form

$$\begin{cases} \frac{\mathrm{d}\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \frac{\mathrm{d}\mathbf{v}_i}{dt} = \alpha(\overline{\mathbf{v}}_i(t) - \mathbf{v}_i(t)), \end{cases} \tag{2.5}$$

where

$$\bar{\mathbf{v}}_i(t) = \sum_{j=1}^N a_{ij}(\mathbf{x}(t-\tau_{ij}))\mathbf{v}_j(t-\tau_{ij}), \sum_{j=1}^N a_{ij}(\mathbf{x}(t-\tau_{ij})) = 1.$$

In the remaining part of this paper, we focus on the normalized self-organized system (2.5).

We first identify candidate flocking solutions of the self-organized system (2.2) and define the following set

$$\begin{aligned} &E = \{(\mathbf{x},\mathbf{v}) : \mathbf{x} = \{\mathbf{x}_i\}_{i=1}^N, \mathbf{v} = \{\mathbf{v}_i\}_{i=1}^N, \mathbf{x}_i, \mathbf{v}_i \in C([-\tau,+\infty), \mathbb{R}^d), \\ &\mathbf{x}_i(s) = \mathbf{f}_i(s), \mathbf{v}_i(s) = \mathbf{g}_i(s), \quad if \quad s \in [-\tau, 0], \\ &\sup_{t\geq 0,i,j} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| < +\infty, \sup_{t\geq 0,i} |\mathbf{v}_i(t)| \leq \sup_i |\mathbf{g}_i(0)|\mathrm{e}^\tau\}. \end{aligned}$$

We make the following assumption:

Assumption 2.1. *There exists a constant L_a such that, for all $\mathbf{x}, \mathbf{p} \in \mathbb{R}^{dN}$,*

$$|a_{ij}(\mathbf{x}) - a_{ij}(\mathbf{p})| \leq L_a|\mathbf{x} - \mathbf{p}|.$$

Assumption 2.2. *When $t \in [-\tau, 0]$, for all i, j, we have $\sup_{i,j} |\mathbf{f}_i(t) - \mathbf{f}_j(t)| < +\infty$ and $\sup_i |\mathbf{g}_i(t)| \leq \sup_i |\mathbf{g}_i(0)|\mathrm{e}^\tau$.*

Setting

$$c = (\alpha+1)N(L_a \sup_i |\mathbf{g}_i(0)|\mathrm{e}^\tau + 1) + 1.$$

We introduce a metric D on the E by

$$D((\mathbf{x},\mathbf{v}),(\mathbf{p},\mathbf{q})) = \sup_{t\geq 0}\{\mathrm{e}^{-ct}\max\{|\mathbf{x}(t) - \mathbf{p}(t)|, |\mathbf{v}(t) - \mathbf{q}(t)|\}\}.$$

For $(\mathbf{x},\mathbf{v}),(\mathbf{p},\mathbf{q}) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$, assume $\{(\mathbf{x}^n,\mathbf{v}^n)\}$ be a convergent sequence in (E,D) with the limit $(\mathbf{x}^0,\mathbf{v}^0)$, then we conclude that $(\mathbf{x}^0,\mathbf{v}^0) \in E$. In fact, for $\mathbf{v}^n$ are bounded by $\sup_i |\mathbf{g}_i(0)|\mathrm{e}^{\tau}$, we see that $\mathbf{v}^n$ are convergent on $[-\tau,+\infty)$ uniformly. Thus $\mathbf{v}^0$ is continuous and also, bounded by $\sup_i |\mathbf{g}_i(0)|\mathrm{e}^{\tau}$. On the other hand, we see that

$$\begin{aligned}
&\sup_{t\geq 0,i,j} |\mathbf{x}_i^0(t)-\mathbf{x}_j^0(t)| \\
=&\sup_{t\geq 0,i,j} |\mathbf{x}_i^0(t)-\mathbf{x}_i^n(t)+\mathbf{x}_i^n(t)-\mathbf{x}_j^n(t)+\mathbf{x}_j^n(t)-\mathbf{x}_j^0(t)| \\
\leq&\sup_{t\geq 0,i,j} |\mathbf{x}_i^0(t)-\mathbf{x}_i^n(t)|+\sup_{t\geq 0,i,j} |\mathbf{x}_i^n(t)-\mathbf{x}_j^n(t)|+\sup_{t\geq 0,i,j} |\mathbf{x}_j^n(t)-\mathbf{x}_j^0(t)| \\
<&+\infty.
\end{aligned}$$

Thus, (E,D) is a complete metric space.

3. Flocking solution of multi-delay C-S model

Theorem 3.1. *If the Assumption 2.1 and Assumption 2.2 hold, then the self-organized system (2.2) with the initial value (2.3) has a unique conditional flocking solution* $\{\mathbf{x}_i(t),\mathbf{v}_i(t)\}_{i=1}^N$ *in* E *.*

Proof. By using the variation-of-constants formula, we see that the solution of system (2.2) with the initial value (2.3) can be translated as a fixed point of operator T : $C([-\tau,+\infty),\mathbb{R}^{dN}) \times C([-\tau,+\infty),\mathbb{R}^{dN}) \to C([-\tau,+\infty),\mathbb{R}^{dN}) \times C([-\tau,+\infty),\mathbb{R}^{dN})$ given by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \Phi(\mathbf{x},\mathbf{v}) \\ \Psi(\mathbf{x},\mathbf{v}) \end{bmatrix},$$

where $(\mathbf{x},\mathbf{v}) \in E, \Phi(\mathbf{x},\mathbf{v}) = [\phi_1,\phi_2,\cdots,\phi_N]^{\mathrm{T}}, \Psi(\mathbf{x},\mathbf{v}) = [\psi_1,\psi_2,\cdots,\psi_N]^{\mathrm{T}}$,

$$\phi_i(\mathbf{x},\mathbf{v})(t) = (1-\mathrm{e}^{-\alpha t})\frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) + \int_0^t (1-\mathrm{e}^{-\alpha(t-s)})\sum_{j=1}^N a_{ij}\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s,$$

and

$$\psi_i(\mathbf{x},\mathbf{v})(t) = \mathrm{e}^{-\alpha t}\mathbf{g}_i(0) + \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^N a_{ij}\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s.$$

For $t \geq 0$, we will finish the proof in two steps.

Step 1: T is a self-mapping on E : For the given $(\mathbf{x},\mathbf{v}) \in E$, then we have

$$\begin{aligned}
|\psi_i(\mathbf{x},\mathbf{v})(t)| &\leq \mathrm{e}^{-\alpha t}\sup_i |\mathbf{g}_i(0)| + \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^N a_{ij}|\mathbf{v}_j(s-\tau_{ij})|\mathrm{d}s \\
&\leq \mathrm{e}^{-\alpha t}\sup_i |\mathbf{g}_i(0)| + \sup_i |\mathbf{g}_i(0)|\mathrm{e}^{\tau}\alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{d}s \\
&= \mathrm{e}^{-\alpha t}\sup_i |\mathbf{g}_i(0)| + \sup_i |\mathbf{g}_i(0)|\mathrm{e}^{\tau}(1-\mathrm{e}^{-\alpha t})
\end{aligned}$$

$$< \sup_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau}.$$

Thus

$$\sup_{t\ge 0, i} |\psi_i(\mathbf{x},\mathbf{v})(t)| \le \sup_i |\mathbf{g}_i(0)|\mathrm{e}^{\tau}.$$

Also, for any $a \in [0,\tau]$, we have

$$\begin{aligned}
&|\psi_i(\mathbf{x},\mathbf{v})(t) - \psi_j(\mathbf{x},\mathbf{v})(t+a)| \\
\le\ & \mathrm{e}^{-\alpha t}|\mathbf{g}_i(0) - \mathrm{e}^{-\alpha a}\mathbf{g}_j(0)| \\
&+\alpha\Big|\int_0^t \mathrm{e}^{-\alpha(t-s)} \sum_{k=1}^N a_{ik}\mathbf{v}_k(s-\tau_{ik})\mathrm{d}s - \int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)} a_{jk}\mathbf{v}_k(s-\tau_{jk})\mathrm{d}s\Big| \\
\le\ & \mathrm{e}^{-\alpha t}|\mathbf{g}_i(0) - \mathrm{e}^{-\alpha a}\mathbf{g}_j(0)| + \alpha \int_0^t \mathrm{e}^{-\alpha(t-s)} \sum_{k=1}^N a_{ik}|\mathbf{v}_k(s-\tau_{ik}) - \mathbf{v}_i(s)|\mathrm{d}s \\
&+\alpha \int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)} \sum_{k=1}^N a_{jk}|\mathbf{v}_k(s-\tau_{jk}) - \mathbf{v}_i(s-a)|\mathrm{d}s \\
&+\alpha\Big|\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathbf{v}_i(s)\mathrm{d}s - \int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)}\mathbf{v}_i(s-a)\mathrm{d}s\Big| \\
\le\ & \mathrm{e}^{-\frac{\alpha}{2}t}|\mathbf{g}_i(0) - \mathrm{e}^{-\alpha a}\mathbf{g}_j(0)| + \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-\frac{\alpha}{2}s} \sup_{i,k,s\ge -\tau, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s+a) - \mathbf{v}_i(s)|\}\mathrm{d}s \\
&+\alpha \int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)}\mathrm{e}^{-\frac{\alpha}{2}s} \sup_{i,k,s\ge -\tau, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s+a) - \mathbf{v}_i(s)|\}\mathrm{d}s \\
&+\alpha\Big|\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathbf{v}_i(s)\mathrm{d}s - \int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)}\mathbf{v}_i(s-a)\mathrm{d}s\Big| \\
\le\ & \mathrm{e}^{-\frac{\alpha}{2}t}|\mathbf{g}_i(0) - \mathrm{e}^{-\alpha a}\mathbf{g}_j(0)| + \Big|\alpha \mathrm{e}^{-\alpha t}\int_{-a}^0 \mathrm{e}^{\alpha s}\mathbf{g}_i(s)\mathrm{d}s\Big| \\
&+\Big[\alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{d}s + \alpha\int_0^{t+a} \mathrm{e}^{-\alpha(t+a-s)}\mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{d}s\Big] \sup_{i,k,s\ge -\tau, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s+a) - \mathbf{v}_i(s)|\} \\
\le\ & \mathrm{e}^{-\frac{\alpha}{2}t}|\mathbf{g}_i(0) - \mathrm{e}^{-\alpha a}\mathbf{g}_j(0)| + \Big|\alpha \mathrm{e}^{-\alpha t}\int_{-a}^0 \mathrm{e}^{\alpha s}\mathbf{g}_i(s)\mathrm{d}s\Big| \\
&+4\mathrm{e}^{-\frac{\alpha}{2}t} \sup_{i,k,s\ge -\tau, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s+a) - \mathbf{v}_i(s)|\}.
\end{aligned}$$

Thus

$$\sup_{i,j,t\ge 0, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}t}|\psi_i(\mathbf{x},\mathbf{v})(t) - \psi_j(\mathbf{x},\mathbf{v})(t+a)|\} < +\infty.$$

Moreover, we have

$$\begin{aligned}
&|\phi_i(\mathbf{x},\mathbf{v})(t) - \phi_j(\mathbf{x},\mathbf{v})(t)| \\
\le\ & \Big|(1-\mathrm{e}^{-\alpha t})\frac{\mathbf{g}_i(0)}{\alpha} - (1-\mathrm{e}^{-\alpha t})\frac{\mathbf{g}_j(0)}{\alpha}\Big| + |\mathbf{f}_i(0) - \mathbf{f}_j(0)| \\
&+\Big|\int_0^t (1-\mathrm{e}^{-\alpha(t-s)}) \sum_{k=1}^N [a_{ik}\mathbf{v}_k(s-\tau_{ik}) - a_{jk}\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{|\mathbf{g}_i(0)-\mathbf{g}_j(0)|}{\alpha}+|\mathbf{f}_i(0)-\mathbf{f}_j(0)|+\int_0^t(1-\mathrm{e}^{-\alpha(t-s)})\sum_{k=1}^N a_{ik}|\mathbf{v}_k(s-\tau_{ik})-\mathbf{v}_i(s)|\mathrm{d}s\\
&\quad+\int_0^t(1-\mathrm{e}^{-\alpha(t-s)})\sum_{k=1}^N a_{jk}|\mathbf{v}_k(s-\tau_{jk})-\mathbf{v}_i(s)|\mathrm{d}s\\
&\le \frac{|\mathbf{g}_i(0)-\mathbf{g}_j(0)|}{\alpha}+|\mathbf{f}_i(0)-\mathbf{f}_j(0)|+2\int_0^t \mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{d}s\sup_{i,k,s\ge-\tau,a\in[0,\tau]}\{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s+a)-\mathbf{v}_i(s)|\}.
\end{aligned}$$

Thus

$$\sup_{i,j,t\ge0}\{|\phi_i(\mathbf{x},\mathbf{v})(t)-\phi_j(\mathbf{x},\mathbf{v})(t)|\}<+\infty.$$

The above arguments show that T is a self-mapping on E.

Step 2: T is a contraction operator on E: In fact, for $(\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})\in E$, we have

$$\begin{aligned}
&|\psi_i(\mathbf{p},\mathbf{q})(t)-\psi_i(\bar{\mathbf{p}},\bar{\mathbf{q}})(t)|\\
&\le \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^N|a_{ij}(\mathbf{p}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\bar{\mathbf{q}}_j(s-\tau_{ij})|\mathrm{d}s\\
&\le \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^N[|a_{ij}(\mathbf{p}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})|\\
&\quad+|a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\bar{\mathbf{q}}_j(s-\tau_{ij})|]\mathrm{d}s\\
&\le \alpha\int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^N(L_a\sup_i|\mathbf{g}_i(0)|\mathrm{e}^{\tau}|\mathbf{p}(s-\tau_{ij}))-\bar{\mathbf{p}}(s-\tau_{ij}))|\\
&\quad+|\mathbf{q}_j(s-\tau_{ij})-\bar{\mathbf{q}}_j(s-\tau_{ij})|)\mathrm{d}s\\
&\le \alpha\sum_{j=1}^N \mathrm{e}^{-c\tau_{ij}}(L_a\sup_i|\mathbf{g}_i(0)|\mathrm{e}^{\tau}+1)\int_0^t \mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{cs}\mathrm{d}sD((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}}))\\
&< \frac{\alpha\sum_{j=1}^N \mathrm{e}^{-c\tau_{ij}}(L_a\sup_i|\mathbf{g}_i(0)|\mathrm{e}^{\tau}+1)}{c+\alpha}\mathrm{e}^{ct}D((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&|\phi_i(\mathbf{p},\mathbf{q})(t)-\phi_i(\bar{\mathbf{p}},\bar{\mathbf{q}})(t)|\\
&\le \int_0^t(1-\mathrm{e}^{-\alpha(t-s)})\sum_{j=1}^N|a_{ij}(\mathbf{p}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\bar{\mathbf{q}}_j(s-\tau_{ij})|\mathrm{d}s\\
&\le \int_0^t\sum_{j=1}^N|a_{ij}(\mathbf{p}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})|\\
&\quad+|a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\mathbf{q}_j(s-\tau_{ij})-a_{ij}(\bar{\mathbf{p}}(s-\tau_{ij}))\bar{\mathbf{q}}_j(s-\tau_{ij})|\mathrm{d}s\\
&\le \int_0^t\sum_{j=1}^N(L_a\sup_i|\mathbf{g}_i(0)|\mathrm{e}^{\tau}|\mathbf{p}(s-\tau_{ij}))-\bar{\mathbf{p}}(s-\tau_{ij}))|+|\mathbf{q}_j(s-\tau_{ij})-\bar{\mathbf{q}}_j(s-\tau_{ij})|)\mathrm{d}s\\
&\le \sum_{j=1}^N \mathrm{e}^{-c\tau_{ij}}(L_a\sup_i|\mathbf{g}_i(0)|\mathrm{e}^{\tau}+1)\int_0^t \mathrm{e}^{cs}\mathrm{d}sD((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}}))
\end{aligned}$$

$$< \frac{\sum_{j=1}^{N} \mathrm{e}^{-c\tau_{ij}} (L_a \sup_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau} + 1)}{c} \mathrm{e}^{ct} D((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})).$$

Thus

$$|\Psi(\mathbf{p},\mathbf{q})(t) - \Psi(\bar{\mathbf{p}},\bar{\mathbf{q}})(t)| \le \frac{\alpha \sup_i \{\sum_{j=1}^{N} \mathrm{e}^{-c\tau_{ij}}\} (L_a \sup_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau} + 1)}{c+\alpha} \mathrm{e}^{ct} D((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})),$$

and

$$|\Phi(\mathbf{p},\mathbf{q})(t) - \Phi(\bar{\mathbf{p}},\bar{\mathbf{q}})(t)| \le \frac{\sup_i \{\sum_{j=1}^{N} \mathrm{e}^{-c\tau_{ij}}\} (L_a \sup_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau} + 1)}{c} \mathrm{e}^{ct} D((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})).$$

Then we have

$$D\left(T\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}, T\begin{pmatrix}\bar{\mathbf{p}}\\ \bar{\mathbf{q}}\end{pmatrix}\right) \le \frac{(\alpha+1) \sup\limits_i \{\sum\limits_{j=1}^{N} \mathrm{e}^{-c\tau_{ij}}\} (L_a \sup\limits_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau} + 1)}{c} D((\mathbf{p},\mathbf{q}),(\bar{\mathbf{p}},\bar{\mathbf{q}})).$$

It follows from $\frac{(\alpha+1) \sup_i \{\sum_{j=1}^{N} \mathrm{e}^{-c\tau_{ij}}\} (L_a \sup_i |\mathbf{g}_i(0)| \mathrm{e}^{\tau} + 1)}{c} < 1$, that T is a strict contractive operator on E. Thus, there exists a unique point $(\mathbf{x},\mathbf{v}) \in E$ such that

$$T\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{v}(t)\end{bmatrix}.$$

Then the fixed point satisfies the following equation

$$\mathbf{x}_i(t) = (1 - \mathrm{e}^{-\alpha t}) \frac{\mathbf{g}_i(0)}{\alpha} + \mathbf{f}_i(0) + \int_0^t (1 - \mathrm{e}^{-\alpha(t-s)}) \sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij})) \mathbf{v}_j(s-\tau_{ij}) \mathrm{d}s, \tag{3.1}$$

and

$$\mathbf{v}_i(t) = \mathrm{e}^{-\alpha t} \mathbf{g}_i(0) + \alpha \int_0^t \mathrm{e}^{-\alpha(t-s)} \sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij})) \mathbf{v}_j(s-\tau_{ij}) \mathrm{d}s. \tag{3.2}$$

□

Corollary 3.1. *If* $a_{ij}(\mathbf{x}) = \frac{\psi(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{x}_i - \mathbf{x}_k|)}$, ψ *is a positive Lipschitz continuous function and Assumption 2.2 hold, then the self-organized system (2.2) with the initial value (2.3) has a unique conditional flocking solution.*

Proof. Since $||t| - |s|| \le |t - s|$ for all $t, s \in \mathbb{R}$, we see that there exists a constant $M > 0$ such that $||\mathbf{x}_i - \mathbf{x}_j| - |\mathbf{p}_i - \mathbf{p}_j|| \le M|\mathbf{x} - \mathbf{p}|$ hold for $\mathbf{x}, \mathbf{p} \in \mathbb{R}^{dN}$. Thus

$$\begin{aligned}
|a_{ij}(\mathbf{x}) - a_{ij}(\mathbf{p})| &= \left|\frac{\psi(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{x}_i - \mathbf{x}_k|)} - \frac{\psi(|\mathbf{p}_i - \mathbf{p}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{p}_i - \mathbf{p}_k|)}\right| \\
&\le \left|\frac{\psi(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{x}_i - \mathbf{x}_k|)} - \frac{\psi(|\mathbf{p}_i - \mathbf{p}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{x}_i - \mathbf{x}_k|)}\right| \\
&\quad + \left|\frac{\psi(|\mathbf{p}_i - \mathbf{p}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{x}_i - \mathbf{x}_k|)} - \frac{\psi(|\mathbf{p}_i - \mathbf{p}_j|)}{\sum_{k=1}^{N} \psi(|\mathbf{p}_i - \mathbf{p}_k|)}\right|
\end{aligned}$$

$$\leq \frac{NML_\psi}{\psi(0)}|\mathbf{x}-\mathbf{p}|,$$

where L_ψ is the Lipschitz constant for ψ. This means that $a_{ij}(\mathbf{x})$ is also a Lipschitz function for all i, j. Then Corollary 3.1 follows from Theorem 3.1 immediately. $\square$

Corollary 3.2. *If* $a_{ij}(\mathbf{x}) = \frac{\psi(|\mathbf{x}_i-\mathbf{x}_j|)}{\sum_{k=1}^N \psi(|\mathbf{x}_i-\mathbf{x}_k|)}$, $\psi(r) = (1+r^2)^{-\beta}$ *for* $r \geq 0$, $\beta > 0$ *and Assumption 2.2 hold, then the self-organized system (2.2) with the initial value (2.3) has a unique conditional flocking solution.*

Proof. Since $|\psi'(r)| = 2\beta r(1+r^2)^{-\beta-1}$ is bounded for $\beta > 0$ on $[0, +\infty)$, then we conclude that $\psi(r)$ is a Lipschitz function. With the similar arguments in Corollary 3.1, we see that $a_{ij}(\mathbf{x})$ is also a Lipschitz function for all i, j. Then Corollary 3.2 follows from Theorem 3.1 immediately. $\square$

4. Asymptotic flocking velocity formula

In this subsection, we investigate the asymptotic flocking velocities and the final relative position between agents. We state our results as follows.

Lemma 4.1. *If the Assumption 2.1 and Assumption 2.2 hold,* $\{(\mathbf{x}_i(t), \mathbf{v}_i(t))\}_{i=1}^N$ *were the flocking solution of the system (2.2) -(2.3) in* E*, then the two limits*

$$\mathbf{w}_i = \lim_{t\to+\infty}\int_0^t \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij}) - \mathbf{v}_i(s-\tau_i))\mathrm{d}s \tag{4.1}$$

and

$$\mathbf{c}_{ij} = \lim_{t\to+\infty}\int_0^t \sum_{k=1}^N [a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_k(s-\tau_{ik}) - a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s \tag{4.2}$$

exists.

Proof. It follows from

$$\begin{aligned}
&|\int_0^t \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij}) - \mathbf{v}_i(s-\tau_i))\mathrm{d}s| \\
&\leq \int_0^t \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))|\mathbf{v}_j(s-\tau_{ij}) - \mathbf{v}_i(s-\tau_i)|\mathrm{d}s \\
&\leq \int_0^t \mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{d}s \sup_{i,j,s\geq-\tau,a\in[0,\tau]}\{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_j(s) - \mathbf{v}_i(s+a)|\} < +\infty.
\end{aligned}$$

That the limit

$$\lim_{t\to\infty}\int_0^t \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij}) - \mathbf{v}_i(s-\tau_i))\mathrm{d}s$$

exists. Note that

$$\sum_{k=1}^N a_{ik}(\mathbf{x}(s-\tau_{ik})) = \sum_{k=1}^N a_{jk}(\mathbf{x}(s-\tau_{jk})) = 1,$$

then

$$
\begin{aligned}
&|\int_0^t \sum_{k=1}^N [a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_k(s-\tau_{ik}) - a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s| \\
=& |\int_0^t \sum_{k=1}^N [a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_k(s-\tau_{ik}) - a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_i(s-\tau) \\
&+a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_i(s-\tau) - a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s| \\
\le& \int_0^t \sum_{k=1}^N a_{ik}(\mathbf{x}(s-\tau_{ik}))|\mathbf{v}_k(s-\tau_{ik}) - \mathbf{v}_i(s-\tau)|\mathrm{d}s \\
&+ \int_0^t \sum_{k=1}^N a_{jk}(\mathbf{x}(s-\tau_{jk}))|\mathbf{v}_i(s-\tau) - \mathbf{v}_k(s-\tau_{jk})|\mathrm{d}s \\
\le& 2N \int_0^t \mathrm{e}^{-\frac{\alpha}{2}s}\mathrm{d}s \sup_{i,k,s\ge -\tau, a\in[0,\tau]} \{\mathrm{e}^{\frac{\alpha}{2}s}|\mathbf{v}_k(s) - \mathbf{v}_i(s+a)|\} < +\infty.
\end{aligned}
$$

Thus the limit

$$
\lim_{t\to+\infty} \int_0^t \sum_{k=1}^N [a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_k(s-\tau_{ik}) - a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s
$$

exists. □

Theorem 4.1. *If the Assumption 2.1 and Assumption 2.2 hold and $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^N$ were given by formula (3.1) and (3.2), respectively, then there is a $\mathbf{v}_\infty$ such that*

$$
\begin{aligned}
&\lim_{t\to+\infty} \mathbf{v}_i(t) = \frac{\mathbf{g}_i(0) + \alpha\mathbf{w}_i}{1+\alpha\tau_i} + \frac{\alpha}{1+\alpha\tau_i}[\mathbf{f}_i(0) - \mathbf{f}_i(-\tau_i)] \triangleq \mathbf{v}_\infty. \\
&\lim_{t\to+\infty} [\mathbf{x}_i(t) - \mathbf{x}_i(t-a)] = a\mathbf{v}_\infty, \text{for given constant } a \in \mathbb{R}.
\end{aligned}
$$

The final position relationship between agents can be expressed as follows:

$$
\lim_{t\to+\infty} [\mathbf{x}_i(t) - \mathbf{x}_j(t)] = \frac{\mathbf{g}_i(0) - \mathbf{g}_j(0)}{\alpha} + [\mathbf{f}_i(0) - \mathbf{f}_j(0)] + \mathbf{c}_{ij},
$$

where $\mathbf{w}_i$ and $\mathbf{c}_{ij}$ are given by (4.1) and (4.2) respectively.

Proof. If the Assumption 2.1 and Assumption 2.2 hold, following Theorem 3.1, we have $\lim_{t\to+\infty} |\mathbf{v}_j(t) - \mathbf{v}_i(t)| = 0$ for all i, j. Thus there is a vector $\mathbf{v}_\infty$ such that $\lim_{t\to+\infty} \mathbf{v}_i(t) = \mathbf{v}_\infty$ for all i. For given constant $a > 0$, noting the (3.1), we have

$$
\begin{aligned}
&\mathbf{x}_i(t) - \mathbf{x}_i(t-a) \\
=& \mathrm{e}^{-\alpha t}(\mathrm{e}^{\alpha a} - 1)\frac{\mathbf{g}_i(0)}{\alpha} + \int_0^t (1 - \mathrm{e}^{-\alpha(t-s)}) \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&- \int_0^{t-a} (1 - \mathrm{e}^{-\alpha(t-a-s)}) \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{-\alpha t}(\mathrm{e}^{\alpha a}-1)\frac{\mathbf{g}_i(0)}{\alpha} + \int_{t-a}^{t}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&\quad - \int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&\quad + \int_0^{t-a} \mathrm{e}^{-\alpha(t-a-s)}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s.
\end{aligned}
$$

It follows from

$$
\lim_{t\to+\infty}\int_{t-a}^{t}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij})-\mathbf{v}_\infty)\mathrm{d}s = 0.
$$

That

$$
\begin{aligned}
&\lim_{t\to+\infty}\int_{t-a}^{t}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&= \lim_{t\to+\infty}\int_{t-a}^{t}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij})-\mathbf{v}_\infty)\mathrm{d}s + a\mathbf{v}_\infty \\
&= a\mathbf{v}_\infty.
\end{aligned}
$$

Also, by

$$
\begin{aligned}
&\lim_{t\to+\infty}\int_0^{t} \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&= \lim_{t\to+\infty}\int_0^{t-a} \mathrm{e}^{-\alpha(t-a-s)}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&= \frac{\mathbf{v}_\infty}{\alpha}.
\end{aligned}
$$

We conclude that

$$
\lim_{t\to+\infty}(\mathbf{x}_i(t)-\mathbf{x}_i(t-a)) = a\mathbf{v}_\infty.
$$

On the other hand, by direct computation, we have

$$
\begin{aligned}
&\int_0^t (1-\mathrm{e}^{-\alpha(t-s)})\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s \\
&= \int_0^t \sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij})-\mathbf{v}_i(s-\tau_i))\mathrm{d}s + \mathbf{x}_i(t-\tau_i) - \mathbf{f}_i(-\tau_i) \\
&\quad - \int_0^t \mathrm{e}^{-\alpha(t-s)}\sum_{j=1}^{N} a_{ij}(\mathbf{x}(s-\tau_{ij}))\mathbf{v}_j(s-\tau_{ij})\mathrm{d}s.
\end{aligned}
$$

Thus, substituting it into (3.1) and letting t go to infinity, we have:

$$
\tau_i\mathbf{v}_\infty = \frac{\mathbf{g}_i(0)+\alpha\mathbf{w}_i}{\alpha} + \mathbf{f}_i(0) - \mathbf{f}_i(-\tau_i) - \frac{\mathbf{v}_\infty}{\alpha}.
$$

Then we have

$$\lim_{t\to+\infty}\mathbf{v}_i(t)=\mathbf{v}_\infty=\frac{\mathbf{g}_i(0)+\alpha\mathbf{w}_i}{1+\alpha\tau_i}+\frac{\alpha}{1+\alpha\tau_i}[\mathbf{f}_i(0)-\mathbf{f}_i(-\tau_i)].$$

At last, by direct calculation, we see that

$$\begin{aligned}\mathbf{x}_i(t)-\mathbf{x}_j(t)&=(1-\mathrm{e}^{-\alpha t})\frac{\mathbf{g}_i(0)-\mathbf{g}_j(0)}{\alpha}+[\mathbf{f}_i(0)-\mathbf{f}_j(0)]\\&\quad+\int_0^t\sum_{k=1}^N[a_{ik}(\mathbf{x}(s-\tau_{ik}))\mathbf{v}_k(s-\tau_{ik})-a_{jk}(\mathbf{x}(s-\tau_{jk}))\mathbf{v}_k(s-\tau_{jk})]\mathrm{d}s,\end{aligned}$$

letting $t\to+\infty$, we have

$$\lim_{t\to+\infty}[\mathbf{x}_i(t)-\mathbf{x}_j(t)]=\frac{\mathbf{g}_i(0)-\mathbf{g}_j(0)}{\alpha}+[\mathbf{f}_i(0)-\mathbf{f}_j(0)]+\mathbf{c}_{ij}.$$

□

Remark 4.1. In Theorem 4.1, if $a_{ij}=a_{ji},\tau_{ij}=\tau$ for all i,j, then $\sum_{i=1}^N\mathbf{w}_i=0$. Thus

$$\mathbf{v}_\infty=\frac{\sum_{i=1}^N\mathbf{g}_i(0)}{N(1+\alpha\tau)}+\frac{\alpha}{N(1+\alpha\tau)}\sum_{i=1}^N[\mathbf{f}_i(0)-\mathbf{f}_i(-\tau)].$$

Especially, $\mathbf{v}_\infty=\frac{1}{N}\sum_{i=1}^N\mathbf{g}_i(0)$ if time delay free.

5. Conclusion

In this paper, we investigated the flocking problem of a modified C-S model with multiple time delays. By the fixed point theorem, we show that the existence and uniqueness of the flocking solution for our delayed C-S model when the influence function has the property of Lipschitz and the initial value satisfies certain conditions. We present the asymptotic flocking velocity and the final relative position between agents of the unique flocking solution.

Acknowledgements. The authors are grateful to the anonymous referees for their useful comments and suggestions.

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