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MULTI-CLUSTER FLOCKING BEHAVIOR FOR A CLASS OF CUCKER-SMALE MODEL WITH A PERTURBATION

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Abstract In this paper, we study a Cucker-Smale-type system with a perturbation in which agents interact with each other by given communication weights. By using a Lyapunov functional approach and some induction arguments we will prove that every agent flocks to the leader, and the flocking of the model depends on the perturbed conditions and initial conditions. Finally, we also provide several numerical examples and compare them with analytical results.

Keywords Flocking behavior, perturbation, leader, Cucker-Smale model.

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1. Introduction

The purpose of this paper is to study the flocking behavior of the perturbed Cucker-Smale model (in short C-S model). The terminology "flocking" represents the phenomena that all birds, fish and other biological agents, e.g., flocking of birds, swarming of locusts, surging of fish, etc. These behaviors have been gained increasing interests from research communities in biology, mathematics and engineering. In recent years, many scientists and scholars have devoted themselves to studying synchronized behaviors, predation behaviors, and animal tracking behaviors in biological populations. As shown in Figure 1, the flocking behaviors are common in nature.

In 2007, Cucker and Smale proposed a C-S model for group behavior research in [3,4], which revealed the mathematical principles and operational mechanisms of the flocking phenomenon described above. Later, there are many results of the C-S model, such as random noise effects [5,15], time delay [24], free-will [25] and mean-field limit [10,11,17,18,38], and other researches can refer [6,8–10,12,20,26–29,38, 40–42]. In [43], Shen firstly introduced the hierarchy to the C-S model and obtained flocking under directed interactions. In the hierarchy C-S model, there exists an important constant β capturing the rate of decay of the influences between agents when they separate in the space. The main results show that unconditional flocking occurs when $0 \leq \beta < \frac{1}{2}$. Meanwhile, flocking occurs under some initial conditions for $\beta \geq \frac{1}{2}$. Some research on this aspect can refer [16,30,31,34].

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(a) Flocking of birds

(b) Shark and panicked fish

Figure 1. the flocking behaviors are common in nature

In 2011, Motsch and Tadmor proposed a mathematical model of self organizing dynamics in [33]. The improved model considers not only the relative distances between agents, but also the impacts between agents which could solve several shortcomings of the C-S model. In [40], the authors considered the multi-cluster flocking behavior of the hierarchical Cucker-Smale model. However, there are no results available about C-S multi-flocking under perturbation in the general case.

Motivated by the above discussion, we consider multi-flocking behavior of the C-S model with a small perturbation. Our main results can describe as follow: when $\beta > \frac{1}{2}$, the flocking would occur which only depend on the initial states of the flock; when $\beta = \frac{1}{2}$, the model produces flocking under appropriate perturbation conditions.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries about C-S-model. The main content is shown in the third section, which is divided into two parts for demonstration (i.e., mono-cluster flocking and multi-cluster flocking). The fourth section verifies our conclusions through simulation experiments. The fifth section summarizes some conclusions and prospects for the future.

2. Preliminaries and problem formulation

In this section, we give some preliminaries and some symbolic explanations of the C-S model.

A class of C-S model with hierarchical leadership:

Consider then a system of n agents with positions and velocities denoted by $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ and $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$, respectively. Here $x_i(t) \in \mathbb{R}^3$ and $v_i(t) \in \mathbb{R}^3$ denoted the position and velocity of $i-$th agent at time t. Assume that the system evolves under hierarchical leadership, following the continuous-time dynamics:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = v_i(t), \\ \dfrac{dv_i(t)}{dt} = \displaystyle\sum_{j \in \tau(i)} a_{ij}(\|x_i(t) - x_j(t)\|)(v_j(t) - v_i(t)), \end{cases} \tag{2.1}$$

where,

(a) $i \in \Omega = \{1, 2, \ldots, n\}$, the symbol"$i$" is defined as the $i-$th agent.

(b) $\tau(i) = \{j : j < i\}$ denotes the subgroup of agents that directly influence agent i. Particularly, $\tau(1) = \emptyset$ and $v_1(t) = v_1(0)$, for all $t \geq 0$. We can regards agent 1 as a virtual leader, which maintains a constant velocity movement, and it only affects the other agents unilaterally.

(c) The weighting coefficient $a_{ij}(t)$ in the current paper take the form:

$$a_{ij}(t) = a_{ij}(\|x_i(t) - x_j(t)\|) = \frac{K}{n}\psi(\|x_i(t) - x_j(t)\|).$$

Here, K is the non-negative coupling strength, $\psi(r) = \frac{1}{(1+r)^{2\beta}}$ is the influence function, and $\psi(r)$ is a decreasing function. $\|\cdot\|$ represents l_1-norm. At the same time, the symbol "$\|\cdot\|_1$" in this paper also denotes "l_1-norm".

Remark 2.1 ([40]). From the definition of $\psi(r)$ and by recalling $a_{ij}(t)$ if necessary, let $a_{ii}(t) = 1 - \sum_{j<i} a_{ij}(t) \geq 0$, we have

$$\sum_{j \leq i} a_{ij}(t) = 1 \text{ and } a_{i1}(t) + \sum_{j \leq i, j \neq 1} a_{ij}(t) = 1,$$

where $\psi(r) \leq 1$, $\sum_{j<i} a_{ij}(t) \leq 1$.

A class of C-S models with a perturbation:

We added a perturbation term $G_i(t) \in \mathbb{R}^3$ to model (2.1) for analyzing the flocking behaviors of the C-S system in a disturbing environment. In this situation, we assume the following dynamic model:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = v_i(t), \\ \dfrac{dv_i(t)}{dt} = K \displaystyle\sum_{j \in \tau(i)} a_{ij}(\|x_i(t) - x_j(t)\|)(v_j(t) - v_i(t)) + G_i(t), \end{cases} \tag{2.2}$$

where, $G_i(t) \in \mathbb{R}^3$. We choose the weighting coefficient (the alignment coefficients) $a_{ij}(t)$ (by [33]), i.e.,

$$a_{ij}(t) = a_{ij}(\|x_i(t) - x_j(t)\|) = \frac{\psi(\|x_j(t) - x_i(t)\|)}{\sum_{k=1}^{n} \psi(\|x_k(t) - x_i(t)\|)}, \tag{2.3}$$

where $\psi(r) = \frac{1}{(1+r)^{2\beta}}, r \geq 0$. $a_{ij}(t)$ is an asymmetric weight coefficient, which has been explained in [33]. At the same time, this is also to show that the C-S system under asymmetric $a_{ij}(t)$ structure can also form flocking behavior.

Particularly, $\tau(1) = \emptyset$ and $v_1(t) = v_1(0) \Rightarrow G_1(t) = 0$, for all $t \geq 0$.

Remark 2.2. Note that, the weight coefficient $a_{ij}(t)$ in model (2.1) is symmetrical, i.e., $a_{ij} = a_{ji}$. Furthermore, the weighting factor a_{ij} in model (2.2) is asymmetrical, i.e., $a_{ij} \neq a_{ji}$.

The goal in this paper is to prove that the flocking would occur almost surely under some conditions on problem (2.2). We next give the definition of flocking.

Definition 2.1 ([40]). Mono-cluster flocking of the system (2.2) occurs if and only if the solutions $\{x_i(t), v_i(t)\}_{i=1}^n$, to Eq. (2.2) satisfy the following two conditions:

(2.1a) the velocity diameter of the set $\{v_i(t)\}_{i=1}^n$ goes to zero as time goes to infinity (velocity alignment):

$$\lim_{t\to\infty} \|v_i(t) - v_j(t)\| = 0, \text{ for } 1 \le i, j \le n.$$

(2.1b) the position diameter of the set $\{x_i(t)\}_{i=1}^n$ is uniformly bounded in time t (forming a group):

$$\sup_{t\ge 0} \|x_i(t) - x_j(t)\| < \infty, \text{ for } 1 \le i, j \le n.$$

Definition 2.2 ([40]). Suppose that $\Omega = \{1, 2, \ldots, n\}$ and $\{\Omega_i\}_{i=1}^m (m \ge 2)$ is a partition of Ω. The system (2.2) is divided into m clusters (or $m-$cluster flocking occurs) if and only if the solutions $\{x_i(t), v_i(t)\}_{i=1}^n$, to Eq. (2.2) satisfy the following two conditions:

(2.2a) for each Ω_i, flocking occurs:

$$(i) \lim_{t\to\infty} \|v_j(t) - v_l(t)\| = 0, \text{ for } j, l \in \Omega_i;$$

$$(ii) \sup_{t\ge 0} \|x_j(t) - x_l(t)\| < \infty, \text{ for } j, l \in \Omega_i.$$

(2.2b) for any two distinct sets Ω_{i_1}, Ω_{i_2}, they satisfy at least one of the following two conditions:

$$(i) \liminf_{t\to\infty} \|v_{j_1}(t) - v_{j_2}(t)\| > 0, \text{ for } j_1 \in \Omega_{i_1}, j_2 \in \Omega_{i_2};$$

$$(ii) \sup_{t\ge 0} \|x_{j_1}(t) - x_{j_2}(t)\| = \infty, \text{ for } j_1 \in \Omega_{i_1}, j_2 \in \Omega_{i_2}.$$

Definition 2.3 ([33, 40]). The diameter form of the position $x_i(t)$ and velocity $v_i(t)$ between the agent "1"(Leader) and the other agents is defined as follows:

$$d_X(t) = \max_{1\le i\le n} \|x_i(t) - x_1(t)\|_2 \text{ and } d_V(t) = \max_{1\le i\le n} \|v_i(t) - v_1(t)\|_2.$$

Here $\|\cdot\|_2$ denotes a l_2-norm, $\|x\|_2 = (\sum_{s=1}^3 \mid x_s \mid^2)^{\frac{1}{2}}$, for $x \in \mathbb{R}^3$.

Remark 2.3 ([33,40]). Combined with Definition 2.1 and Definition 2.3, Definition 2.1 is equivalent to the following conditions:

$$\lim_{t\to\infty} d_V(t) = 0 \text{ and } \sup_{t\ge 0} d_X(t) < \infty.$$

Remark 2.4. There exist two constants $c_1 > 0, c_2 > 0$, such that

$$c_1\|x\|_2 \le \|x\|_1 \le c_2\|x\|_2.$$

3. Main results

In [43], Shen has investigated the hierarchical leadership flocking under a free-will leader. More precisely, unconditional flocking would occur for $\beta < 1/2$, while for $\beta \ge 1/2$ conditional flocking would occur under some condition on the initial positions and velocities of the agents only. So we will begin to analyze the main research content of this paper, which is divided into two cases: $\beta = 1/2$ and $\beta > 1/2$.

Remark 3.1. By Definition 2.3, we can easily obtain the results similar to [33, Theorem 3.5] and [40, Theorem 2.1], i.e.,

$$\frac{d}{dt}d_X(t) \leq d_V(t). \tag{3.1}$$

3.1. Mono-cluster flocking behavior

Theorem 3.1. *Let $\beta = \frac{1}{2}$. Consider the system (2.2) with the connectivity coefficients given by Eq. (2.3), and the system (2.2) satisfies the following conditions:*

(i) there exist constants $a > 0, b > 0, M_0 > 0$ such that

$$\|G_i(t)\|_2 \leq M_0 e^{-a(t+b)},\ \forall t \geq 0.$$

(ii) there exists two constants $c_2 > 0, T > 0$ such that

$$d_V(0) + \int_0^{Td_X(0)} d_V(\nu)d\nu < \frac{K}{nc_2}\int_{c_2 d_X(0)}^{\infty} \psi(\nu)d\nu.$$

Moreover, assume that

$$\int_0^{Td_X(0)} d_V(\nu)d\nu \geq \frac{M_0}{ae^{ab}}.$$

Then there exists a positive constant $\xi \in [c_2 d_X(0), +\infty)$ satisfying the inequality

$$\frac{K}{n}\psi(\xi) > a > 0,$$

and such that the estimate holds

$$\sup_{t>0} d_X(t) \leq \frac{\xi}{c_2},\ d_V(t) < \left(d_V(0) + \frac{B_0}{A_0 - a}\right)e^{-at}, \tag{3.2}$$

where $A_0 = \frac{K}{n}\psi(\xi) > a > 0, B_0 = \frac{M_0}{e^{ab}}$.

Proof. At first, we will find $t-$derivative for $\|v_i(t) - v_1(t)\|_2^2$. For any $t \geq 0$, fix agent i and denote $d_V(t) = \|v_i(t) - v_1(t)\|_2$, then by (2.2), we have

$$\begin{aligned}
\frac{d}{dt}d_V^2(t) =& \frac{d}{dt}\|v_i(t) - v_1(t)\|_2^2 \\
=& 2\langle \dot{v_i}(t) - \dot{v_1}(t), v_i(t) - v_1(t)\rangle \\
=& 2\left\langle K\sum_{j<i} a_{ij}(\|x_i(t) - x_j(t)\|_1)(v_j(t) - v_i(t)) + G_i(t), v_i(t) - v_1(t)\right\rangle \\
=& 2K\left\langle \sum_{j<i} a_{ij}(t)v_j(t) - \sum_{j<i} a_{ij}(t)v_i(t) + v_1(t) - v_1(t), v_i(t) - v_1(t)\right\rangle \\
&+ 2\left\langle G_i(t), v_i(t) - v_1(t)\right\rangle,
\end{aligned} \tag{3.3}$$

owning to $v_1(t)$ is a constant and $\dot{v}_1(t)=0$. Applying Remark 2.1 to (3.3) yields

$$\begin{aligned}
\frac{d}{dt}d_V^2(t) =& 2K\left\langle \sum_{j<i} a_{ij}(t)v_j(t) - \left(\sum_{j\le i} a_{ij}(t) - a_{ii}(t)\right)v_i(t)\right.\\
&\left. + v_1(t) - v_1(t), v_i(t) - v_1(t)\right\rangle + 2\left\langle G_i(t), v_i(t) - v_1(t)\right\rangle\\
=& 2K\left\langle \left(\sum_{j\le i} a_{ij}(t)v_j(t) - 1\times v_1(t)\right) - \left(1\times v_i(t) - v_1(t)\right), v_i(t) - v_1(t)\right\rangle\\
&+ 2\left\langle G_i(t), v_i(t) - v_1(t)\right\rangle\\
=& 2K\left\langle \sum_{j\le i} a_{ij}(t)\left(v_j(t) - v_1(t)\right), v_i(t) - v_1(t)\right\rangle\\
&- 2K\left\langle v_i(t) - v_1(t), v_i(t) - v_1(t)\right\rangle + 2\left\langle G_i(t), v_i(t) - v_1(t)\right\rangle\\
=& 2K\left\langle \sum_{j\le i, j\ne 1} a_{ij}(t)\left(v_j(t) - v_1(t)\right) + a_{i1}(t)\left(v_1(t) - v_1(t)\right), v_i(t) - v_1(t)\right\rangle\\
&- 2K\left\langle v_i(t) - v_1(t), v_i(t) - v_1(t)\right\rangle + 2\left\langle G_i(t), v_i(t) - v_1(t)\right\rangle.
\end{aligned} \tag{3.4}$$

Using condition (i) and Cauchy-Schwartz inequality lead to

$$\begin{aligned}
\frac{d}{dt}d_V^2(t) \le& 2K\sum_{j\le i, j\ne 1} a_{ij}(t)\times\|v_j(t) - v_1(t)\|_2\times\|v_i(t) - v_1(t)\|_2\\
&- 2Kd_V^2(t) + 2\|G_i(t)\|_2\times\|v_i(t) - v_1(t)\|_2\\
\le& 2K\left(\sum_{j\le i, j\ne 1} a_{ij}(t) - 1\right)d_V^2(t) + 2\|G_i(t)\|_2\times d_V(t)\\
=& -2Ka_{i1}(t)d_V^2(t) + 2\|G_i(t)\|_2 d_V(t).
\end{aligned} \tag{3.5}$$

According to Remark 2.4, $\|x_i(t) - x_1(t)\|_1$ and $d_X(t)$ are equivalent, that is, there must be two constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 d_X(t) \le \|x_i(t) - x_1(t)\|_1 \le c_2 d_X(t).$$

By $a_{ij}(t) = \frac{\psi(\|x_j(t)-x_i(t)\|_1)}{\sum_{k=1}^n \psi(\|x_k(t)-x_i(t)\|_1)}$, we have

$$\begin{aligned}
&0 < \psi\left(c_2 d_X(t)\right) \le \psi\left(\|x_k(t) - x_i(t)\|_1\right) \le 1\\
\Longrightarrow& \sum_{k=1}^n \psi\left(\|x_k(t) - x_i(t)\|_1\right) \le \sum_{k=1}^n 1 = n.
\end{aligned}$$

$$\Longrightarrow a_{i1}(t) = \frac{\psi\Bigg(\|x_1(t)-x_i(t)\|_1\Bigg)}{\sum_{k=1}^n \psi\Bigg(\|x_k(t)-x_i(t)\|_1\Bigg)} \geq \frac{\psi\Bigg(c_2 d_X(t)\Bigg)}{n} > 0.$$

Thence, by inequality (3.5), we obtain

$$\frac{d}{dt}d_V^2(t) \leq -\frac{2K}{n}\psi\Bigg(c_2 d_X(t)\Bigg)d_V^2(t) + 2\|G_i(t)\|_2 d_V(t). \tag{3.6}$$

$$\Longrightarrow \frac{d}{dt}d_V(t) \leq -\frac{K}{n}\psi\Bigg(c_2 d_X(t)\Bigg)d_V(t) + \|G_i(t)\|_2.$$

We can construct the energy function of system (2.2) as

$$L(t) = d_V(t) - \int_0^t \|G_i(\nu)\|_2 d\nu + \frac{K}{nc_2}\int_0^{c_2 d_X(t)} \psi(\nu)d\nu. \tag{3.7}$$

Then, combining with the estimate of (3.7), we calculate

$$\begin{aligned}\frac{d}{dt}L(t) =& \frac{d}{dt}d_V(t) - \|G_i(t)\|_2 + \frac{K}{n}\psi\Bigg(c_2 d_X(t)\Bigg)\frac{d}{dt}d_X(t)\\ \leq& -\frac{K}{n}\psi\Bigg(c_2 d_X(t)\Bigg)d_V(t) + \|G_i(t)\|_2 - \|G_i(t)\|_2\\ &+\frac{K}{n}\psi(c_2 d_X(t))\frac{d}{dt}d_X(t)\\ \leq& 0,\end{aligned} \tag{3.8}$$

where used the inequality $\frac{d}{dt}d_X(t) \leq d_V(t)$ (see Remark 3.1). Therefore, $L(t) \leq L(0)$ for any $t \geq 0$, thus,

$$\begin{aligned}&d_V(t) - \int_0^t \|G_i(\nu)\|_2 d\nu + \frac{K}{nc_2}\int_0^{c_2 d_X(t)} \psi(\nu)d\nu\\ \leq& d_V(0) + \frac{K}{nc_2}\int_0^{c_2 d_X(0)} \psi(\nu)d\nu.\end{aligned} \tag{3.9}$$

The above inequality (3.9) is simplified to

$$d_V(t) \leq d_V(0) + \frac{K}{nc_2}\int_{c_2 d_X(t)}^{c_2 d_X(0)} \psi(\nu)d\nu + \int_0^t \|G_i(\nu)\|_2 d\nu. \tag{3.10}$$

Using $\beta = \frac{1}{2}$ in (2.3), we get

$$\int_0^\infty \psi(r)dr = \int_0^\infty \frac{1}{1+r}dr = \infty.$$

From assumption (ii) it follows that there exists a $\gamma \geq c_2 d_X(0)$ such that

$$d_V(0) + \int_0^{Td_X(0)} d_V(\nu)d\nu = \frac{K}{nc_2}\int_{c_2 d_X(0)}^{\gamma} \psi(\nu)d\nu. \tag{3.11}$$

Substituting (3.11) into (3.10) and using the assumption (i), we obtain

$$\begin{aligned} 0 \leq & d_V(t) \leq d_V(0) + \frac{K}{nc_2}\int_{c_2 d_X(t)}^{c_2 d_X(0)} \psi(\nu)d\nu + \int_0^t M_0 e^{-a(\nu+b)}d\nu \\ =& \frac{K}{nc_2}\int_{c_2 d_X(0)}^{\gamma} \psi(\nu)d\nu - \int_0^{Td_X(0)} d_V(\nu)d\nu \\ &+ \frac{K}{nc_2}\int_{c_2 d_X(t)}^{c_2 d_X(0)} \psi(\nu)d\nu + \frac{M_0}{ae^{ab}}(1-e^{-at}) \\ =& \frac{K}{nc_2}\int_{c_2 d_X(t)}^{\gamma} \psi(\nu)d\nu - \left(\int_0^{Td_X(0)} d_V(\nu)d\nu - \frac{M_0}{ae^{ab}}\right) - \frac{M_0}{ae^{ab}}e^{-at}. \end{aligned} \tag{3.12}$$

Note that $-\frac{M_0}{ae^{ab}}e^{-at} \leq 0, \forall t \geq 0$ and $-\left(\int_0^{Td_X(0)} d_V(\nu)d\nu - \frac{M_0}{ae^{ab}}\right) \leq 0$. Hence, we have

$$\begin{aligned} c_2 d_X(t) \leq \gamma \Rightarrow & \sup_{1\leq i\leq n} \|x_i(t) - x_1(t)\| \leq \frac{\gamma}{c_2}. \\ & \Rightarrow \psi(\gamma) \leq \psi(c_2 d_X(t)). \end{aligned} \tag{3.13}$$

Using inequality (3.13) and condition (i), we obtain the following estimate

$$\begin{aligned} \frac{d}{dt}d_V(t) \leq & -\frac{K}{n}\psi\left(c_2 d_X(t)\right)d_V(t) + \|G_i(t)\|_2 \\ \leq & -\frac{K}{n}\psi(\xi)d_V(t) + M_0 e^{-a(t+b)} \\ = & -A_0 d_V(t) + B_0 e^{-at}, \end{aligned} \tag{3.14}$$

where

$$A_0 = \frac{K}{n}\psi(\xi) > a > 0, B_0 = \frac{M_0}{e^{ab}}.$$

Applying Gronwall's inequality to inequality (3.14), we have

$$\begin{aligned} d_V(t) \leq & e^{-A_0 t}d_V(0) + \frac{B_0}{A_0 - a}e^{-at}\left(1 - e^{-(A_0-a)t}\right) \\ \leq & e^{-A_0 t}d_V(0) + \frac{B_0}{A_0 - a}e^{-at} \\ < & \left(d_V(0) + \frac{B_0}{A_0 - a}\right)e^{-at}. \end{aligned} \tag{3.15}$$

The proof is completed. □

3.2. Multi-cluster flocking behavior

We use the position and velocity errors in [40] and give the following definition:

Definition 3.1. Let

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)); v(t) = (v_1(t), v_2(t), \ldots, v_n(t)).$$

The error is defined as

$$\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_{n-1}(t)) \in \mathbb{R}^{3(n-1)},$$

here $\hat{x}_i(t) = x_i(t) - x_{i+1}(t)$;

$$\hat{v}(t) = (\hat{v}_1(t), \hat{v}_2(t), \dots, \hat{v}_{n-1}(t)) \in \mathbb{R}^{3(n-1)},$$

here $\hat{v}_i(t) = v_i(t) - v_{i+1}(t)$;

$$\hat{G}_i(t) = G_i(t) - G_{i+1}(t), \text{ for } i, i+1 \in \Omega = \{1, 2, \dots, n\}.$$

In the following discussion, we choose $a_{ij}(t) = \frac{\psi(\|x_i(t)-x_j(t)\|_1)}{n}$.

According to the Definition 3.1 of the position and velocity errors, we can change the form of the C-S model (2.2) into

$$\begin{cases} \dot{\hat{x}}_i(t) = \dot{x}_i(t) - \dot{x}_{i+1}(t), \\ \dot{\hat{v}}_i(t) = \dot{v}_i(t) - \dot{v}_{i+1}(t). \end{cases} \tag{3.16}$$

By (2.2) and (3.16), we have

$$\begin{cases} \dot{\hat{x}}_i(t) = \hat{v}_i(t), \\ \dot{\hat{v}}_i(t) = K\left(\sum_{j<i} a_{ij}(t) \sum_{m=j}^{i-1} \hat{v}_m(t) - \sum_{j<i+1} a_{i+1,j}(t) \sum_{m=j}^{i} \hat{v}_m(t) \right) + \hat{G}_i(t) \end{cases} \tag{3.17}$$

where, $v_j(t) - v_i(t) = \sum_{m=j}^{i-1} \hat{v}_m(t)$, $v_j(t) - v_{i+1}(t) = \sum_{m=j}^{i} \hat{v}_m(t)$ and $a_{ij}(t) = \frac{\psi(\|x_i(t)-x_j(t)\|_1)}{n}$. In this subsection, we will demonstrate that the agent can be flocked at $\beta > 1/2$ in a disturbing environment. In order to get this result, we need the following lemma.

Lemma 3.1. *Assume that the initial data of the position and velocity of the system (3.17) satisfies the following order condition in the case of $\beta > 1/2$.*

$$\begin{aligned} &x_1(0) \geq x_2(0) \geq \dots \geq x_n(0), \\ &v_1(0) > v_2(0) > \dots > v_n(0), \\ &G_1(t) = G_2(t) \geq G_3(t) \geq \dots \geq G_n(t), \end{aligned} \tag{3.18}$$

where $G_i(t)$ is a continuous and differentiable function. Then, for any $1 \leq i \leq n-1$ and $t \geq 0$, we have

$$\hat{x}_i(t) \geq \hat{x}_i(0), \hat{v}_i(t) \geq \Theta, \tag{3.19}$$

here we write $y > z$ $(y \geq z)$ when each entry of y is larger (not smaller) than that of z, i.e., $y^k > z^k (y^k \geq z^k)$ for $1 \leq k \leq 3$. Also, $\Theta \in \mathbb{R}^3$ denotes the vector in $\mathbb{R}^3$, whose all of entries are equal to zero.

Proof. Let $x_i(t) = (x_i^1(t), x_i^2(t), x_i^3(t)), v_i(t) = (v_i^1(t), v_i^2(t), v_i^3(t)) \in \mathbb{R}^3$. For $i = 1$, using $v_1(t) = v_1(0), \hat{G}_1^\ell(t) = G_1^\ell(t) - G_2^\ell(t) = 0$ $(\ell = 1, 2, 3)$, we have

$$\begin{aligned} \dot{\hat{v}}_1^\ell(t) = &- a_{21}(t)\hat{v}_1^\ell(t) = -\frac{K\hat{v}_1^\ell(t)}{n\left(1 + \|\hat{x}_1(t)\|\right)^{2\beta}} + \hat{G}_1^\ell(t) \\ = &- \frac{K\hat{v}_1^\ell(t)}{n\left(1 + \|\hat{x}_1(t)\|\right)^{2\beta}}. \end{aligned} \tag{3.20}$$

According to [40, Lemma 3.1], we have

$$\hat{v}_1(t) = v_1(t) - v_2(t) \geq \Theta, \text{for any } t \geq 0.$$

Since $x_1(t) = x_1(0) + \int_0^t v_1(s)ds, x_2(t) = x_2(0) + \int_0^t v_2(s)ds$, we have

$$\hat{x}_1(t) = x_1(t) - x_2(t) = x_1(0) - x_2(0) + \int_0^t \Bigg(v_1(s) - v_2(s) \Bigg) ds \geq \hat{x}_1(0).$$

Finally, it is clearly that

$$\hat{x}_1(t) \geq \hat{x}_1(0), \hat{v}_1(t) = v_1(t) - v_2(t) \geq \Theta.$$

For $i > 1$, on the one hand, by continuity and differentiation of $\hat{v}_i^\ell(t)$, there exists t_4 satisfying:

1. $\hat{v}_i^\ell(t_4) = 0$, that is to say $v_i^\ell(t_4) = v_{i+1}^\ell(t_4)$;
2. There exists a $\delta' > 0$ such that $\hat{v}_i^\ell(t) < 0, t \in (t_4, t_4 + \delta']$.

Obviously, at the moment t_4 we have

$$\dot{\hat{v}}_i^\ell(t_4) \leq 0. \tag{3.21}$$

On the other hand, according to (3.18), we have $\hat{v}_j^\ell(t) \geq \Theta$, for $t \leq t_4$ and $j < i$. According to [40, Lemma 3.1] and $\hat{G}_i^\ell(t) = G_i^\ell(t) - G_{i+1}^\ell(t) \geq 0$, Eq. (3.17) takes the form at the moment t_4

$$\begin{aligned}
\hat{v}_i^\ell(t_4) =& \sum_{j<i} a_{ij}(t_4)\left(v_j^\ell(t_4) - v_i^\ell(t_4)\right) \\
& - \sum_{j<i+1} a_{i+1,j}(t_4)\left(v_j^\ell(t_4) - v_{i+1}^\ell(t_4)\right) + \hat{G}_i^\ell(t) \\
=& \sum_{j<i} \left(a_{ij}(t_4) - a_{i+1,j}(t_4)\right)\left(v_j^\ell(t_4) - v_i^\ell(t_4)\right) \\
>& 0,
\end{aligned}$$

which is contrary to Eq. (3.21). Therefore, $\hat{v}_i^\ell(t) \geq 0$. □

Remark 3.2. Since $\hat{x}_i(t) \in \mathbb{R}^3, \hat{v}_i(t) \in \mathbb{R}^3$, then $\hat{x}_i(t) = (\hat{x}_i^1(t), \hat{x}_i^2(t), \hat{x}_i^3(t))$, $\hat{v}_i(t) = (\hat{v}_i^1(t), \hat{v}_i^2(t), \hat{v}_i^3(t))$, for $1 \leq i \leq n-1$, and

$$\|x_j(t) - x_i(t)\|_1 = \sum_{\ell=1}^{3} \mid x_j^\ell(t) - x_i^\ell(t) \mid = \sum_{\ell=1}^{3}\sum_{m=j}^{i-1}(x_m^\ell(t) - x_{m+1}^\ell(t)) = \sum_{\ell=1}^{3}\sum_{m=j}^{i-1}\hat{x}_m^\ell(t),$$

$$\|v_j(t) - v_i(t)\|_1 = \sum_{\ell=1}^{3} \mid v_j^\ell(t) - v_i^\ell(t) \mid = \sum_{\ell=1}^{3}\sum_{m=j}^{i-1}(v_m^\ell(t) - v_{m+1}^\ell(t)) = \sum_{\ell=1}^{3}\sum_{m=j}^{i-1}\hat{v}_m^\ell(t).$$

For convenience, we need to introduce some mathematical expressions of the article [40]. According to equation (3.17) and Remark 3.2, we have

$$\begin{aligned}
\int_0^t \dot{\hat{v}}_i^\ell(t) =& K\int_0^t \Bigg(\sum_{j<i}(a_{ij}(\nu)\sum_{m=j}^{i-1}\hat{v}_m^\ell(\nu)) \\
& - \sum_{j<i+1}(a_{i+1,j}(\nu)\sum_{m=j}^{i}\hat{v}_m^\ell(\nu))\Bigg)d\nu + \int_0^t \hat{G}_i^\ell(\nu)d\nu.
\end{aligned} \tag{3.22}$$

Similarly, we also can deduce that

$$\begin{aligned}\sum_{\ell=1}^{3}\int_0^t \dot{\hat{v}}_i^\ell(t) =&\frac{K}{n}\sum_{\ell=1}^{3}\int_0^t \Bigg(\sum_{j<i}\frac{(\sum_{m=j}^{i-1}\hat{x}_m^\ell(\nu))'}{(1+\sum_{\ell=1}^{3}\sum_{m=j}^{i-1}\hat{x}_m^\ell(\nu))^{2\beta}}\\ &-\sum_{j<i+1}\frac{(\sum_{m=j}^{i}\hat{x}_m^\ell(\nu))'}{(1+\sum_{\ell=1}^{3}\sum_{m=j}^{i}\hat{x}_m^\ell(\nu))^{2\beta}}\Bigg)d\nu \\ &+\sum_{\ell=1}^{3}\int_0^t \hat{G}_i^\ell(\nu)d\nu.\end{aligned} \tag{3.23}$$

Moreover, by a simple calculation we have

$$\|\hat{v}_i(t)\|_1 = \frac{f_i(t)-f_{i-1}(t)}{2\beta-1} - \rho_i + \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu, \tag{3.24}$$

where

$$\begin{aligned}f_i(t) &= \frac{K}{n}\sum_{j<i+1}\frac{1}{(1+\sum_{\ell=1}^{3}\sum_{m=j}^{i}\hat{x}_m(t))^{2\beta-1}},\\ \rho_i &= \frac{f_i(0)-f_{i-1}(0)}{2\beta-1} - \|\hat{v}_i(0)\|.\end{aligned}$$

where

$$f_i(0) = \frac{K}{n}\sum_{j<i+1}\frac{1}{(1+\sum_{\ell=1}^{3}\sum_{m=j}^{i}\hat{x}_m(0))^{2\beta-1}},$$

When $i=0$, $f_0(t)$ does not exist and is set to $f_0(t)=0$, Particularly, if $a_{ij}(t) = a_{ij}(\|x_i(t)-x_j(t)\|) = \frac{\psi(\|x_j(t)-x_i(t)\|)}{\sum_{k=1}^{n}\psi(\|x_k(t)-x_i(t)\|)}$, the right side of this equality is not integrable, which causes us not to obtain an equality similar to Eq. (3.24). Thus, we cannot obtain the similar results about multi-cluster flocking.

Now, we are ready to prove our main result. The proof idea is mainly due to Ru and Xue [40, Theorem 3.1], who found a multi-cluster flocking behavior for hierarchical Cucker-Smale model. For the completeness, we also give the details for the readers here.

Theorem 3.2. *Suppose that the initial data of the system (3.17) with $\beta > 1/2$ satisfy Eq. (3.18). Defines $\hat{\Omega} = \{\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_{n-1}\}$, and divides $\hat{\Omega}$ into $\hat{\Omega} = \hat{\Omega}_1 \cup \hat{\Omega}_2 \cup \hat{\Omega}_3$. Here $\hat{i}_k$ denotes the relative initial state of the k-th agent, that is, $\rho_i = \frac{f_i(0)-f_{i-1}(0)}{2\beta-1} - \|\hat{v}_i(0)\|$. Sets $\hat{\Omega}_1$, $\hat{\Omega}_2$, and $\hat{\Omega}_3$ are defined as*

$$\begin{aligned}\hat{\Omega}_1 &= \left\{\hat{i}_i \in \hat{\Omega} : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu > 0, \forall t \geq 0\right\},\\ \hat{\Omega}_2 &= \Bigg\{\hat{i}_i \in \hat{\Omega} : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu \leq 0,\\ &\qquad \exists j \in \mathbb{N}, 1 \leq j \leq i, \sum_{k=j}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) > 0, \forall t \geq 0\Bigg\},\end{aligned}$$

$$\hat{\Omega}_3 = \Bigg\{ \hat{i}_i \in \hat{\Omega} : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu \leq 0, \\ \forall j \in \mathbb{N}, 1 \leq j \leq i, \sum_{k=j}^{i} (\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \leq 0, \forall t \geq 0 \Bigg\},$$

where $\rho_i = \frac{f_i(0) - f_{i-1}(0)}{2\beta - 1} - \|\hat{v}_i(0)\|$. *The perturbation* $\hat{G}_i(t)$ *in the above set satisfies the following conditions:*

(H1) $\hat{G}_i(t)$ *is a derivable function with respect to time* t*, and*

$$\begin{aligned} &\|\hat{G}_i(t)\| \leq M_0 e^{-a(t+b)}, \ (a > 0, b > 0, M_0 > 0), \ \text{for any } t \geq 0, \\ &\text{and } \rho_i > \frac{M_0}{ae^{ab}} = M, \ i = 1, 2, \cdots, n. \end{aligned} \tag{3.25}$$

(H2) for any $t \geq 0$ *and* $i, j \in \hat{\Omega}$*, there exist a positive constant* δ *such that*

$$| \hat{G}_i(t) - \hat{G}_j(t) | \leq \delta. \tag{3.26}$$

Then the system (3.17) can be divided into $| \hat{\Omega}_3 | + 1$ *clusters. The meaning of* $| \hat{\Omega}_3 |$ *here is the cardinality of* $\hat{\Omega}_3$.

Remark 3.3 (*Barbalat's* lemma)**.** If $F : [0, \infty) \to \mathbb{R}$ is uniformly continuous (*Lipschitz* continuous), and the $\lim_{t\to\infty} \int_0^t F(\nu) d\nu$ exists and is bounded, then

$$\lim_{t\to\infty} F(t) = 0. \tag{3.27}$$

Now, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. To complete the proof of the main result, we need to consider the following four steps.

Step 1: If $\hat{i}_i \in \hat{\Omega}_1$, then $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$ and $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 < \infty$.

Based on the ordered conditions (3.18) and (3.19) of the initial data in Lemma 3.1, when $m = j < i$, then

$$\hat{x}_m^\ell(t) = \hat{x}_j^\ell(t) \geq \hat{x}_i^\ell(t) = x_i^\ell(t) - x_{i+1}^\ell(t) \geq 0.$$

Consequently, for all $t \geq 0, i = 1, 2, \ldots, n-1$, we obtain $f_i(t) \geq 0, f_0(t) = 0$, and $\|\hat{x}_m(t)\|_1 \geq \|\hat{x}_i(t)\|_1 \Longrightarrow \frac{K}{n(1+\|\hat{x}_m(t)\|_1)^{2\beta-1}} \leq \frac{K}{n(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}}$, for $\beta > 1/2$, thus

$$\begin{aligned} & f_i(t) - f_{i-1}(t) \leq \frac{K}{n(1 + \|\hat{x}_i(t)\|_1)^{2\beta-1}} \\ \Longrightarrow \quad & f_i(t) - f_{i-1}(t) - \frac{K}{n(1 + \|\hat{x}_i(t)\|_1)^{2\beta-1}} \leq 0. \end{aligned} \tag{3.28}$$

Thus, equation (3.24) can be further written as

$$\begin{aligned} 0 \leq \|\hat{v}_i(t)\|_1 = & \frac{1}{2\beta - 1} \left(f_i(t) - f_{i-1}(t) - \frac{K}{n(1 + \|\hat{x}_i(t)\|_1)^{2\beta-1}} \right) \\ & + \frac{K}{n(2\beta - 1)(1 + \|\hat{x}_i(t)\|_1)^{2\beta-1}} - \rho_i + \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu. \end{aligned} \tag{3.29}$$

From the estimate of (3.28) and equation (3.29), we have

$$\frac{K}{n(2\beta-1)(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}}-\rho_i+\int_0^t\|\hat{G}_i(\nu)\|_1d\nu\geq 0. \tag{3.30}$$

In addition, according to the disturbance conditions $(H1)$, we have

$$\|\hat{G}_i(\nu)\|\leq M_0e^{-a(t+b)}\Longrightarrow\int_0^t\|\hat{G}_i(\nu)\|_1d\nu\leq\frac{M_0}{ae^{ab}}=M<\infty. \tag{3.31}$$

For $\hat{i}_i\in\hat{\Omega}_1$, the condition $\rho_i-\int_0^t\|\hat{G}_i(\nu)\|_1d\nu>0$ and (H1) of Theorem 3.2, we deduce that

$$\begin{aligned}&\frac{K}{n(2\beta-1)(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}}\geq\rho_i-\int_0^t\|\hat{G}_i(\nu)\|_1d\nu>0\\ \Longrightarrow\quad&\|\hat{x}_i(t)\|_1\leq\left(\frac{K}{n(2\beta-1)(\rho_i-\int_0^t\|\hat{G}_i(\nu)\|_1d\nu)}\right)^{\frac{1}{2\beta-1}}-1\\ &\qquad\quad\leq\left(\frac{K}{n(2\beta-1)(\rho_i-M)}\right)^{\frac{1}{2\beta-1}}-1.\end{aligned} \tag{3.32}$$

Therefore, $\|\hat{x}_i(t)\|_1$ remains bounded at $t\geq 0$, which means that $\hat{x}_i^\ell(t)$ is also bounded on $t\geq 0$. Since $\dot{\hat{x}}_i^\ell(t)=\hat{v}_i^\ell(t)\geq 0$, $\hat{x}_i^\ell(t)$ is monotonically increasing.

In summary, it is indicated that there is $\hat{x}_i^\ell<+\infty$ such that $\hat{x}_i^\ell(t)\to\hat{x}_i^\ell$ when $t\to\infty$. From this, it can be seen that when $t\to\infty$, $\|\hat{x}_i(t)\|_1\to\|\hat{x}_i\|_1=\sum_{\ell=1}^3\mid\hat{x}_i^\ell\mid$.

The following proves $\lim_{t\to\infty}\hat{v}_i^\ell(t)=0$. According to equation (2.1) and condition (H2) in Theorem 3.2, applying Lemma 3.1, we have

$$\begin{aligned}\mid\dot{\hat{v}}_i^\ell(t)\mid=&\mid\dot{v}_i^\ell(t)-\dot{v}_{i+1}^\ell(t)\mid\\ =&\mid K\sum_{j=1}^n a_{ij}(t)\left(v_j^\ell(t)-v_i^\ell(t)\right)+G_i^\ell(t)\\ &-K\sum_{j=1}^n a_{i+1,j}(t)\left(v_j^\ell(t)-v_{i+1}^\ell(t)\right)-G_{i+1}^\ell(t)\mid\\ \leq&2K\sum_{j=1}^n v_1^\ell(0)\left(a_{i,j}(0)+a_{i+1,j}(0)\right)+\mid G_i^\ell(t)-G_{i+1}^\ell(t)\mid\\ \leq&2K\sum_{j=1}^n v_1^\ell(0)\left(a_{i,j}(0)+a_{i+1,j}(0)\right)+\delta\\ =&L<\infty.\end{aligned} \tag{3.33}$$

This implies for any $t_1\geq 0,t_2\geq 0$, applying differential mean value theorem to $\hat{v}_i^\ell(t)$ between t_1 and t_2, we obtain

$$\mid\hat{v}_i^\ell(t_1)-\hat{v}_i^\ell(t_2)\mid=\mid\dot{\hat{v}}_i^\ell(t)(t_1-t_2)\mid\leq L\mid t_1-t_2\mid.$$

Note that $\hat{v}_i^\ell(t)\geq 0$, $\hat{v}_i^\ell(t)\in C^1[0,+\infty)$ and $\int_0^\infty\hat{v}_i^\ell(\nu)d\nu=\hat{x}_i^\ell(t)-\hat{x}_i^\ell(0)$, then by Remark 3.3, we have $\lim_{t\to\infty}\hat{v}_i^\ell(t)=0$. Means $\hat{v}_i(t)\to\Theta\in\mathbb{R}^3$, when $t\to\infty$, which proves the conclusion of Step 1.

Step 2: If $\hat{i}_i \in \hat{\Omega}_2$, then $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$ and $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 < \infty$.

Similarly we can also prove Step 2, according to the estimation of Step 1, for any $t \geq 0$ and $j-1 < i$, we have

$$\begin{aligned} & f_i(t) - f_{j-1}(t) \leq \frac{K(i-j+1)}{n(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}}, \\ \Longrightarrow \quad & f_i(t) - f_{j-1}(t) - \frac{K(i-j+1)}{n(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}} \leq 0. \end{aligned}$$

By $\hat{i}_i \in \hat{\Omega}_2$, we have

$$\begin{aligned} \sum_{k=j}^{i} \|\hat{v}_k(t)\|_1 =& \frac{1}{2\beta-1} \sum_{k=j}^{i} \left(f_k(t) - f_{k-1}(t) \right) \\ & - \sum_{k=j}^{i} \left(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu \right) \\ =& \frac{1}{2\beta-1} \left(f_i(t) - f_{j-1}(t) \right) \\ & - \sum_{k=j}^{i} \left(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu \right) \geq 0. \end{aligned} \tag{3.34}$$

Similar to inequality (3.30), (3.32), we have

$$\begin{aligned} & \frac{K(i-j+1)}{n(2\beta-1)(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}} - \sum_{k=j}^{i} \left(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu \right) \geq 0 \\ \Longrightarrow \quad & \|\hat{x}_i(t)\|_1 \leq \left(\frac{K(i-j+1)}{n(2\beta-1)\sum_{k=j}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu)} \right)^{\frac{1}{2\beta-1}} - 1. \end{aligned} \tag{3.35}$$

For the proof of $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$, it is the same as Step 1. Thus, we prove the conclusion of Step 2.

For convenience, we put the specific demonstration process of Step 3 in the form of Appendix A at the end of the paper.

Next we briefly explain the sequence of the entire proof of Step 3. In order to assist the better and faster reading of the specific demonstration process of Appendix A.

Step 3: If $\hat{i}_i \in \hat{\Omega}_3 \neq \emptyset$, then $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$. Since the set $\hat{\Omega}_3$ constructed in Step 3 is special, we need to divide the set $\hat{\Omega}_3$.
Let

$$\begin{aligned} \hat{\Omega}_3 = \hat{\Omega}_4 \cup \hat{\Omega}_5 = & \Bigg\{ \hat{i}_i \in \hat{\Omega} : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu \leq 0, \forall j \in \mathbb{N}, 1 \leq j \leq i, \\ & \sum_{k=j}^{i} (\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \leq 0, \forall t \geq 0 \Bigg\}, \end{aligned}$$

where

$$\begin{aligned}\hat{\Omega}_4 =&\Bigg\{\hat{i}_i \in \hat{\Omega}_3 : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu \le 0,\\ &\forall j \in \mathbb{N}, 1 \le j \le i, \sum_{k=j}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \le 0,\\ &\text{and } \exists j^{'} \in \mathbb{N}, 1 \le j^{'} \le i, \sum_{k=j^{'}}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0, \forall t \ge 0\Bigg\},\\ \hat{\Omega}_5 =&\Bigg\{\hat{i}_i \in \hat{\Omega}_3 : \rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu \le 0,\\ &\forall j \in \mathbb{N}, 1 \le j \le i, \sum_{k=j}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) < 0, \forall t \ge 0.\Bigg\}\end{aligned}$$

The proof of Step 3 is proved in three cases.

Step 3.1: If $\hat{i}_i \in \hat{\Omega}_4 \ne \emptyset$, then $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$.

Step 3.2: If $\hat{i}_i \in \hat{\Omega}_5 \ne \emptyset$, then

$$\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 > 0 \text{ and } \lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty.$$

Step 3.3: If $\hat{i}_i \in \hat{\Omega}_4 \ne \emptyset$, then $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$.

Next we will prove the last step of the Theorem 3.2.

Step 4: System (3.17) can be divided into $\mid \hat{\Omega}_3 \mid +1$ clusters.

The proof of Step 4 is demonstrated in conjunction with Steps 1, 2, and 3. Thus, we can easily find that after the results of Steps 1 and 2 above are proved, we obtain if the agent $\hat{i}_i \in \hat{\Omega}_1 \cup \hat{\Omega}_2$, then

$$\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0 \text{ and } \lim_{t\to\infty} \|\hat{x}_i(t)\|_1 < \infty.$$

By Step 3, if $\hat{i}_i \in \hat{\Omega}_3$, then $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$. Let $\mid \hat{\Omega}_3 \mid= m$, accordingly $\hat{\Omega}_3 = \{\hat{i}_{i_1}, \hat{i}_{i_2}, \dots, \hat{i}_{i_m}\}$, where $m \le n-1 \Rightarrow m+1 \le n$.

Combined with the $\Omega = \{1, 2, \dots, n\}$ in the previous Definition 2.2, we have $\Omega_k = \{i_{k-1}+1, i_{k-1}+2, \dots, i_k\}$, for $k = 1, 2, \dots, m+1$, where $i_0 = 0$ and $i_{m+1} = n$. Thus, $k-1 = 1, 2, \dots, m \Longrightarrow i_{k-1} = i_0, i_1, i_2, \dots, i_m \Longrightarrow$

$$\begin{gathered} i_{k-1}+1 = i_0+1, i_1+1, i_2+1, \dots, i_m+1,\\ i_{k-1}+2 = i_0+2, i_1+2, i_2+2, \dots, i_m+2,\\ i_{k-1}+3 = i_0+3, i_1+3, i_2+3, \dots, i_m+3,\\ \cdots\cdots\\ i_k = i_1, i_2, \dots, i_{m+1}.\end{gathered}$$

By $\hat{\Omega}_3 = \{\hat{i}_{i_1}, \hat{i}_{i_2}, \dots, \hat{i}_{i_m}\}$ and the above equations $i_{k-1}+1, i_{k-1}+2, i_{k-1}+3, \dots, i_k$, we find that $\hat{i}_i = \hat{i}_{i_{k-1}+1}, \hat{i}_{i_{k-1}+2}, \hat{i}_{i_{k-1}+3}, \dots, \hat{i}_{i_k-1} \in \hat{\Omega}_1 \cup \hat{\Omega}_2$. Thus, for each $\Omega_k = \{i_{k-1}+1, i_{k-1}+2, \dots, i_k\}$, ordered conditions using Lemma 3.1, we can obtain

$$\lim_{t\to\infty} \sup_{i,j\in\Omega_k} \|v_i(t) - v_j(t)\|_1 = \lim_{t\to\infty} \|v_{i_k}(t) - v_{i_{k-1}-1}(t)\|_1$$

$$= \lim_{t\to\infty} \sum_{i=i_{k-1}+1}^{i_k+1} \|\hat{v}_i(t)\|_1 = 0.$$

Similarly, we can also get

$$\sup_{0\le t\le\infty, i,j\in\Omega_k} \|x_i(t) - x_j(t)\|_1 < \infty.$$

In the following we will take two sets Ω_k and Ω_{k+1}, we have

$$\lim_{t\to\infty} \|x_{i_k}(t) - x_{i_k+1}(t)\|_1 = \lim_{t\to\infty} \|\hat{x}_{i_k}(t)\|_1 = \infty,$$

this is because $\hat{i}_{i_k} \in \hat{\Omega}_3 = \{\hat{i}_{i_1}, \hat{i}_{i_2}, \ldots, \hat{i}_{i_m}\}$ causes the above situation to occur between the agents.

For the same reason, take $i_1 \in \Omega_k$ and $i_2 \in \Omega_{k+1}$, according to Lemma 3.1, we have

$$\sup_{t\ge 0} \|x_{i_1}(t) - x_{i_2}(t)\|_1 \ge \sup_{t\ge 0} \|x_{i_k}(t) - x_{i_k+1}(t)\|_1 = \infty.$$

Looking back at our previous Definition 2.2, it is not difficult to find any different segmentation set $i_1 \in \Omega_{k_1}, i_2 \in \Omega_{k_2}, k_1 < k_2$, we have

$$\sup_{t\ge 0} \|x_{i_1}(t) - x_{i_2}(t)\|_1 = \infty.$$

Therefore, by Definition 2.2, we obtain the system (2.2) is divided into $| \Omega_3 | + 1$ clusters.

In summary, we prove the Theorem 3.2. $\square$

4. Numerical results

In this section, we use numerical examples with 40 agents to illustrate our theoretical results. In order to test the results of single flocking cluster behavior and multi-flocking cluster behavior of our models (2.2) and (3.17), we take the number of agents as $n = 40$ and the step size is $h = 0.5$. Next we test the Theorem 3.1 and the result of Theorem 3.2.

Experiment A: Verification Theorem 3.1.

$$x(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_{40}(t)) \in \mathbb{R}^{3*40},$$
$$v(t) = (v_1(t), v_2(t), v_3(t), \ldots, v_{40}(t)) \in \mathbb{R}^{3*40}.$$

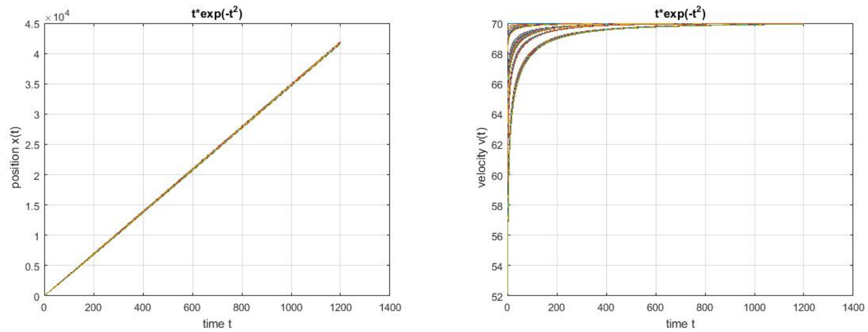
Take $\beta = 1/2, K = 14$, $G_i(t) = (G_i^1(t), G_i^2(t), G_i^3(t)) \in \mathbb{R}^3$, where $G_i^1(t) = \frac{1}{(t+1)^2} + \frac{1}{3}t\exp(-t^2)$, $G_i^2(t) = 1 - \frac{1}{(t+1)^2}$, and $G_i^3(t) = \frac{2}{3}t\exp(-t^2) - 1$.

For the interference $G_i(t)$, we have

$$\|G_i(t)\|_1 = G_i^1(t) + G_i^2(t) + G_i^3(t) = t\exp(-t^2).$$

For the integration of $\|G_i(t)\|_1$, we obtain

$$\int_0^{+\infty} \|G_i(\nu)\|_1 d\nu = \int_0^{+\infty} \nu\exp(-\nu^2)d\nu = \frac{1}{2} < +\infty.$$



(a) $x(t) = \{x_1(t), x_2(t), \ldots, x_{40}(t)\} \in \mathbb{R}^{3*40}$ (b) $v(t) = \{v_1(t), v_2(t), \ldots, v_{40}(t)\} \in \mathbb{R}^{3*40}$

Figure 2. Dynamic trends of velocity and position of 40 agents with disturbance systems

Obviously, the disturbance $\|G_i(t)\|_1$ satisfies the condition of Theorem 3.1. Therefore, after experimental analysis, we obtained the phenomenon of Mono-cluster flocking behavior as shown in Figure 2.

Thus, we verify the conclusion of Theorem 3.1.

Experiment B: Verification Theorem 3.2.

$$x(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_{40}(t)) \in \mathbb{R}^{3*40},$$
$$v(t) = (v_1(t), v_2(t), v_3(t), \ldots, v_{40}(t)) \in \mathbb{R}^{3*40}.$$

In this experiment, based on the convenience and significance of the experiment, we still take $K = 14$, and $n = 40$. For the model (3.17), $\psi(r) = \frac{1}{(1+r)^{2\beta}}$, and the attenuation coefficient $\beta > 1/2$.

Experiment B1: For the interference factor $\hat{G}_i(t) = G_i(t) - G_{i+1}(t)$, here, $G_i(t) = (G_i^1(t), G_i^2(t), G_i^3(t)) \in \mathbb{R}^3$. We assume

$$\begin{aligned}
G_i^1(t) &= \frac{1}{3}\frac{1}{(t^4+1)^{1/2}} + \frac{1}{(1+t)^{2\beta}}, (\beta > \frac{1}{2}),\\
G_{i+1}^1(t) &= -\frac{2}{3}\frac{1}{(t^4+1)^{1/2}} - \exp(-2t)\cos(t),\\
G_i^2(t) &= \frac{sign(\sin(t))}{1+t^2} + \frac{2}{3}\exp(-2t) - \frac{1}{100}\frac{1}{(\exp(t))^{1/2}},\\
G_{i+1}^2(t) &= \frac{sign(\sin(t))}{1+t^2} + \frac{1}{4}\exp(-2t) + \frac{3}{100}\frac{1}{(\exp(t))^{1/2}},\\
G_i^3(t) &= \frac{3}{100}\frac{1}{(\exp(t))^{1/2}} + \frac{1}{5}\exp(-2t)cos(t) - \frac{1}{(t^4+1)^{1/2}},\\
G_{i+1}^3(t) &= -\frac{1}{100}\frac{1}{(\exp(t))^{1/2}} + (\frac{4}{5}\cos(t) + \frac{5}{12})\exp(-2t).
\end{aligned}$$

Therefore, we have

$$\|\hat{G}_i(t)\|_1 = \sum_{p=1}^{3} \mid G_i^p(t) - G_{i+1}^p(t) \mid = \frac{1}{(1+t)^{2\beta}}.$$

On both sides of the $\|\hat{G}_i(t)\|_1$ integral, we obtain

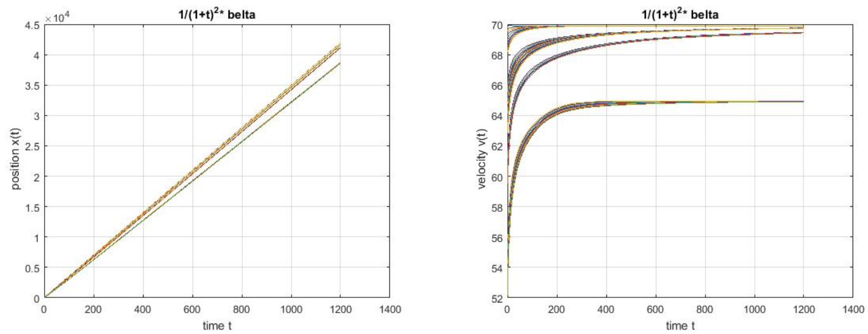
$$\int_0^{+\infty} \|\hat{G}_i(\nu)\|_1 d\nu = \int_0^{+\infty} \frac{1}{(1+\nu)^{2\beta}} d\nu = \frac{1}{2\beta - 1} < +\infty.$$

Therefore, the interference function $\hat{G}_i(t)$ satisfies the requirements of Theorem 3.2. At the same time, we give the initial values of the position and velocity of 40 agents, and require these initial data to satisfy Lemma 3.1, i.e.,

$$x_1(0) \geq x_2(0) \geq x_3(0) \geq, \ldots, \geq x_{40}(0),$$

$$v_1(0) > v_2(0) > v_3(0) >, \ldots, > v_{40}(0).$$

Take $\beta = 0.60 > 1/2$, we obtained the results shown in Figure 4 after experimental simulation. From Figure 3, we can easily find that among the experimental phenomena of system (3.17) with 40 agents, 40 agents can be divided into 3 clusters and synchronized into clusters.



(a) $x(t) = \{x_1(t), x_2(t), \ldots, x_{40}(t)\} \in \mathbb{R}^{3*40}$ (b) $v(t) = \{v_1(t), v_2(t), \ldots, v_{40}(t)\} \in \mathbb{R}^{3*40}$

Figure 3. Dynamic trends of velocity and position of 40 agents with disturbance systems

After the given initial data, we calculate that the three clusters are $\Omega_1 = \{1, 2, 3, \ldots, 24\}$, $\Omega_2 = \{25, 26, \ldots, 30\}$, $\Omega_3 = \{31, 32, \ldots, 40\}$, and $\Omega = \{1, 2, 3, \ldots, 40\} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, which further embodies the meaning of Definition 2.2 in this paper. This presents a phenomenon of multiple flocking cluster behavior between agents in an environment with interference factors.

Experiment B2: Similarly, in the case of satisfying Theorem 3.2, we take $\|\hat{G}_i(t)\|_1 = t\exp(-t^2)$ again. Here, $\beta = 0.60 > 1/2$, and we can divide 40 agents into 4 clusters, e.i.,

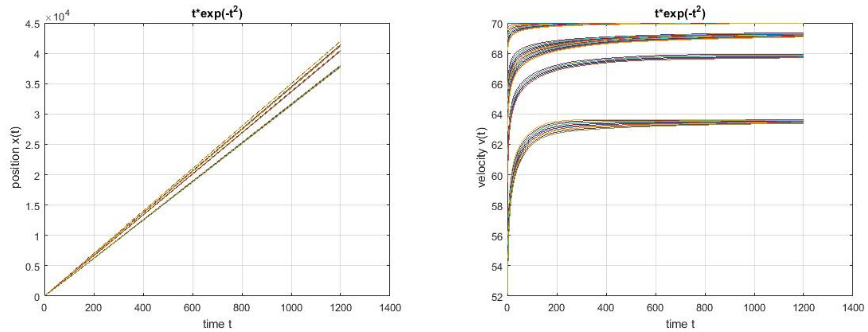
$$\Omega = \{1, 2, 3, \ldots, 40\} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,$$

where $\Omega_1 = \{1, 2, 3, \ldots, 10\}$, $\Omega_2 = \{11, 12, \ldots, 34\}$, $\Omega_3 = \{35, 36, \ldots, 40\}$ and $\Omega_4 = \{31, 32, \ldots, 40\}$, as shown in Figure 4.

In summary, we have fully verified the results of Theorem 3.1 and Theorem 3.2.

5. Conclusions

In this paper, we study the perturbed C-S model, which contains the structure of hierarchical leadership. The research in this paper shows that flocking (Mono-cluster



(a) $x(t) = \{x_1(t), x_2(t), \ldots, x_{40}(t)\} \in \mathbb{R}^{3*40}$ (b) $v(t) = \{v_1(t), v_2(t), \ldots, v_{40}(t)\} \in \mathbb{R}^{3*40}$

Figure 4. Dynamic trends of velocity and position of 40 agents with disturbance systems

flocking) occurs when the perturbation in system (2.2) satisfies some conditions and $\beta = \frac{1}{2}$, while for $\beta > \frac{1}{2}$ conditional flocking (Multi-cluster flocking) would occur provided the initial data [40, Lemma 3.1] and perturbation satisfy some given conditions. Moreover, we verify the main results of this paper through numerical simulation experiments. Our results maybe applied to aircraft formations, team coordinated operations, financial fluctuations, etc.

We leave a few far-reaching questions in this paper, since the conditions for analyzing flocking behavior are limited, we hope to try to improve it in the future research process, and we can also consider the multi-flocking behavior of the C-S model with fractional order and time delay.

Appendix A

In this part, we will prove Step 3 in Theorem 3.2. The proof of Step 3 is divided in three cases.

Step 3.1: If $\hat{i}_i \in \hat{\Omega}_4 \neq \emptyset$, then $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$. We divide Step 3.1 into two parts:

$$\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu = 0 \text{ and } \rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu < 0.$$

Step 3.1.1: Assume $\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu = 0$. Thus for equation (3.24) we obtain the following form:

$$\|\hat{v}_i(t)\|_1 = \frac{f_i(t) - f_{i-1}(t)}{2\beta - 1}.$$

Note that, we obtain $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$. The proof process is the same as in case I of article [40]. Thence, the following contradiction

$$0 < \lim_{t\to\infty} \|\hat{v}_i(t)\|_1 \leq \liminf_{t\to\infty} \frac{-f_{i-1}(t)}{2\beta - 1} \leq 0.$$

Therefore, $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$.

Step 3.1.2: Assume $\forall t \geq 0, \rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu < 0$, and $\exists j^{'} \in \mathbb{N}, 1 \leq j^{'} \leq i, \sum_{k=j'}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0$, for $\hat{i}_i \in \hat{\Omega}_4 \Longrightarrow \lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$. Suppose not, we have $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 > 0 \Longrightarrow \lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$. Combining (3.24) and (3.35), we have

$$
\begin{aligned}
0 < \sum_{k=j'}^{i} \|\hat{v}_k(t)\|_1 &= \frac{1}{2\beta-1}\sum_{k=j'}^{i}(f_k(t) - f_{k-1}(t)) \\
&\quad - \sum_{k=j'}^{i}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \\
&= \frac{1}{2\beta-1}(f_i(t) - f_{j'-1}(t)) \\
&\leq \frac{K(i-j^{'}+1)}{n(2\beta-1)(1+\|\hat{x}_i(t)\|_1)^{2\beta-1}} \to 0, \ (\text{as } t\to\infty).
\end{aligned} \tag{5.1}
$$

As a result, there is a contradiction, hence $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 = 0$.

Next, we use mathematical induction to prove Step 3.2 and 3.3.

Step 3.2: If $\hat{i}_i \in \hat{\Omega}_5 \neq \emptyset$, then $\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 > 0$, $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$. For any $\hat{i}_i \in \hat{\Omega}_5$, Assuming that there are j elements belonging to $\hat{\Omega}_5$, and these j elements are smaller than $\hat{i}_i$, then, we have

$$\{\hat{i}_{i_1}, \hat{i}_{i_2}, \ldots, \hat{i}_{i_j}, \hat{i}_{i_{j+1}} = \hat{i}_i\} \subset \hat{\Omega}_5 \Longrightarrow \hat{i}_{i_1} < \hat{i}_{i_2} <, \ldots, < \hat{i}_{i_j} < \hat{i}_{i_{j+1}} = \hat{i}_i.$$

Under the premise of the above assumptions, we will prove

$$\lim_{t\to\infty} \|\hat{v}_i(t)\|_1 > 0 \text{ and } \lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty.$$

For any $1 \leq m \leq j+1$, we use mathematical induction to prove this situation.

(3.2.1) If $j = 0 \Rightarrow \hat{i}_{i_1} = \hat{i}_i \Rightarrow i_1 = i$. Therefore, we need to prove the situation of $m = 1$. For $j < i_1$, since $\hat{i}_{i_1}$ is the first element of $\hat{\Omega}_5$, thus $\hat{i}_j \notin \hat{\Omega}_5$.

Since $\hat{i}_j \notin \hat{\Omega}_5$, then we only have $\lim_{t\to\infty} \|\hat{v}_j(t)\|_1 = 0$, for $j < i_1$. By (5.1), similar we can get

$$
\begin{aligned}
\sum_{k=1}^{i_1} \|\hat{v}_k(t)\|_1 =& \frac{1}{2\beta-1}\sum_{k=1}^{i_1}(f_k(t) - f_{k-1}(t)) \\
& - \sum_{k=1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \\
=& \frac{1}{2\beta-1} f_{i_1}(t) - \sum_{k=1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \\
\geq& -\sum_{k=1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) > 0.
\end{aligned} \tag{5.2}
$$

$\Longrightarrow \|\hat{x}_{i_1}(t)\|_1 \to \infty$, as $t \to \infty$, this proves the conclusion of $m = 1$.

(3.2.2) If $j \neq 0 \Rightarrow m \geq 1$. Under the induction hypothesis, we assume that if $m \geq 1$, $\lim_{t\to\infty} \|\hat{x}_{i_m}(t)\|_1 = \infty$ and $\lim_{t\to\infty} \|\hat{v}_{i_m}(t)\|_1 > 0$ hold, which means $f_{i_m}(t) \to 0$, (as $t \to \infty$).

(3.2.3) If the assumption of (3.2.2) is reasonable and meaningful, we next prove that $\lim_{t\to\infty}\|\hat{x}_{i_{m+1}}(t)\|_1=\infty$ and $\lim_{t\to\infty}\|\hat{v}_{i_{m+1}}(t)\|_1>0$ are hold. According to the derivation of equation (5.2), we can obtain the equation

$$\begin{aligned}\sum_{k=1}^{i_m}\|\hat{v}_k(t)\|_1=&\frac{1}{2\beta-1}\sum_{k=1}^{i_m}(f_k(t)-f_{k-1}(t))\\&-\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1d\nu)\\=&\frac{1}{2\beta-1}f_{i_m}(t)-\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1d\nu).\end{aligned}\tag{5.3}$$

Combining the conclusions in (3.2.2), we have $\lim_{t\to\infty}\sum_{k=1}^{i_m}\|\hat{v}_k(t)\|_1=-\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1d\nu)$ on both sides of equation (5.3) with respect to t tending to infinity. Thus, we obtain

$$\begin{aligned}&\lim_{t\to\infty}\sum_{k=1}^{i_{m+1}}\|\hat{v}_k(t)\|_1\\=&\frac{1}{2\beta-1}\lim_{t\to\infty}f_{i_{m+1}}(t)-\sum_{k=1}^{i_{m+1}}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1d\nu)\\=&\frac{1}{2\beta-1}\lim_{t\to\infty}f_{i_{m+1}}(t)-\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1)\\&-(\rho_{i_{m+1}}-\int_0^t\|\hat{G}_{i_{m+1}}(\nu)\|_1)\\>&0+0-(\rho_{i_{m+1}}-\int_0^t\|\hat{G}_{i_{m+1}}(\nu)\|_1)\\=&-\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1)>0.\end{aligned}\tag{5.4}$$

When $\hat{i}_i\in\hat{\Omega}_5\neq\emptyset$, the above inequality holds. Thus, we have $\rho_{i_{m+1}}-\int_0^t\|\hat{G}_{i_{m+1}}(\nu)\|_1\leq 0$, $\sum_{k=1}^{i_m}(\rho_k-\int_0^t\|\hat{G}_k(\nu)\|_1)<0$, and $f_{i_{m+1}}(t)\geq 0$.

Accordingly, we obtain $\lim_{t\to\infty}\|\hat{v}_{i_{m+1}}(t)\|_1>0$. Applying the ordered relationship of Lemma 3.1, it is easy to obtain

$$\lim_{t\to\infty}\|\hat{x}_{i_{m+1}}(t)\|_1=\infty.$$

According to the inductive hypothesis, our conclusions are further proved by (3.2.1), (3.2.2) and (3.2.3) above.

Similar to the proof in Step 3.3 of [40] and Step 3.2 above, we can show the following Step 3.3.

Step 3.3: If $\hat{i}_i\in\hat{\Omega}_4\neq\emptyset$, then $\lim_{t\to\infty}\|\hat{x}_i(t)\|_1=\infty$. For any $\hat{i}_i\in\hat{\Omega}_4$, Assuming that there are j elements belonging to $\hat{\Omega}_4$, and these j elements are smaller than $\hat{i}_i$, then we have

$$\{\hat{i}_{i_1},\hat{i}_{i_2},\ldots,\hat{i}_{i_j},\hat{i}_{i_{j+1}}=\hat{i}_i\}\subset\hat{\Omega}_4\Longrightarrow 1\leq\hat{i}_{i_1}<\hat{i}_{i_2}<,\ldots,<\hat{i}_{i_j}<\hat{i}_{i_{j+1}}=\hat{i}_i.$$

Under the premise of the above assumptions, we will prove $\lim_{t\to\infty}\|\hat{v}_i(t)\|_1 = 0$ and $\lim_{t\to\infty}\|\hat{x}_i(t)\|_1 = \infty$.

For any $1 \le m \le j+1$, we use mathematical induction to prove this situation.

(3.3.1) If $j = 0 \Rightarrow \hat{i}_{i_1} = \hat{i}_i \Rightarrow i_1 = i$ and $m = 1$. Therefore, we need to prove the situation of $m = 1$. Since $\hat{i}_{i_1}$ is the first element of $\hat{\Omega}_4$, there is a unique $j^{'}$ that satisfies $1 \le j^{'} \le i_1$ and $\sum_{k=j^{'}}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0$.

Next we will prove that $\hat{i}_i \in \hat{\Omega}_4 \ne \emptyset$ is true in both $\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu = 0$ and $\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu < 0$ cases.

(3.3.1A) Suppose $\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu = 0$.

When $i_1 = 1$, we can get $\lim_{t\to\infty}\|\hat{v}_1(t)\|_1 = \frac{K}{(2\beta-1)(1+\|\hat{x}_1(t)\|_1)^{2\beta-1}}$, by equation (3.24). According to the previously proven Step 3.1, we obtain

$$\lim_{t\to\infty}\|\hat{v}_1(t)\|_1 = 0.$$

Thus, it is proved that $\lim_{t\to\infty}\|\hat{x}_1(t)\|_1 = \infty$.
When $i_1 > 1$, for $1 \le j \le i_1 - 1$, we deduce that

$$\begin{aligned}
&\sum_{k=j}^{i_1-1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu)\\
=&0 + \sum_{k=j}^{i_1-1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu)\\
=&\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu + \sum_{k=j}^{i_1-1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu)\\
=&\sum_{k=j}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1) < 0.
\end{aligned} \tag{5.5}$$

The reason why the above inequality holds is because i_1 is the first element of the set $\hat{\Omega}_4$, then $i_1 - 1$ is not in the $\hat{\Omega}_4$. Obviously, the inequality meets the definition of $\hat{\Omega}_5$, hence $\hat{i}_{i_1-1} \in \hat{\Omega}_5$.

According to the previously proven Step 3.2, we obtain

$$\lim_{t\to\infty}\|\hat{x}_{i_1-1}(t)\|_1 = \infty \Longrightarrow \lim_{t\to\infty} f_{i_1-1}(t) = 0.$$

By equation (3.24), we obtain

$$\begin{aligned}
\lim_{t\to\infty}\|\hat{v}_{i_1}(t)\|_1 =&\frac{1}{2\beta-1}\lim_{t\to\infty}(f_{i_1}(t) - f_{i_1-1}(t))\\
&-(\rho_i - \int_0^t \|\hat{G}_i(\nu)\|_1 d\nu)\\
=&\frac{1}{2\beta-1}\lim_{t\to\infty} f_{i_1}(t).
\end{aligned} \tag{5.6}$$

In a similar way, we can also prove

$$\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu = 0 \Longrightarrow \rho_1 - \int_0^t \|\hat{G}_1(\nu)\|_1 d\nu = 0,$$

thus $\lim_{t\to\infty} \|\hat{x}_i(t)\|_1 = \infty$.

(3.3.1B) Suppose $\rho_{i_1} - \int_0^t \|\hat{G}_{i_1}(\nu)\|_1 d\nu < 0$.

When $i_1 \neq 1 \Rightarrow i_1 \geq 2$, and $i_1 \in \hat{\Omega}_4$. Due to specialization of i_1, there exists a unique $j^{'}$ satisfying $\sum_{k=j^{'}}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0$ which use the counter-evidence. Suppose not, then

$$\sum_{k=j^{'}}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) < 0, \text{ for } j \neq j^{'}.$$

When $j^{'} = 1 \Longrightarrow \sum_{k=1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0$ and $\sum_{k=2}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) < 0 \Longrightarrow \rho_1 - \int_0^t \|\hat{G}_1(\nu)\|_1 d\nu) > 0$, since

$$(\rho_1 - \int_0^t \|\hat{G}_1(\nu)\|_1 d\nu) + \sum_{k=2}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) = 0,$$

we deduce that

$$\rho_1 - \int_0^t \|\hat{G}_1(\nu)\|_1 d\nu = -\sum_{k=2}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) > 0.$$

Moreover, for $1 \leq j \leq i_1 - 1$, we have $\sum_{k=1}^{j}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) > 0$. Suppose not, then $\exists j^{''} \in [1, i_1 - 1]$ such that $\sum_{k=1}^{j^{''}}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \leq 0$. From the above ordered relationship, we get the following equation

$$\begin{aligned}
0 &= \sum_{k=1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \\
&= \sum_{k=1}^{j^{''}}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) + \sum_{k=j^{''}+1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) \\
&\leq \sum_{k=j^{''}+1}^{i_1}(\rho_k - \int_0^t \|\hat{G}_k(\nu)\|_1 d\nu) < 0,
\end{aligned} \tag{5.7}$$

this contradicts the facts of the above hypothesis. Since (5.7) meets the definition of Ω_1 and Ω_2, thus, for $1 \leq j \leq i_1 - 1 \Rightarrow \hat{i}_j \in \hat{\Omega}_1 \cup \hat{\Omega}_2$. Therefore, by Step 1 and Step 2, we have $\lim_{t\to\infty} \|\hat{v}_j(t)\|_1 = 0$, for $1 \leq j \leq i_1 - 1$. By equation (3.24), we obtain

$$\begin{aligned}
\lim_{t\to\infty} \|\hat{v}_{i_1}(t)\|_1 =& 0 + 0 +, \ldots, + \lim_{t\to\infty} \|\hat{v}_{i_1}(t)\|_1 \\
=& \lim_{t\to\infty}(\|\hat{v}_1(t)\|_1 + \|\hat{v}_2(t)\|_1 +, \ldots, + \|\hat{v}_{i_1}(t)\|_1) \\
=& \frac{1}{2\beta - 1}\lim_{t\to\infty}\sum_{k=1}^{i_1}(f_k(t) - f_{k-1}(t)) \\
& - \sum_{k=1}^{i_1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \\
=& \frac{1}{2\beta - 1}\lim_{t\to\infty} f_{i_1}(t).
\end{aligned} \tag{5.8}$$

By Step 3.1, it follows that $\lim_{t\to\infty}\|\hat{v}_1(t)\|_1 = 0$. Then we have $\lim_{t\to\infty} f_{i_1}(t) = 0$, meaning $\lim_{t\to\infty}\|\hat{x}_1(t)\|_1 = \infty$.

When $j^{'} \geq 2$, for $1 \leq j \leq j^{'} - 1$, we have

$$\begin{aligned}
&\sum_{k=j}^{i_1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \\
&= \sum_{k=j}^{j^{'}-1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) + \sum_{k=j^{'}}^{i_1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \\
&= \sum_{k=j}^{j^{'}-1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) < 0.
\end{aligned} \tag{5.9}$$

Since $\hat{i}_{i_1} \in \hat{\Omega}_4$, and $\hat{i}_{i_1}$ is the first element in $\hat{\Omega}_4$. By Step 3.2, we have $\lim_{t\to\infty}\|\hat{x}_{j^{'}-1}(t)\|_1 = \infty \Rightarrow \lim_{t\to\infty} f_{j^{'}-1}(t) = 0$. This proves the case of $j^{'} \geq 2$ and $m = 1$.

(3.3.2) Similarly, under the induction hypothesis, we assume that

$$\lim_{t\to\infty}\|\hat{x}_{i_m}(t)\|_1 = \infty \text{ holds for } m \geq 1.$$

(3.3.3) Next we will prove the case of $m+1$, and also consider the following two cases $\rho_{i_{m+1}} - \int_0^t \|G_{i_{m+1}}(\nu)\|_1 d\nu = 0$ and $\rho_{i_{m+1}} - \int_0^t \|G_{i_{m+1}}(\nu)\|_1 d\nu < 0$ to prove $\lim_{t\to\infty}\|\hat{x}_{i_{m+1}}(t)\|_1 = \infty$.

(3.3.3A) Suppose $\rho_{i_{m+1}} - \int_0^t \|G_{i_{m+1}}(\nu)\|_1 d\nu = 0$.

For $1 \leq j \leq i_{m+1} - 1$, we again get a similar result of

$$\begin{aligned}
&\sum_{k=j}^{i_{m+1}-1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \\
&= \sum_{k=j}^{i_{m+1}}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \leq 0.
\end{aligned} \tag{5.10}$$

This result conforms to the definition of $\hat{\Omega}_3$, thus $\hat{i}_{i_{m+1}-1} \in \hat{\Omega}_3 \Longrightarrow \sum_{k=j}^{i_{m+1}-1}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) = \sum_{k=j}^{i_{m+1}}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) \leq 0$.

On the one hand, we find that if $\hat{i}_{i_{m+1}-1} \in \hat{\Omega}_4$, then $i_{m+1} - 1 = i_m$. Thus, the result of $\lim_{t\to\infty}\|\hat{x}_{i_{m+1}-1}(t)\|_1 = \infty$ can also be obtained according to the induction method.

On the other hand, if $\hat{i}_{i_{m+1}-1} \in \hat{\Omega}_5$, then $\lim_{t\to\infty}\|\hat{x}_{i_{m+1}-1}(t)\|_1 = \infty$ can be obtained by Step 3.2. According to the $f_{i_{m+1}-1}(t)$ defined in equation (3.24), we can know that $\lim_{t\to\infty} f_{i_{m+1}-1}(t) = 0$, of course, the prerequisite for reaching this limit is $\lim_{t\to\infty}\|\hat{x}_{i_{m+1}-1}(t)\|_1 = \infty$. By equation (3.24), we can also get a similar (5.8) result, i.e.,

$$\begin{aligned}
\lim_{t\to\infty}\|\hat{v}_{i_{m+1}}(t)\|_1 &= \frac{1}{2\beta - 1}\lim_{t\to\infty}(f_{i_{m+1}}(t) - f_{i_{m+1}-1}(t)) \\
&= \frac{1}{2\beta - 1}\lim_{t\to\infty} f_{i_{m+1}}(t).
\end{aligned} \tag{5.11}$$

In the same way as the case of $j' = 1$ in (3.3.1B), we have

$$\lim_{t\to\infty} \|\hat{x}_{i_{m+1}}(t)\|_1 = \infty.$$

(3.3.3B) Suppose $\rho_{i_{m+1}} - \int_0^t \|G_{i_{m+1}}(\nu)\|_1 d\nu < 0$. We have $\sum_{k=i_m+1}^{i_{m+1}}(\rho_k - \int_0^t \|G_k(\nu)\|_1 d\nu) = 0$. We now prove that the process is the same as in the case of $j^{'} \geq 2$ of (3.3.1B) in (3.3.1), thus proving our conclusion of $\lim_{t\to\infty} \|\hat{x}_{i_{m+1}}(t)\|_1 = \infty$.

In summary, we use the method in [40] to prove that $\lim_{t\to\infty} \|\hat{x}_{i_{j+1}}(t)\|_1 = \infty$, further illustrating that the results of similar [40] can also be obtained in the case of disturbance.

Finally, the conclusion of Step 3 is proved in conjunction with Steps 1, 2 and 3 of the inductive hypothesis in Step 3.3.

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