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A GEOMETRICALLY CONVERGENT PSEUDO–SPECTRAL METHOD FOR MULTI–DIMENSIONAL TWO–SIDED SPACE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract In this study, we present a geometrically convergent numerical method for solving multi–dimensional two–sided space–time fractional differential equations. The approach represents the solutions of the differential equations in terms of the shifted Chebyshev polynomials. The expansions are evaluated at the shifted Chebyshev–Gauss–Lobatto nodes. We present the approximations for left–sided integration, and the left– and right– sided differentiation. The performance of the method is demonstrated using some two–sided space fractional partial differential equations in one and two dimensions. The numerical results obtained show that the method is accurate and computationally efficient. A theoretical analysis of the convergence of the method is presented, where it is shown that, given a continuously differentiable solution, the numerical solution converges for a sufficiently large number of grid points.

Keywords Shifted Chebyshev polynomials, Chebyshev–Gauss–Lobatto quadrature, two–sided space–time fractional partial differentials, fractional calculus, convergence analysis.

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1. Introduction

Interest in the description of physical processes using models with arbitrary non–integer order derivatives has grown tremendously over the last few decades. Physical models, including the diffusion equation [9,19], the wave equation [6,29], viscoelastic fluid models [18,24,27,28], flows in porous medium [12,25], hydrological models [2] and chaotic systems [13,30], have been generalized using fractional order derivatives. Arbitrary non–integer order differential equations can model and describe physical parameters of intricate systems [21,26]. The global nature of the time fractional derivative is fundamental in describing long term memory characteristics of dynamical systems, where these are inherently present. In physical applications, particularly in diffusion and dispersion processes, space fractional derivatives are

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used to model anomalous diffusion, in which there is a nonlinear relationship between the mean squared displacement and time. Of interest, is the two–sided space fractional derivatives which allow the effects from either side of the physical domain of a process to be measured [22]. The literature shows that physical processes that are modelled with two–sided space fractional derivatives have been studied extensively.

In this study, we present a numerical scheme for approximating the solution of two–sided fractional differential equations. Due to the complexity of fractional differential operators, obtaining analytical solutions of fractional differential equations can be demanding, thus necessitating the use of numerical methods. Published articles on the numerical methods for solving two–sided fractional partial differential equations are relatively few. Meerschaert and Tedjeran [20] used an implicit Euler method with the shifted Grünwald formulae to solve the two–sided space time dependent fractional partial differential equations. The shifted Grünwald formulae was used to circumvent the instability that is associated with the finite difference schemes based on the left– and right–sided Grünwald–Letnikov formulas. In Chen and Deng [9], an alternating direct implicit method was proposed for two–sided space fractional convection diffusion equations. The numerical scheme was based on the linear spline approximation of both the right– and left–sided Riemann–Liouville fractional operators. In Liu et. al [17], a semi implicit form of the method proposed by Chen and Deng [9] was used to solve a two–dimensional FitzHugh–Nagumo monodomain model defined by the Riesz space fractional differential operator. Using the shifted Grünwald formulae proposed by Meerschaert and Tedjeran [20], explicit and implicit difference schemes for the space–time fractional advection–diffusion equation were described by Liu et. al [16]. In Feng et. al [15], a finite volume method for the two–sided space fractional diffusion equation based on nodal basis functions was presented. An implicit finite volume method was used, and the resulting discrete system was solved in matrix form. Chen et. al [10] used a semi–implicit difference method, and an iterative procedure developed by decomposing the dense coefficient matrix into a combination of Toeplitz–like matrices, thus diminishing the storage requirement and high computational cost associated with difference methods.

One of the many challenges associated with obtaining accurate numerical solutions of fractional differential equations is the high computational costs. Owing to the non–local nature of fractional differential operators, non–local methods are best suited in approximating solutions of fractional differential equations. One such non–local method is the spectral method. Several versions of the spectral method have been developed and applied to solve one–sided fractional differential equations [see Refs. [3, 14, 23, 31] and the references therein]. However, studies on spectral methods for two–sided fractional differential equation are comparatively uncommon. Among the few such studies are Bhrawy et. al [4], Mao and Shen [19] and Bhrawy et. al [5].

In this study, we consider a general two–sided space time fractional partial differential equations of the form

$$ {}_0D_t^{\gamma}u(\mathbf{x},t) = \sum_{\iota=1}^{d}\left[c_{+,\iota}(\mathbf{x})\,{}_0D_{x_\iota}^{\alpha_\iota}u(\mathbf{x},t) + c_{-,\iota}(\mathbf{x})\,{}_{x_\iota}D_{\mathsf{L}_\iota}^{\alpha_\iota}u(\mathbf{x},t)\right] + s(\mathbf{x},t), \quad 0 < t \le \mathsf{T}, \tag{1.1}$$

on the finite domain $0 < x_\iota < \mathsf{L}_\iota$, such that $x_\iota \in \mathbb{R}^d$, where $c_{+,\iota}(\mathbf{x}), c_{-,\iota}(\mathbf{x})$ are variable coefficients and $s(\mathbf{x},t)$ is the source function.

Equation (1.1) above has been adopted for modelling concentration distribution in a Levy diffusion process, which has applications in modelling flows through a porous medium and diffusion of contaminants. Equation (1.1) is defined in its general form, with $d = 1, d = 2$ respectively corresponding to the one and two–dimensional cases. For two–dimensional cases, we redefine the spatial variables $x_1 = x$ and $x_2 = y$, and the fractional orders $\alpha_1 = \alpha$ and $\alpha_2 = \beta$. In the case of the one–dimensional two–sided space time fractional differential equation, all subscripts on α, x, L are dropped. The left–sided fractional derivative ${}_0D_x^\alpha$ of any function at a point x depends on all the values of the function to the left of x. In like manner, the right–sided fractional derivative ${}_xD_{\mathsf{L}}^\alpha$ of any function at a point x depends on all values of the function to the right of x. These two fractional operators are not identical, except, on the one hand, when α is an even integer; in which case, the derivatives are classical and equal. On the other hand, when α is an odd integer, in this case, the two operators are local and opposite in sign [20]. In this study, we introduce a spectral method to solve two–sided space–time fractional differential equations of the form in Equation (1.1). The method involves approximating $u(\mathbf{x}, t)$ in terms of shifted first kind Chebyshev polynomials integrated at the Gauss–Lobatto quadrature. We perform a convergence analysis of the method for the general two–sided fractional differential equation. We present some examples to illustrate the effectiveness of the method and compare the results obtained with those from closed-form solutions, and where possible, with those obtained in past studies.

The remainder of the article is structured as follows. In Section 2, we present preliminary notations of fractional operators and properties of the shifted first kind Chebyshev polynomials. In Section 3, we described the two–sided fractional differential approximations of the left and right–sided Caputo fractional operators. In Section 4, we present the general convergence of the numerical scheme, and in Section 5, we demonstrated the numerical method with selected examples. The article is concluded in Section 6.

2. Preliminaries and Notations

In this section, we provide definitions of fractional operators, the properties of the shifted Chebyshev polynomial of the first kind and other important concepts that are relevant to this study.

2.1. Fractional operators

Definition 2.1. Consider a function $f(x) : [0, \mathsf{L}] \to \mathbb{R}$, which is integrable and continuously differentiable in $\mathbb{R}$. The left–sided Riemann–Liouville fractional integrals of $f(x)$ of any arbitrary non–integer order α are defined as [21,26]

$$ {}_0I_x^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\tilde{x})}{(x-\tilde{x})^{1-\alpha}} d\tilde{x}, \quad \alpha > 0,\ x > 0, \tag{2.1} $$

where Γ is the Euler gamma function. For the case $\alpha = 0$, we have

$$ {}_0I_x^0 f(x) := f(x). \tag{2.2} $$

We define the fractional integral of a power function x^j as

$$ {}_0I_x^\alpha x^j = \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} x^{j+\alpha}. \tag{2.3} $$

Definition 2.2. For any $\alpha \in \mathbb{R}^+$, such that $n-1 \leq \alpha < n$, with $n \in \mathbb{N}^+$, the left–sided Riemann–Liouville and left–sided Caputo fractional derivatives of $f(x)$, a continuously differentiable function, are defined as [21, 26]

$$ {}_0^{RL}D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(\tilde{x})}{(x-\tilde{x})^{\alpha+1-n}} d\tilde{x} \tag{2.4} $$

and

$$ {}_0^{C}D_x^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^n(\tilde{x})}{(x-\tilde{x})^{\alpha+1-n}} d\tilde{x}, \quad x > 0, \tag{2.5} $$

respectively. The corresponding right–sided Riemann–Liouville and right–sided Caputo fractional derivatives are defined as [21, 26]

$$ {}_x^{RL}D_{\mathsf{L}}^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^{\mathsf{L}} \frac{f(\tilde{x})}{(\tilde{x}-x)^{\alpha+1-n}} d\tilde{x} \tag{2.6} $$

and

$$ {}_x^{C}D_{\mathsf{L}}^\alpha f(x) := \frac{1}{\Gamma(n-\alpha)} \int_x^{\mathsf{L}} \frac{f^n(\tilde{x})}{(\tilde{x}-x)^{\alpha+1-n}} d\tilde{x}, \quad x < \mathsf{L}, \tag{2.7} $$

respectively.

In both the Riemann–Liouville and Caputo definitions, the fractional operators ${}_0^{RL}D_x^\alpha, {}_0^{C}D_x^\alpha \to d^n/dx^n$ as $\alpha \to n$, thus recapturing the classical derivative with respect to x. The fractional operators defined above can be rewritten for multivariate functions. Additionally, from the Caputo derivatives, we have the following properties, which are important in the proposition of the numerical scheme

$$ {}_0^{C}D_x^\alpha x^j = \begin{cases} 0 & j \in \mathbb{N}_0,\ j < \lceil \alpha \rceil, \\ \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}, & j \in \mathbb{N}_0,\ j \geq \lceil \alpha \rceil, \end{cases} \tag{2.8} $$

and

$$ {}_x^{C}D_{\mathsf{L}}^\alpha (x-\mathsf{L})^j = \begin{cases} 0 & j \in \mathbb{N}_0,\ j < \lceil \alpha \rceil, \\ \\ (-1)^j \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} (\mathsf{L}-x)^{j-\alpha}, & j \in \mathbb{N}_0,\ j \geq \lceil \alpha \rceil. \end{cases} \tag{2.9} $$

2.2. The shifted Chebyshev polynomials

The Chebyshev polynomials $T_n(x) = \cos(n \arccos x), n = 0, 1, \ldots$ are eigenvalue solutions of the Sturm–Liouville eigenvalue problem [1]

$$ \left(\sqrt{1-x^2}\, T_n'(x)\right)' + \frac{n^2}{\sqrt{1-x^2}} T_n(x) = 0, \quad -1 \leq x \leq 1,\ n = 0, 1, \ldots. \tag{2.10} $$

If we consider a change in variable $x = 2\tilde{x}/\mathsf{L} - 1$, such that $x \in [-1, 1] \mapsto x \in [0, \mathsf{L}]$, then we have the shifted Chebyshev polynomials generated through recursive formula

$$\tilde{T}_{\mathsf{L},n+1}(\tilde{x}) = 2\left(\frac{2\tilde{x}}{\mathsf{L}} - 1\right)\tilde{T}_{\mathsf{L},n}(\tilde{x}) - \tilde{T}_{\mathsf{L},n-1}(\tilde{x}), \quad 0 \le \tilde{x} \le \mathsf{L},\ n = 1, 2, \dots, \tag{2.11}$$

where $\tilde{T}_{\mathsf{L},0}(\tilde{x}) = 1$ and $\tilde{T}_{\mathsf{L},1}(\tilde{x}) = 2\tilde{x}/\mathsf{L} - 1$. The shifted Chebyshev polynomials is given in series form as (tilde dropped for simplicity) [1,14]

$$T_{\mathsf{L},n}(x) = n\sum_{j=0}^{n} \frac{(-1)^{n-j}(n+j-1)!2^{2j}}{(n-j)!(2j)!\mathsf{L}^j} x^j \tag{2.12}$$

$$= n\sum_{j=0}^{n} \frac{(n+j-1)!2^{2j}}{(n-j)!(2j)!\mathsf{L}^j} (x-\mathsf{L})^j. \tag{2.13}$$

These sets of polynomials satisfy the weighted $L^2_{w_\mathsf{L}}(0, \mathsf{L})$ – orthogonality condition

$$\int_0^{\mathsf{L}} T_{\mathsf{L},n}(x)T_{\mathsf{L},m}(x)w_{\mathsf{L}}(x)dx = \delta_{mn}h_n, \tag{2.14}$$

where $w_{\mathsf{L}}(x) = 1/\sqrt{\mathsf{L}x - x^2}$ is the weight function of the shifted polynomial, $h_n = c_n\pi/2$, such that $c_0 = 2, c_n = 1$ for $n \ge 1$. In approximating on the Chebyshev–Gauss–Lobatto quadrature, we define the Christoffel number as $\varpi_j = \pi/c_jN,\ 0 \le j \le N$, with $c_0 = c_N = 2$ and $c_j = 1$ for $j = 1, 2, \dots, N-1$.

3. Shifted Chebyshev spectral approximation

In this section, we describe the left and right sided fractional differential approximations. The approximation of the left–sided fractional differential operator is used whenever we approximate the temporal derivative. In general, we seek a solution in terms of the shifted Chebyshev polynomials:

$$u(\mathbf{x}, t) \approx U(\mathbf{x}, t) = \sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t} \hat{U}_{n_1,n_2}T_{\mathsf{L},n_1}(\mathbf{x})T_{\mathsf{T},n_2}(t), \quad \mathbf{x} \in [0, \mathsf{L}],\ t \in [0, \mathsf{T}], \tag{3.1}$$

where $\hat{U}_{n_1,n_2} : 0 \le n_1 \le N_{\mathbf{x}}, 0 \le n_2 \le N_t$ satisfies the orthogonality condition

$$\hat{U}_{n_1,n_2} = \frac{1}{h_{n_1}}\frac{1}{h_{n_2}}\int_0^{\mathsf{L}}\int_0^{\mathsf{T}} U(\mathbf{x}, t)T_{\mathsf{L},n_1}(\mathbf{x})T_{\mathsf{T},n_2}(t)w_{\mathsf{L}}(\mathbf{x})w_{\mathsf{T}}(t)d\mathbf{x}dt, \tag{3.2}$$

and can be represented in discrete form as

$$\hat{U}_{n_1,n_2} = \frac{1}{h_{n_1}}\frac{1}{h_{n_2}}\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t} \varpi_{j_1}\varpi_{j_2}U(\mathbf{x}_{j_1}, t_{j_2})T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{T},n_2}(t_{j_2}). \tag{3.3}$$

Therefore, the approximate solution for $u(\mathbf{x}, t)$ is

$U(\mathbf{x}, t) =$

$$\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t}\left[\varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{L},n_1}(\mathbf{x}_{p_1})T_{\mathsf{T},n_2}(t_{j_2})T_{\mathsf{T},n_2}(t_{p_2})\right]U(\mathbf{x}_{j_1},t_{j_2})$$
$$p_1=0,1,\ldots,N_{\mathbf{x}},\ p_2=0,1,\ldots,N_t. \tag{3.4}$$

Equation (3.4) can be rewritten to approximate a function of three variables by simply expanding it in terms of shifted Chebyshev polynomials in three variables.

Theorem 3.1. *Consider a smooth function $u(\mathbf{x},t)$ approximated in terms of the truncated shifted Chebyshev polynomials. The left–sided fractional integration of $u(\mathbf{x},t)$ with respect to t is given by [24].*

$$\begin{aligned}{}_0I_t^{\gamma}U(\mathbf{x},t)=\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t}\Bigg[\varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\sum_{k=0}^{N_t}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{L},n_1}(\mathbf{x}_{p_1})T_{\mathsf{T},n_2}(t_{j_2})\\ {}^{left}I^{(\gamma)}_{n_2,k}T_{\mathsf{T},k}(t_{p_2})\Bigg]U(\mathbf{x}_{j_1},t_{j_2}),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}{}^{left}I^{(\gamma)}_{n_2,k}=&n_2\sum_{j_2=0}^{n_2}\frac{(-1)^{n_2-j_2}(n_2+j_2-1)!2^{2j_2}\mathsf{T}^{\gamma}}{(n_2-j_2)!(2j_2)!}\frac{\Gamma(j_2+1)}{\Gamma(j_2+\gamma+1)}\\ &\times\frac{2k}{\sqrt{\pi}c_k}\sum_{r=0}^{k}\frac{(-1)^{k-r}(k+r-1)!2^{2r}\Gamma(j_2+r+\gamma+1/2)}{(k-r)!(2r)!\Gamma(j_2+\gamma+r+1)}.\end{aligned} \tag{3.6}$$

Proof. The proof follows from applying the fractional integral operator (2.1) to the approximate solution Equation (3.4), such that the arbitrary non–integer order integration of the shifted first kind Chebyshev polynomial is given as

$${}^{left}I^{(\gamma)}_{n_2,k}T_{\mathsf{T},k}(t)=n_2\sum_{j_2=0}^{n_2}\frac{(-1)^{n_2-j_2}(n_2+j_2-1)!2^{2j_2}}{(n_2-j_2)!(2j_2)!\mathsf{T}^{j_2}}{}_0I_t^{\gamma}t^{j_2} \tag{3.7}$$

$$=n_2\sum_{j_2=0}^{n_2}\frac{(-1)^{n_2-j_2}(n_2+j_2-1)!2^{2j_2}}{(n_2-j_2)!(2j_2)!\mathsf{T}^{j_2}}\frac{\Gamma(j_2+1)}{\Gamma(j_2+\gamma+1)}t^{j_2+\gamma}. \tag{3.8}$$

We can approximate $t^{j_2+\gamma}$ in terms of series expansion of the Chebyshev polynomials of the first kind, so that we have

$$t^{j_2+\gamma}=\sum_{k=0}^{\infty}\Theta_{j_2,k}T_{\mathsf{T},k}(t), \tag{3.9}$$

where $\Theta_{j_2,k}$ satisfies the orthogonality condition in (3.2) and is given as

$$\Theta_{j,k}=\frac{2k}{\sqrt{\pi}c_k}\sum_{r=0}^{k}\frac{(-1)^{k-r}(k+r-1)!2^{2r}T^{j_2+\gamma}}{(k-r)!(2r)!}\frac{\Gamma(j_2+r+\gamma+1/2)}{\Gamma(j_2+r+\gamma+1)}. \tag{3.10}$$

Substituting Equation (3.10) into Equation (3.8) results in Equation (3.6).

We define the approximation of the left–sided integration of $u(\mathbf{x},t)$ as

$${}_0I_t^{\gamma}U(\mathbf{x},t)=\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t}{}^{left}\mathbf{I}^{\gamma}_{j2,p2}U(\mathbf{x}_{j_1},t_{j_2}),\quad p_2=0,\ldots,N_t, \tag{3.11}$$

and the left–sided fractional integration approximation is defined as

$$
\begin{aligned}
&{}^{left}\mathbf{I}^{\gamma}_{j2,p2}\\
=&\varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\sum_{k=0}^{N_t}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{L},n_1}(\mathbf{x}_{p_1})T_{\mathsf{T},n_2}(t_{j_2})^{left}I^{(\gamma)}_{n_2,k}T_{\mathsf{T},k}(t_{p_2})\\
&p_1,j_1=0,1,\dots N_{\mathbf{x}},\ p_2,j_2=0,1,\dots,N_t.
\end{aligned}
\tag{3.12}
$$

This completes the proof of the theorem. □

Theorem 3.2. *The left–sided fractional derivative of $u(\mathbf{x},t)$ with respect to x, where $u(\mathbf{x},t)$ is a continuously bounded and integrable function is given as*

$$
\begin{aligned}
{}_0^C D^{\alpha}_{\mathbf{x}}U(\mathbf{x},t)=\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t}\Bigg[&\varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\sum_{k=0}^{N_{\mathbf{x}}}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{T},n_2}(t_{j_2})T_{\mathsf{T},n_2}(t_{p_2})\\
&{}^{left}D^{(\alpha)}_{n_1,k}T_{\mathsf{L},k}(\mathbf{x}_{p_1})\Bigg]U(\mathbf{x}_{j_1},t_{j_2}),
\end{aligned}
\tag{3.13}
$$

where

$$
{}^{left}D^{(\alpha)}_{n_1,k}=n_1\sum_{j_1=0}^{n_1}\frac{(-1)^{n_1-j_1}(n_1+j_1-1)!2^{2j_1}}{(n_1-j_1)!(2j_1)!\mathsf{L}^{j_1}}\frac{\Gamma(j_1+1)}{\Gamma(j_1-\alpha+1)}q_{j_1,k},
\tag{3.14}
$$

and

$$
q_{j_1,k}=\begin{cases}0, & j_1=0,\dots\lceil\alpha\rceil-1,\\ \frac{k\sqrt{\pi}}{h_k}\sum_{r=0}^{k}\frac{(-1)^{k-r}(k+r-1)!2^{2r}}{(k-r)!(2r)!}\mathsf{L}^{j_1-\alpha}\frac{\Gamma(j_1-\alpha+r+1/2)}{\Gamma(j_1-\alpha+r+1)}, \\ j_1=\lceil\alpha\rceil,\dots,N,k=0,\dots,N.\end{cases}
\tag{3.15}
$$

Proof. See [14, 23] for the derivation of $q_{j_1,k}$ and ${}^{left}D^{(\alpha)}_{n_1,k}$. Therefore, the left–sided derivative of $u(\mathbf{x},t)$ with respect to $\mathbf{x}$, approximated on the shifted Chebyshev–Gauss–Lobatto quadrature, is defined as

$$
{}_0^C D^{\alpha}_{\mathbf{x}}U(\mathbf{x},t)=\sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t}{}^{left}\mathbf{D}^{\alpha}_{j_1,p_1}U(\mathbf{x}_{j_1},t_{j2}),\quad p_1=0,\dots,N_{\mathbf{x}},
\tag{3.16}
$$

such that the left–sided fractional differential approximation with respect to the variable $\mathbf{x}$ is given as

$$
\begin{aligned}
&{}^{left}\mathbf{D}^{\alpha}_{j_1,p_1}\\
=&\varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\sum_{k=0}^{N_{\mathbf{x}}}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{T},n_2}(t_{j_2})T_{\mathsf{T},n_2}(t_{p_2})^{left}D^{(\alpha)}_{n_1,k}T_{\mathsf{L},k}(\mathbf{x}_{p_1}).
\end{aligned}
\tag{3.17}
$$

□

Theorem 3.3. *Let $T_{\mathsf{L},n}(x)$ be a shifted Chebyshev polynomial of order n, then any arbitrary non–integer order derivative based on the right–sided Caputo's operator is defined as*

$$ {}_{x}^{C}D_{\mathsf{L}}^{\alpha} T_{\mathsf{L},n}(x) = \sum_{k=0}^{N} {}^{right}D_{n,k}^{(\alpha)} T_{\mathsf{L},k}(x), \tag{3.18}$$

where

$$\begin{aligned} {}^{right}D_{n,k}^{(\alpha)} =& n\sum_{j=0}^{n} \frac{(-1)^j (n+j-1)! 2^{2j}}{(n-j)!(2j)!\mathsf{L}^{\alpha}} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} \\ &\times \frac{2k}{\pi c_k} \sum_{r=0}^{k} \frac{(-1)^r (k+r-1)! 2^{2r}}{(k-r)!(2r)!} \frac{\sqrt{\pi}\Gamma(j-\alpha+r+1/2)}{\Gamma(j-\alpha+r+1)}. \end{aligned} \tag{3.19}$$

Proof. Consider the series form of the shifted Chebyshev polynomials $T_{\mathsf{L},n}(x)$ of order n, given in Equation (2.13). By taking the right–sided Caputo derivative, we have

$$ {}_{x}^{C}D_{\mathsf{L}}^{\alpha} T_{\mathsf{L},n}(x) = n\sum_{j=0}^{n} \frac{(n+j-1)! 2^{2j}}{(n-j)!(2j)!\mathsf{L}^{j}} {}_{x}^{C}D_{\mathsf{L}}^{\alpha} (x-\mathsf{L})^j \tag{3.20}$$

$$ = n\sum_{j=\lceil\alpha\rceil}^{n} \frac{(n+j-1)! 2^{2j}}{(n-j)!(2j)!\mathsf{L}^{j}} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} (\mathsf{L}-x)^{j-\alpha}, \; n = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \ldots, \tag{3.21}$$

where $(\mathsf{L}-x)^{j-\alpha}$ can be expressed in terms of the truncated shifted Chebyshev polynomials as

$$(\mathsf{L}-x)^{j-\alpha} = \sum_{k=0}^{N} \phi_{j,k} T_{\mathsf{L},k}(x), \tag{3.22}$$

and $\phi_{j,k}$ satisfies the orthogonality condition of the shifted Chebyshev polynomials. This is given as

$$\phi_{j,k} = \frac{2}{\pi c_k} \int_0^{\mathsf{L}} (\mathsf{L}-x)^{j-\alpha} T_{\mathsf{L},k}(x) w_{\mathsf{L}}(x) dx \tag{3.23}$$

$$ = \frac{2k}{\pi c_k} \sum_{r=0}^{k} \frac{(k+r-1)! 2^{2r}}{(k-r)!(2r)!\mathsf{L}^{r}} \int_0^{\mathsf{L}} \frac{(\mathsf{L}-x)^{j-\alpha}}{\sqrt{\mathsf{L}x-x^2}} (x-\mathsf{L})^r dx. \tag{3.24}$$

After evaluating the integral in Equation (3.24) above, we have

$$\phi_{j,k} = \frac{2k}{\pi c_k} \sum_{r=0}^{k} \frac{(k+r-1)! 2^{2r}}{(k-r)!(2r)!\mathsf{L}^{r}} \frac{\mathsf{L}^{j-\alpha+r}\sqrt{\pi}\Gamma(j-\alpha+r+1/2)}{\Gamma(j-\alpha+r+1)}. \tag{3.25}$$

Therefore,

$$(\mathsf{L}-x)^{j-\alpha} = \sum_{k=0}^{N} \frac{2k}{\pi c_k} \sum_{r=0}^{k} \frac{(k+r-1)! 2^{2r}}{(k-r)!(2r)!} \frac{\mathsf{L}^{j-\alpha}\sqrt{\pi}\Gamma(j-\alpha+r+1/2)}{\Gamma(j-\alpha+r+1)} T_{\mathsf{L},k}(x). \tag{3.26}$$

Substituting Equation (3.26) in Equation (3.21) completes the proof. □

Theorem 3.4. *Assume that $u(\mathbf{x},t)$ is a continuously bounded and smooth function, approximated in terms of the truncated shifted Chebyshev polynomials, and integrated over the Gauss–Lobatto quadrature, then its right–sided derivative of arbitrary non–integer order in the Caputo sense is approximated as*

$$ {}_{\mathbf{x}}^{C}D_{\mathsf{L}}^{\alpha}U(\mathbf{x},t) = \sum_{j_1=0}^{N_{\mathbf{x}}}\sum_{j_2=0}^{N_t} {}^{right}\mathbf{D}_{j_1,p_1}^{\alpha} U(\mathbf{x}_{j_1}, t_{j2}), \quad p_1 = 0,\dots,N_{\mathbf{x}} \tag{3.27}$$

where

$$\begin{aligned} &{}^{right}\mathbf{D}_{j_1,p_1}^{\alpha} \\ &= \varpi_{j_1}\varpi_{j_2}\sum_{n_1=0}^{N_{\mathbf{x}}}\sum_{n_2=0}^{N_t}\sum_{k=0}^{N_{\mathbf{x}}}\frac{1}{h_{n_1}}\frac{1}{h_{n_2}}T_{\mathsf{L},n_1}(\mathbf{x}_{j_1})T_{\mathsf{T},n_2}(t_{j_2})T_{\mathsf{T},n_2}(t_{p_2})^{right}D_{n_1,k}^{(\alpha)}T_{\mathsf{L},k}(\mathbf{x}_{p_1}). \end{aligned} \tag{3.28}$$

Proof. Using the result of Theorem 3.3 and Equation (3.4), the theorem is proven. □

Remark 3.1. Each approximation leads to a system of algebraic equations that are approximated at the shifted Chebyshev–Gauss–Lobatto nodes. The approximations here are presented for functions of two variables, and this can be extended for trivariate functions.

Using the results in this section, we can obtain the discretization of the two–sided space–time fractional differential Equation (1.1) in both the (1+1) and (2+1) dimensions.

4. Convergence analysis

This section discusses the general convergence of the numerical scheme for two–sided fractional differential equation. We demonstrate the convergence in the weighted $L^2_{w_\mathsf{T}}(\Omega_t; L^2_{w_\mathsf{L}}(\Omega_{\mathbf{x}}))$ norm, where $\Omega_t = [0,\mathsf{T}]$ and $\Omega_{\mathbf{x}} = [0,\mathsf{L}]$. We restrict the discussion to the general two–sided fractional partial differential equation (1.1). It is however important to establish certain results first.

Definition 4.1. We define $H^p_{w_\mathsf{L}}(\Omega_{\mathbf{x}}), p \geq 0$, the weighted Sobolev space of a function $u(\mathbf{x})$ [7]

$$H^p_{w_\mathsf{L}}(\Omega_{\mathbf{x}}) = \left\{ u \in L^2_{w_\mathsf{L}}(\Omega_{\mathbf{x}}) : \frac{\partial^i u(\mathbf{x})}{\partial \mathbf{x}^i} \in L^2_{w_\mathsf{L}}(\Omega_{\mathbf{x}}), 0 \leq i \leq p \right\}, \tag{4.1}$$

such that the semi–norm and norm associated with the space are, respectively,

$$|u|_{H^p_{w_\mathsf{L}}(\Omega_{\mathbf{x}})} = \left(\left\| \frac{\partial^p u}{\partial \mathbf{x}^p} \right\|^2_{L^2_{w_\mathsf{L}}(\Omega_{\mathbf{x}})} \right)^{1/2}, \ \|u\|_{H^p_{w_\mathsf{L}}(\Omega_{\mathbf{x}})} = \left(\sum_{i=0}^{p} \left\| \frac{\partial^i u}{\partial \mathbf{x}^i} \right\|^2_{L^2_{w_\mathsf{L}}(\Omega_{\mathbf{x}})} \right)^{1/2}. \tag{4.2}$$

Definition 4.2. We define the weighted norm [7, 8]

$$\|u\|_{L^2_w(\Omega)} = \left(\int_0^{\mathsf{T}} \int_0^{\mathsf{L}} |u(\mathbf{x},t)|^2 \, w_\mathsf{L}(\mathbf{x}) w_\mathsf{T}(t) d\mathbf{x} dt \right)^{1/2}, \tag{4.3}$$

where $L^2_w(\Omega) = L^2_{w_{\mathsf{T}}}(\Omega_t; L^2_{w_{\mathsf{L}}}(\Omega_{\mathbf{x}}))$. We define the weighted Hilbert space

$$\begin{aligned} H^{p,q}_w(\Omega) &= H^q_{w_{\mathsf{T}}}(\Omega_t; H^p_{w_{\mathsf{L}}}(\Omega_{\mathbf{x}})) \\ &= \left\{ u(\mathbf{x},t) \in L^2_w(\Omega) : \frac{\partial^{i+j} u}{\partial \mathbf{x}^i \partial t^j} \in L^2_w(\Omega), i \in [0,p], j \in [0,q] \right\}. \end{aligned} \tag{4.4}$$

The associated norm is defined as

$$\|u\|_{H^{p,q}_w(\Omega)} = \left(\sum_{i=0}^{p} \sum_{j=0}^{q} \left\| \frac{\partial^{i+j} u}{\partial \mathbf{x}^i \partial t^j} \right\|^2_{L^2_w(\Omega)} \right)^{1/2}. \tag{4.5}$$

Lemma 4.1. *Let $T_{N_{\mathbf{x}},N_t} u(\mathbf{x},t)$ be the orthogonal projection of $u(\mathbf{x},t)$ onto $T_{N_{\mathbf{x}},N_t}$ and assume that the Gauss–Lobatto points relative to the shifted Chebyshev weights are used for integration. Then, for all $p, q \geq 0$, the truncation error estimate holds [4, 7]*

$$\|u(\mathbf{x},t) - T_{N_{\mathbf{x}},N_t} u(\mathbf{x},t)\|_{L^2_w(\Omega)} \leq C_1 N_{\mathbf{x}}^{-p} \|u\|_{H^{p,0}_w(\Omega)} + C_2 N_t^{-q} \|u\|_{H^{0,q}_w(\Omega)}. \tag{4.6}$$

Here, the C's are independent of $N_{\mathbf{x}}$ and N_t and are positive constants.

Theorem 4.1. *Consider the 1–d case of Equation* (1.1)*. Assume that $u(x,t)$ is the exact solution of Equation* (1.1) *and $T_{N_x,N_t} u(x,t) = U(x,t)$ is the solution obtained through approximation in terms of the shifted Chebyshev polynomials. Then, for sufficiently large N_x, N_t,*

$$\|u(x,t) - U(x,t)\|_{L^2_w(\Omega)} \to 0, \tag{4.7}$$

provided $c_+(x), c_-(x), s(x,t)$ from Equation (1.1) *are smooth functions.*

Proof. Consider the one–dimensional case of Equation (1.1). By direct integration, the exact solution is given by

$$u(x,t) = c_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} u(x,t) + c_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} u(x,t) + {}_0I_t^{\gamma} s(x,t), \tag{4.8}$$

and we define the general approximation using the N_x–truncated Chebyshev expansion in x and N_t–truncated Chebyshev expansion in t as

$$\begin{aligned} U(x,t) =& T_{N_x,N_t} C_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} U(x,t) + T_{N_x,N_t} C_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} U(x,t) \\ &+ T_{N_x,N_t} {}_0I_t^{\gamma} S(x,t). \end{aligned} \tag{4.9}$$

If we define $E(x,t) = u(x,t) - U(x,t)$ and subtract Equation (4.9) from (4.8), we have

$$\begin{aligned} u(x,t) - U(x,t) =& c_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} u(x,t) - T_{N_x,N_t} C_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} U(x,t) \\ &+ c_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} u(x,t) - T_{N_x,N_t} C_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} U(x,t) \\ &+ {}_0I_t^{\gamma} s(x,t) - T_{N_x,N_t} {}_0I_t^{\gamma} S(x,t) \qquad (4.10) \\ =& c_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} u(x,t) - T_{N_x,N_t} c_+(x) {}_0I_t^{\gamma} {}^C_0D_x^{\alpha} u(x,t) \\ &+ c_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} u(x,t) - T_{N_x,N_t} c_-(x) {}_0I_t^{\gamma} {}^C_xD_{\mathsf{L}}^{\alpha} u(x,t) \\ &+ {}_0I_t^{\gamma} s(x,t) - T_{N_x,N_t} {}_0I_t^{\gamma} s(x,t) + R(x,t), \qquad (4.11) \end{aligned}$$

where $R(x,t)$ is the residual. Taking the weighted norm of Equation (4.11), we have

$$\begin{aligned}
&\|u(x,t)-U(x,t)r\|_{L^2_w(\Omega)}\\
\le&\|c_+(x)_0I_t^{\gamma}{}_0^CD_x^{\alpha}u(x,t)-T_{N_x,N_t}c_+(x)_0I_t^{\gamma}{}_0^CD_x^{\alpha}u(x,t)\|_{L^2_w(\Omega)}\\
&+\|c_-(x)_0I_t^{\gamma}{}_x^CD_{\mathsf{L}}^{\alpha}u(x,t)-T_{N_x,N_t}c_-(x)_0I_t^{\gamma}{}_x^CD_{\mathsf{L}}^{\alpha}u(x,t)\|_{L^2_w(\Omega)}\\
&+\|_0I_t^{\gamma}s(x,t)-T_{N_x,N_t}{}_0I_t^{\gamma}s(x,t)\|_{L^2_w(\Omega)}+\|R(x,t)\|_{L^2_w(\Omega)} \qquad (4.12)\\
\le&C_l\left(N_x^{2\alpha-p}\|u\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|u\|_{H_w^{0,q}(\Omega)}\right)\\
&+C_r\left(N_x^{2\alpha-p}\|u\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|u\|_{H_w^{0,q}(\Omega)}\right)\\
&+C\left(N_x^{-p}\|s\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|s\|_{H_w^{0,q}(\Omega)}\right)\\
&+C_2\left(N_x^{-p}\|u\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|u\|_{H_w^{0,q}(\Omega)}\right) \qquad (4.13)\\
\le&C_l\left(N_x^{2\alpha-p}\|u\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|u\|_{H_w^{0,q}(\Omega)}\right)\\
&+C_r\left(N_x^{2\alpha-p}\|u\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|u\|_{H_w^{0,q}(\Omega)}\right)\\
&+C\left(N_x^{-p}\|s\|_{H_w^{p,0}(\Omega)}+N_t^{-q}\|s\|_{H_w^{0,q}(\Omega)}\right). \qquad (4.14)
\end{aligned}$$

Each term in Equation (4.14), given that $0\le\alpha\le p$, tends to zero for sufficiently large N_x and N_t, therefore, the proof is complete. □

The convergence of the numerical solution $U(x,t)$ to the exact solution $u(x,t)$ is dependent on the number of times that $u(x,t)$ is continuously differentiable with respect to x and t.

5. Numerical Examples

Having demonstrated the convergence of the numerical scheme in the previous section, we now present some numerical examples to test the numerical scheme. In this section, we solve selected one– and two–dimensional two–sided fractional differential equations to demonstrate the performance, accuracy and efficiency of the proposed numerical method.

Example 5.1. We consider a one–dimensional two–sided space fractional partial differential equation (FPDE) [20]

$$\frac{\partial u}{\partial t}=c_+\,{}_0^CD_x^{1.8}u+c_-\,{}_x^CD_2^{1.8}u+s(x,t) \qquad (5.1)$$

on a finite domain $\Omega_x=0<x<2$ and $t>0$, with the coefficient functions defined as

$$c_+(x)=\Gamma(1.2)x^{1.8},\qquad c_-(x)=\Gamma(1.2)(2-x)^{1.8}. \qquad (5.2)$$

If we consider the forcing function

$$s(x,t)=32e^{-t}\left[\frac{1}{8}x^2(2-x)^2+x^2+(2-x)^2-2.5(x^3+(2-x)^3)+\frac{25}{22}(x^4+(2-x)^4)\right], \qquad (5.3)$$

the closed form solution of Equation (5.1) is given as $u(x,t) = 4e^{-t}x^2(2-x)^2$, and the initial condition is defined as

$$u(x,0) = 4x^2(2-x)^2, \tag{5.4}$$

while the boundary condition is defined as

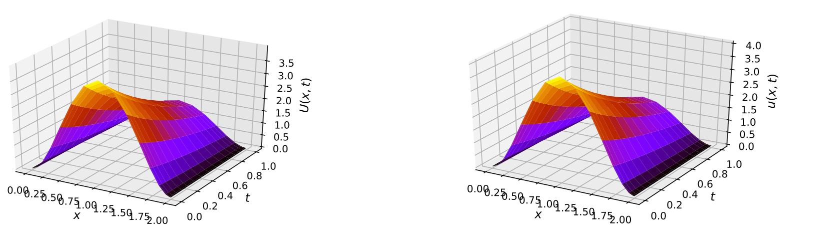
$$u(x,t)|_{\partial\Omega_x} = 0. \tag{5.5}$$

In general, we seek a solution of the form in Equation (3.4). The left– and right–sided derivatives are approximated accordingly. Upon approximating the function and its derivatives using the proposed method, the expansions as well as the initial and boundary conditions result in a consistent system of algebraic equations evaluated on the shifted Chebyshev–Gauss–Lobatto points.

Table 1 shows the performance of the proposed scheme in terms of the error norm for Example 5.1. We compute the error norms across the spatio–temporal points and we compared the result with the maximum error norm obtained by Meerschaert and Tedjeran [20] through a finite difference scheme. As the number of points in the space and time variables increases geometrically, the L_2 error norm decay geometrically. This attribute is synonymous with spectral based methods, unlike the linear decrease in the error norm obtained in the finite difference scheme of Meerschaert and Tedjeran [20]. The order of convergence and the error of the proposed scheme demonstrate the convergence error estimate presented in Theorem 4.1. Figure 1 shows the plots of the exact and numerical solutions on the spatio–temporal grid. The similarity in the plots validate the accuracy of the numerical scheme presented in this study.

Table 1. Error norm (L_2–error) for Example 5.1 compared with the maximum error norm in Meerschaert and Tedjeran [20].

	Current results			Meerschaert and Tedjeran [20]			
$N_x = N_t$	4	8	16	10	20	40	80
Error	0.8291	0.0342	2.5244×10^{-5}	0.1417	0.0571	0.0249	0.0113
Order	–	4.599	10.404				



(a) Numerical solution (b) Exact solution

Figure 1. Numerical and exact solutions of Example 5.1 on the temporal and spatial grids for $N_x = 16$ and $N_t = 16$.

Example 5.2. In this example, we consider a one–dimensional space–time FPDE of the form

$$ {}_0^C D_t^\gamma u = c_+ \, {}_0^C D_x^\alpha u + c_- \, {}_x^C D_1^\alpha u + s(x,t), \; 0 < \gamma \leq 1, \; 1 < \alpha \leq 2 \tag{5.6} $$

on the domain $\Omega_x = 0 < x < 1$, $t > 0$ and the variable coefficients given as

$$ c_+(x) = \frac{x^\alpha}{2}, \quad c_-(x) = \frac{(1-x)^\alpha}{2}, \tag{5.7} $$

and the source function defined as

$$ \begin{aligned} s(x,t) =& 2x^2(1-x)^2 \left[\frac{t^{2-\gamma}}{\Gamma(3-\gamma)} - \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \right] - (t-1)^2 \left[\frac{1}{\Gamma(3-\alpha)} (x^2 + (1-x)^2) \right. \\ & \left. - \frac{6}{\Gamma(4-\alpha)} (x^3 + (1-x)^3) + \frac{12}{\Gamma(5-\alpha)} (x^4 + (1-x)^4) \right]. \end{aligned} \tag{5.8} $$

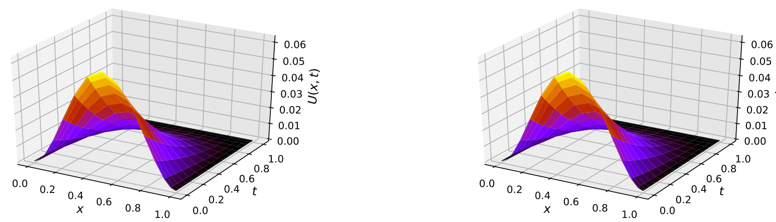
The exact solution is given as $u(x,t) = (t-1)^2 x^2 (1-x)^2$, so that the initial condition is defined as $u(x,0) = x^2(1-x)^2$ and the boundary condition as

$$ u(x,t)|_{\partial \Omega_x} = 0. \tag{5.9} $$

Again, we seek a solution of the form in Equation (3.4), and all derivatives are approximated as discussed in Section 3. In Table 2, we present the L_2 error norm using approximations in terms of different orders of the shifted Chebyshev polynomials in the two variables (x,t) for different values of the fractional orders (γ, α) at $t = 1.0$. The order of convergence of the errors shows the geometrical convergence of the numerical scheme, and the order of the error norms points to the accuracy of the method. Figure 2 shows a comparison between the exact solution and the numerical solution on the spatio–temporal grid for $N_x = N_t = 16$ and $\gamma = 0.5, \alpha = 1.5$. The figure shows that the solutions are similar, and this attests to the accuracy of the numerical scheme.

Table 2. L_2 error norm for Example 5.2 for different values of the arbitrary non–integer orders of derivatives (γ, α).

			$N_x = N_t$		
γ	α		4	8	16
0.1	1.1	Error	0.0331	0.0025	4.3034×10^{-6}
		Order	–	3.727	9.182
	1.5	Error	0.0079	0.0010	2.4993×10^{-6}
		Order	–	3.203	8.644
0.5	1.1	Error	0.0175	0.0019	5.9794×10^{-6}
		Order	–	3.203	8.312
	1.5	Error	0.0078	0.0009	7.3358×10^{-6}
		Order	–	3.115	6.939
1	1.1	Error	0.0152	0.0014	1.4488×10^{-6}
		Order	–	3.441	9.916
	1.5	Error	0.0083	0.0009	2.5627×10^{-7}
		Order	–	3.205	11.778



(a) Numerical solution (b) Exact solution

Figure 2. Solutions of Example 5.2 on the spatial–temporal grid for $N_x = N_t = 16$ with $\gamma = 0.5$ and $\alpha = 1.5$.

Example 5.3. Consider the general two–dimensional two–sided space fractional partial differential equation of the form [11]

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} =& c_{+,1}\, {}_0^C D_x^{\alpha} u + c_{-,1}\, {}_x^C D_1^{\alpha} u + c_{+,2}\, {}_0^C D_y^{\beta} u + c_{-,2}\, {}_y^C D_1^{\beta} u + e u_x + f u_y + s(x,y,t),\\ & 1 < \alpha, \beta \le 2, \end{aligned} \tag{5.10}$$

defined on $\Omega = \Omega_x \times \Omega_y = [0,1] \times [0,1]$, $t > 0$. The variable coefficients are given as

$$\begin{aligned} & c_{+,1}(x) = \Gamma(3-\alpha), x^{\alpha},\ c_{-,1}(x) = \Gamma(3-\alpha)(1-x)^{\alpha}, c_{+,2}(y) = \Gamma(3-\beta) y^{\beta},\\ & c_{-,2}(y) = \Gamma(3-\beta)(1-y)^{\beta}, e(x) = \frac{x}{2},\ f(y) = \frac{y}{2}, \end{aligned} \tag{5.11}$$

and the analytical solution is obtained as

$$u(x,y,t) = \sin(\pi t) x^2 (1-x)^2 y^2 (1-y)^2, \tag{5.12}$$

for which the forcing function can be obtained by simply substituting the exact solution and the coefficients into the Equation (5.10). The initial conditions are given as

$$u(x,y,0) = 0, \quad u_t(x,y,0) = \pi x^2 (1-x)^2 y^2 (1-y)^2, \tag{5.13}$$

and boundary condition as

$$u(x,y,t)|_{\partial\Omega} = 0. \tag{5.14}$$

We seek a solution as an expansion in terms of the shifted Chebyshev polynomials in three variables of the form

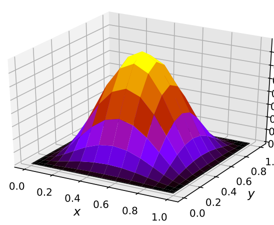
$$\begin{aligned} U(x,y,t) = \sum_{j_1=0}^{N_x} \sum_{j_2=0}^{N_y} \sum_{j_3=0}^{N_t} \Bigg[& \varpi_{j_1} \varpi_{j_2} \varpi_{j_3} \sum_{n_1=0}^{N_x} \sum_{n_2=0}^{N_y} \sum_{n_3=0}^{N_t} \frac{1}{h_{n_1}} \frac{1}{h_{n_2}} \frac{1}{h_{n_3}} T_{1,n_1}(x_{j_1}) T_{1,n_1}(x_{p_1}) \\ & \times T_{1,n_2}(y_{j_2}) T_{1,n_2}(y_{p_2}) T_{\mathsf{T},n_3}(t_{j_3}) T_{\mathsf{T},n_3}(t_{p_3}) \Bigg] U(x_{j_1}, y_{j_2}, t_{j_3}), \\ & p_1 = 0,1,\ldots,N_x,\ p_2 = 0,1,\ldots,N_y,\ p_3 = 0,1,\ldots,N_t, \end{aligned} \tag{5.15}$$

and the derivatives are approximated accordingly.

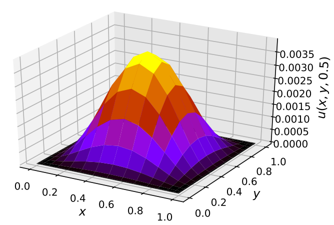
In Table 3, we present the error norms and CPU time for different number of grid–points in x, y, which demonstrate the accuracy and efficiency of the proposed numerical scheme. The error norms and CPU time obtained are compared with the average absolute errors and computational time obtained in Cheng et. al [11] using the moving least square approximation. While the approximation in Cheng et. al [11] and the method proposed in this study perform well in terms of accuracy, as evidenced by the error norm, the approximation in this study performs better in terms of computational time, both approximations were carried out on computers with similar processors and speed. From the table, the geometric convergence of the numerical scheme is evident, as it can be seen that the order of convergence increases geometrically as the number of collocation points increases. In Figure 3, we present the surface plots of the numerical solutions on the spatial grids and compare the solutions with the given closed form solution. Solutions at $t = 0.5$ and $t = 1.5$ are presented.

Table 3. L_2 error norm for Example 5.3 at $t = 0.5, \alpha = \beta = 1.5, N_t = 16$ compared with the average absolute errors obtained in Cheng et. al [11]

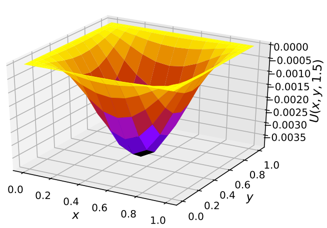
	Current results			Cheng et. al [11]			
$N_x \times N_y$	4×4	8×8	16×16	11×11	16×16	21×21	26×26
Error	0.0034	9.9636×10^{-5}	5.1381×10^{-7}	4.337×10^{-5}	2.345×10^{-5}	1.409×10^{-5}	8.78×10^{-6}
CPU time (secs)	0.053	0.370	6.835	26.25	15.45	8.08	3.46
Order	–	5.093	7.599				



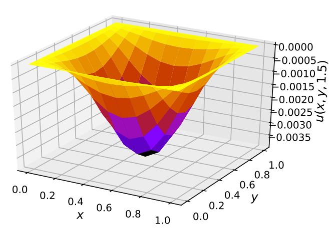
(a) $t = 0.5$



(b) $t = 0.5$



(c) Numerical solution, $t = 1.5$



(d) Exact solution, $t = 1.5$

Figure 3. Solutions of Example 5.3 for $N_x = N_y = N_t = 16$ for $\alpha = \beta = 1.5$ at $t = (0.5, 1.5)$.

Example 5.4. Consider the general two–dimensional two–sided space–time differential equations of arbitrary non–integer order

$$ {}_0^C D_t^{\gamma} u = c_{+,1}\, {}_0^C D_x^{\alpha} u + c_{-,1}\, {}_x^C D_1^{\alpha} u + c_{+,2}\, {}_0^C D_y^{\beta} u + c_{-,2}\, {}_y^C D_1^{\beta} u + s(x, y, t), \tag{5.16} $$

$$0 < \gamma \leq 1, \; 1 < \alpha, \beta \leq 2, \tag{5.17}$$

whose exact solution is given as $u(x, y, t) = (t^3 + 1)x^2(1 - x)^2 y^2(1 - y)^2$. The differential equation satisfies the boundary condition

$$u(x, y, t)|_{\partial\Omega} = 0, \tag{5.18}$$

where $\Omega = \Omega_x \times \Omega_y = [0, 1] \times [0, 1]$, and initial condition is given by

$$u(x, y, 0) = x^2(1 - x)^2 y^2(1 - y)^2. \tag{5.19}$$

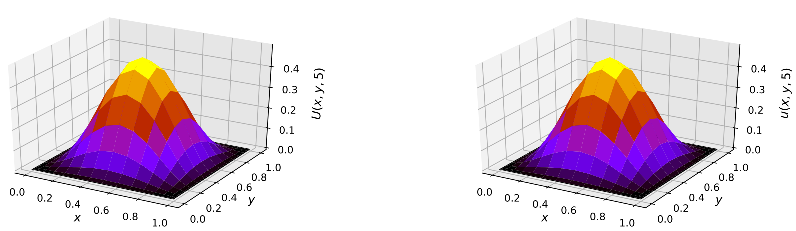
The variable coefficients are defined as

$$c_{+,1}(x) = x^{\alpha}, \; c_{-,1}(x) = (1 - x)^{\alpha}, \; c_{+,2}(y) = y^{\beta}, \; c_{-,2}(y) = (1 - y)^{\beta} \tag{5.20}$$

and the source term is defined to satisfy the differential equation.

Table 4. Error norm for Example 5.4 at $t = 2, N_t = 12$ for different sizes of x, y grid points and fractional orders.

			$N_x \times N_y$		
γ	$\alpha = \beta$		4×4	8×8	16×16
0.5	1.5	Error	0.0048	0.0009	1.7575×10^{-7}
		Order	–	2.415	12.322
	1.9	Error	0.0048	8.5615×10^{-5}	8.2109×10^{-8}
		Order	–	5.809	12.250
0.9	1.5	Error	0.0048	0.0004	1.3885×10^{-7}
		Order	–	3.585	11.492
	1.9	Error	0.0047	8.4207×10^{-5}	2.0854×10^{-7}
		Order	–	5.803	8.657



(a) Numerical solution (b) Exact solution

Figure 4. Exact and numerical solutions of Example 5.4 with $\gamma = 0.9$ and $\alpha = \beta = 1.5$ at $t = 5$.

We seek a solution of the form in Equation (5.15). Table 4 shows the error norm between the approximation and exact solution at $t = 1$ for different values of the fractional orders $(\gamma, \alpha = \beta)$. It can be seen that the approximated solution is in agreement with the exact solution. The table also shows the order of convergence of the numerical scheme. The result is consistent with the theoretical analysis of the convergence error norm. Figure 4 demonstrates the accuracy of the numerical solution. The figures presented are surface plots of the exact and numerical solutions over the interval $[0,1]^2$ on a 17×17 grid.

6. Conclusion

In this study, we have presented a geometrically convergent numerical method in terms of the shifted Chebyshev polynomials for two–sided partial differential equations with arbitrary non–integer temporal and/or spatial orders. The fractional derivatives were defined in the Caputo sense and the approximations of fractional differentiations follow the Caputo fractional operators. The approximations in terms of the shifted Chebyshev polynomials were evaluated at the Gauss–Lobatto quadrature. To demonstrate the applicability of the method, we solved several partial differential equations of arbitrary non–integer orders, which included one– and two–dimensional two–sided space fractional equations and one– and two–dimensional two–sided space time fractional partial differential equations. In addition to the accurate results obtained, the convergence of the scheme was shown to be geometrically convergent similar to other spectral based methods. The method is computationally efficient which is evident in the computational time reported in this study.

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