Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 1779–1801
Website:http://www.jaac-online.com
DOI:10.11948/20200190

RANK-ONE CHAOS IN A DELAYED SIR EPIDEMIC MODEL WITH NONLINEAR INCIDENCE AND TREATMENT RATES*

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Abstract The rank one chaos in a SIR model with two time-delays is studied in this paper. By using center manifold theorem, normal form theory and Hassard's method, the existence, direction and stability of Hopf bifurcation are discussed. Based on the rank-one chaos theory for delayed differential equations, the conditions for the existence of rank-one strange attractor in disturbed system are obtained. Finally, numerical simulations are given to verify the theoretical analysis results.

Keywords Delayed SIR model, Hopf bifurcation, periodically kicked, rank-one strange attractor.

MSC(2010) 34K18, 37D45.

1. Introduction

With the development of chaos theory, more and more scholars pay attention to chaos. Rank-one chaos is an important branch of chaos phenomena. Wang and Yang [19] studied the strange attractors, which has an unstable single direction and some controlled behaviors, and they obtained the existence condition of the strange attractors. In [20], Wang and Yang proved that for a class of second order ODEs, there were global strange attractors with fully stochastic properties. In 2003, Wang and Yang [21] proved the emergence of chaotic behavior in the form of horseshoes and strange attractors with SRB measures when certain simple dynamical systems are kicked at periodic time intervals. In 2005, Wang and Oksasoglu [22] applied the above theory and results to the Chua's circuit, and they confirmed the existence of strange attractors in Chua's circuit. Since then, more and more scholars have paid attention to the rank-one strange attractor. In 2009, Chen and Han [1] verified the existence of rank-one chaos in a plane systems with heteroclinic cycles. In 2010, Wang and Oksasoglu [15] studied a switch-controlled Chua's circuit, and then they found strange attractors in the system. Recently, some researchers introduced the rank one chaos theory into the delay differential equation [14, 25–27]. In 2015, Dai and Lin et al. [2] developed rank-one theory for delayed differential equations and applied the Chen system with time-delay. They showed that there was rank-one strange attractor in time-delayed Chen system. In [24], Yang and Fang et al. found

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*The authors were supported by National Natural Science Foundation of China (11761040) and National Natural Science Foundation of China (11461036).

the existence of rank-one strange attractor in a periodically kicked delayed three-species food chain.

Since the SIR model was proposed by Kermack and McKendrick [9] in 1927, many scholars have studied the dynamics of epidemic models. In the process of studying infectious diseases, the transmission mode and the therapeutic effect of diseases are the key issues. In [9], authors used bilinear incidence rate βSI and linear treatment rate γI to describe the dynamics of an infectious disease. With the deepening of the research on infectious diseases, authors found that simple bilinear incidence and treatment rate were not enough to describe the increasingly complex infectious disease system, many researchers have modified the incidence and treatment rate of the diseases. In 1986, Liu et al. [13] proposed a general incidence rate $g(I)S = \frac{kI^pS}{1+\alpha I^q}$ which considered psychological effects. In 2013, Holling type III treatment rate $T(I) = \frac{aI^2}{1+bI^2}$ was proposed by Dubey et al. [3]. Recently, the authors [4–6, 10, 16, 18] have studied epidemic model with nonlinear incidence and treatment rates.

A SIR model with Monod-Haldane functional-type incidence rate and Holling type III saturated treatment rate was suggested by Kumar and Nilam [11] as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) - \dfrac{\beta S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)}, \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)} - (\mu + d + \sigma)I(t) - \dfrac{aI^2(t)}{1+bI^2(t)}, \\ \dfrac{dR(t)}{dt} = \dfrac{aI^2(t)}{1+bI^2(t)} + \sigma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

where the term $\frac{\beta S(t)I(t)}{1+\alpha I^2(t)}$ represents the Monod-Haldane functional-type incidence rate. The total population $N(t)$ at time t was divided into three parts: susceptible individuals $S(t)$, infected individuals $I(t)$, recovered individuals $R(t)$. The total population moved to susceptible individuals at A constant rate, μ is natural mortality rate, β is the disease transmission rate(The disease can only be transmitted from the infected to the susceptible), α is the psychological inhibitory effect, d is the disease mortality, σ is the recovery rate of infected individuals, a is the cure rate of the disease, b is the effectiveness of the treatment, τ is incubation period of disease.

Considering the treatment provided to infected individuals, many factors lead to the time delay, such as the infected individuals cannot be treated at once, or drugs do not take effect immediately due to the physical condition of the infected individuals. Considering the influence of the above factors, in order to make the mathematical model more realistic, we take into account the time delay during the treatment and add a time delay τ_2 to the treatment term, as following form:

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) - \dfrac{\beta S(t)I(t-\tau_1)}{1+\alpha I^2(t-\tau_1)}, \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t-\tau_1)}{1+\alpha I^2(t-\tau_1)} - (\mu + d + \sigma)I(t) - \dfrac{aI^2(t-\tau_2)}{1+bI^2(t-\tau_2)}, \\ \dfrac{dR(t)}{dt} = \dfrac{aI^2(t-\tau_2)}{1+bI^2(t-\tau_2)} + \sigma I(t) - \mu R(t), \end{cases} \tag{1.2}$$

where τ_1 is incubation period of disease, τ_2 is the time delay during the treatment of the infected individuals. We give the initial conditions of system (1.2):

$$S(\vartheta) = \varphi_1(\vartheta), I(\vartheta) = \varphi_2(\vartheta), R(\vartheta) = \varphi_3(\vartheta), \varphi_i(\vartheta) \geq 0, (i = 1, 2, 3), -\tau \leq \vartheta \leq 0, \tag{1.3}$$

where $(\varphi_1(\vartheta), \varphi_2(\vartheta), \varphi_3(\vartheta)) \in C([-\tau, 0], R^3_+)$ is the Banach space of continuous functions mapping $[-\tau, 0] \longrightarrow R^3_+$, and $R^3_+ = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$. Then system (1.2) has a unique solution $(S(t), I(t), R(t))$ that satisfies (1.3) according to the fundamental theory of functional differential equations [7].

In this paper, we think about the fact that in real life some very small periodic external factors will have some effect on the epidemic. For example, some animals carrying the virus will return to the same place in the same season every year. In terms of the spread of infectious diseases, this phenomenon will cause the outbreak of infectious diseases with a certain periodicity, which is consistent with the fact that the high incidence of infectious diseases usually occurs in autumn and winter. From the perspective of disease prevention and control, the actual immunization strategy is to conduct pulsed immunization at a fixed cycle T. For example, we will prevent the flu by taking an annual vaccination before the flu season every year. This kind of periodic immunity measure can cause certain influence to the prevention and control of infectious disease. So we consider the effect of the time delay and an external periodic force as an input on the system, we devote our attention to rank-one strange attractor in a periodically kicked system with two time-delays. Then we consider the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = A - \mu S(t) - \dfrac{\beta S(t) I(t-\tau_1)}{1 + \alpha I^2(t-\tau_1)} + \varepsilon S(t) P_T(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t) I(t-\tau_1)}{1 + \alpha I^2(t-\tau_1)} - (\mu + d + \sigma) I(t) - \dfrac{a I^2(t-\tau_2)}{1 + b I^2(t-\tau_2)} + \varepsilon I(t) P_T(t), \\ \dfrac{dR(t)}{dt} = \dfrac{a I^2(t-\tau_2)}{1 + b I^2(t-\tau_2)} + \sigma I(t) - \mu R(t) + \varepsilon R(t) P_T(t), \end{cases} \tag{1.4}$$

where $\varepsilon > 0$ is small enough, $P_T = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\delta(\cdot)$ is Dirac-delta function.

The structure of the paper is as follows: In Section 2, we deduced the conditions for local asymptotic stability and Hopf bifurcation at positive equilibrium. In Section 3, the direction and stability of Hopf bifurcation are given. In Section 4, the conditions for generating rank one chaos in a disturbed system are discussed. Numerical simulations are presented in Section 5. Finally, the conclusions are given in Section 6.

2. Stability and existence of Hopf bifurcation

We firstly consider the existence of positive equilibrium of system (1.2). By calculation,

$$E(S^*, I^*, R^*) = \Big(\frac{(1 + \alpha {I^*}^2)[(1 + b{I^*}^2)(\mu + d + \sigma) + a I^*]}{\beta(1 + b{I^*}^2)}, I^*, \frac{I^*[\sigma(1 + b{I^*}^2) + a I^*]}{\mu(1 + b{I^*}^2)}\Big),$$

where I^* satisfies the following equation:

$$K_1 {I^*}^4 + K_2 {I^*}^3 + K_3 {I^*}^2 + K_4 I^* + K_5 = 0, \tag{2.1}$$

with

$$\begin{aligned}
K_1 &= \mu\alpha b(\mu + d + \sigma),\\
K_2 &= \mu\alpha a + \beta b(\mu + d + \sigma),\\
K_3 &= \mu(\alpha + b)(\mu + d + \sigma) + \beta(a - Ab),\\
K_4 &= \mu a + \beta(\mu + d + \sigma),\\
K_5 &= \mu(\mu + d + \sigma) - \beta A.
\end{aligned}$$

According to Descartes' rule of sign [23], equation (2.1) has a positive real root if the following holds true:

(H_1): $K_5 < 0$.

Then there is a positive equilibrium $E(S^*, I^*, R^*)$ of system (1.2) if (H_1) holds true.

Next, we can obtain the characteristic matrix of the linearized system of (1.2) at the positive equilibrium E, denoted as: $\lambda I - B - e^{-\lambda\tau_1}C - e^{-\lambda\tau_2}D$, where

$$B = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ 0 & b_{32} & b_{33} \end{pmatrix}, C = \begin{pmatrix} 0 & c_{12} & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & d_{32} & 0 \end{pmatrix}, \tag{2.2}$$

and

$$\begin{aligned}
&b_{11} = -\mu - \frac{\beta I^*}{1 + \alpha I^{*2}}, b_{21} = \frac{\beta I^*}{1 + \alpha I^{*2}}, b_{22} = -(\mu + d + \sigma),\\
&b_{32} = \sigma, \qquad b_{33} = -\mu, \qquad c_{12} = -\frac{\beta S^*(1 - \alpha {I^*}^2)}{(1 + \alpha I^{*2})^2},\\
&c_{22} = \frac{\beta S^*(1 - \alpha {I^*}^2)}{(1 + \alpha I^{*2})^2}, d_{22} = -\frac{2aI^*}{(1 + bI^{*2})^2}, d_{32} = \frac{2aI^*}{(1 + bI^{*2})^2}.
\end{aligned}$$

The characteristic equation is given as:

$$\lambda^3 + P_1\lambda^2 + Q_1\lambda + R_1 + (P_2\lambda^2 + Q_2\lambda + R_2)e^{-\lambda\tau_1} + (P_3\lambda^2 + Q_3\lambda + R_3)e^{-\lambda\tau_2} = 0, \tag{2.3}$$

where

$$\begin{aligned}
&P_1 = 3\mu + d + \sigma + \frac{\beta I^*}{1 + \alpha I^{*2}}, \quad P_2 = -\frac{\beta S^*(1 - \alpha {I^*}^2)}{(1 + \alpha I^{*2})^2},\\
&R_1 = \mu(\mu + d + \sigma)(\mu + \frac{\beta I^*}{1 + \alpha I^{*2}}),\\
&Q_1 = \mu(3\mu + 2d + 2\sigma) + \frac{(2\mu + d + \sigma)\beta I^*}{1 + \alpha I^{*2}}, \quad Q_2 = -\frac{2\mu\beta S^*(1 - \alpha {I^*}^2)}{(1 + \alpha I^{*2})^2},\\
&R_2 = -\frac{\mu^2\beta S^*(1 - \alpha {I^*}^2)}{(1 + \alpha I^{*2})^2},\\
&P_3 = \frac{2aI^*}{(1 + bI^{*2})^2}, \quad Q_3 = \frac{4\mu a I^*}{(1 + bI^{*2})^2} + \frac{2a\beta {I^*}^2}{(1 + \alpha I^{*2})(1 + bI^{*2})^2},\\
&R_3 = \frac{2aI^*}{(1 + bI^{*2})^2}(\mu^2 + \frac{\beta I^*\mu}{1 + \alpha I^{*2}}).
\end{aligned}$$

Next, we will discuss these four cases: (1) $\tau_1 = \tau_2 = 0$; (2) $\tau_1 = 0, \tau_2 > 0$; (3) $\tau_1 = \tau_2 = \tau$; (4) $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$.

Case (1): $\tau_1 = \tau_2 = 0$.

Equation (2.3) is reduced to

$$\lambda^3 + (P_1 + P_2 + P_3)\lambda^2 + (Q_1 + Q_2 + Q_3)\lambda + (R_1 + R_2 + R_3) = 0. \tag{2.4}$$

Based on the Routh-Hurwitz criteria, we give the following condition:

(H_2): $P_1 + P_2 + P_3 > 0,\ R_1 + R_2 + R_3 > 0,$
$(P_1 + P_2 + P_3)(Q_1 + Q_2 + Q_3) - (R_1 + R_2 + R_3) > 0.$

Assuming that conditions (H_1) and (H_2) are true, then all the roots of equation (2.4) have a negative real part. Then we can get:

Theorem 2.1. *For $\tau_1 = \tau_2 = 0$, if (H_1), (H_2) hold, then $E(S^*, I^*, R^*)$ is locally asymptotically stable.*

Case (2): $\tau_1 = 0,\ \tau_2 > 0$.

Then equation (2.3) becomes

$$\lambda^3 + (P_1 + P_2)\lambda^2 + (Q_1 + Q_2)\lambda + R_1 + R_2 + (P_3\lambda^2 + Q_3\lambda + R_3)e^{-\lambda\tau_2} = 0. \tag{2.5}$$

Assuming that $i\omega_2(\omega_2 > 0)$ is a root of equation (2.5). Substituting it to equation (2.5), we can get

$$\begin{cases} (R_3 - P_3\omega_2^2)\cos\omega_2\tau_2 + Q_3\omega_2\sin\omega_2\tau_2 = (P_1 + P_2)\omega_2^2 - (R_1 + R_2), \\ -(R_3 - P_3\omega_2^2)\sin\omega_2\tau_2 + Q_3\omega_2\cos\omega_2\tau_2 = \omega_2^3 - (Q_1 + Q_2)\omega_2. \end{cases} \tag{2.6}$$

From equation (2.6), we have

$$\omega_2^6 + P_{12}\omega_2^4 + Q_{12}\omega_2^2 + R_{12} = 0, \tag{2.7}$$

where

$$\begin{aligned} &P_{12} = (P_1 + P_2 + P_3)(P_1 + P_2 - P_3) - 2(Q_1 + Q_2), \\ &Q_{12} = (Q_1 + Q_2 + Q_3)(Q_1 + Q_2 - Q_3) + 2[P_3R_3 - (P_1 + P_2)(R_1 + R_2)], \\ &R_{12} = (R_1 + R_2 + R_3)(R_1 + R_2 - R_3). \end{aligned}$$

Denote $z_2 = \omega_2^2$, equation (2.7) is reduced to

$$z_2^3 + P_{12}z_2^2 + Q_{12}z_2 + R_{12} = 0. \tag{2.8}$$

Let

$$h_2(z_2) = z_2^3 + P_{12}z_2^2 + Q_{12}z_2 + R_{12}, \tag{2.9}$$

thus

$$\frac{dh_2(z_2)}{dz_2} = 3z_2^2 + 2P_{12}z_2 + Q_{12}.$$

Next, let

$$3z_2^2 + 2P_{12}z_2 + Q_{12} = 0,$$

when $\triangle_2 = P_{12}^2 - 3Q_{12} > 0$, it has two real roots: $z_{21}^* = \frac{-P_{12}+\sqrt{\triangle_2}}{3}$, $z_{22}^* = \frac{-P_{12}-\sqrt{\triangle_2}}{3}$.

On the one hand, when $R_{12} = (R_1+R_2+R_3)(R_1+R_2-R_3) < 0$, since $h_2(0) < 0$, $\lim_{z_2 \longrightarrow +\infty} h_2(z_2) = +\infty$, equation (2.8) has at least one positive root. On the other hand, when $R_{12} = (R_1 + R_2 + R_3)(R_1 + R_2 - R_3) \geq 0$, if $\triangle_2 = P_{12}^2 - 3Q_{12} \leq 0$, then equation (2.8) has no positive root; if $\triangle_2 = P_{12}^2 - 3Q_{12} > 0$, $z_{21}^* > 0$ and $h_2(z_{21}^*) \leq 0$ hold, then equation (2.8) has at least one positive root.

For the general case, since equation (2.8) has at most three positive roots, respectively as z_{21}, z_{22} and z_{23}, so correspondingly there are three positive roots $\omega_{2k} = \sqrt{z_{2k}}, k = 1, 2, 3$ in equation (2.7). According to equation (2.6), we have

$$\begin{aligned}\cos\omega_{2k}\tau_{2k} =& \frac{[Q_3 - P_3(P_1+P_2)]\omega_{2k}^4}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2} \\ &+ \frac{[R_3(P_1+P_2) + P_3(R_1+R_2) - Q_3(Q_1+Q_2)]\omega_{2k}^2}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2} \\ &- \frac{R_3(R_1+R_2)}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2}\end{aligned}$$

and

$$\begin{aligned}\tau_{2k}^{(j)} =& \frac{1}{\omega_{2k}}\Bigg\{ \arccos\Bigg(\frac{[Q_3 - P_3(P_1+P_2)]\omega_{2k}^4}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2} \\ &+ \frac{[R_3(P_1+P_2) + P_3(R_1+R_2) - Q_3(Q_1+Q_2)]\omega_{2k}^2}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2} \\ &- \frac{R_3(R_1+R_2)}{P_3\omega_{2k}^4 + (Q_3^2 - 2P_3R_3)\omega_{2k}^2 + R_3^2}\Bigg) + 2\pi j\Bigg\}\end{aligned}$$

where $j = 0, 1, 2, \cdots$.

Denote

$$\tau_{20} = \tau_{2k_0}^{(0)} = \min_{k\in\{1,2,3\}}\{\tau_{2k}^{(0)}\}, \;\; \omega_{20} = \omega_{2k_0}.$$

Supposing that equation (2.5) has a root $\lambda(\tau_2) = \alpha_2(\tau_2) + i\omega_2(\tau_2)$ near $\tau_2 = \tau_{2k}^{(j)}$, and the root satisfies the following conditions:

$$\alpha_2(\tau_{2k}^{(j)}) = 0, \;\; \omega_2(\tau_{2k}^{(j)}) = \omega_{2k}.$$

Then we have

$$[\frac{d\lambda}{d\tau_2}]^{-1} = \frac{[3\lambda^2 + 2(P_1+P_2)\lambda + (Q_1+Q_2)]e^{\lambda\tau_2}}{\lambda(P_3\lambda^2 + Q_3\lambda + R_3)} + \frac{2P_3\lambda + Q_3}{\lambda(P_3\lambda^2 + Q_3\lambda + R_3)} - \frac{\tau_2}{\lambda}. \tag{2.10}$$

From (2.10), we can get

$$\begin{aligned}&\left[Re\left(\frac{d}{d\tau_2}(\lambda(\tau_2))\right)\right]^{-1}_{\tau_2=\tau_{2k}^{(j)}} \\ &= Re\left[\frac{[3\lambda^2 + 2(P_1+P_2)\lambda + (Q_1+Q_2)]e^{\lambda\tau_2}}{\lambda(P_3\lambda^2 + Q_3\lambda + R_3)} + \frac{2P_3\lambda + Q_3}{\lambda(P_3\lambda^2 + Q_3\lambda + R_3)}\right]_{\tau_2=\tau_{2k}^{(j)}} \\ &= \frac{1}{\Lambda_2}\{-Q_3\omega_{2k}^2[(Q_1 + Q_2 - 3\omega_{2k}^2)\cos(\omega_{2k}\tau_{2k}^{(j)}) - 2(P_1+P_2)\omega_{2k}\sin(\omega_{2k}\tau_{2k}^{(j)})]\end{aligned}$$

$$\begin{aligned}
&+ (R_3\omega_{2k} - P_3\omega_{2k}^3)[2(P_1+P_2)\omega_{2k}\cos(\omega_{2k}\tau_{2k}^{(j)}) + (Q_1+Q_2-3\omega_{2k}^2)\sin(\omega_{2k}\tau_{2k}^{(j)})] \\
&- Q_3^2\omega_{2k}^2 + 2P_3\omega_{2k}(R_3\omega_{2k} - P_3\omega_{2k}^3)\} \\
=& \frac{1}{\Lambda_2}\{3\omega_{2k}^6 + 2[(P_1+P_2)^2 - P_3^2 - 2(Q_1+Q_2)]\omega_{2k}^4 \\
&+ [(Q_1+Q_2)^2 - Q_3^2 + 2P_3R_3 - 2(P_1+P_2)(R_1+R_2)]\omega_{2k}^2\} \\
=& \frac{1}{\Lambda_2}[z_{2k}(3z_{2k}^2 + 2P_{12}z_{2k} + Q_{12})] \\
=& \frac{1}{\Lambda_2}z_{2k}h_2'(z_{2k}),
\end{aligned}$$

where $\Lambda_2 = Q_3^2\omega_{2k}^4 + (R_3\omega_{2k} - P_3\omega_{2k}^3)^2 > 0$, $z_{2k} > 0$. Therefore

$$\begin{aligned}
sign\left\{\left[Re\left(\frac{d}{d\tau_2}(\lambda(\tau_2))\right)\right]_{\tau_2=\tau_{2k}^{(j)}}\right\} &= sign\left\{\left[Re\left(\frac{d}{d\tau_2}(\lambda(\tau_2))\right)\right]^{-1}_{\tau_2=\tau_{2k}^{(j)}}\right\} \\
&= sign\left\{\frac{1}{\Lambda_2}z_{2k}h_2'(z_{2k})\right\},
\end{aligned}$$

then the sign of $Re\left(\frac{d}{d\tau_2}(\lambda(\tau_{2k}^{(j)}))\right)$ is consistent with the sign of $h_2'(z_{2k})$.

According to [17, Corollary 2.4] and the above analysis, we can obtain:

Theorem 2.2. *For $\tau_1 = 0$, $\tau_2 > 0$, then*

(i) If (H_1), (H_2), $R_{12} = (R_1+R_2+R_3)(R_1+R_2-R_3) \geq 0$ and $\triangle_2 = P_{12}^2 - 3Q_{12} \leq 0$ hold, $E(S^*, I^*, R^*)$ is locally asymptotically stable for $\tau_2 > 0$.

(ii) If (H_1), (H_2) hold, and either $R_{12} = (R_1+R_2+R_3)(R_1+R_2-R_3) < 0$ or $R_{12} = (R_1+R_2+R_3)(R_1+R_2-R_3) \geq 0$ and $\triangle_2 = P_{12}^2 - 3Q_{12} > 0$, $z_{21}^* = \frac{-P_{12}+\sqrt{\triangle_2}}{3} > 0$ and $h_2(z_{21}^*) \leq 0$, $E(S^*, I^*, R^*)$ is locally asymptotically stable for $\tau_2 \in (0, \tau_{20})$.

(iii) If (ii) *and $h_2'(z_{2k}) \neq 0$ hold, then Hopf bifurcations occurs in system* (1.2) *at $E(S^*, I^*, R^*)$ for $\tau_2 = \tau_{2k}^{(j)}$.*

Case (3): $\tau_1 = \tau_2 = \tau > 0$.

The characteristic equation changes to the following form:

$$\lambda^3 + P_1\lambda^2 + Q_1\lambda + R_1 + [(P_2+P_3)\lambda^2 + (Q_2+Q_3)\lambda + R_2 + R_3]e^{-\lambda\tau} = 0. \quad (2.11)$$

Assuming that equation (2.11) has a root of $\lambda = i\omega_3(\omega_3 > 0)$, we have

$$\begin{cases} -\omega_3^3 + Q_1\omega_3 = [R_2+R_3-(P_2+P_3)\omega_3^2]\sin\omega_3\tau - (Q_2+Q_3)\omega_3\cos\omega_3\tau, \\ R_1 - P_1\omega_3^2 = -[R_2+R_3-(P_2+P_3)\omega_3^2]\cos\omega_3\tau - (Q_2+Q_3)\omega_3\sin\omega_3\tau. \end{cases} \quad (2.12)$$

Next, we can get

$$\omega_3^6 + P_{13}\omega_3^4 + Q_{13}\omega_3^2 + R_{13} = 0, \quad (2.13)$$

where

$$P_{13} = (P_1+P_2+P_3)(P_1-P_2-P_3) - 2Q_1,$$

$$Q_{13}=(Q_1+Q_2+Q_3)(Q_1-Q_2-Q_3)+2(R_2+R_3)(P_2+P_3)-2P_1R_1,$$
$$R_{13}=(R_1+R_2+R_3)(R_1-R_2-R_3).$$

Then, equation (2.13) becomes

$$z_3^3+P_{13}z_3^2+Q_{13}z_3+R_{13}=0, \tag{2.14}$$

where $z_3=\omega_3^2$.

Let (H_3): Equation (2.14) has at least one positive real root.

If (H_3) holds, similar to the case (2), the three roots of equation (2.14) are expressed as z_{31}, z_{32} and z_{33}, so correspondingly there are three positive roots $\omega_{3k}=\sqrt{z_{3k}}, k=1,2,3$ in equation (2.13). From equation (2.12), we can get

$$\begin{aligned}\cos\omega_{3k}\tau=&\frac{[Q_2+Q_3-P_1(P_2+P_3)]\omega_{3k}^4}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\\&+\frac{[R_1(P_2+P_3)+P_1(R_2+R_3)-Q_1(Q_2+Q_3)]\omega_{2k}^2}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\\&-\frac{R_1(R_2+R_3)}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\end{aligned}$$

and

$$\begin{aligned}\tau^{(j)}=&\frac{1}{\omega_{3k}}\Bigg\{\arccos\Bigg(\frac{[Q_2+Q_3-P_1(P_2+P_3)]\omega_{3k}^4}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\\&+\frac{[R_1(P_2+P_3)+P_1(R_2+R_3)-Q_1(Q_2+Q_3)]\omega_{2k}^2}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\\&-\frac{R_1(R_2+R_3)}{(P_2+P_3)^2\omega_{3k}^4-[2(P_2+P_3)(R_2+R_3)-(Q_2+Q_3)^2]\omega_{3k}^2+(R_2+R_3)^2}\Bigg)+2\pi j\Bigg\}\end{aligned}$$

where $j=0,1,2,\cdots$.

Define

$$\tau_{30}=\tau_0=\min_{k\in\{1,2,3\}}\{\tau^{(0)}\},\ \ \omega_{30}=\omega_{3k_0}.$$

Assuming that equation (2.11) has a root $\lambda(\tau)=\alpha_3(\tau)+i\omega_3(\tau)$ near $\tau=\tau^{(j)}$, and the root satisfies the following conditions:

$$\alpha_3(\tau^{(j)})=0, \omega_3(\tau^{(j)})=\omega_{3k}.$$

Then we have

$$[\frac{d}{d\tau}(\lambda(\tau))]^{-1}=\frac{3\lambda^2+2P_1\lambda+Q_1+[2(P_2+P_3)\lambda+Q_2+Q_3]e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}[(P_2+P_3)\lambda^2+(Q_2+Q_3)\lambda+R_2+R_3]}-\frac{\tau}{\lambda}. \tag{2.15}$$

Similar to the discussion in case (2), we can get

$$\left[Re\left(\frac{d}{d\tau}(\lambda(\tau))\right)\right]^{-1}_{\tau=\tau^{(j)}}=\frac{1}{\Lambda_3}z_{3k}h_3'(z_{3k})$$

where $\Lambda_3=(Q_2+Q_3)^2\omega_{3k}^4+[(R_2+R_3)\omega_{3k}-(P_2+P_3)\omega_{3k}^3]^2>0, h_3(z_3)=z_3^3+P_{13}z_3^2+Q_{13}z_3+R_{13}$. Since $\Lambda_3>0,\ \ z_{3k}>0$, then

$$sign\Bigg\{\left[Re\left(\frac{d}{d\tau}(\lambda(\tau))\right)\right]_{\tau=\tau^{(j)}}\Bigg\}=sign\Bigg\{\left[Re\left(\frac{d}{d\tau}(\lambda(\tau))\right)\right]^{-1}_{\tau=\tau^{(j)}}\Bigg\}$$

$$= sign\left\{\frac{1}{\Lambda_3}z_{3k}h_3'(z_{3k})\right\},$$

thus the sign of $Re\left(\frac{d}{d\tau}(\lambda(\tau^{(j)}))\right)$ is consistent with the sign of $h_3'(z_{3k})$.

Obviously, we have the following theorem.

Theorem 2.3. *For $\tau_1 = \tau_2 = \tau > 0$, if $(H_1 - H_3)$ and $h_3'(z_{3k}) \neq 0$ hold, then*

(i) $E(S^*, I^*, R^*)$ is asymptotically stable for $\tau \in (0, \tau_0)$.

(ii) $E(S^*, I^*, R^*)$ is unstable for $\tau > \tau_0$.

(iii) System (1.2) undergoes Hopf bifurcations at $E(S^*, I^*, R^*)$ for $\tau = \tau^{(j)}$.

Case (4): $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$, $\tau_1 \neq \tau_2$.

In system (1.2), τ_1 is considered as a parameter. Assuming that equation (2.3) has a root of $\lambda = i\omega_4(\omega_4 > 0)$, then we have

$$\begin{cases} \quad -\omega_4^3 + Q_1\omega_4 + (P_3\omega_4^2 - R_3)\sin(\omega_4\tau_2^*) + Q_3\omega_4\cos(\omega_4\tau_2^*) \\ = -(P_2\omega_4^2 - R_2)\sin(\omega_4\tau_1) - Q_2\omega_4\cos(\omega_4\tau_1), \\ \quad -P_1\omega_4^2 + R_1 - (P_3\omega_4^2 - R_3)\cos(\omega_4\tau_2^*) + Q_3\omega_4\sin(\omega_4\tau_2^*) \\ = (P_2\omega_4^2 - R_2)\cos(\omega_4\tau_1) - Q_2\omega_4\sin(\omega_4\tau_1). \end{cases} \tag{2.16}$$

According to equation (2.16), we can get

$$\omega_4^6 + P_{14}\omega_4^4 + Q_{14}\omega_4^2 + R_{14} + M_{14}\sin(\omega_4\tau_2^*) + N_{14}\cos(\omega_4\tau_2^*) = 0 \tag{2.17}$$

where

$$\begin{aligned} P_{14} &= P_1^2 + P_3^2 - P_2^2 - 2Q_1, \\ Q_{14} &= Q_1^2 + Q_3^2 - Q_2^2 + 2P_2R_2 - 2P_1R_1 - 2P_3R_3, \\ R_{14} &= R_1^2 + R_3^2 - R_2^2, \\ M_{14} &= 2[Q_3\omega_4(R_1 - P_1\omega_4^2) + (Q_1\omega_4 - \omega_4^3)(P_3\omega_4^2 - R_3)], \\ N_{14} &= 2[Q_3\omega_4(Q_1\omega_4 - \omega_4^3) - (R_1 - P_1\omega_4^2)(P_3\omega_4^2 - R_3)]. \end{aligned}$$

Similar to the discussion in case (3), let

(H_4): Equation (2.17) has at least one positive real root.

Supposing that the positive real roots of equation (2.17) are $\omega_{4k}(k = 1, 2, \cdots, 6)$. From (2.16), we have

$$\begin{aligned} \tau_{1k}^{*\,(j)} =& \frac{1}{\omega_{4k}}\Bigg\{\arccos\Bigg(\frac{(Q_2 - P_1P_2)\omega_{4k}^4 + (P_1R_2 + P_2R_1 - Q_1Q_2)\omega_{4k}^2 - R_1R_2}{P_2^2\omega_{4k}^4 + (Q_2^2 - 2P_2R_2)\omega_{4k}^2 + R_2^2} \\ &+ \frac{[(P_2Q_3 - P_3Q_2)\omega_{4k}^3 + (Q_2R_3 - Q_3R_2)\omega_{4k}]\sin(\omega_{4k}\tau_2^*)}{P_2^2\omega_{4k}^4 + (Q_2^2 - 2P_2R_2)\omega_{4k}^2 + R_2^2} \\ &- \frac{[P_2P_3\omega_{4k}^4 + (Q_2Q_3 - P_2R_3 - P_3R_2)\omega_{4k}^2 + R_2R_3]\cos(\omega_{4k}\tau_2^*)}{P_2^2\omega_{4k}^4 + (Q_2^2 - 2P_2R_2)\omega_{4k}^2 + R_2^2}\Bigg) + 2\pi j\Bigg\} \end{aligned}$$

where $j = 0, 1, 2, \cdots$.

Define

$$\tau_{10}^* = {\tau_{1k_0}^*}^{(0)} = \min_{k\in\{1,2,\cdots,6\}}\{{\tau_{1k}^*}^{(0)}\}, \ \ \omega_{40} = \omega_{4k_0}. \tag{2.18}$$

Then we can obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{Pe^{\lambda\tau_1} + Q + Re^{\lambda(\tau_1-\tau_2)}}{M} - \frac{\tau_1}{\lambda} \tag{2.19}$$

where

$$\begin{aligned} P =& 3\lambda^2 + 2P_1\lambda + Q_1, \\ Q =& 2P_2\lambda + Q_2, \\ R =& (2P_3\lambda + Q_3) - \tau_2(P_3\lambda^2 + Q_3\lambda + R_3), \\ M =& \lambda(P_2\lambda^2 + Q_2\lambda + R_2). \end{aligned}$$

From equation (2.19), we have

$$\left[Re\left(\tfrac{d}{d\tau_1}(\lambda({\tau_{1k}^*}^{(j)}))\right)\right]^{-1}_{\lambda=i\omega_{4k}} = \tfrac{M_1N_1+M_2N_2}{M_1^2+M_2^2}$$

where

$$\begin{aligned} M_1 =& -Q_2\omega_{4k}^2, \\ M_2 =& R_2\omega_{4k} - P_2\omega_{4k}^3, \\ N_1 =& (-3\omega_{4k}^2 + Q_1)\cos(\omega_{4k}{\tau_{1k}^*}^{(j)}) - 2P_1\omega_{4k}\sin(\omega_{4k}{\tau_{1k}^*}^{(j)}) + Q_2 + (P_3\omega_{4k}^2\tau_2^* + Q_3 \\ & - R_3\tau_2^*)\cos(\omega_{4k}({\tau_{1k}^*}^{(j)} - \tau_2^*)) - (2P_3\omega_{4k} - Q_3\omega_{4k}\tau_2^*)\sin(\omega_{4k}({\tau_{1k}^*}^{(j)} - \tau_2^*)), \\ N_2 =& 2P_1\omega_{4k}\cos(\omega_{4k}{\tau_{1k}^*}^{(j)}) + (-3\omega_{4k}^2 + Q_1)\sin(\omega_{4k}{\tau_{1k}^*}^{(j)}) + 2P_2\omega_{4k} + (P_3\omega_{4k}^2\tau_2^* \\ & + Q_3 - R_3\tau_2^*)\sin(\omega_{4k}({\tau_{1k}^*}^{(j)} - \tau_2^*)) + (2P_3\omega_{4k} - Q_3\omega_{4k}\tau_2^*)\cos(\omega_{4k}({\tau_{1k}^*}^{(j)} - \tau_2^*)). \end{aligned}$$

Let

(H_5): $M_1N_1 + M_2N_2 \neq 0$.

We can get the following conclusion.

Theorem 2.4. *For $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$, if (H_1), (H_2), (H_4), (H_5) hold, then*

(i) $E(S^*, I^*, R^*)$ is asymptotically stable for $\tau_1 \in (0, \tau_{10}^*)$.

(ii) $E(S^*, I^*, R^*)$ is unstable for $\tau_1 > \tau_{10}^*$.

(iii) System (1.2) undergoes Hopf bifurcations at $E(S^*, I^*, R^*)$ for $\tau_1 = {\tau_{1k}^*}^{(j)}$.

3. Direction and stability of the Hopf bifurcation

In the previous section, we obtained the conditions for the local stability and Hopf bifurcation of the system (1.2) at the positive equilibrium. In this section, we discuss the direction and stability of the Hopf bifurcation of the system (1.2) when $\tau_1 > 0$, $\tau_2 = \tau_2^* \in (0, \tau_{20})$ by [7, 8, 12].

Let $x_1(t) = S(t) - S^*$, $x_2(t) = I(t) - I^*$, $x_3(t) = R(t) - R^*$, $t = t/\tau_1$, and $\tau_1 = \tau_{10}^* + \mu$ (μ is bifurcation parameters of system (1.2)). Then the system (1.2) can be written as a FDE in $C = C([-1, 0], R^3)$,

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.1}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T \in R^3$. $L_\mu(\phi) : C \to R^3$ and $f(\mu, x_t)$ are shown below:

$$L_\mu(\phi) = (\tau_{10}^* + \mu)B \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tau_{10}^* + \mu)C \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} + (\tau_{10}^* + \mu)D \begin{pmatrix} \phi_1(-\frac{\tau_2^*}{\tau_1}) \\ \phi_2(-\frac{\tau_2^*}{\tau_1}) \\ \phi_3(-\frac{\tau_2^*}{\tau_1}) \end{pmatrix}, \tag{3.2}$$

and

$$f(\mu, \phi) = (\tau_{10}^* + \mu) \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{3.3}$$

with $\phi = (\phi_1, \phi_2, \phi_3)^T \in C([-1, 0], R^3)$, and

$$\begin{aligned} f_1 =& k_{11}\phi_2^2(-1) + k_{12}\phi_1(0)\phi_2(-1) + k_{13}\phi_1(0)\phi_2^2(-1) + k_{14}\phi_2^3(-1) + \cdots, \\ f_2 =& -k_{11}\phi_2^2(-1) - k_{12}\phi_1(0)\phi_2(-1) + k_{21}\phi_2^2(-\frac{\tau_2^*}{\tau_1}) - k_{13}\phi_1(0)\phi_2^2(-1) - k_{14}\phi_2^3(-1) \\ & + k_{22}\phi_2^3(-\frac{\tau_2^*}{\tau_1}) + \cdots, \\ f_3 =& -k_{21}\phi_2^2(-\frac{\tau_2^*}{\tau_1}) - k_{22}\phi_2^3(-\frac{\tau_2^*}{\tau_1}) + \cdots, \end{aligned}$$

and

$$\begin{aligned} k_{11} &= \frac{\alpha\beta S^* I^*(3 - 2\alpha I^{*^2})}{(1 + \alpha I^{*2})^3}, \; k_{12} = -\frac{\beta(1 - \alpha I^{*^2})}{(1 + \alpha I^{*2})^2}, \; k_{13} = \frac{\alpha\beta I^*(3 - 2\alpha I^{*^2})}{(1 + \alpha I^{*2})^3}, \\ k_{14} &= \frac{\alpha\beta S^*(2\alpha^2 I^{*^4} - 7\alpha I^{*^2} + 1)}{(1 + \alpha I^{*2})^4}, \; k_{21} = -\frac{a(1 - 3bI^{*^2})}{(1 + bI^{*2})^3}, \; k_{22} = \frac{4abI^*(1 - bI^{*^2})}{(1 + bI^{*2})^4}. \end{aligned}$$

According to the Riesz representation theorem, there exists a $\eta(\theta, \mu)(-1 \le \theta \le 0)$, $\eta(\theta, \mu)$ is a bounded variation function, such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \;\; \phi \in C([-1, 0], R^3). \tag{3.4}$$

Then, we choose

$$\eta(\theta, \mu) = (\tau_{10}^* + \mu)[B\delta(\theta) + C\delta(\theta + 1) + D\delta(\theta + \frac{\tau_2^*}{\tau_1})], \tag{3.5}$$

where δ is the Dirac delta function.

Next, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0, \end{cases}$$

where $\phi \in C^1([-1, 0], R^3)$.

For $\theta = 0$, equation (3.1) is equivalent to the following form:

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{3.6}$$

where $x_t = x(t + \theta) = (x_1(t + \theta),\, x_2(t + \theta),\, x_3(t + \theta))$ for $\theta \in [-1, 0]$.

Define

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t, 0)\psi(-t), & s = 0. \end{cases}$$

and

$$\langle \psi(s), \phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.7}$$

where $\eta(\theta) = \eta(\theta, 0)$, A and A^* are adjoint operators. Let $A = A(0)$, then $\pm i\omega_{40}\tau_{10}^*$ are eigenvalues of A and A^* respectively. Assuming that $q(\theta) = (1, \gamma, \zeta)^T e^{i\omega_{40}\tau_{10}^*\theta}$ is the eigenvector of A corresponding to $i\omega_{40}\tau_{10}^*$. Then, according to the definition of A and (3.5), we have

$$\tau_{10}^* \begin{pmatrix} i\omega_{40} - b_{11} & -c_{12}e^{-i\omega_{40}\tau_{10}^*} & 0 \\ -b_{21} & i\omega_{40} - b_{22} - c_{22}e^{-i\omega_{40}\tau_{10}^*} - d_{22}e^{-i\omega_{40}\tau_2^*} & 0 \\ 0 & -b_{32} - d_{32}e^{-i\omega_{40}\tau_2^*} & i\omega_{40} - b_{33} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields

$$q(0) = (1, \gamma, \zeta)^T = \Big(1, \frac{i\omega_{40} - b_{11}}{c_{12}e^{-i\omega_{40}\tau_{10}^*}}, \frac{(b_{32} + d_{32}e^{-i\omega_{40}\tau_2^*})(i\omega_{40} - b_{11})}{(i\omega_{40} - b_{33})c_{12}e^{-i\omega_{40}\tau_{10}^*}}\Big)^T.$$

Similar to the discussion in $q(\theta)$, assuming that $q^*(s) = D'(1, \gamma^*, \zeta^*)e^{i\omega_{40}\tau_{10}^* s}$ is the eigenvector of A^* corresponding to $-i\omega_{40}\tau_{10}^*$, which yields

$$q^*(0) = D'(1, \gamma^*, \zeta^*) = D'(1, \frac{-i\omega_{40} - b_{11}}{b_{21}}, 0).$$

From (3.7), we can obtain

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \bar{D}'(1, \bar{\gamma^*}, \bar{\zeta^*})(1, \gamma, \zeta)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}'(1, \bar{\gamma^*}, \bar{\zeta^*})e^{-i(\xi-\theta)\omega_{40}\tau_{10}^*}d\eta(\theta)(1, \gamma, \zeta)^T e^{i\xi\omega_{40}\tau_{10}^*}d\xi \\ &= \bar{D}'\{1 + \gamma\bar{\gamma^*} + \zeta\bar{\zeta^*} - \int_{-1}^{0}(1, \bar{\gamma^*}, \bar{\zeta^*})\theta e^{i\theta\omega_{40}\tau_{10}^*}d\eta(\theta)(1, \gamma, \zeta)^T\} \\ &= \bar{D}'\{1 + \gamma\bar{\gamma^*} + \zeta\bar{\zeta^*} + e^{-i\omega_{40}\tau_{10}^*}(1, \bar{\gamma^*}, \bar{\zeta^*}) \cdot C(1, \gamma, \zeta)^T \\ &\quad + \frac{\tau_2^*}{\tau_{10}^*}e^{-i\omega_{40}\tau_2^*}(1, \bar{\gamma^*}, \bar{\zeta^*}) \cdot D(1, \gamma, \zeta)^T\} \end{aligned}$$

$$
\begin{aligned}
&= \bar{D}'\{1 + \gamma\bar{\gamma}^* + \zeta\bar{\zeta}^* + e^{-i\omega_{40}\tau_{10}^*}\gamma(c_{12} + \bar{\gamma}^* c_{22}) \\
&\quad + \frac{\tau_2^*}{\tau_{10}^*} e^{-i\omega_{40}\tau_2^*}\gamma(\bar{\gamma}^* d_{22} + \bar{\zeta}^* d_{32})\}.
\end{aligned}
$$

Thus, we have

$$
\bar{D}' = \{1+\gamma\bar{\gamma}^*+\zeta\bar{\zeta}^*+e^{-i\omega_{40}\tau_{10}^*}\gamma(c_{12}+\bar{\gamma}^* c_{22})+\frac{\tau_2^*}{\tau_{10}^*} e^{-i\omega_{40}\tau_2^*}\gamma(\bar{\gamma}^* d_{22}+\bar{\zeta}^* d_{32})\}^{-1}. \quad (3.8)
$$

In order to describe the center manifold C_0 at $\mu = 0$, assuming that x_t is the solution of equation (3.1) at $\mu = 0$ and defining

$$
z(t) = \langle q^*, x_t\rangle, \quad W(t,\theta) = x_t(\theta) - 2Re\{z(t)q(\theta)\}. \quad (3.9)
$$

On the center manifold C_0, we have

$$
W(t,\theta) = W(z(t),\bar{z}(t),\theta) = W_{20}(0)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots,
$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q and $\bar{q}$.

Then we have

$$
\begin{aligned}
\dot{z}(t) &= \langle q^*(\theta), \dot{x}_t(\theta)\rangle \\
&= \langle q^*(\theta), Ax_t(\theta) + Rx_t(\theta)\rangle \\
&= \langle A^* q^*(\theta), x_t(\theta)\rangle + \langle q^*(\theta), Rx_t(\theta)\rangle \\
&= iw_{40}\langle q^*(\theta), x_t(\theta)\rangle + \langle q^*(\theta), Rx_t(\theta)\rangle.
\end{aligned} \quad (3.10)
$$

When $\theta = 0$,

$$
\begin{aligned}
\dot{z} &= i\omega_{40}\tau_{10}^* z + \langle q^*(\theta), f(0, W(z(t),\bar{z}(t),\theta) + 2Re\{z(t)q(\theta)\}\rangle \\
&= i\omega_{40}\tau_{10}^* z + \bar{q}^*(0)f(0, W(z(t),\bar{z}(t),0) + 2Re\{z(t)q(0)\} \\
&= i\omega_{40}\tau_{10}^* z + \bar{q}^*(0)f_0(z,\bar{z}) \triangleq i\omega_{40}\tau_{10}^* z + g(z,\bar{z}),
\end{aligned} \quad (3.11)
$$

where

$$
g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{\bar{z}z^2}{2} + \cdots. \quad (3.12)
$$

Since $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))^T = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (1,\gamma,\zeta)^T e^{i\omega_{40}\tau_{10}^*\theta}$, we have

$$
\begin{aligned}
x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{2t}(0) &= z\gamma + \bar{z}\bar{\gamma} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{3t}(0) &= z\zeta + \bar{z}\bar{\zeta} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3),
\end{aligned}
$$

$$
\begin{aligned}
x_{1t}(-1) &= ze^{-i\omega_{40}\tau_{10}^*} + \bar{z}e^{i\omega_{40}\tau_{10}^*} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} \\
&\quad + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{2t}(-1) &= z\gamma e^{-i\omega_{40}\tau_{10}^*} + \bar{z}\bar{\gamma}e^{i\omega_{40}\tau_{10}^*} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} \\
&\quad + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{3t}(-1) &= z\zeta e^{-i\omega_{40}\tau_{10}^*} + \bar{z}\bar{\zeta}e^{i\omega_{40}\tau_{10}^*} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} \\
&\quad + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{1t}(-\frac{\tau_2^*}{\tau_{10}^*}) &= ze^{-i\omega_{40}\tau_2^*} + \bar{z}e^{i\omega_{40}\tau_2^*} + W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{z^2}{2} + W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_{10}^*})z\bar{z} \\
&\quad + W_{02}^{(1)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{2t}(-\frac{\tau_2^*}{\tau_{10}^*}) &= z\gamma e^{-i\omega_{40}\tau_2^*} + \bar{z}\bar{\gamma}e^{i\omega_{40}\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{z^2}{2} + W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*})z\bar{z} \\
&\quad + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\
x_{3t}(-\frac{\tau_2^*}{\tau_{10}^*}) &= z\zeta e^{-i\omega_{40}\tau_2^*} + \bar{z}\bar{\zeta}e^{i\omega_{40}\tau_2^*} + W_{20}^{(3)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{z^2}{2} + W_{11}^{(3)}(-\frac{\tau_2^*}{\tau_{10}^*})z\bar{z} \\
&\quad + W_{02}^{(3)}(-\frac{\tau_2^*}{\tau_{10}^*})\frac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3).
\end{aligned}
$$

Then

$$
\begin{aligned}
g(z,\bar{z}) &= \bar{q^*}(0)f_0(z,\bar{z}) = \bar{D}'\tau_{10}(1,\bar{\gamma^*},\bar{\zeta^*})(f_1^{(0)}\quad f_2^{(0)}\quad f_3^{(0)})^T \\
&= \bar{D}'\tau_{10}\{[k_{11}x_{2t}^2(-1) + k_{12}x_{1t}(0)x_{2t}(-1) + k_{13}x_{1t}(0)x_{2t}^2(-1) \\
&\quad + k_{14}x_{2t}^3(-1) + \cdots] + \bar{\gamma^*}[-k_{11}x_{2t}^2(-1) - k_{12}x_{1t}(0)x_{2t}(-1) \\
&\quad + k_{21}x_{2t}^2(-\frac{\tau_2^*}{\tau_{10}^*}) - k_{13}x_{1t}(0)x_{2t}^2(-1) - k_{14}x_{2t}^3(-1) \\
&\quad + k_{22}x_{2t}^3(-\frac{\tau_2^*}{\tau_{10}^*}) + \cdots] + \bar{\zeta^*}[-k_{21}x_{2t}^2(-\frac{\tau_2^*}{\tau_{10}^*}) - k_{22}x_{2t}^3(-\frac{\tau_2^*}{\tau_{10}^*}) + \cdots]\}.
\end{aligned}
$$

According to (3.12), we can get

$$
\begin{aligned}
g_{20} =& 2\bar{D}'\tau_{10}^*[(k_{11}\gamma^2e^{-2i\omega_{40}\tau_{10}^*} + k_{12}\gamma e^{-i\omega_{40}\tau_{10}^*}) + \bar{\gamma^*}(k_{21}\gamma^2e^{-2i\omega_{40}\tau_2^*} \\
&- k_{11}\gamma^2e^{-2i\omega_{40}\tau_{10}^*} - k_{12}\gamma e^{-i\omega_{40}\tau_{10}^*}) + \bar{\zeta^*}(-k_{21}\gamma^2e^{-2i\omega_{40}\tau_2^*})], \\
g_{11} =& \bar{D}'\tau_{10}^*\{[2k_{11}\gamma\bar{\gamma} + k_{12}(\bar{\gamma}e^{i\omega_{40}\tau_{10}^*} + \gamma e^{-i\omega_{40}\tau_{10}^*})] + \bar{\gamma^*}[2k_{21}\gamma\bar{\gamma} - 2k_{11}\gamma\bar{\gamma} \\
&- k_{12}(\bar{\gamma}e^{i\omega_{40}\tau_{10}^*} + \gamma e^{-i\omega_{40}\tau_{10}^*})] + \bar{\zeta^*}(-2k_{21}\gamma\bar{\gamma})\}, \\
g_{02} =& 2\bar{D}'\tau_{10}^*[(k_{11}\bar{\gamma}^2e^{2i\omega_{40}\tau_{10}^*} + k_{12}\bar{\gamma}e^{i\omega_{40}\tau_{10}^*}) + \bar{\gamma^*}(k_{21}\bar{\gamma}^2e^{2i\omega_{40}\tau_2^*} - k_{11}\bar{\gamma}^2e^{2i\omega_{40}\tau_{10}^*} \\
&- k_{12}\bar{\gamma}e^{i\omega_{40}\tau_{10}^*}) + \bar{\zeta^*}(-k_{21}\bar{\gamma}^2e^{2i\omega_{40}\tau_2^*})], \\
g_{21} =& 2\bar{D}'\tau_{10}^*\{[k_{11}(2\gamma e^{-i\omega_{40}\tau_{10}^*}W_{11}^{(2)}(-1) + \bar{\gamma}e^{i\omega_{40}\tau_{10}^*}W_{20}^{(2)}(-1))
\end{aligned}
$$

$$
\begin{aligned}
&+ k_{12}(W_{11}^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1) + \frac{1}{2}\bar{\gamma}e^{i\omega_{40}\tau_{10}^*}W_{20}^{(1)}(0) + \gamma e^{-i\omega_{40}\tau_{10}^*}W_{11}^{(1)}(0)) \\
&+ k_{13}(2\gamma\bar{\gamma} + \gamma^2 e^{-2i\omega_{40}\tau_{10}^*}) + k_{14}(3\gamma^2\bar{\gamma}e^{-i\omega_{40}\tau_{10}^*})] \\
&+ \bar{\gamma^*}[k_{21}(2\gamma e^{-i\omega_{40}\tau_2^*}W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*}) + \bar{\gamma}e^{i\omega_{40}\tau_2^*}W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*})) \\
&+ k_{22}(3\gamma^2\bar{\gamma}e^{-i\omega_{40}\tau_2^*}) - k_{11}(2\gamma e^{-i\omega_{40}\tau_{10}^*}W_{11}^{(2)}(-1) + \bar{\gamma}e^{i\omega_{40}\tau_{10}^*}W_{20}^{(2)}(-1)) \\
&- k_{12}(W_{11}^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1) + \frac{1}{2}\bar{\gamma}e^{i\omega_{40}\tau_{10}^*}W_{20}^{(1)}(0) + \gamma e^{-i\omega_{40}\tau_{10}^*}W_{11}^{(1)}(0)) \\
&- k_{13}(2\gamma\bar{\gamma} + \gamma^2 e^{-2i\omega_{40}\tau_{10}^*}) - k_{14}(3\gamma^2\bar{\gamma}e^{-i\omega_{40}\tau_{10}^*})] + \bar{\zeta^*}[-k_{22}(3\gamma^2\bar{\gamma}e^{-i\omega_{40}\tau_2^*}) \\
&- k_{21}(2\gamma e^{-i\omega_{40}\tau_2^*}W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*}) + \bar{\gamma}e^{i\omega_{40}\tau_2^*}W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_{10}^*}))]\},
\end{aligned}
$$

where

$$
\begin{aligned}
W_{20}(\theta) =& \frac{ig_{20}}{\omega_{40}\tau_{10}^*}q(0)e^{i\omega_{40}\tau_{10}^*\theta} + \frac{i\bar{g}_{02}}{3\omega_{40}\tau_{10}^*}\bar{q}(0)e^{-i\omega_{40}\tau_{10}^*\theta} + E_1 e^{2i\omega_{40}\tau_{10}^*\theta}, \\
W_{11}(\theta) =& -\frac{ig_{11}}{\omega_{40}\tau_{10}^*}q(0)e^{i\omega_{40}\tau_{10}^*\theta} + \frac{i\bar{g}_{11}}{\omega_{40}\tau_{10}^*}\bar{q}(0)e^{-i\omega_{40}\tau_{10}^*\theta} + E_2,
\end{aligned}
$$

and

$$
\begin{aligned}
E_1 =& 2\begin{pmatrix} 2i\omega_{40} - b_{11} & -c_{12}e^{-2i\omega_{40}\tau_{10}^*} & 0 \\ -b_{21} & 2i\omega_{40} - b_{22} - c_{22}e^{-2i\omega_{40}\tau_{10}^*} - d_{22}e^{-2i\omega_{40}\tau_2^*} & 0 \\ 0 & -b_{32} - d_{32}e^{-2i\omega_{40}\tau_2^*} & 2i\omega_{40} - b_{33} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \\
E_2 =& \begin{pmatrix} -b_{11} & -c_{12} & 0 \\ -b_{21} & -b_{22} - c_{22} - d_{22} & 0 \\ 0 & -b_{32} - d_{32} & -b_{33} \end{pmatrix}^{-1} \cdot \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
F_1 &= k_{11}\gamma^2 e^{-2i\omega_{40}\tau_{10}^*} + k_{12}\gamma e^{-i\omega_{40}\tau_{10}^*}, \\
F_2 &= k_{21}\gamma^2 e^{-2i\omega_{40}\tau_2^*} - k_{11}\gamma^2 e^{-2i\omega_{40}\tau_{10}^*} - k_{12}\gamma e^{-i\omega_{40}\tau_{10}^*}, \\
F_3 &= -k_{21}\gamma^2 e^{-2i\omega_{40}\tau_2^*}, \\
G_1 &= 2k_{11}\gamma\bar{\gamma} + k_{12}(\bar{\gamma}e^{i\omega_{40}\tau_{10}^*} + \gamma e^{-i\omega_{40}\tau_{10}^*}), \\
G_2 &= 2k_{21}\gamma\bar{\gamma} - 2k_{11}\gamma\bar{\gamma} - k_{12}(\bar{\gamma}e^{i\omega_{40}\tau_{10}^*} + \gamma e^{-i\omega_{40}\tau_{10}^*}), \\
G_3 &= -2k_{21}\gamma\bar{\gamma}.
\end{aligned}
$$

Next, we can calculate the following values:

$$
C_1(0) = \frac{i}{2\omega_{40}\tau_{10}^*}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \;\; \mu_2 = -\frac{Re(C_1(0))}{Re(\lambda'(\tau_{10}^*))},
$$

$$T_2 = -\frac{ImC_1(0) + \mu_2 Im\lambda^{'}(\tau_{10}^*)}{\omega_{40}\tau_{10}^*}, \quad \beta_2 = 2Re(C_1(0)).$$

Where $\mu_2 < 0$ ($\mu_2 > 0$), the Hopf bifurcation is subcritical (supercritical); $\beta_2 > 0$ ($\beta_2 < 0$), the bifurcating periodic solutions are unstable (stable); $T_2 > 0$ ($T_2 < 0$), the period of the bifurcating periodic solutions is increasing (decreasing).

4. Rank-one strange attractor

In the previous section, we discussed the direction and stability of Hopf bifurcation. In this section, we always assumed that the bifurcating periodic solutions of system (1.2) is supercritical. System (1.4) can be rewritten as:

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t) + \varepsilon\Phi(x_t)P_T(t), \tag{4.1}$$

where

$$\Phi(\phi) = (\tau_{10}^* + \mu)\begin{pmatrix} \phi_1(0) + S^* \\ \phi_2(0) + I^* \\ \phi_3(0) + R^* \end{pmatrix}, \tag{4.2}$$

and $\varepsilon > 0$ is small enough, $P_T = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $\delta(\cdot)$ is Dirac-delta function.

In section 3, we defined $A(\mu)$ and $R(\mu)$. In this section, we redefine

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-r, 0), \\ f(\mu, \phi) + \varepsilon\Phi(t)P_T(t), & \theta = 0. \end{cases}$$

For $\theta = 0$, system (4.1) is equivalent to

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t. \tag{4.3}$$

According to the Harssard's method, we have

$$\begin{cases} \dot{z} = i\omega_{40}z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + g_{21}z^2\bar{z} + \cdots \\ \qquad + \varepsilon P_T \bar{q^*}(0)\Phi(W(z, \bar{z}, 0) + 2Rezq(0)), \\ \dot{W} = AW + H(z, \bar{z}, 0). \end{cases} \tag{4.4}$$

Let $W \in \mathcal{B}$, $\mathcal{B}$ is a *Banach* space. In equation (4.4), let $z = x + yi$, define

$$\begin{aligned} \Psi_x(x, y) &= Re\{\bar{q^*}(0)\Phi(W(x, y, 0) + 2Re(x + yi)q(0))\}, \\ \Psi_y(x, y) &= Im\{\bar{q^*}(0)\Phi(W(x, y, 0) + 2Re(x + yi)q(0))\}. \end{aligned} \tag{4.5}$$

Next, in equation (4.5), let $x = \cos\theta$, $y = \sin\theta$, $W = 0$, then

$$\{\hat{\mathbf{s}}_0 = (\cos\theta, \sin\theta, \mathbf{0}) \in S \times \mathcal{B}, \theta \in [0, 2\pi)\}$$

is the unit circle in (x, y)-*plane* in $(x, y, \mathbf{W})$-*space*, and define

$$\varphi(\theta) = \cos\theta\Psi_x(s_0) + \sin\theta\Psi_y(s_0). \tag{4.6}$$

The following results are given in Dai et al. [2, Theorem 1].

Lemma 4.1 ([2, Theorem 1]). *Assume that the bifurcating periodic solutions of system (1.2) is supercritical, $\varphi(\theta)$ is a Morse function and $0<\mu\ll 1$, $0<\varepsilon\ll 1$. Regard the period T of the forcing as a parameter and denote $F_T=F_{\mu,\varepsilon,T}$. Then there exists a constant K_2, determined exclusively by $\varphi(\theta)$, such that if $\left|\varepsilon\frac{Imc_1(0)}{Rec_1(0)}\right|>K_2$, then there exists a positive measure $\Delta\subset(\mu^{-1},\infty)$ for T, so that for $T\in\Delta$, F_T has a strange attractor Λ. This is to say that there exists an open neighborhood U of Λ such that F_T has a positive Lyapunov exponent for Lebesgue almost every point in U. Furthermore, F_T admits an ergodic SRB measure, with respect to which almost every point of U is generic.*

Next we verify that $\varphi(\theta)$ is a Morse function according to Lemma 4.1. We let $z=x+yi$, $W=0$.

According to equation (4.5) and (4.6), we have

$$
\begin{aligned}
&\bar{q}^*(0)\Phi(W(z,\bar{z},0)+2Re\{zq(0)\}\\
=&\bar{q^*}(0)\Phi(2Rezq(0))\\
=&\bar{q^*}(0)\Phi(zq(0)+\bar{z}\bar{q}(0))\\
=&\bar{D'}\tau_{10}^*(1,\bar{\gamma^*},\bar{\zeta^*})\begin{pmatrix} z+\bar{z}+S^* \\ z\gamma+\bar{z}\bar{\gamma}+I^* \\ z\zeta+\bar{z}\bar{\zeta}+R^*\end{pmatrix}\\
=&\tau_{10}^*(d_1+d_2i)\{[2x+S^*+r_1^*(2xr_1-2yr_2+I^*)+s_1^*(2xs_1-2ys_2+R^*)]\\
&-[r_2^*(2xr_1-2yr_2+I^*)+s_2^*(2xs_1-2ys_2+R^*)]i\},
\end{aligned}
$$

where $\bar{D'}=d_1+d_2i, \gamma=r_1+r_2i, \zeta=s_1+s_2i, \gamma^*=r_1^*+r_2^*i, \zeta^*=s_1^*+s_2^*i$.

Then, we can obtain

$$
\begin{aligned}
\Psi_x(x,y)=&Re\{\bar{q^*}(0)\Phi(2Rezq(0))\}\\
=&\tau_{10}^*d_1[2x+S^*+r_1^*(2xr_1-2yr_2+I^*)+s_1^*(2xs_1-2ys_2+R^*)]\\
&+\tau_{10}^*d_2[r_2^*(2xr_1-2yr_2+I^*)+s_2^*(2xs_1-2ys_2+R^*)],
\end{aligned}\tag{4.7}
$$

$$
\begin{aligned}
\Psi_y(x,y)=&Im\{\bar{q^*}(0)\Phi(2Rezq(0))\}\\
=&-\tau_{10}^*d_1[r_2^*(2xr_1-2yr_2+I^*)+s_2^*(2xs_1-2ys_2+R^*)]\\
&+\tau_{10}^*d_2[2x+S^*+r_1^*(2xr_1-2yr_2+I^*)+s_1^*(2xs_1-2ys_2+R^*)].
\end{aligned}\tag{4.8}
$$

Let $x=\cos\theta, y=\sin\theta$, then we have

$$
\begin{aligned}
\varphi(\theta)=&\cos\theta\Psi_x(s_0)+\sin\theta\Psi_y(s_0)\\
=&\cos\theta\{\tau_{10}^*d_1[2\cos\theta+S^*+r_1^*(2\cos\theta r_1-2\sin\theta r_2+I^*)+s_1^*(2\cos\theta s_1\\
&-2\sin\theta s_2+R^*)]+\tau_{10}^*d_2[r_2^*(2\cos\theta r_1-2\sin\theta r_2+I^*)+s_2^*(2\cos\theta s_1\\
&-2\sin\theta s_2+R^*)]\}+\sin\theta\{-\tau_{10}^*d_1[r_2^*(2\cos\theta r_1-2\sin\theta r_2+I^*)\\
&+s_2^*(2\cos\theta s_1-2\sin\theta s_2+R^*)]+\tau_{10}^*d_2[2\cos\theta+S^*+r_1^*(2\cos\theta r_1\\
&-2\sin\theta r_2+I^*)+s_1^*(2\cos\theta s_1-2\sin\theta s_2+R^*)]\}.
\end{aligned}\tag{4.9}
$$

Then we can prove that $\varphi(\theta)$ is a Morse function in system (4.1). From Lemma 4.1, if $Re(c_1(0))<0$, $\exists$ a constant K_2, such that a rank-one strange attractor is observable in system (4.1) when $\left|\varepsilon\frac{Imc_1(0)}{Rec_1(0)}\right|>K_2$.

5. Numerical simulation

In this section, we choose a set of parameter values as [11] $A = 5$, $\mu = 0.05$, $a = 0.295$, $b = 0.0387$, $\alpha = 1.2$, $\beta = 0.54$, $d = 0.001$, $\sigma = 0.002$.

As an example, we consider the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = 5 - 0.05S(t) - \dfrac{0.54S(t)I(t-\tau_1)}{1+1.2I^2(t-\tau_1)} + \varepsilon S(t)P_T(t), \\ \dfrac{dI(t)}{dt} = \dfrac{0.54S(t)I(t-\tau_1)}{1+1.2I^2(t-\tau_1)} - (0.05+0.001+0.002)I(t) - \dfrac{0.295I^2(t-\tau_2)}{1+0.0387I^2(t-\tau_2)} \\ \qquad + \varepsilon I(t)P_T(t), \\ \dfrac{dR(t)}{dt} = \dfrac{0.295I^2(t-\tau_2)}{1+0.0387I^2(t-\tau_2)} + 0.002I(t) - 0.05R(t) + \varepsilon R(t)P_T(t). \end{cases} \tag{5.1}$$

When $\varepsilon = 0$, system (5.1) is undisturbed system (1.2). We have the following four cases.

(i) $\tau_1 = \tau_2 = 0$

We have $K_5 = \mu(\mu+d+\sigma) - \beta A = -2.6974 < 0$, (H_1) is satisfied. Then system (5.1) has a positive equilibrium $E = (33.0056, 4.2373, 62.6723)$. Moreover, we can obtain $P_1+P_2+P_3 = 1.8452 > 0$, $R_1+R_2+R_3 = 0.0088 > 0$, $(P_1+P_2+P_3)(Q_1+Q_2+Q_3) - (R_1+R_2+R_3) = 0.4814 > 0$. (H_2) also holds true, then E is locally asymptotically stable from Theorem 2.1 (see Figure 1).

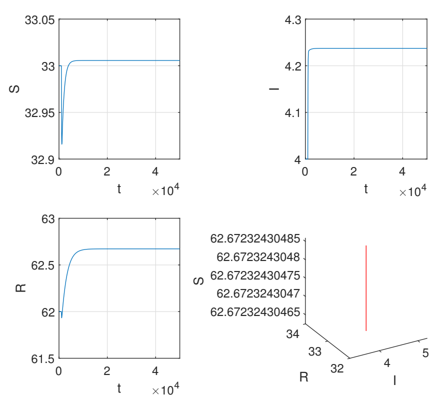

Figure 1. E is locally asymptotically stable with $\varepsilon = 0$, $\tau_1 = \tau_2 = 0$.

(ii) $\tau_1 = 0, \tau_2 > 0$

By calculation we can get $R_{12} = (R_1+R_2+R_3)(R_1+R_2-R_3) = -3.8607e{-}05 < 0$. Furthermore, we have $\omega_{20} = 0.3456$, $\tau_{20} = 6.8303$. According to Theorem 2.2, E is asymptotically stable when $\tau_2 \in (0, 6.8303)$ (see Figure 2). E loses its stability when τ_2 passes through the critical value $\tau_{20} = 6.8303$. And a Hopf bifurcation occurs at E when $\tau_2 = 7.14 > \tau_{20}$ (see Figure 3).

(iii) $\tau_1 = \tau_2 = \tau > 0$

By calculations, $\omega_{30} = 1.5861$,$\tau_0 = 1.0295$. From Theorem 2.3, E is asymptotically stable when $\tau \in (0, 1.0295)$ (see Figure 4). A periodic solutions occurs at E when $\tau = 1.04 > \tau_0$ (see Figure 5).

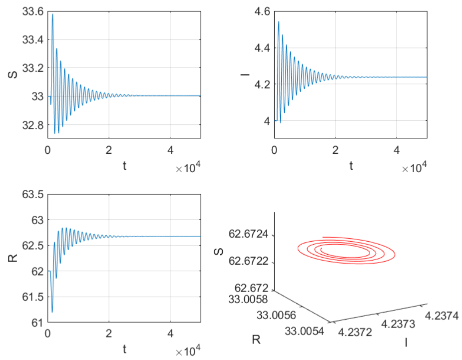

Figure 2. E is asymptotically stable with $\varepsilon = 0$, $\tau_1 = 0$, $\tau_2 = 5$.

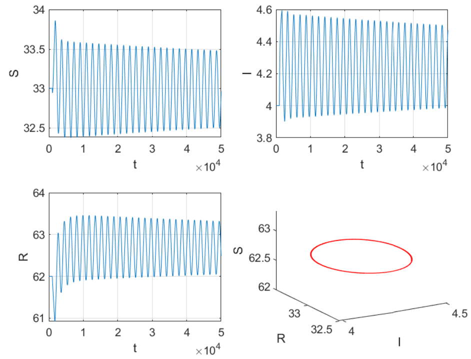

Figure 3. A Hopf bifurcation occurs from E with $\varepsilon = 0$, $\tau_1 = 0$, $\tau_2 = 7.14$.

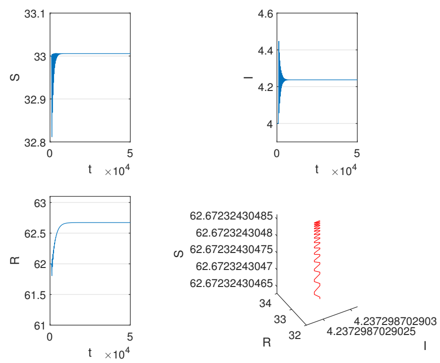

Figure 4. E is asymptotically stable with $\varepsilon = 0$, $\tau = 0.9$.

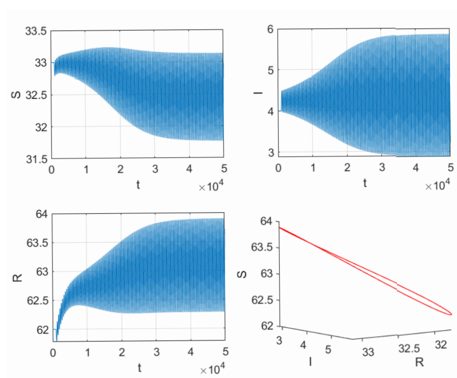

Figure 5. A Hopf bifurcation occurs from E with $\varepsilon = 0$, $\tau = 1.04$.

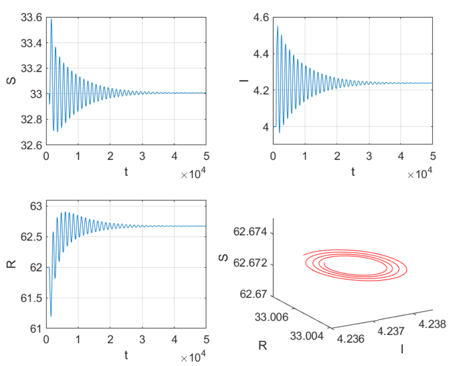

Figure 6. E is asymptotically stable with $\varepsilon = 0$, $\tau_1 = 0.2$, $\tau_2 = 5$.

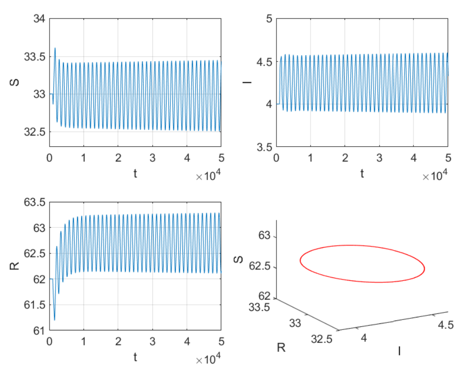

Figure 7. Stable periodic solutions bifurcate from E with $\varepsilon = 0$, $\tau_1 = 0.56$, $\tau_2 = 5$.

(iv) $\tau_1 > 0, \tau_2 \in (0, 6.8303)$

We chose τ_1 as a parameter and let $\tau_2 = 5$. Then, we can get $\omega_{40} = 0.5193$, $\tau_{10}^* = 0.5516$. From Theorem 2.4, E is asymptotically stable when $\tau_1 \in (0, \tau_{10}^*)$ (see Figure

6). Moreover, we can calculate the following values: $c_1(0) = -0.0020 - 0.0049i$, $\mu_2 = 0.0465$, $T_2 = 0.0066$, $\beta_2 = -0.0040$. System (5.1) undergoes a Hopf bifurcation at E when $\tau_1 = 0.56 > \tau_{10}^*$, and the Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable (see Figure 7).

When $\varepsilon > 0$, system (5.1) is disturbed system (1.4). According to (iv), we can obtain $|\frac{Im(c_1(0))}{Re(c_1(0))}| = 2.4071$, $\mu = \tau_1 - \tau_{10}^* = 0.0084$, $\varphi(\theta) = -0.1888\sin\theta - 0.7556\cos\theta - 0.0761\cos 2\theta + 0.0809\sin\theta\cos\theta + 0.0428$. It's easy to verify that $\varphi(\theta)$ is a Morse function. There $\exists\ \Delta \subset (0.0084^{-1}, \infty)$ of T (Δ is a positive measure set), such that the time-T map F_T has a strange attractor for $T \in \Delta$. For system (5.1), we choose $\tau_1 = 0.56$, $\tau_2 = 5$, $\varepsilon \in (0, 1)$ and $T \in (1000, 6000)$, we show a rank-one strange attractor with $\varepsilon = 0.1$, $T = 2000$ in Figure 8. A rank-one strange attractor occurs with $\varepsilon = 0.1$, $T = 4000$ (see Figure 9). In Figure 10 and 11, largest Lyapunov exponent λ versus ε with $T = 2000$, and $T = 4000$ are given. In Figure 12, we give the largest Lyapunov exponent λ versus T with $\varepsilon = 0.1$, where T varying from 1000 to 6000.

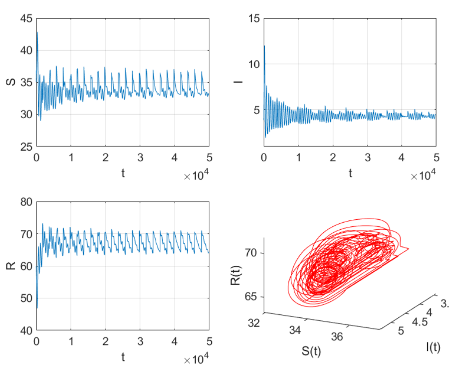

Figure 8. E is unstable, and a rank-one strange attractor occurs with $\tau_1 = 0.56$, $\tau_2 = 5$, $\varepsilon = 0.1$, $T = 2000$.

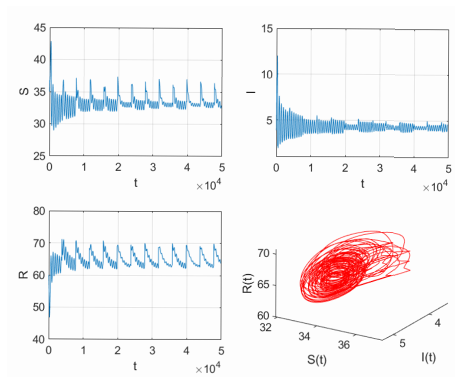

Figure 9. E is unstable, and a rank-one strange attractor occurs with $\tau_1 = 0.56$, $\tau_2 = 5$, $\varepsilon = 0.1$, $T = 4000$.

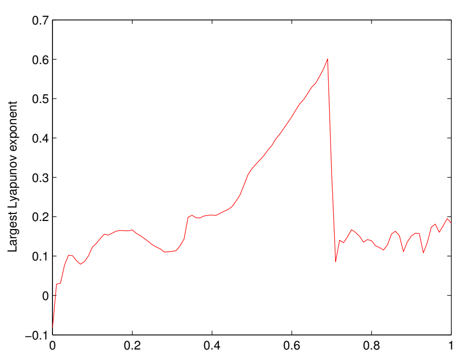

Figure 10. Largest Lyapunov exponent λ versus ε with $\tau_1 = 0.56$, $\tau_2 = 5$, $T = 2000$, ε varying from 0 to 1.

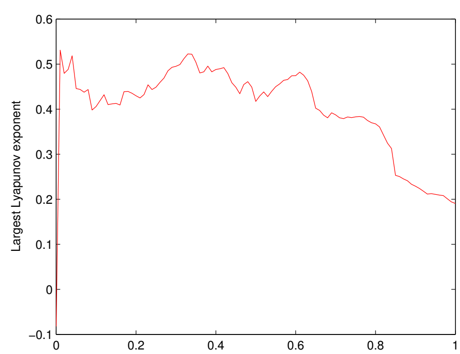

Figure 11. Largest Lyapunov exponent λ versus ε with $\tau_1 = 0.56$, $\tau_2 = 5$, $T = 4000$, ε varying from 0 to 1.

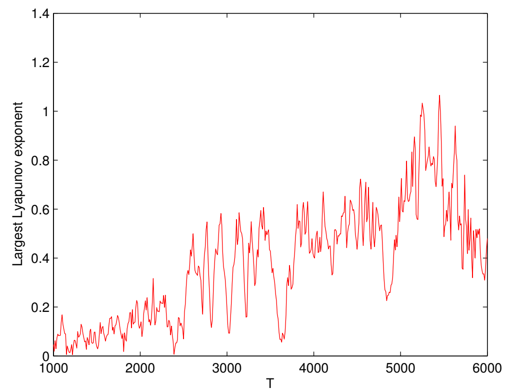

Figure 12. Largest Lyapunov exponent λ versus T with $\tau_1 = 0.56$, $\tau_2 = 5$, $\varepsilon = 0.1$, T varying from 1000 to 6000.

6. Conclusion

In this paper, the rank-one chaos theory for delayed differential equations is applied to a SIR epidemic model with two time-delays. By using Hopf bifurcation theory, we discuss the local stability of the positive equilibrium and the existence of Hopf bifurcation for the following four cases: (1) $\tau_1 = \tau_2 = 0$, (2) $\tau_1 = 0, \tau_2 > 0$, (3) $\tau_1 = \tau_2 = \tau$ and (4) $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$. When $\tau_1 > 0, \tau_2 \in (0, \tau_{20})$, regarding τ_1 as a parameter, we study the direction and stability of the Hopf bifurcation. We add periodic kicks to the susceptible, the infected and the recovered individuals. It is shown that when the system undergoing supercritical Hopf bifurcation is subjected periodic kicks, rank one strange attractor is observable. It means that if susceptible individuals, infected individuals, recovered individuals suffered a periodic external force, then chaotic behaviors will occur. Finally, the numerical simulation results are consistent with the theoretical results.

Acknowledgements

We greatly appreciate the editors' and the anonymous referees' careful reading and helpful suggestions which have made the manuscript a real significant improvement.

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