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# STATIC OUTPUT FEEDBACK DESIGN USING MODEL REDUCTION METHODS FOR SECOND-ORDER SYSTEMS*

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**Abstract** In this paper, the static output feedback design is investigated using model reduction methods for large-scale second-order systems. First, based on the second-order Krylov subspace method, a low dimensional structure-preserving second-order system is derived. Then, applying matrix transformation, the relationship between input variables and output variables is directly established in the low dimensional system. We design output feedback controller for this system. Finally, using the argument principle, a computable stability criterion is presented to check the stability of the closed-loop system. Furthermore, a numerical algorithm is provided to design the output feedback controller for large-scale second-order systems. Numerical examples are given to illustrate the effectiveness of the algorithms.

**Keywords** Second-order systems, static output feedback, model reduction methods, argument principle

**MSC(2010)** 93D15, 93B52, 93D20.

## 1. Introduction

Second-order systems or higher-order systems often appear in the field of engineering and physics, such as circuit simulation [7,8], structural dynamics [12,15], mechanical systems [23], electromagnetics [26], microelectromechanical and nanoelectromechanical systems (MEMS and NEMS) [17,21,25]. However, with the increase of system complexity, the dimension of system becomes very large, which makes it difficult to simulate and control the system. Especially for large-scale distributed flexible systems, the dynamic model in the form of second-order system has hundreds of coordinates. If we apply common controller design methods [1–3, 22, 30], the resulting controller will have the same number of states as the original system, which means that all of states need to be measured. This is generally difficult to achieve in practical engineering. Therefore, it is necessary to obtain a low dimensional system, which retains some important properties of the original second-order system and has practical value for controller design. In order to obtain this low dimensional system, we need to use model reduction methods.

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In recent years, model reduction methods of second-order systems have been widely studied; see [4, 6, 9, 11, 13, 16, 18, 20, 24, 27, 28]. Most of these methods can be divided into two classes. One is to transform the second-order system into a mathematically equivalent first-order system, and then apply model reduction methods to this first-order system; see [4, 9, 16] and references therein. The other is to deal with the second-order system directly and obtain a low dimensional second-order system, such as the balanced truncation approach [11, 18], the second-order Krylov subspace method [6, 20, 24], the data-driven method [13], iterative rational Krylov method [27] and the proper orthogonal decomposition method [28].

In particular, Bai and Su [6] presented a structure-preserving dimension reduction algorithm for large-scale single-input single-output(SISO) second-order systems, which was a projection method based on a second-order Krylov subspace. Lin et al. [20] extended the result of [6] to multi-input multi-output(MIMO) second-order systems and gave the corresponding dimension reduction algorithm. Sailimbahrami and Lohmann [24] introduced two methods based on Krylov subspace for order reduction of large-scale second-order systems, both of which preserved the specific structure of original second-order systems.

Inspired by the work of Bai and Su [6], the aim of this paper is to consider static output feedback problem for large-scale second-order systems by using model reduction methods. That is, we first apply model reduction methods to the original second-order system to obtain a low dimensional system. Then, we directly establish the relationship between input variables and output variables through matrix transformation. Hence we can design the output feedback controller for this system. Finally, using the argument principle, we demonstrate whether the resulting controller can stabilize the original system.

This paper is organized as follows. In Section 2, we review model reduction methods and feedback stabilization problem of second-order systems. In Section 3, a stability criterion is given for second-order closed-loop systems. In Section 4, we present a static output feedback design algorithm for large-scale unstable second-order systems. In Section 5, several numerical examples are provided to demonstrate the main results. Some conclusions are given in Section 6.

Throughout the paper, $\|A\|$ represents any matrix norm of $A$. 0 denotes the zero vector or matrix. The space spanned by the vector sequence $q_0, q_1, ..., q_l$ and the columns of the matrix $Q$ are denoted by $\text{span}\{q_0, q_1, ..., q_l\}$ and $\text{span}\{Q\}$, respectively. $\text{Re}\, z$ and $\text{Im}\, z$ stand for the real part and the imaginary part of a complex number $z$, respectively.

# 2. Problem formulation and preliminaries

In this section, we review model reduction methods of second-order systems. In particular, we introduce the second-order Krylov subspace method. Moreover, we present the feedback stabilization problem of second-order systems.

## 2.1. Dimension reduction of second-order systems

Consider a large-scale linear time-invariant second-order system:

$$\begin{cases} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{2.1}$$

where $M, D, K \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$ and $C \in \mathbb{R}^{p\times N}$. $x(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. The initial conditions are $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. In structural dynamics, the matrices $M, D$ and $K$ are known as mass, damping and stiffness, respectively. We assume that the system is controllable in this paper.

In general, the dimension of system (2.1) is extremely large, which makes it difficult to design controller directly for this system. Therefore, one approach to overcome this problem is through model reduction. Here, model reduction is a computational method to approximate large-scale complex systems by using low dimensional systems. For the second-order system (2.1), the model reduction technique is to find a reduced system which can keep the second-order form:

$$\begin{cases} \tilde{M}\ddot{z}(t) + \tilde{D}\dot{z}(t) + \tilde{K}z(t) = \tilde{B}u(t), \\ \tilde{y}(t) = \tilde{C}z(t), \end{cases} \tag{2.2}$$

where the dimension of $z(t)$ is far less than that of $x(t)$.

The transfer matrix of system (2.1) is given by

$$G(s) = C(Ms^2 + Ds + K)^{-1}B,$$

and the power series expansion of $G(s)$ can be expressed as

$$G(s) = M_0 + M_1 s + M_2 s^2 + \cdots = \sum_{i=0}^{\infty} M_i s^i,$$

where $M_i (i \geq 0)$ are called $i$th order moments of system (2.1).

For model reduction methods, it always requires that the reduced system (2.2) can match the moments of the transfer matrix of the original system (2.1) as much as possible. The second-order Krylov subspace method is one of the most important methods. It can preserve the structure of the original system and be suitable for the dimension reduction of large-scale second-order systems. Next we recover the definition of the second-order Krylov subspace.

**Definition 2.1** (Definition 2.1, [5]). Let $A_1$ and $A_2$ be square matrices of order $N$, and let $v \neq 0$ be an $N$ vector. Then the sequence

$$r_0, r_1, r_2, ..., r_{n-1},$$

where

$$\begin{aligned} r_0 &= v, \\ r_1 &= A_1 r_0, \\ r_j &= A_1 r_{j-1} + A_2 r_{j-2} \text{ for j} \geq 2, \end{aligned}$$

is called a second-order Krylov sequence based on $A_1$, $A_2$ and $v$. The space

$$\mathcal{G}_n(A_1, A_2; v) = \text{span}\{r_0, r_1, r_2, ..., r_{n-1}\}$$

is called an $n$th second-order Krylov subspace.

For system (2.1), the second-order Krylov subspace is defined in [6, p5] as follows:

$$\mathcal{G}_n(A_1, A_2; v) = \mathcal{G}_n(-K^{-1}D, -K^{-1}M; K^{-1}B) = \text{span}\{r_0, r_1, r_2, ..., r_{n-1}\}.$$

In order to generate an orthonormal basis of $\mathcal{G}_n(A_1, A_2; v)$, a second-order Arnoldi method was given in [6, Algorithm 1] for single-input single-output system (2.1), and a block second-order Arnoldi method was proposed in [20, Algorithm 3.1] for multi-input multi-output system (2.1). By performing these algorithms, an orthogonal matrix $Q_n \in \mathbb{R}^{N\times n}(n \ll N)$ was obtained, which satisfied

$$\mathrm{span}\{Q_n\} = \mathcal{G}_n(A_1, A_2; v).$$

Next taking an approximation $x(t) \approx Q_n z(t)$, and substituting it into system (2.1), we can obtain the low dimensional system (2.2). Here, the transfer matrix and $i$th order moment of the reduced system (2.2) are denoted as $G_n(s)$ and $M_l^{(n)}$, respectively. To illustrate the moment-matching property between system (2.1) and (2.2), we need the following lemma.

**Lemma 2.1** (Theorem 4.1, [6]). *The first $n$ moments of the original system* (2.1) *and the reduced system* (2.2) *are matched, i.e., $M_l = M_l^{(n)}$ for $l = 0, 1, 2, ..., n-1$. Hence $G_n(s)$ is an $n$th Padé-type approximant of the transfer matrix $G(s)$*:

$$G(s) = G_n(s) + \mathcal{O}(s^n).$$

Furthermore, we hope to get a special low dimensional system in practical application, which can directly establish the relationship between input variables and output variables. That is, we obtain the following system:

$$\hat{M}\ddot{y}(t) + \hat{D}\dot{y}(t) + \hat{K}y(t) = \hat{B}u(t),$$

where matrices $\hat{M}, \hat{D}, \hat{K}$ and $\hat{B}$ need to be determined. Since the dimension of the output variables is smaller than that of the state variables, it is possible to convert the state of the low dimensional system to the output variables of the system. If this transformation can be realized, then the input-output relationship of the system is established. Hence, we can design output feedback controller for this system.

## 2.2. Feedback stabilization of second-order systems

For a large-scale unstable second-order system (2.1), our main work is to design an output feedback controller of the form:

$$u(t) = -F_1 y(t) - F_2 \dot{y}(t), \tag{2.3}$$

which makes the system (2.1) asymptotically stable. Here, $F_1, F_2 \in \mathbb{R}^{m\times p}$ are feedback gain matrices to be determined. Since all the output variables of second-order systems can be measured directly, the output feedback controller (2.3) is easy to realize in engineering practice. Meanwhile, the corresponding state feedback controller has the form:

$$u(t) = -F_1 C x(t) - F_2 C \dot{x}(t).$$

Let $K_1 = F_1 C$, $K_2 = F_2 C$. Then, we have

$$u(t) = -K_1 x(t) - K_2 \dot{x}(t). \tag{2.4}$$

Substituting the controller (2.4) into system (2.1), we have the closed-loop system:

$$M\ddot{x}(t) + (D + BK_2)\dot{x}(t) + (K + BK_1)x(t) = 0. \tag{2.5}$$

# 3. Stability criterion of second-order systems

In this section, we will discuss the stability criterion of second-order closed-loop system (2.5).

For the second-order closed-loop system (2.5), we can get the corresponding matrix polynomial:

$$P(s) = Ms^2 + (D + BK_2)s + (K + BK_1). \tag{3.1}$$

The location of the zeros of $\det P(s)$ plays an important role in checking the stability of the system. The following lemma which was given in [14, p1926] demonstrates this point.

**Lemma 3.1.** *The second-order closed-loop system* (2.5) *is stable if and only if* $\det P(s)$ *has all its zeros in the open left half-plane.*

The characteristic polynomial

$$g(s) = \det P(s),$$

whose root is called an eigenvalue of the matrix polynomial (3.1).

If $M$ is nonsingular in system (2.5), upper and lower bounds were derived in [19, Theorem 3.2] for the eigenvalues of matrix polynomial (3.1).

**Lemma 3.2.** *Every eigenvalue $\xi$ of matrix polynomial* (3.1) *satisfies*

$$\frac{1}{\max\{1, a\}} \leq |\xi| \leq \max\{1, b\},$$

*where the scalars $a$ and $b$ are defined by*

$$a = \|(K + BK_1)^{-1}M\| + \|(K + BK_1)^{-1}(D + BK_2)\|,$$
$$b = \|M^{-1}(D + BK_2)\| + \|M^{-1}(K + BK_1)\|.$$

**Remark 3.1.** All unstable eigenvalues of matrix polynomial (3.1) are bounded and lie in a semicircular region

$$\Omega = \{\xi : |\xi| \leq \beta \text{ and } \operatorname{Re} \xi \geq 0\},$$

where $\beta = \max\{1, b\}$, $b$ is given by Lemma 3.2.

Now we denote the boundary of the closed semicircular region $\Omega$ as $\Gamma$. See Fig.1, where $d_1 = i\beta$ and $d_2 = -i\beta$. Furthermore, a stability criterion of second-order closed-loop system (2.5) was given in [19, Theorem 4.1] by using the argument principle [10, p289].

**Theorem 3.1.** *The second-order closed-loop system* (2.5) *is stable if and only if*

$$g(s) \neq 0 \text{ for } s \in \Gamma$$

*and*

$$\Delta_\Gamma \arg g(s) = 0,$$

*where $\Delta_\Gamma \arg g(s)$ stands for the change of the argument of $g(s)$ along the boundary $\Gamma$.*

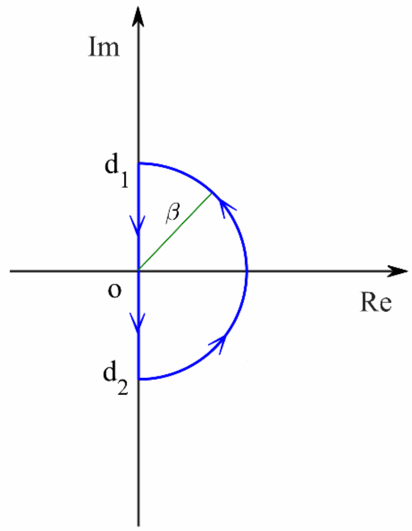


**Figure 1.** Region $\Omega$

Meanwhile, the number of unstable eigenvalues of matrix polynomial (3.1) was investigated.

**Theorem 3.2** (Theorem 4.2, [19]). *If*

$$g(s) \neq 0 \text{ for } s \in \Gamma$$

*and*

$$\frac{1}{2\pi}\Delta_\Gamma \arg g(s) = z$$

*hold, then the number of the unstable eigenvalues of matrix polynomial* (3.1) *is* $z$.

According to Theorem 3.1, we give an algorithm to judge the stability of second-order closed-loop system (2.5).

**Algorithm 3.1.** Stability criterion of second-order closed-loop system (2.5).

**Step 1.** Calculate the radius $\beta$ of semi-circle region $\Omega$. Here, $\beta$ is determined by Lemma 3.2. Then we have the closed contour $\Gamma$ of region $\Omega$.

**Step 2.** Given a sufficiently large integer $N_1$, and divide $\Gamma$ into $N_1$ node points $s_1, s_2, \ldots, s_{N_1}$ as uniformly as possible. For each $s_j$ $(j = 1, 2, \ldots, N_1)$, we calculate the values of $g(s_j)$

$$g(s_j) = \det[M s_j^2 + (D + BK_2)s_j + (K + BK_1)].$$

**Step 3.** In order to check whether $g(s_j) = 0$ for each $s_j$ $(j = 1, 2, \ldots, N_1)$, we need to evaluate its magnitude satisfies $|g(s_j)| \leq \gamma_1$ with the preassigned tolerance $\gamma_1$. If it holds for some $s_j$, then the closed-loop system (2.5) is not asymptotically stable, and we stop the algorithm. Otherwise, we go to the next step.

**Step 4.** Compute whether $\Delta_\Gamma \arg g(s) = 0$ along the sequence $\{g(s_j)\}$ for $s_j (j = 1, 2, \ldots, N_1)$ by evaluating $|\Delta_\Gamma \arg g(s)| \leq \gamma_2$ with the preassigned tolerance $\gamma_2$. If it holds, according to Theorem 3.1, the closed-loop system (2.5) is stable, otherwise not stable.

# 4. Static output feedback design

In this section, we will discuss static output feedback design by using model reduction methods for large-scale unstable second-order systems (2.1).

For a large-scale second-order system, if we directly use standard controller design methods, it will require unrealistic amounts of calculation. Hence, we first utilize model reduction methods to obtain a low dimensional approximation of the original system. Then, applying matrix transformation, the relationship between input variables and output variables is directly established in the low dimensional system. We design output feedback controller for the low dimensional system. Finally, we need to demonstrate whether the output feedback controller can stabilize the original system. According to this design idea, we give the specific design algorithm.

**Algorithm 4.1.** Static output feedback design for second-order system (2.1).

**Step 1.** Applying the second-order Krylov subspace method to the original system (2.1), we can generate an orthonormal basis $Q_n$ of the second-order Krylov subspace $\mathcal{G}_n(A_1, A_2; v)$.

**Step 2.** We employ the subspace projection technique to obtain a special reduced system, which directly establishes the input-output relationship of the system. The key point is that we transform the state of low dimensional system to the output variables $y(t)$ in the process of dimension reduction.

**Step 3.** By solving a linear matrix equation, we design the output feedback controller to stabilize the reduced system.

**Step 4.** We demonstrate whether the resulting controller can stabilize the original system by using the argument principle.

Next, we explain Algorithm 4.1 in detail.

In Step 1, let $A_1 = -K^{-1}D,\ A_2 = -K^{-1}M,\ v = K^{-1}B$ in the second-order Krylov subspace $\mathcal{G}_n(A_1, A_2; v)$. By running the second-order Arnoldi method or the block second-order Arnoldi method, we obtain an orthonormal basis of the second-order Krylov subspace

$$\operatorname{span}\{Q_n\} = \mathcal{G}_n(-K^{-1}D, -K^{-1}M; K^{-1}B).$$

In Step 2, we use the orthogonal matrix $Q_n$ as a projection matrix to generate a reduced model. Let $Q_n \in \mathbb{R}^{N\times n}(n \ll N)$. Taking an approximation $x(t) \approx Q_n z(t)$, we get the reduced model as follows:

$$\begin{cases} M_n\ddot{z}(t) + D_n\dot{z}(t) + K_n z(t) = B_n u(t), \\ \tilde{y}(t) = C_n z(t), \end{cases} \tag{4.1}$$

where $M_n, D_n, K_n \in \mathbb{R}^{n\times n}$ satisfy $M_n = Q_n^{\mathrm{T}} M Q_n, D_n = Q_n^{\mathrm{T}} D Q_n$ and $K_n = Q_n^{\mathrm{T}} K Q_n$. $B_n \in \mathbb{R}^{n\times m}$ and $C_n \in \mathbb{R}^{p\times n}$ satisfy $B_n = Q_n^{\mathrm{T}} B$ and $C_n = CQ_n$, respectively.

Lemma 2.1 ensures that the reduced system (4.1) can match the first $n$ moments of the original system (2.1).

From the equation $y(t) = Cx(t) \approx CQ_n z(t)$, we get

$$z(t) \approx (CQ_n)^\dagger y(t) = (C_n)^\dagger y(t), \tag{4.2}$$

where $(C_n)^\dagger$ denotes the Moore-Penrose inverse of $C_n$, i.e., $(C_n)^\dagger$ satisfies the following four equations:

$$C_n X C_n = C_n,\ X C_n X = X,\ (C_n X)^{\mathrm{H}} = C_n X,\ (X C_n)^{\mathrm{H}} = X C_n.$$

Here $(C_n)^{\mathrm{H}}$ is the conjugate transpose of $C_n$. If $C_n$ is a square matrix and invertible, then † represents an ordinary matrix inverse.

Substituting (4.2) into (4.1), we obtain an approximate equation

$$\tilde{M}_n \ddot{y}(t) + \tilde{D}_n \dot{y}(t) + \tilde{K}_n y(t) \approx \tilde{B}_n u(t), \tag{4.3}$$

where $\tilde{M}_n, \tilde{D}_n, \tilde{K}_n \in \mathbb{R}^{p\times p}$ satisfy $\tilde{M}_n = C_n M_n (C_n)^\dagger, \tilde{D}_n = C_n D_n (C_n)^\dagger$, $\tilde{K}_n = C_n K_n (C_n)^\dagger$. $\tilde{B}_n$ is $p\times m$ matrix such that $\tilde{B}_n = C_n B_n$. This special low dimensional system directly establishes the relationship between input variables and output variables.

In Step 3, we design an output feedback controller to stabilize the reduced system (4.3) by solving a linear matrix equation.

For the second-order system (4.3), the traditional controller design method is to transform this system into an equivalent first-order system, and then design the optimal feedback gain matrices by solving the nonlinear matrix Riccati equation. However, it is difficult to solve directly a nonlinear matrix equation. Thus, Zhang et al. [30] transformed a generalized nonlinear Riccati equation into a linear matrix equation by using matrix's singular value decomposition [29, Theorem 3.6] and matrix transformation. Then the optimal control gain can be obtained by solving the linear matrix equation. Inspired by the idea of Zhang et al., we design the output feedback controller by solving the following linear matrix equation:

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ * & \Sigma_{22} \end{bmatrix} - \begin{bmatrix} \tilde{D}_n^{\mathrm{T}} R_2 \tilde{D}_n & \tilde{D}_n^{\mathrm{T}} R_2 \tilde{M}_n \\ \tilde{M}_n^{\mathrm{T}} R_2 \tilde{D}_n & \tilde{M}_n^{\mathrm{T}} (R_1 + R_2) \tilde{M}_n \end{bmatrix} = 0, \tag{4.4}$$

where $*$ represents the corresponding symmetric matrix, and

$$\begin{aligned}
\Sigma_{11} &= \tilde{K}_n^{\mathrm{T}} P_2 \tilde{D}_n + \tilde{D}_n^{\mathrm{T}} P_2 \tilde{K}_n + \tilde{D}_n^{\mathrm{T}} X_1 Q \tilde{D}_n, \\
\Sigma_{12} &= \tilde{K}_n^{\mathrm{T}} P_2 \tilde{M}_n + \tilde{K}_n^{\mathrm{T}} P_1 \tilde{M}_n + \tilde{D}_n^{\mathrm{T}} X_2 Q \tilde{M}_n, \\
\Sigma_{22} &= \tilde{D}_n^{\mathrm{T}} P_1 \tilde{M}_n + \tilde{M}_n^{\mathrm{T}} P_1 \tilde{D}_n + \tilde{M}_n^{\mathrm{T}} X_3 Q \tilde{M}_n, \\
P_2 &= V \begin{bmatrix} \mathcal{P}_{11} & 0 \\ 0 & \mathcal{P}_{22} \end{bmatrix} V^{\mathrm{T}}, \; P_1 + P_2 = V \begin{bmatrix} \mathcal{W}_{11} & 0 \\ 0 & \mathcal{W}_{22} \end{bmatrix} V^{\mathrm{T}}.
\end{aligned}$$

Here, $R_1, R_2 \in \mathbb{R}^{p\times p}$, $R_3 \in \mathbb{R}^{m\times m}$ are given symmetric positive definite matrices, and $Q = \tilde{B}_n R_3^{-1} \tilde{B}_n^{\mathrm{T}}$ with $\operatorname{rank} \mathrm{Q} = \mathrm{r}$. The singular value decomposition of matrix $Q$ is defined by

$$Q = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^{\mathrm{T}},$$

where $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_i\,(i = 1, 2, \ldots, r)$ are the nonzero singular values of $Q$. If there exist the symmetric matrices $P_1, P_2 \in \mathbb{R}^{p\times p} \geq 0$, $\mathcal{P}_{11}, \mathcal{W}_{11} \in \mathbb{R}^{r\times r}$, $\mathcal{P}_{22}, \mathcal{W}_{22} \in \mathbb{R}^{(p-r)\times(p-r)} \geq 0$ and $X_1, X_2, X_3 \in \mathbb{R}^{p\times p}$ such that equation (4.4) holds. Then, the associated output feedback controller is given by

$$u(t) = -F_1 y(t) - F_2 \dot{y}(t) = -R_3^{-1} \tilde{B}_n^{\mathrm{T}} P_2 \tilde{D}_n y(t) - R_3^{-1} \tilde{B}_n^{\mathrm{T}} (P_1 + P_2) \tilde{M}_n \dot{y}(t).$$

The corresponding state feedback controller is

$$u(t) = -K_1 x(t) - K_2 \dot{x}(t) = -R_3^{-1} \tilde{B}_n^{\mathrm{T}} P_2 \tilde{D}_n C x(t) - R_3^{-1} \tilde{B}_n^{\mathrm{T}} (P_1 + P_2) \tilde{M}_n C \dot{x}(t).$$

In Step 4, we substitute the controller into the original system to obtain the closed-loop system (2.5). According to Algorithm 3.1, we check the stability of the closed-loop system (2.5). If the closed-loop system (2.5) is stable, the output feedback controller design is completed.

**Remark 4.1.** In order to avoid transforming the system into a first-order system, we directly deal with the second-order system in the process of system reduction and controller design. This design method can not only reduce the amount of calculation, but also keep the special structure and physical properties of the second-order system.

## 5. Numerical examples

In this section, we present three numerical examples to demonstrate the effectiveness of the algorithms.

**Example 5.1.** Consider a single-input second-order system in the form of (2.1), where parameter matrices are given by

$$M = \begin{bmatrix} 0.5356 & -0.1642 & 0.0734 & 0.0290 & -0.1075 \\ -0.1642 & 0.6635 & 0.1334 & -0.0593 & 0.1530 \\ 0.0734 & 0.1334 & 0.1753 & -0.1298 & 0.0436 \\ 0.0290 & -0.0593 & -0.1298 & 0.4643 & -0.0233 \\ -0.1075 & 0.1530 & 0.0436 & -0.0233 & 0.6018 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.4063 & -0.3953 & 0.1998 & -0.0667 & 0.4032 \\ -0.2443 & 1.3914 & 0.1995 & -0.2160 & 1.4751 \\ 0.0813 & -0.1658 & 0.5131 & 0.5272 & -1.5943 \\ -0.3347 & -0.0908 & -1.0557 & 2.8520 & 1.3295 \\ -0.9728 & -0.2744 & 0.6865 & 0.9021 & 0.9606 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to evaluate that $M$ is nonsingular. From Lemma 3.2, we have $b = \|M^{-1}D\|_2 + \|M^{-1}K\|_2 = 40.0658$ and the radius of the semicircular region is $\beta = \max\{1, b\} = 40.0658$. Here, the 2-matrix norm $\|A\|_2 = (\lambda_{\max}(A^{\mathrm{H}}A))^{\frac{1}{2}}$ is used. By Algorithm 3.1, we use uniform step-size $h = 0.01$ to divide boundary $\Gamma$, and then we can calculate $\frac{1}{2\pi}\Delta_\Gamma \arg g(s) = 2$. According to Theorem 3.1 and Theorem 3.2, we know that the original system is not stable and there are two unstable eigenvalues in its matrix polynomial. Hence we will use Algorithm 4.1 to design an output feedback controller to stabilize the original system. From Step 1 and Step 2, we can obtain the reduced model (4.3) and its coefficient matrices are as follows:

$$\tilde{M}_n = \begin{bmatrix} 0.5356 & -0.7616 & -0.5549 \\ -0.0489 & 0.6286 & 0.1866 \\ 0.0318 & 0.0042 & 0.4252 \end{bmatrix}, \tilde{K}_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{D}_n = \begin{bmatrix} -0.4063 & 1.899 & 2.5441 \\ -0.2443 & 8.518 & -6.2715 \\ -0.0813 & -7.6069 & -5.4525 \end{bmatrix}, \ \tilde{B}_n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By solving the linear matrix equation (4.4), we get the output feedback controller:

$$\begin{aligned} u(t) &= -F_1 y(t) - F_2 \dot{y}(t) \\ &= -\begin{bmatrix} -0.8039 & 3.7571 & 5.0334 \end{bmatrix} y(t) - \begin{bmatrix} 1.0597 & -1.5069 & -1.0978 \end{bmatrix} \dot{y}(t). \end{aligned}$$

For the original system, the state feedback controller is

$$\begin{aligned} u(t) &= -K_1 x(t) - K_2 \dot{x}(t) \\ &= -\begin{bmatrix} -0.8039 & 3.7571 & 5.0334 & 0 & 0 \end{bmatrix} x(t) - \begin{bmatrix} 1.0597 & -1.5069 & -1.0978 & 0 & 0 \end{bmatrix} \dot{x}(t). \end{aligned}$$

Combining with Algorithm 3.1, we check the stability of the closed-loop system (2.5). From Lemma 3.2, we calculate the radius $\beta$ of semicircular region $\Omega$ as follows

$$\beta = \|M^{-1}(D + BK_2)\|_2 + \|M^{-1}(K + BK_1)\|_2 = 46.7867.$$

Next, we can compute that $g(s_j) \neq 0$ for all $j = 1, 2, ..., N_1$, and $\Delta_\Gamma \arg g(s) = 0$, and hence the closed-loop system (2.5) is stable. The output feedback controller design is completed.

Furthermore, we choose the initial values $x_0 = [0.35\ 0.3\ -0.06\ -0.69\ -1.15]$, $\dot{x}_0 = [1.41\ -0.19\ -1.27\ -0.19\ -0.49]$. Figure 2 and Figure 3 display the state response of the original system (2.1) and the closed-loop system (2.5), respectively. Here, the velocity variable $\dot{x}(t)$ is represented by $v(t)$. We can find that the closed-loop system is stable, which means that the output feedback controller can stabilize the original system.

**Example 5.2.** Consider a multi-input multi-output second-order system with system parameters

$$M = \begin{bmatrix} 0.3915 & 0.0813 & -0.1007 & -0.1319 & 0.1026 & -0.1990 \\ 0.0813 & 0.5030 & 0.1024 & -0.2072 & 0.1166 & 0.1779 \\ -0.1007 & 0.1024 & 0.5451 & 0.0286 & 0.1523 & -0.0643 \\ -0.1319 & -0.2072 & 0.0286 & 0.5073 & 0.0394 & -0.0121 \\ 0.1026 & 0.1166 & 0.1523 & 0.0394 & 0.2635 & -0.1152 \\ -0.1990 & 0.1779 & -0.0643 & -0.0121 & -0.1152 & 0.4351 \end{bmatrix}, \ K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

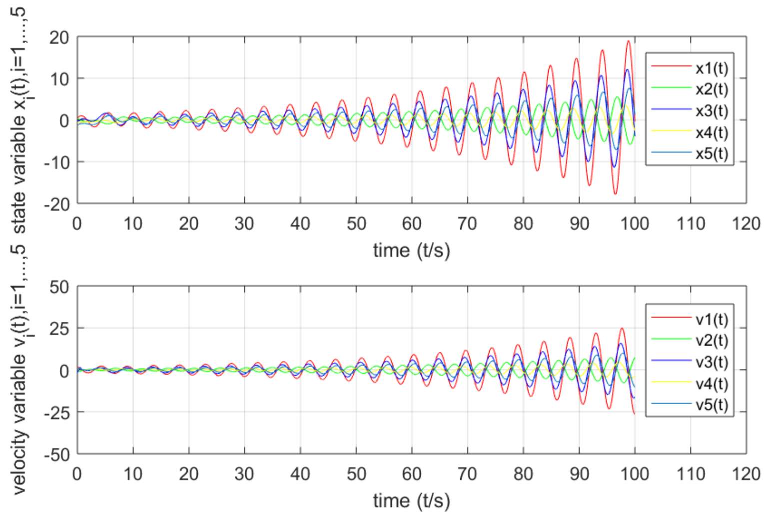


**Figure 2.** The state response of the original system in Example 5.1

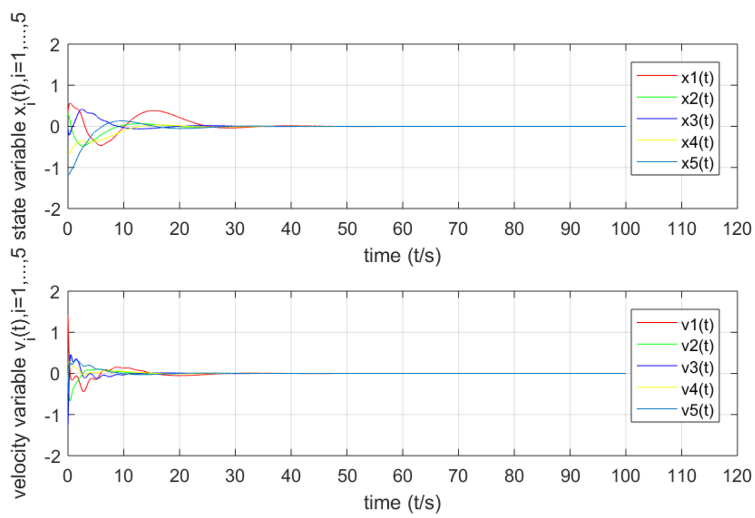


**Figure 3.** The state response of the closed-loop system in Example 5.1

$$D=\begin{bmatrix} 0.8442 & 0.7483 & -0.8234 & 1.2447 & 0.5095 & 2.4697 \\ -0.6909 & 1.0960 & -0.8759 & -0.2342 & 0.1740 & 0.8259 \\ 1.2131 & 0.3353 & 0.4042 & 0.7944 & 1.1806 & 0.6478 \\ -1.1429 & 1.2381 & 0.9816 & 0.8137 & 0.8111 & -0.8683 \\ -0.4282 & 0.6964 & 1.2663 & 0.0315 & 2.6088 & 0.0838 \\ -1.1338 & 0.7303 & 0.4912 & 0.0252 & 1.4695 & 0.1476 \end{bmatrix}, \; B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; C=\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{bmatrix}.$$

According to Algorithm 3.1, we first check the stability of this second-order system. We can calculate that $\beta = 51.9099$ and $\frac{1}{2\pi}\Delta_\Gamma \arg g(s) = 2$, which implies the original system is not stable and the number of unstable eigenvalues of its matrix polynomial is two. Next, we need to design an output feedback controller to stabilize this system. We first get a low dimensional approximation of it by using the second-order Krylov subspace method. Then, we solve the linear matrix equation (4.4) to obtain the output feedback controller:

$$u(t) = -F_1 y(t) - F_2 \dot{y}(t) = -\begin{bmatrix} 8.6242 & -1.3750 & -3.0151 & 16.5168 \\ -9.6237 & 4.2458 & -0.3274 & -13.6596 \end{bmatrix} y(t)$$
$$-\begin{bmatrix} 1.9187 & -2.2920 & -0.5187 & -1.1340 \\ -1.5246 & 3.5232 & 0.6591 & 0.9621 \end{bmatrix} \dot{y}(t),$$

and the state feedback controller is

$$u(t) = -K_1 x(t) - K_2 \dot{x}(t) = -\begin{bmatrix} 8.6242 & -1.3750 & -3.0151 & 16.5168 & 0 & 0 \\ -9.6237 & 4.2458 & -0.3274 & -13.6596 & 0 & 0 \end{bmatrix} x(t)$$
$$-\begin{bmatrix} 1.9187 & -2.2920 & -0.5187 & -1.1340 & 0 & 0 \\ -1.5246 & 3.5232 & 0.6591 & 0.9621 & 0 & 0 \end{bmatrix} \dot{x}(t).$$

Finally, based on Lemma 3.2, we get the radius $\beta$ of semicircular region $\Omega$ is

$$\beta = \|M^{-1}(D + BK_2)\|_2 + \|M^{-1}(K + BK_1)\|_2 = 314.4683,$$

From Algorithm 3.1, we obtain $g(s_j) \neq 0$ for all $j = 1, 2, ..., N_1$, and $\Delta_\Gamma \arg g(s) = 0$. Thus the closed-loop system (2.5) is stable.

Moreover, we give the initial values $x_0 = [1.11\ -0.51\ -1.82\ -1.14\ -0.33\ 0.24]$, $\dot{x}_0 = [1.92\ 0.23\ 0.10\ 0.79\ -1.54\ 0.15]$. By traditional Riccati equation method, Figure 4 shows the simulation result of the closed-loop system (2.5). Figure 5 shows the state response of the closed-loop system (2.5) with our method. Notice that the convergence time by Riccati equation method is longer than that of our method.

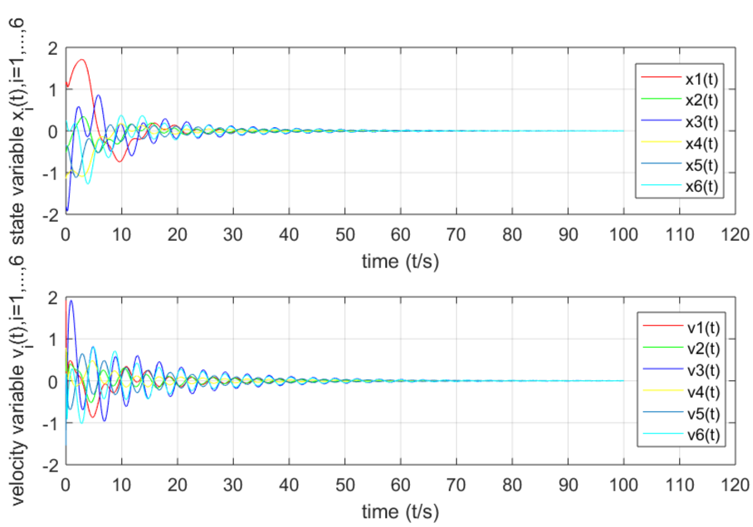


**Figure 4.** The state response of the closed-loop system by Riccati equation method in Example 5.2

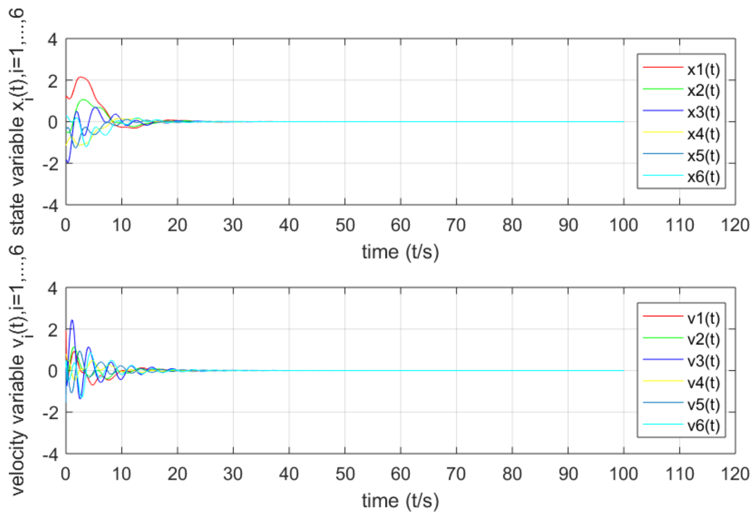


**Figure 5.** The state response of the closed-loop system in Example 5.2

**Example 5.3.** Consider a vibrating spring-mass system with damping in linear connection [5]. $M$ and $D$ are diagonal matrices, and $K$ is tridiagonal. We choose $100 \times 100$ matrices, and set $M = 0.1 \times \mathrm{I}$, $D = \mathrm{diag}(-0.25, 0.01, \ldots, 0.01)$,

$$K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 \; 0 \; 0 \; 0 \cdots 0 \\ 0 \; 1 \; 0 \; 0 \cdots 0 \\ 0 \; 0 \; 1 \; 0 \cdots 0 \end{bmatrix}.$$

It is obvious that matrix $M$ is nonsingular. By means of Algorithm 3.1, we have $\beta = 42.4902$ and $\frac{1}{2\pi}\Delta_\Gamma \arg g(s) = 2$. This means that the original system is not stable. In addition, we can evaluate that the unstable eigenvalues are $0.2395 \pm 4.3560i$. Next, we design an output feedback controller to stabilize this system. We first obtain a reduced system (4.3) to establish the input-output relationship of the system. Then, by solving the linear matrix equation (4.4), we design the output feedback controller as follows:

$$\begin{aligned} u(t) &= -\, F_1 y(t) - F_2 \dot{y}(t) \\ &= -\left[-0.0013 \; 0.0001 \; -0.0004\right] y(t) - \left[0.0954 \; -0.0024 \; -0.0918\right] \dot{y}(t), \end{aligned}$$

and the state feedback controller is

$$\begin{aligned} u(t) &= -\, K_1 x(t) - K_2 \dot{x}(t) \\ &= -\left[-0.0013 \; 0.0001 \; -0.0004 \; 0 \cdots \; 0\right] x(t) \\ &\quad - \left[0.0954 \; -0.0024 \; -0.0918 \; 0 \cdots \; 0\right] \dot{x}(t). \end{aligned}$$

Finally, according to Lemma 3.2, the radius $\beta$ of semicircular region $\Omega$ is

$$\beta = \|M^{-1}(D + BK_2)\|_2 + \|M^{-1}(K + BK_1)\|_2 = 41.7885.$$

We can calculate that $g(s_j) \neq 0$ for all $j = 1, 2, ..., N_1$, and $\Delta_\Gamma \arg g(s) = 0$. From Theorem 3.1, we have the closed-loop system (2.5) is stable.

Furthermore, we randomly generate a set of initial conditions $x_0$ and $\dot{x}_0$. The two figures in Figure 6 represent the norm of the state response of the original system and closed-loop system, respectively. Here, the norm of the state response denotes $\|x\|_2 = (\sum_{i=1}^{100} |x_i|^2)^{\frac{1}{2}}$. It is clear that the closed-loop system's state can stabilize to the origin, which implies our method is effective.

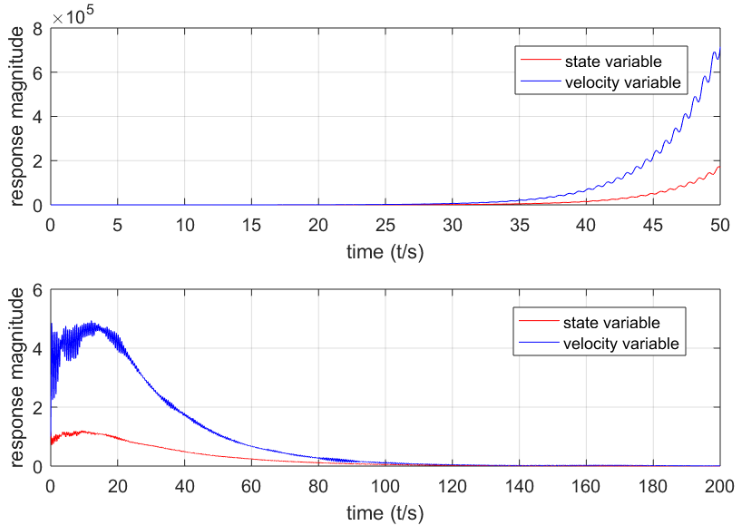


**Figure 6.** The state responses of the original system and closed-loop system in Example 5.3

# 6. Conclusions

In this paper, we present a static output feedback design algorithm for large-scale unstable second-order systems. This algorithm uses model reduction methods to construct a special reduced system, which directly establishes the relationship between input variables and output variables. For this low dimensional system, we can design the output feedback controller. Furthermore, three numerical examples are given to illustrate the effectiveness and feasibility of the proposed design algorithm.

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