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EXISTENCE OF WEAK SOLUTIONS FOR ψ-CAPUTO FRACTIONAL BOUNDARY VALUE PROBLEM VIA VARIATIONAL METHODS

Adnan Khaliq[1,†] and Mujeeb ur Rehman[1]

Abstract This article is about a ψ-Caputo fractional boundary value problem which is investigated with the help of variational methods and critical point theory. A touchstone is obtained on the existence of the solution of the problem with the help of a functional. The problem is reduced into an equivalent form such that the solutions of the problem coincide with the critical points of a functional. Using aid from critical point theory, sufficient conditions are obtained for the existence of at least one solution. In the end, an example is also given to enrich our results.

Keywords ψ-Caputo fractional differential operator, weak solution, critical point theory.

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1. Introduction

Fractional calculus changes the discrete nature of order of derivative and integration as in classical calculus to continues one. For the recent developments about the fractional calculus, one may get aid from the monographs of [20, 21, 26, 30]. Fractional calculus is a rich field as it has a variety of applications in almost every field of science and engineering, see for example [8, 9, 23, 27, 33].

In literature, one can find a number of definitions given for differential and integral operators of fractional order [16, 31]. Almost all the types of fractional boundary and initial value problems from various fields are discussed by using different forms of definitions [1–6, 10–15, 17–19, 22, 28, 29, 32, 34]. Therefore there is a strong need of an approach which generalizes the maximum number of definitions of fractional differentiation and integration. This deficiency has been covered by Almeida [7] by introducing the notion of ψ-Caputo differential operator. Although this concept is first given by Osler [25] and Kilbas etc [20] but Almeida [7] enriched the concept with basic properties.

The fractional boundary value problems where the corresponding integral equations are not easy to obtain, Critical point theory along with variational methods is used effectively there. First time this technique is applied by Jiao and Zhou [17] to investigate a type of such problems. They applied the same technique in [18] to

[†]The corresponding author. Email address: adnankhaliqsns@gmail.com (Adnan Khaliq)
[1]Department of Mathematics, School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan

investigate a fractional boundary value problem using variational methods. Getting motivation from the above cited groundwork and especially from [18], in this paper we investigated following family of ψ-Caputo fractional boundary value problem:

$$\begin{cases} {}^C D_{T-}^{\alpha,\psi}(\psi'(t) {}^C D_{0+}^{\alpha,\psi} u(t)) = \nabla G(t, u(t)), \\ u(0) = 0 = u(T), \end{cases} \tag{1.1}$$

for almost every $t \in [0, T]$. Here ${}^C D_{T-}^{\alpha,\psi}$ and ${}^C D_{0+}^{\alpha,\psi}$ denote right and left ψ-Caputo derivatives of fractional order $0 < \alpha \leq 1$ respectively, $\psi : [0, T] \to \mathbb{R}$ is an increasing function and $\psi'(t) \neq 0$, for all $t \in [0, T]$ and $\nabla G(t, x)$ is gradient of a function $G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ which satisfies some assumptions. To the best of our knowledge, ψ-Caputo fractional boundary value problem using critical point theory and variational methods has never been discussed before. So in this paper, we investigated this untouched novel issue.

Remark 1.1. For the different choices of $\psi(t)$, (1.1) will reduced in different types of fractional boundary value problem and especially for $\psi(t) = t$, it reduces to the problem studied in [18].

The remaining part of the article is coordinated as follows. Section 2 contains relevant definitions and results. Section 3 includes the variational structure of the problem (1.1) along with the main results. At the end, an example is given to highlight the applicability of our outcomes in the Section 4.

2. Preliminaries

This section includes all the necessary definitions and results which will be used as helping tools in the proof of main theorems.

Definition 2.1 ([7,20,25])**.** Let $\psi \in C^1(J)$ for a finite interval $J = [a, b]$. Suppose u is an integrable function defined on J, then the right and left fractional integrals with respect to function ψ of u are given by

$${}_z I_b^{\alpha,\psi} u(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (\psi(\xi) - \psi(z))^{\alpha-1} \psi'(\xi) u(\xi) d\xi,$$

and

$${}_a I_z^{\alpha,\psi} u(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (\psi(z) - \psi(\xi))^{\alpha-1} \psi'(\xi) u(\xi) d\xi,$$

respectively.

Definition 2.2 ([7,20,25])**.** Let $\psi \in C^1(J)$ for a finite interval $J = [a, b]$. Suppose u is an integrable function defined on J. The right and left Riemann Liouville type fractional derivatives of u with respect to function ψ are given as

$$\begin{aligned} {}_z D_b^{\alpha,\psi} u(z) &= \left(-\frac{1}{\psi'(z)} \frac{d}{dz}\right)^n {}_z I_b^{n-\alpha,\psi} u(z) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(z)} \frac{d}{dz}\right)^n \int_z^b (\psi(\xi) - \psi(z))^{n-\alpha-1} \psi'(\xi) u(\xi) d\xi, \end{aligned}$$

and

$$
\begin{aligned}
{}_aD_z^{\alpha,\psi}u(z) &= \left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n {}_aI_z^{n-\alpha,\psi}u(z)\\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n \int_a^z (\psi(z)-\psi(\xi))^{n-\alpha-1}\psi'(\xi)u(\xi)d\xi,
\end{aligned}
$$

respectively. Here $n=[\alpha]+1$, $[\cdot]$ denotes the bracket function.

Definition 2.3 ([7]). Let $u,\psi \in C^n(J)$, then right and left ψ-Caputo fractional derivatives of u are defined as

$$
\begin{aligned}
{}^CD_{b-}^{\alpha,\psi}u(z) &= {}_zI_b^{n-\alpha,\psi}\left(-\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n u(z)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_z^b (\psi(\xi)-\psi(z))^{n-\alpha-1}\psi'(\xi)\left(-\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^n u(\xi)d\xi
\end{aligned}
$$

and

$$
\begin{aligned}
{}^CD_{a+}^{\alpha,\psi}u(z) &= {}_aI_z^{n-\alpha,\psi}\left(\frac{1}{\psi'(z)}\frac{d}{dz}\right)^n u(z)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^z (\psi(z)-\psi(\xi))^{n-\alpha-1}\psi'(\xi)\left(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^n u(\xi)d\xi,
\end{aligned}
$$

respectively. Here $n=[\alpha]+1$; $[\cdot]$ denotes the bracket function.

For the operator $D_{a+}^{\alpha,\psi}$, by using [18], following integration by parts formula can be easily derived.

$$
\int_a^b \psi'(z)D_{b-}^{\alpha,\psi}\left(\psi'(z)D_{a+}^{\alpha,\psi}u(z)\right)v(z)dz = \int_a^b |\psi'(z)|^2 D_{a+}^{\alpha,\psi}u(z)D_{a+}^{\alpha,\psi}v(z)dz, \quad (2.1)
$$

provided $v\in L([a,b],\mathbb{R}^n)$ and $u'\in L^\infty([a,b],\mathbb{R}^n)$ with $u(a)=u(b)=0$.

Or $u\in L([a,b],\mathbb{R}^n)$ and $v'\in L^\infty([a,b],\mathbb{R}^n)$ with $v(a)=v(b)=0$.

ψ-Caputo fractional derivative space

In order to examine the existence of solutions of the boundary value problem (1.1), we need a solution space. Let $C_0^\infty([0,T],\mathbb{R}^n)$ be a space of functions u such that $u\in C^\infty[0,T]$ and $u(0)=0=u(T)$. Let $1<p<\infty$ and $0<\alpha\le 1$. Define ψ-Caputo fractional derivative space ${}_0E_{\alpha,\psi}^p$ by the closure of $C_0^\infty([0,T],\mathbb{R}^n)$ with respect to the norm

$$
\|u\|_{\alpha,\psi}^p = \left(\int_0^T |u(t)|^p dt + \int_0^T |\psi'(t)^C D_{a+}^{\alpha,\psi}u(t)|^p dt\right)^{\frac{1}{p}}, \quad \forall u\in {}_0E_{\alpha,\psi}^p. \qquad (2.2)
$$

It can be seen easily that ${}_0E_{\alpha,\psi}^p$ $(\psi - \mathit{Caputo\ fractional\ derivative\ space})$ is the space of the functions u having ψ-Caputo fractional derivative of order α such that $\psi'(t)^C D_{a+}^{\alpha,\psi}u\in L^p([0,T],\mathbb{R}^n)$ and $u\in L^p([0,T],\mathbb{R}^n)$.

As we all know that $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$, $\|u\|_{L^p([0,t])} = \left(\int_0^t |u(\eta)|^p d\eta\right)^{\frac{1}{p}}$ and $\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}}$ are well known norms for $1\le p<\infty$ and any fixed $t\in[0,T]$.

Following [18, Proposition 3.1], below result can be easily obtained.

Proposition 2.1. *Let $1 < p < \infty$ and $0 < \alpha \leq 1$, then the ψ-Caputo fractional derivative space ${}_0E^p_{\alpha,\psi}$ is a separable and reflexive Banach space.*

From end to end of this paper, we consider the following notations and assumptions.

Assumption 2.1. $M_{\psi'} = \max\limits_{t\in[0,T]}\{\psi'(t)\}, \quad M_{\psi} = \max\limits_{t\in[0,T]}\{\psi(t)\},$
$m_{\psi'} = \min\limits_{t\in[0,T]}\{\psi'(t)\}$ *and* $m_{\psi} = \min\limits_{t\in[0,T]}\{\psi(t)\}$

Assumption 2.2. *There exists a constant $A_\psi > 0$ for $0 \leq t_1 < t_2 \leq T$ such that $|\psi(t_2) - \psi(t_1)| = A_\psi |t_2 - t_1|$.*

Assumption 2.3. *Let $G : [0,T] \times \mathbb{R}^n \to \mathbb{R}$. For almost every $t \in [0,T]$, it is continuously differentiable in x and for all $x \in \mathbb{R}^n$, it is measurable in t and for $n_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $|G(t,x)| \leq n_1(|x|)n_2(t)$ and for $n_2 \in L^1([0,T], \mathbb{R}^+)$, $|\nabla G(t,x)| \leq n_1(|x|)n_2(t)$ for all $x \in \mathbb{R}^+$ and a.e. $t \in [0,T]$.*

Lemma 2.1. *Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$, then*

$$||{}_0I^{\alpha,\psi}_{\xi} f||_{L^p[0,t]} \leq \frac{M_{\psi'}[\psi(t)]^\alpha}{\Gamma(\alpha+1)}||f||_{L^p[0,t]}, \tag{2.3}$$

for $f \in L^p([0,T], \mathbb{R})$, $t \in [0,T]$ and $\xi \in [0,t]$.

Proof. First we prove the inequality for $p = 1$. By changing the order of integration, we have,

$$\begin{aligned}
||{}_0I^{\alpha,\psi}_{\xi} f||_{L[0,t]} &= \frac{1}{\Gamma(\alpha)}\int_0^t \left(\int_0^\xi (\psi(\xi) - \psi(z))^{\alpha-1}\psi'(z)f(z)dz\right)d\xi \\
&\leq \frac{M_{\psi'}}{\Gamma(\alpha)}\int_0^t \left(\int_z^t (\psi(\xi) - \psi(z))^{\alpha-1}f(z)d\xi\right)dz \\
&\leq \frac{M_{\psi'}(\psi(t))^\alpha}{\Gamma(\alpha+1)}||f||_{L[0,t]}.
\end{aligned} \tag{2.4}$$

Now, for $1 < p < \infty$, after applying the substitution $\psi(\xi) - \psi(z) = \psi(\eta)$ and changing the order of integration, choosing q such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $g \in L^q[(0,T), \mathbb{R}]$

$$\begin{aligned}
&\left|\int_0^t g(\xi)\left(\int_0^\xi (\psi(\xi) - \psi(z))^{\alpha-1}\psi'(z)f(z)dz\right)d\xi\right| \\
&= \left|\int_0^t g(\xi)\left(\int_{\psi^{-1}(0)}^{\psi^{-1}(\psi(\xi)-\psi(0))} (\psi(\eta))^{\alpha-1}\psi'(\eta)f(\psi^{-1}(\psi(\xi) - \psi(\eta)))d\eta\right)d\xi\right| \\
&\leq M_{\psi'}\int_{\psi^{-1}(0)}^{\psi^{-1}(\psi(t)-\psi(0))} (\psi(\eta))^\alpha \left|\int_{\psi^{-1}(\psi(\eta)+\psi(0))}^t g(\xi)f(\psi^{-1}(\psi(\xi) - \psi(\eta)))d\xi\right| d\eta \\
&\leq \frac{M_{\psi'}[\psi(t)]^\alpha}{\alpha}\|f\|_{L^p[0,t]}\|g\|_{L^q[0,t]}.
\end{aligned} \tag{2.5}$$

Suppose the following functional $M_{\xi * f} : L^q([0,T],\mathbb{R}) \to \mathbb{R}$

$$M_{\xi * f} = \int_0^t \left(\int_0^\xi (\psi(\xi) - \psi(z))^{\alpha-1} \psi'(z) f(z) dz \right) g(\xi) d\xi, \quad \text{for any fixed } t \in [0,T]. \tag{2.6}$$

From estimate (2.5), it is clear that $M_{\xi * f} \in (L^q([0,T],\mathbb{R}))^* \{\text{dual space of } L^q([0,T],\mathbb{R})\}$. According Riesz representation theorem, from (2.5) and (2.6), there is a $m \in L^p([0,T],\mathbb{R})$ such that

$$\int_0^t m(\xi) g(\xi) d\xi = \int_0^t g(\xi) \left(\int_0^\xi \psi'(z) (\psi(\xi) - \psi(z))^{\alpha-1} f(z) dz \right) d\xi, \tag{2.7}$$

and

$$\|m\|_{L^p([0,t])} \leq \frac{M_{\psi'} [\psi(t)]^\alpha}{\alpha} \|f\|_{L^p[0,t]}. \tag{2.8}$$

Using (2.7), we have

$$\frac{1}{\Gamma(\alpha)} m(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\psi(\xi) - \psi(z))^{\alpha-1} \psi'(z) f(z) dz = {}_0 I_\xi^{\alpha,\psi} f(\xi). \tag{2.9}$$

Now from (2.8), we have

$$||{}_0 I_\xi^{\alpha,\psi} f||_{L^p[0,t]} = \frac{1}{\Gamma(\alpha)} \|m\|_{L^p[0,t]} \leq \frac{M_{\psi'} [\psi(t)]^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p[0,t]}. \tag{2.10}$$

Connecting (2.4) and (2.10), we get (2.3) and proof is completed. □

Proposition 2.2. *Let $1 < p < \infty$ and $0 < \alpha \leq 1$. For all $u \in_0 E^p_{\alpha,\psi}$, if $\alpha > \frac{1}{p}$ then*

$$||u||_{L^p} \leq \frac{M_{\psi'} [\psi(T)]^\alpha}{\Gamma(\alpha+1)} \|\psi'(t) D_{0+}^{\alpha,\psi} u(t)\|_{L^p}. \tag{2.11}$$

Moreover the following inequality also holds if $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > \frac{1}{p}$:

$$||u||_\infty \leq \frac{[\psi(T)]^{\alpha - \frac{1}{p}}}{[(\alpha-1)q+1]^{\frac{1}{q}} \Gamma(\alpha)} \|\psi'(t) D_{0+}^{\alpha,\psi} u(t)\|_{L^p}. \tag{2.12}$$

Proof. By using inequality (2.3) and [7, Theorem 4], inequality (2.11) can be proved easily.

Now we can prove inequality (2.12). For $\alpha > \frac{1}{p}$, choose q such that $\frac{1}{p} + \frac{1}{q} = 1$. $\forall u \in E_0^{\alpha_\psi, p}$ using Hölder inequality and [7, Theorem 4] we have

$$\begin{aligned}
|u(t)| &= |{}_0 I_t^{\alpha,\psi} {}_0 D_t^{\alpha,\psi} u(t)| \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(z))^{\alpha-1} \psi'(z) {}_0 D_t^{\alpha,\psi} u(z) dz \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (\psi(t) - \psi(z))^{(\alpha-1)^q} dz \right)^{\frac{1}{q}} \|\psi'(t) D_{0+}^{\alpha,\psi} u(t)\|_{L^p} \\
&\leq \frac{[u(T)]^{\alpha - \frac{1}{p}}}{[(\alpha-1)q+1]^{\frac{1}{q}} \Gamma(\alpha)} \|\psi'(t) D_{0+}^{\alpha,\psi} u(t)\|_{L^p},
\end{aligned}$$

and this completes the proof. □

Remark 2.1. According to (2.11) and [7, Theorem 3], in the remaining part of analysis, space ${}_0E^p_{\alpha,\psi}$ can be considered w. r. t. the norm

$$\|u\|^p_{\alpha,\psi} = \left(\int_0^T |\psi'(t)^C D^{\alpha,\psi}_{a+} u(t)|^p dt\right)^{\frac{1}{p}}. \tag{2.13}$$

Proposition 2.3. *Let $1 < p < \infty$ and $0 < \alpha \leq 1$, consider $\alpha > \frac{1}{p}$ and the sequence $\{u_k\}$ is such that $u_k \rightharpoonup u$ (converges weakly) in ${}_0E^p_{\alpha,\psi}$, then $u_k \to u$ (converges strongly) in $C([0,T],\mathbb{R}^n)$, i.e. $\|u_k - u\| \to 0$ as $k \to \infty$.*

Proof. If $\alpha > \frac{1}{p}$, then using (2.12) and (2.13) it is clear that with the usual norm $\|\cdot\|_\infty$, space ${}_0E^p_{\alpha,\psi}$ has a continuous injection into $C([0,T],\mathbb{R}^n)$. So $u_k \to u$ in $C([0,T],\mathbb{R}^n)$ for given $u_k \to u$ in ${}_0E^p_{\alpha,\psi}$.

For $u_k \to u$ in ${}_0E^p_{\alpha,\psi}$ and $g \in (C([0,T],\mathbb{R}^n))^*$, $g(u_k) \to g(u)$, which shows that $g \in \left({}_0E^p_{\alpha,\psi}\right)^*$. Thus $(C([0,T],\mathbb{R}^n))^* \subseteq \left({}_0E^p_{\alpha,\psi}\right)^*$. Hence $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R}^n)$ if $u_k \rightharpoonup u$ in ${}_0E^p_{\alpha,\psi}$. $\{u_k\}$ is bounded in $C([0,T],\mathbb{R}^n)$. This is because, using Banach-Steinhaus theorem, $\{u_k\}$ is bounded in ${}_0E^p_{\alpha,\psi}$.

Now to show that $\{u_k\}$ is uniformly equi-continuous, we have the following result. Let $0 \leq t_1 < t_2 \leq T$ and $\frac{1}{p} + \frac{1}{q} = 1$. As $\alpha > \frac{1}{p}$, using Hölder inequality for all $f \in L^p([0,T],\mathbb{R}^n)$ and assumption 2.2, we have

$$\begin{aligned}
&\left|{}_0I^{\alpha,\psi}_{t_1} f(t_1) - {}_0I^{\alpha,\psi}_{t_2} f(t_2)\right| \\
=&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} (\psi(t_1)-\psi(z))^{\alpha-1}\psi'(z)f(z)dz - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(\psi(t_2)-\psi(z))^{\alpha-1}\psi'(z)f(z)dz\right| \\
\leq&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(\psi(t_1)-\psi(z))^{\alpha-1} - (\psi(t_2)-\psi(z))^{\alpha-1}\right]\psi'(z)f(z)dz\right| \\
&+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(\psi(t_2)-\psi(z))^{\alpha-1}\psi'(z)f(z)dz\right| \\
\leq&\frac{2||\psi' f||_{L^p}}{((\alpha-1)q+1)^{1/q}\Gamma(\alpha)}(\psi(t_2)-\psi(t_1))^{\alpha-1+1/q} \\
\leq&\frac{2A_\psi||\psi' f||_{L^p}}{((\alpha-1)q+1)^{1/q}\Gamma(\alpha)}(t_2-t_1)^{\alpha-1/p}.
\end{aligned} \tag{2.14}$$

Now we can prove that $\{u_k\}$ is uniformly equi-continuous. Let $0 \leq t_1 < t_2 \leq T$, by using (2.13) and (2.14)

$$\begin{aligned}
|u_k(t_1) - u_k(t_2)| &= \left|{}_0I^{\alpha,\psi}_{t_1}\left({}^C_0D^{\alpha,\psi}_{t_1}u(t_1)\right) - {}_0I^{\alpha,\psi}_{t_1}\left({}^C_0D^{\alpha,\psi}_{t_1}u(t_1)\right)\right| \\
&\leq \frac{2A_\psi\,(t_2-t_1)^{\alpha-1/p}}{((\alpha-1)q+1)^{1/q}\Gamma(\alpha)}\left\|\psi'^C D^{\alpha,\psi}\right\|_{L^p} \\
&= \frac{2A_\psi\,(t_2-t_1)^{\alpha-1/p}}{((\alpha-1)q+1)^{1/q}\Gamma(\alpha)}\|u\|^p_{\alpha,\psi} \\
&\leq c\,(t_2-t_1)^{\alpha-1/p},
\end{aligned}$$

where $c \in \mathbb{R}^+$ is a constant. So $\{u_k\}$ is relatively compact in $C([0,T],\mathbb{R}^n)$ as a consequence of Ascoli-Arzela theorem. Also every uniformly convergent sub sequence of $\{u_k\}$ converges uniformly to u on $[0,T]$. This completes the proof. $\square$

Theorem 2.1 ([24]). *Let H be a real and reflexive Banach space. If the functional $\Omega : H \to \mathbb{R}^n$ is coercive, i.e. $\Omega(u) = +\infty$ as $\|u\| \to \infty$ and weakly lower semi-continuous, Then there is a $u_0 \in H$ such that $\Omega(u_0) = \inf_{u\in H} \Omega(u)$. Furthermore, $\Omega'(u_0) = 0$, provided Ω is Fréchet differentiable on H.*

3. Variational structure and main results

The coming theorem is a backbone in constructing the variational structure for the problem (1.1) on the space ${}_0E^p_{\alpha,\psi}$. Here we are leaving the proof to the reader which is very similar to [24, Theorem 1.4]. We only have to change weak derivatives of u and v by $\psi' D^{\alpha,\psi}_{0+}u$ and $\psi' D^{\alpha,\psi}_{0+}v$ respectively in [24, Theorem 1.4].

Theorem 3.1. *Let $0 < p < 1$ and $1/p < \alpha < 1$. Consider $K : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(t, x_1, x_2,) \longmapsto K(t, x_1, x_2)$ is continuously differentiable in $[x_1, x_2]$ for almost every $t \in [0, T]$ and measurable in t for every $[x_1, x_2] \in \mathbb{R}^n \times \mathbb{R}^n$. For $\frac{1}{p} + \frac{1}{q} = 1$, if there exists $n_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $n_2 \in L^1([0, T], \mathbb{R}^+)$ and $n_3 \in L^q([0, T], \mathbb{R}^+)$, $0 < q < \infty$ such that for almost every $t \in [0, T]$ and every $[x_1, x_2] \in \mathbb{R}^+ \times \mathbb{R}^+$, then we have*

$$\begin{aligned}
&|K(t, x_1, x_2)| \le n_1(|x_1|)(n_2(t) + |x_2|^p),\\
&|D_{x_1} K(t, x_1, x_2)| \le n_1(|x|)(n_2(t) + |x_2|^p),\\
&|D_{x_2} K(t, x_1, x_2)| \le n_1(|x_1|)(n_3(t) + |x_2|^{p-1}).
\end{aligned}$$

Define a functional by

$$\Phi(u) = \int_0^T K(t, u(t), \psi'(t)_0 D^{\alpha,\psi}_t u(t))dt,$$

then $\Phi(u)$ is continuously differentiable on ${}_0E^p_{\alpha,\psi}$ and

$$\begin{aligned}
\langle \Phi'(u), v\rangle = \int_0^T &\big[D_x K(t, u(t), \psi'(t)_0 D^{\alpha,\psi}_t u(t))v(t) + D_y K(t, u(t), \psi'(t)_0 D^{\alpha,\psi}_t u(t))\\
&\psi'(t)_0 D^{\alpha,\psi}_t v(t)\big]dt,
\end{aligned}$$

for all $u, v \in_0 E^p_{\alpha,\psi}$.

Equivalent Form
For problem (1.1), an equivalent form can be obtained by multiplying (1.1) with $v \in C_0^\infty([0, T]\mathbb{R}^n)$ and ψ', and by using the formula (2.1).

$$\int_0^T |\psi'(t)|^2\ {}^C D^{\alpha}_{0+} u(t) {}^C D^{\alpha,\psi}_{0+} v(t)dt = \int_0^T \nabla G(t, u(t))v(t)\psi'(t)dt. \tag{3.1}$$

Definition 3.1. Let $u \in {}_0E^p_{\alpha,\psi}$. If u satisfies (3.1) for all $v \in C_0^\infty([0, T], \mathbb{R}^n)$ along with boundary conditions $u(0) = u(T) = 0$, then u is called the weak solution of the problem (1.1).

Corollary 3.1. *Define $K : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$K(t, x_1, x_2) = \frac{1}{2}|x_2|^2 - \psi'(t)G(t, x_1),$$

where G satisfies assumption 2.3. For $\frac{1}{2} < \alpha < 1$ and $\forall u \in {}_0E^p_{\alpha,\psi}$, define a functional Ω as

$$\Omega(u) = \frac{1}{2}\int_0^T \left|\psi'(t)^C D^{\alpha,\psi}_{0+} u(t)\right|^2 dt - \int_0^T G(t,u(t))\psi'(t)dt. \tag{3.2}$$

If u is a solution of $\Omega'(u) = 0$, Then u is weak solution of the problem (1.1).

Proof. By using Theorem 3.1, it can be easily proved. □

Remark 3.1. From Corollary 3.1, we can see that weak solutions of the problem (1.1) are also the critical points of the functional defined in (3.2).

Theorem 3.2. *Let $\frac{1}{2} < \alpha \leq 1$ and suppose that G mollifies assumption 2.3. For $a \in \left[0, \frac{\Gamma^2(\alpha+1)}{2(M_{\psi'})^3[\psi(T)]^{2\alpha}}\right)$, $b \in L^{2/\eta}([0,T],\mathbb{R})$, $\eta \in (0,2)$ and $c \in L^1([0,T],\mathbb{R})$, if*

$$|G(t,z)| \leq a|z|^2 + b(t)|z|^{2-\eta} + c(t), \quad for\ z \in \mathbb{R}^n,\ a.e.\ t \in [0,T]. \tag{3.3}$$

Then the problem (1.1) *has at least one weak solution which also minimizes Ω on ${}_0E^2_{\alpha,\psi}$.*

Proof. Our main task is to apply Theorem 2.1 and the first one is Ω is a weakly lower semi continuous functional. The functional $u \to \int_0^T \left(\left|\psi'(t)^C D^{\alpha,\psi}_{0+} u(t)\right|^2 /2\right) dt$ is continuous and convex on ${}_0E^2_{\alpha,\psi}$ because the ψ-Caputo derivative is a linear operator [7]. Now according to Proposition 2.3, for a.e. $t \in [0,T]$, $G(t,u_k(t)) \to G(t,u(t))$. So $\int_0^T G(t,u_k(t))dt \to \int_0^T G(t,u(t))dt$ with the help of Lebesgue dominated convergence theorem, which shows that the functional $u \to \int_0^T G(t,u(t))dt$ is weakly continuous on ${}_0E^2_{\alpha,\psi}$. So the functional Ω defined in (3.2) is continuously differentiable according to Corollary 3.1 and is sum of a convex and weakly continuous functional. Hence Ω is weakly lower semi continuous functional on ${}_0E^2_{\alpha,\psi}$ according to [24, Theorem 1.2, Propositon 1.2].

Now we shall prove that Ω is coercive on ${}_0E^2_{\alpha,\psi}$. Using inequality (2.11) and (3.3), for $u \in {}_0E^2_{\alpha,\psi}$, we have

$$\begin{aligned}
\Omega(u) &= \frac{1}{2}\int_0^T \left|\psi'(t)^C D^{\alpha,\psi}_{0+} u(t)\right|^2 dt - \int_0^T G(t,u(t))\psi'(t)dt \\
&\geq \frac{1}{2}\left(\|u\|^2_{\alpha,\psi}\right)^2 - M_{\psi'}a\int_0^T |u(t)|^2 dt - M_{\psi'}\int_0^T b(t)|u(t)|^{2-\eta}dt - M_{\psi'}\int_0^T c(t)dt \\
&\geq \frac{1}{2}\left(\|u\|^2_{\alpha,\psi}\right)^2 - M_{\psi'}a\left(\|u\|_{L^2}\right)^2 - M_{\psi'}\left(\int_0^T |b(t)|^{2/\eta}dt\right)^{\eta/2}\left(\int_0^T |u(t)|^2 dt\right)^{1-\eta/2} \\
&\quad - c_1 \\
&= \frac{1}{2}\left(\|u\|^2_{\alpha,\psi}\right)^2 - M_{\psi'}a\left(\|u\|_{L^2}\right)^2 - b_1 M_{\psi'}\left(\|u\|_{L^2}\right)^{2-\eta} - c_1 \\
&\geq \left[\frac{1}{2} - \frac{a(M_{\psi'})^3[\psi(T)]^2\alpha}{\Gamma^2(\alpha+1)}\right]\left(\|u\|^2_{\alpha,\psi}\right)^2 - b_1 M_{\psi'}\left[\frac{M_{\psi'}[\psi(T)]^\alpha}{\Gamma(\alpha+1)}\right]\left(\|u\|^2_{\alpha,\psi}\right)^{2-\eta} - c_1,
\end{aligned}$$

where $b_1 = \left(\int_0^T |b(t)|^{2/\eta}dt\right)^{\eta/2}$ and $c_1 = M_{\psi'}\int_0^T c(t)dt$. Since $a \in \left[0, \frac{\Gamma^2(\alpha+1)}{2(M_{\psi'})^3[\psi(t)]^{2\alpha}}\right)$, $\eta \in (0,2)$, therefore, for $\|u\|_{\alpha,\psi} \to \infty$, $\Omega(u) = +\infty$. Hence Ω is coercive and this completes the proof. □

4. Example

Example 4.1.

$$\begin{cases} {}^{C}D_{1-}^{\frac{3}{4},e^t}(e^{t\,C}D_{0+}^{\frac{3}{4},e^t}u(t)) = \dfrac{1}{200}u + 1, \\ u(0) = 0 = u(1). \end{cases} \tag{4.1}$$

Here $\alpha = \frac{3}{4}$, $\psi(t) = e^t$, $T = 1$ and

$$\nabla G(t, u(t)) = \frac{1}{200}u + 1 \implies G(t, u(t)) = \frac{1}{400}u^2 + u + 1.$$

Simple calculation shows that $\frac{\Gamma^2(\alpha+1)}{2(M_{\psi'})^3[\psi(T)]^{2\alpha}} = \frac{\Gamma^2(1.75)}{2(e)^3(e)^{1.5}} \approxeq 0.0046$.

Since, we can write

$$\begin{aligned} G(t, u(t)) &= \frac{1}{400}u^2 + u + 1, \\ &\leq \frac{7}{2000}u^2 + u + 1, \end{aligned}$$

which shows that $a = \frac{7}{2000} = 0.0035 \in \Big(0, 0.0046\Big) = \Big[0, \frac{\Gamma^2(\alpha+1)}{2(M_{\psi'})^3[\psi(T)]^{2\alpha}}\Big)$, $\eta = 1 \in \Big(0, 2\Big)$, $b(t) = 1 \in L^2\Big([0,1], \mathbb{R}\Big)$ and $c(t) = 1 \in L^2\Big([0,1], \mathbb{R}\Big)$.

So all the requirements of Theorem 3.2 are satisfied, Hence problem (4.1) has at least one weak solution as a consequence of Theorem 3.2.

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