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# MODELING THE SPREAD OF WEST NILE VIRUS IN A SPATIALLY HETEROGENEOUS AND ADVECTIVE ENVIRONMENT*

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**Abstract** In this paper, we put forward and explore a reaction-diffusion-advection system with free boundaries in spatially heterogeneous environment to model the spatial transmission of West Nile virus. The transmission dynamics are given for the model involving mosquitoes and birds, and the free boundaries are introduced to describe the moving fronts of the infected region. The spatial-temporal risk index $R_0^F(t)$ , which depends on time $t$, spatial heterogeneity and advection intensity, is derived by variational method. Sufficient conditions for the virus to extinct or to persist are given. Our results show that, if $R_0^F(\infty) \leq 1$, the virus extinct eventually, and if $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, the virus will spread continuously, while if $R_0^F(0) < 1 < R_0^F(\infty)$, the extinction or persistence of the virus depends on the initial scale of infected mosquitoes and birds, or the size of the infected region, the advection intensity and other factors. Finally, numerical simulations indicate that the advection intensity and the expanding capability affect the spreading fronts of the infected region.

**Keywords** West Nile virus, heterogeneity, advection, free boundary, the risk index.

**MSC(2010)** 35K51, 92D30.

## 1. Introduction

West Nile virus (WNv) is a mosquito-borne flavivirus and human, equine, and avian neuropathogen. The virus is initially detected in Uganda in 1937, then spread across Africa to the Middle East, West Asia, and eastern Europe. Birds are the natural reservoir (amplifying) hosts, and WNv is maintained in nature in a mosquito-bird-mosquito transmission cycle primarily involving *Culex spp* mosquitoes [9]. WNv was recently introduced to North America, where it was first detected in 1999 during an epidemic of meningoencephalitis in New York City. During 1999-2002, the virus extended its range throughout much of the eastern parts of the USA, and its range within the western hemisphere is expected to continue to expand. Since then the virus has kept spreading to its neighboring states. By 2002, WNv was reported

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in 40 states and the District of Columbia with 4,156 human and 14,539 equine cases of infection [12]. As we all know that the WNv still remains an important threat to public health nowadays, it is an important issue to improve our understanding of WNv transmission dynamics. In the absence of vaccines and specific treatments, to explore the mosquitoes'and birds' cross-infection mechanism is the effective and affordable measure to control the viruses' transmission broadly. The transmission of mosquito-borne diseases, such as West Nile virus, malaria, dengue fever, is not from human-to-human, but through infected mosquitoes. The West Nile virus are transmitted from human-to-human via an effective bite from an infected female adult mosquito. Also, the virus can be vertically transmitted from a mosquito to its offspring. While adult male mosquitoes feed on plant liquids such as nectar, honeydew, fruit juices and other sources of sugar for energy, female mosquitoes, in addition to feeding on sugar sources (for energy), feed on the blood meals of human and other mammals solely to acquire the proteins needed for eggs development. Therefore, it is advantageous to predict human WNv risks for cost-effective controls of the disease and optimal allocations of limited resources [4]. Although mathematical modeling of the spread of epidemics poses intriguing challenges, it can provide useful insights and possibly predictive capabilities, which can help public health officials to provide a theoretical basis for decision-making. Mathematical models, which reflect the spatial-independent dynamics, were developed by Kenkre et al. [20], Bowman et al. [7] , Abdelrazec et al. [1], and references therein. Those models considered different aspects of WNv, such as the periodicity of the infection by considering vertical transmission, the full life cycle of the mosquito, and they determined threshold conditions regarding control strategies for prevention and control of the virus. Lewis et al. firstly formulated spatially homogeneous model [22]. They studied WNv propagation using traveling wave solutions as a simplified model, in which they did not consider the vertical transmission, WNv death rate or a recovering avian subpopulation. The effects of vertical transmission and advection movement in the spatial propagation of the WNv for different bird species were investigated by Maidana and Yang [31],

$$\begin{cases} \dfrac{\partial S_a}{\partial t} - \dfrac{\partial^2 S_a}{\partial x^2} + \nu_1 \dfrac{\partial S_a}{\partial x} = \mu_a - \dfrac{\beta_a}{N_a} I_v S_a - \mu_a S_a & t > 0, x \in \Omega, \\ \dfrac{\partial I_a}{\partial t} - \dfrac{\partial^2 I_a}{\partial x^2} + \nu_1 \dfrac{\partial I_a}{\partial x} = \dfrac{\beta_a}{N_a} I_v S_a - (\gamma_a + \mu_a + \alpha_a) I_a & t > 0, x \in \Omega, \\ \dfrac{\partial I_v}{\partial t} - D\dfrac{\partial^2 I_v}{\partial x^2} + \nu_2 \dfrac{\partial I_v}{\partial x} = \dfrac{\beta_v}{N_a} I_a (N_v^* - I_v) - (1-p)\mu_v I_v & t > 0, x \in \Omega, \\ \dfrac{\partial N_a}{\partial t} - \dfrac{\partial^2 N_a}{\partial x^2} + \nu_1 \dfrac{\partial N_a}{\partial x} = \mu_a - \mu_a N_a - \alpha_a I_a & t > 0, x \in \Omega, \end{cases} \tag{1.1}$$

where the avian population was divided into susceptible, infective and recovered subpopulations, $S_a, I_a$ and $R_a$, respectively, while the vector population was divided into susceptible and infected subpopulations $S_v, I_v$. The total populations are $N_a = S_a + I_a + R_a$ and $N_v^* = S_v + I_v$, where the mosquito population $N_v^*$ was regarded as constant. The biting rate $b$ of mosquitoes is defined as the average number of bites per mosquito per day. $\beta_a$ and $\beta_v$ were the transmission probabilities from vector to bird and from bird to vector, respectively. The fraction of progeny of mosquitoes that are infectious is denoted by $p$, with $0 < p < 1$. The advection coefficients were denoted by $\nu_a$ and $\nu_v$ for avian and mosquito populations, respectively, with

$\nu_v \ll \nu_a$. The constant $\mu_v$ accounted for the birth rate of the mosquito. The constants $\mu_a, \alpha_a$, and $\gamma_a$ were defined as the birth rate, the death date induced by disease and the recovery rate of the avian population, respectively. They explored that the spreading wave speed was a function of the model's parameters, which can be used to assess the control strategies. The propagation of West Nile Virus from New York City to California state is viewed as a consequence of the diffusion and advection movements of birds, see Fig. 1. The results revealed the mosquito movements do not play a key role in the disease dissemination, while the bird advection became an important factor for lowering mosquito biting rates.

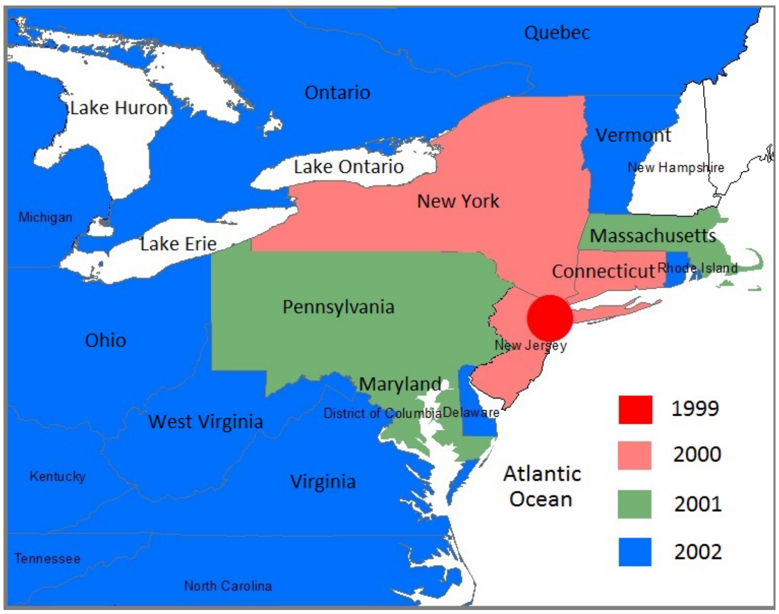


**Figure 1.** The diffusion and advection movements of West Nile virus from New York city to its neighboring states from 1999 to 2002.

To explore the dynamics of disease transmission in a spatially heterogeneous environment, Allen et al. put forward and explored an SIS type epidemic reaction-diffusion model in [3]. The contact transmission rate and the recovery rate are spatially dependent. Their results show that spatial heterogeneity has great influence on the persistence and extinction of the disease. Since then, there are a series of work to describe the disease transmission mechanism in spatial heterogeneous environment. For instance, Lin and Wang [28] proposed and analyzed a nonlocal and time-delayed model in spatially heterogeneous environment to describe the WNv transmission between mosquitoes and birds. We refer interested readers to [13, 16, 17, 21, 23, 25, 26, 32] and the references therein for related research work on disease transmission in spatially heterogeneous environment.

Free boundary problem has been comprehensively applied in many application fields, such as in chemical, biological and ecological problem as well as financial mathematics since the 1950s. In 2010, Du and Lin [14] firstly formulated and explored a diffusive logistic type model in homogeneous environment. The free boundary $x = h(t)$ was introduced to describe the moving front of an invasive species. The spreading-vanishing dichotomy, sharp criteria for spreading and vanishing, and the asymptotic spreading speed of the free boundary problem have been established, where the asymptotic spreading speed is smaller than the minimal speed of the traveling waves of the corresponding Cauchy problem in fixed region. Since then, there has been much more increasing interest in understanding the importance that the free boundary plays in the mathematical ecology [14, 16, 17, 19, 29, 35] and the references therein.

To investigate the spatial transmission mechanism of WNv, Lin and Zhu [29] proposed an reaction-diffusion simplified model with free boundaries in homogeneous environment

$$\begin{cases} \dfrac{\partial V_i}{\partial t} - D_v \dfrac{\partial^2 V_i}{\partial x^2} = \dfrac{\beta_v (N_v^* - V_i) H_i}{N_h^*} - r_v (1-q) V_i, & t > 0, g(t) < x < h(t), \\ \dfrac{\partial H_i}{\partial t} - D_h \dfrac{\partial^2 H_i}{\partial x^2} = \dfrac{\beta_h V_i (N_h^* - H_i)}{N_h^*} - (d_h + \gamma_h) H_i, & t > 0,\ g(t) < x < h(t), \\ V_i(x,t) = H_i(x,t) = 0, & t > 0,\ x = g(t) \,\text{or}\, x = h(t), \\ g(0) = -h_0,\ g'(t) = -\mu D_h \dfrac{\partial H_i}{\partial x}(g(t),t), & t > 0, \\ h(0) = h_0,\ h'(t) = -\mu D_h \dfrac{\partial H_i}{\partial x}(h(t),t), & t > 0, \\ V_i(x,0) = V_{i0}(x),\ H_i(x,0) = H_{i0}(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.2}$$

where the infected area of the disease $[g(t), h(t)] \subset \mathbb{R}$ is a moving interval with its two end points representing the spreading fronts of the disease and to be determined, which models the spatial expanding of the infection (infected area); $V_i(t)$ accounts for infected mosquitoes, $H_i(t)$ represents infected birds, $r_v$ is the recruitment rate of the mosquitoes; $\beta_v$ is the contact transmission rate of hosts to vectors; $d_h$ is the natural death rate of birds; $\beta_h$ is the contact transmission rate of the virus from mosquitoes to birds and $\gamma_h$ is the recovery rate of birds recovering from the infection. $D_v$, $D_h$ represent the diffusion rates for the vector mosquitoes and host birds, respectively, and they assume that $0 < D_v \ll D_h$. The parameter $0 < q \ll 1$ measures the vertical transmission rate of the virus in culex mosquitoes. $h_0$, and $\mu$ are positive constants, and the initial functions $V_{i0}$ and $H_{i0}$ are nonnegative and satisfy

$$\begin{cases} V_{i0} \in C^2([-h_0, h_0]),\ V_{i0}(\pm h_0) = 0 \,\text{and}\, 0 \le V_{i0}(x) \le N_v^*,\ x \in (-h_0, h_0), \\ H_{i0} \in C^2([-h_0, h_0]),\ H_{i0}(\pm h_0) = 0 \,\text{and}\, 0 < H_{i0}(x) \le N_h^*,\ x \in (-h_0, h_0). \end{cases} \tag{1.3}$$

They introduced the spatial-temporal risk index $R_0^F(t)$ for the simplified model with the free boundary, which was associated with the initial scales of infected vector mosquitoes and host birds, the area of the initial infected region, the diffusion rates and other factors. It was combined to develop sufficient conditions for the virus to extinct or to become spatially endemic.

Several environmental factors including landuse, climate, and host community composition can also influence the abundance of WNv hosts and vectors, and subsequently, affect WNv transmission rates. Spatial heterogeneity exists in the risk of human exposure to infectious disease vectors. For example, occurrences of infectious diseases often are spatially correlated; human disease incidence at a location is positively related to incidences at neighboring locations [6, 8, 11].

In reality, some species or diseases prefer to move towards one direction because of appropriate climate, wind direction, etc. For example, in studying the propagation of West Nile Virus in North America, it was observed in [31] that West Nile Virus appeared for the first time in New York City in the summer of 1999. In the second year the wave front travels 187 km to the north and 1100 km to the south, till 2002, it has been spread across almost the whole America continent. Therefore, the propagation of WNv from New York City to California state is a consequence of the diffusion and advection movements of birds. Especially, bird advection becomes an important factor to lower mosquito biting rates.

There have been many research to explore the propagation of WNv in the last two decades, however, advection terms have not obtained much more attention in investigating the extinction and persistence of WNv. Therefore, it is much worthwhile to take into consideration the influence of advection movement on the transmission of West Nile virus. And due to the effect of advection rate on the mosquitoes is small enough, so we will only consider the impact of advection movement on the birds.

Sometime the mosquitoes are viewed as a sessile population, then $D_v \ll D_h$. For instance, the mean dispersal distance for *Aedes* aegypti was ranged from 28 to 199 m, according to Harrington et al. [18]. To better explore the heterogeneity and advection on the movement on birds, in this paper we will ignore the diffusion and advection on mosquitoes, that means we can consider the limiting case $D_v \equiv 0$. This research is devoted to more explicitly describe the transmission mechanism of WNv in the spatially heterogeneous and advection environment on the basis of [29] and the above assumptions. Therefore, we will investigate the spatial degenerate partial systems for $V_i(t,x)$ and $H_i(t,x)$ with free boundaries $x = g(t)$ and $x = h(t)$ as follows

$$\begin{cases} \dfrac{\partial V_i}{\partial t} = \dfrac{\beta_v(x)(N_v^* - V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i, & t>0,\ g(t)<x<h(t), \\ \quad \dfrac{\partial H_i}{\partial t} - D_h \dfrac{\partial^2 H_i}{\partial x^2} + \beta \dfrac{\partial H_i}{\partial x} & \\ = \dfrac{\beta_h(x)V_i(N_h^* - H_i)}{N_h^*} - (d_h(x)+\gamma_h(x))H_i, & t>0,\ g(t)<x<h(t), \\ V_i(t,x) = H_i(t,x) = 0, & t>0,\ x=g(t) \text{ or } x=h(t), \\ g(0) = -h_0,\ g'(t) = -\mu D_h \dfrac{\partial H_i}{\partial x}(t,g(t)), & t>0, \\ h(0) = h_0,\ h'(t) = -\mu D_h \dfrac{\partial H_i}{\partial x}(t,h(t)), & t>0, \\ V_i(x,0) = V_{i0}(x),\ H_i(x,0) = H_{i0}(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.4}$$

where $\beta_v(x)$, $r_v(x)$, $\gamma_h(x)$, $d_h(x)$,and $\beta_h(x)$ are positive Hölder continuous functions, which represent the contact transmission rate of birds to mosquitoes, the natural death rate of infected female mosquitoes, the recovery rate of infected birds recovering from the infection, the natural death rate of infected birds, the contact transmission rate of the virus from mosquitoes to birds, at location $x$, respectively. The variables $N_h^*, N_v^*, V_i, H_i$ are the same as the statements in reference [29]. $N_h^*$ and $N_v^*$ mean the the mosquitoes and the birds remain constant in space for all time. $V_i$ and $H_i$ describe the infected mosquitoes and infected birds, respectively. The constant $\beta$ describe the infected birds move to the gradient of their density at a constant rate. And the functions $V_{i0}$ and $H_{i0}$ are nonnegative and satisfy the initial conditions (1.3).

The free boundary conditions $g'(t) = -\mu D_h \frac{\partial H_i}{\partial x}(t, g(t))$ and $h'(t) = -\mu D_h \frac{\partial H_i}{\partial x}(t, h(t))$ mean that the expanding rate of the infected interval $[g(t), h(t)]$ is proportional to the outward flux of the population across the boundary of the range (see [14] for further explanations and justification). Epidemiologically, it means that beyond the free boundaries $x = g(t)$ and $x = h(t)$, there are only susceptible host birds, and no birds carrying the virus.

Throughout this paper, we will assume

$$
\begin{aligned}
(H) \quad & \lim_{x\to\pm\infty} \beta_v(x) = \beta_v^\infty, \quad \lim_{x\to\pm\infty} \beta_h(x) = \beta_h^\infty, \ \lim_{x\to\pm\infty} r_v(x) = r_v^\infty, \\
& \lim_{x\to\pm\infty} d_h(x) = d_h^\infty, \ \lim_{x\to\pm\infty} \gamma_h(x) = \gamma_h^\infty \text{ and },\\
& \frac{\beta_v^\infty \beta_h^\infty N_v^*}{N_h^* r_v^\infty (1-q)} - d_h^\infty - \gamma_h^\infty > 0,
\end{aligned}
$$

which means that far sites of the habitat are similar and high-risk. Furthermore we will assume $|\beta| < 2\sqrt{D_h(\frac{\beta_v^\infty \beta_h^\infty N_v^*}{N_h^* r_v^\infty (1-q)} - d_h^\infty - \gamma_h^\infty)}$ which means the advection intensity is small.

## 2. Preliminaries

In this section, we firstly present the following local existence and uniqueness results of the degenerate partial system (1.4) with the initial conditions (1.3) by applying the contraction mapping theorem, standard $L^p$ estimate and Sobolev embedding theorem, and then we show global existence with the help of some suitable estimates.

**Theorem 2.1.** *For any given* $(V_{i0}, H_{i0})$ *satisfying* (1.3)*, and any* $\alpha \in (0,1)$*, there exists the constant* $T > 0$ *such that problem* (1.4) *admits a unique solution*

$$
(V_i, H_i; g, h) \in [C^{\frac{1+\alpha}{2}, 1+\alpha}(D_T)]^2 \times [C^{1+\frac{\alpha}{2}}([0,T])]^2;
$$

*and the solution satisfies*

$$
\|H_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + \|V_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}(D_T)} + ||g||_{C^{1+\frac{\alpha}{2}}([0,T])} + ||h||_{C^{1+\frac{\alpha}{2}}([0,T])} \le C, \quad (2.1)
$$

*where* $D_T = \{(t,x) \in \mathbb{R}^2 : t \in [0,T], x \in [g(t), h(t)]\}$*,* $C$ *and* $T$ *depend only on* $\|V_{i0}\|_{C^2([-h_0,h_0])}$*,* $h_0, \alpha$ *and* $\|H_{i0}\|_{C^2([-h_0,h_0])}$*.*

**Proof.** For any given constant $T > 0$, we define

$$
\begin{aligned}
\mathcal{G}_T &= \{g \in C^1([0,T]) : g(0) = -h_0, \ g'(t) \le 0, \ 0 \le t \le T\},\\
\mathcal{H}_T &= \{h \in C^1([0,T]) : h(0) = h_0, \ h'(t) \ge 0, \ 0 \le t \le T\}.
\end{aligned}
$$

Note that the first equation in (1.4) for $V_i$ has no diffusion and advection terms , we can use $g, h$ and $H_i$ to represent $V_i$.

If $g(t) \in \mathcal{G}_T$, $h(t) \in \mathcal{H}_T$ and $H_i(t,x) \in C(D_T)$, then for $(t,x) \in D_T$, the unknown $V_i$ can be rewritten as

$$
V_i(t,x) := H(t, H_i(t,x)) = e^{-\frac{\beta_v(x)}{N_h^*}\int_0^t H_i(s,x)ds - (1-q) r_v(x) t} V_i(0,x)
$$

$$+\frac{N_v^*}{N_h^*}\int_0^t \beta_v(x)H_i(\tau,x)e^{\frac{\beta_v(x)}{N_h^*}\int_t^\tau H_i(s,x)ds+(1-q)r_v(\tau-t)}d\tau.$$

Let

$$\begin{aligned}
&y=\frac{2x}{h(t)-g(t)}-\frac{(h(t)+g(t))}{h(t)-g(t)},\\
&\beta_h^1(t,y)=\beta_h(x),\quad d_h^1(t,y)=d_h(x),\quad \gamma_h^1(t,y)=\gamma_h(x),
\end{aligned}$$

and

$$\begin{aligned}
w(t,y)&=H_i(t,\frac{(h(t)-g(t))y+(h(t)+g(t))}{2}),\\
z(t,y)&=V_i(t,\frac{(h(t)-g(t))y+(h(t)+g(t))}{2}),
\end{aligned}$$

then problem (1.4) can be transformed into

$$\begin{cases}
w_t-D_hA^2w_{yy}+(\beta A-B)w_y & \\
=\beta_h^1(t,y)H(t,w)(1-\frac{w}{N_h^*})-d_h^1(t,y)w-\gamma_h^1(t,y)w, & 0<t<T,\ -1<y<1,\\
w(t,1)=w(t,-1)=0, & 0\le t<T,\\
w(0,y)=H_{i0}(h_0y)=w_0(y), & -1<y<1,
\end{cases}\tag{2.2}$$

and

$$\begin{cases}
g'(t)=-\mu Aw_y(t,-1),\ 0<t<T,\\
h'(t)=-\mu Aw_y(t,1),\quad 0<t<T,\\
g(0)=-h_0,\ h(0)=0,
\end{cases}\tag{2.3}$$

where$A=A(h(t),g(t))=\frac{2}{h(t)-g(t)}$ and $B=B(h(t),g(t),y)=y\frac{h'(t)-g'(t)}{h(t)-g(t)}+\frac{h'(t)+g'(t)}{h(t)-g(t)}$. This transformation changes the free boundary problem (1.4) to the initial boundary problem (2.2)and (2.3) in $[0,T]\times(-1,1)$ with more complex equations.

Similarly as those in [2, 14], the rest of the proof follows from the contraction mapping theorem together with the standard $L^p$ theory and the Sobolev imbedding theorem, we omit it here. □

To prove that the local solution stated in Theorem 2.1 can be extended to all $t>0$, we need the following estimates.

**Theorem 2.2.** *Let $(V_i,H_i;g,h)$ be a solution to problem (1.4) defined for $t\in(0,T_0]$ for some $T_0\in(0,+\infty)$. Then the following conclusion hold.*

*$(i)$ $0<V_i(t,x)\le N_v^*$ and $0<H_i(t,x)\le N_h^*$ for $t\in(0,T_0]$, $g(t)<x<h(t)$;*

*$(ii)$ There exists a constant $C_1$ independent of $T_0$ such that*

$$0<-g'(t),\ h'(t)\le C_1\quad for\quad t\in(0,T_0];$$

**Proof.** $(i)$ is directly from the comparison principle, see Lemma 2.2 in [2]. The proof of $(ii)$ is similar to that of Lemma 2.2 in [14], where

$$C_1:=\max\{\frac{1}{2h_0},\frac{\beta}{D_h}+\sqrt{\frac{N_h^*}{2D_h}},\frac{4||H_{i0}||_{C^1([-h_0,h_0])}}{3N_h^*}\}.$$

□

**Remark 2.1.** From (ii) in Theorem 2.2, we obtain an estimate about the upper bound and lower bound of the asymptotic spreading speeds for the leftward front and the rightward front of the infected region.

Combining Theorems 2.1 and 2.2 as in [14], we obtain the following global existence result.

**Theorem 2.3.** *The solution of* (1.4) *exists and is unique for all* $t \in (0, \infty)$.

**Proof.** In fact, since $V_i, H_i$ and $g'(t), h'(t)$ are bounded in $(g(t), h(t)) \times (0, T_0]$ by constants independent of $T_0$, the maximal existing time of the solution of (1.4) can be extended to infinity. □

In what follows, we exhibit a comparison principle for the free boundary problem (1.4), which can be proved similarly as Lemma 3.5 in [14] and can be used to estimate $V_i, H_i$ and the free boundaries $x = g(t), x = h(t)$.

**Lemma 2.1** (The Comparison Principle)**.** *Assume that* $\overline{g}, \overline{h} \in C^1([0,+\infty))$, $\overline{V}_i(t,x)$, $\overline{H}_i(t,x) \in C([0,+\infty) \times [\overline{g}(t), \overline{h}(t)]) \cap C^{1,2}((0,+\infty) \times (\overline{g}(t), \overline{h}(t)))$, *and*

$$
\begin{cases}
\dfrac{\partial \overline{V}_i}{\partial t} \geq \dfrac{\beta_v(x)(N_v^* - \overline{V}_i)\overline{H}_i}{N_h^*} - (1-q)r_v(x)\overline{V}_i, & t > 0,\ \overline{g}(t) < x < \overline{h}(t),\\
\dfrac{\partial \overline{H}_i}{\partial t} - D_h \dfrac{\partial^2 \overline{H}_i}{\partial x^2} + \beta \dfrac{\partial \overline{H}_i}{\partial x} \geq \dfrac{\beta_h \overline{V}_i (N_h^* - \overline{H}_i)}{N_h^*} - (d_h + \gamma_h)\overline{H}_i, & t > 0,\ \overline{g}(t) < x < \overline{h}(t),\\
\overline{H}_i(t,x) = \overline{V}_i(t,x) = 0, & t > 0,\ x = \overline{g}(t) \text{ or } x = \overline{h}(t),\\
\overline{g}(0) \leq -h_0,\ \overline{g}'(t) \leq -\mu \dfrac{\partial \overline{H}_i}{\partial x}(t, \overline{g}(t)), & t > 0,\\
\overline{h}(0) \geq h_0,\ \overline{h}'(t) \geq -\mu \dfrac{\partial \overline{H}_i}{\partial x}(t, \overline{h}(t)), & t > 0,\\
\overline{V}_i(0,x) \geq V_{i0}(x),\ \overline{H}_i(0,x) \geq H_{i0}(x), & -h_0 \leq x \leq h_0.
\end{cases}
$$

*Then the solution* $(V_i, H_i; g, h)$ *to the free boundary problem* (1.4) *satisfies*

$$
\begin{aligned}
&h(t) \leq \overline{h}(t),\ g(t) \geq \overline{g}(t), \quad t \in [0, +\infty),\\
&V_i(t,x) \leq \overline{V}_i(t,x),\ H_i(t,x) \leq \overline{H}_i(t,x),\ t \geq 0,\ x \in [g(t), h(t)].
\end{aligned}
$$

**Remark 2.2.** The pair $(\overline{V}_i, \overline{H}_i; \overline{g}, \overline{h})$ in Lemma 2.1 is usually called an upper solution (a supersolution) to problem (1.4). Correspondingly, the lower solution $(\underline{V}_i, \underline{H}_i; \underline{g}, \underline{h})$ (or a subsolution) can be defined analogously by reversing all the inequalities in the obvious places, and the similar comparison principle holds.

To emphasize the dependence of the solution on the expanding capability $\mu$, we rewrite the solution as $(V_i^\mu, H_i^\mu; g^\mu, h^\mu)$. As a corollary of Lemma 2.1, we have the following monotonicities:

**Corollary 2.1.** *For fixed* $V_{i0}, H_{i0}, \beta, h_0, r_v(x), \gamma_h(x), d_h(x), \beta_v(x)$ *and* $\beta_h(x)$. *If* $\mu_1 \leq \mu_2$, *then* $V_i^{\mu_1}(t,x) \leq V_i^{\mu_2}(t,x)$, $H_i^{\mu_1}(t,x) \leq H_i^{\mu_2}(t,x)$ *in* $(0,\infty) \times [g^{\mu_1}(t), h^{\mu_1}(t)]$ *and* $g^{\mu_2}(t) \leq g^{\mu_1}(t)$, $h^{\mu_1}(t) \leq h^{\mu_2}(t)$ *in* $(0, \infty)$.

Corollary (2.1) shows that the left free boundary for problem (1.4) is strictly monotone decreasing and the right one is increasing. From the epidemiological point of view, it means that the infection area which contains infected birds is always gradually expanding.

## 3. The risk index

In this section, we will give the risk index for the free boundary problem (1.4), which is similar to the basic reproduction number in a fixed domain. And we will derive its analytical properties.

First, we define the basic reproduction number and present its properties and implications for the following reaction-diffusion-advection model with Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial V_i}{\partial t} = \dfrac{\beta_v(x)(N_v^* - V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i, & t>0,\ x\in(p,q), \\ \dfrac{\partial H_i}{\partial t} - D_h\dfrac{\partial^2 H_i}{\partial x^2} + \beta\dfrac{\partial H_i}{\partial x} = \dfrac{\beta_h V_i(N_h^* - H_i)}{N_h^*} - (d_h+\gamma_h)H_i, & t>0,\ x\in(p,q), \\ V_i(t,x) = H_i(t,x) = 0, & t>0,\ x=p \,\text{or}\, x=q. \end{cases} \tag{3.1}$$

We linearize (3.1) around (0,0) to obtain the following linear system

$$\begin{cases} \xi_t = \beta_v(x)\dfrac{N_v^*}{N_h^*}\eta - (1-q)r_v(x)\xi, & t>0,\ x\in(p,q), \\ \eta_t - D_h\eta_{xx} + \beta\eta_x = \beta_h(x)\xi - (d_h(x)+\gamma_h(x))\eta, & t>0,\ x\in(p,q), \\ \xi(t,x) = \eta(t,x) = 0, & t>0,\ x=p \,\text{or}\, x=q, \end{cases} \tag{3.2}$$

and consider the corresponding eigenvalue problem

$$\begin{cases} 0 = \dfrac{\beta_v(x)N_v^*}{\lambda N_h^*}\phi - (1-q)r_v(x)\psi, & x\in(p,q), \\ -D_h\phi_{xx} + \beta\phi_x = \dfrac{\beta_h(x)}{\lambda}\psi - (d_h(x)+\gamma_h(x))\phi, & x\in(p,q), \\ \psi(x) = \phi(x) = 0, & x=p \,\text{or}\, x=q. \end{cases} \tag{3.3}$$

By the variational method as stated in [3] in a fixed region, we can deduce the eigenvalue $\lambda$ as follows

$$\lambda = \sup_{\psi\in H_0^1(p,q),\psi\neq 0}\left\{\sqrt{\frac{\int_p^q \frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*(1-q)r_v(x)}\psi^2 dx}{\int_p^q (D_h\psi_x^2 + \frac{\beta^2}{4D_h}\psi^2 + (d_h(x)+\gamma_h(x))\psi^2)dx}}\right\}.$$

Define the basic reproduction number $R_0^{DA} = R_0^{DA}((p,q),D_h,\beta)$ of system (3.1) as follows $R_0^{DA} = R_0^{DA}((p,q),D_h,\beta) := \lambda$, where $\lambda$ is the eigenvalue in system (3.3).

The following result follows from variational methods, see Chapter 2 in [10] for datails.

**Lemma 3.1.** $1 - R_0^{DA}$ *has the same sign as* $\lambda_0$, *where* $\lambda_0$ *is the principal eigenvalue* *of the reaction-diffusion-advection problem*

$$\begin{cases} 0 = \dfrac{N_v^*}{N_h^*}\beta_v(x)\phi - (1-q)r_v(x)\psi + \lambda_0\psi, & x \in (p,q), \\ -D_h\phi_{xx} + \beta\phi_x = \beta_h(x)\psi - (d_h(x)+\gamma_h(x))\phi + \lambda_0\phi, & x \in (p,q), \\ \psi(x) = \phi(x) = 0, & x = p \, or \, x = q. \end{cases} \tag{3.4}$$

and the corresponding eigenfunction pair satisfy

$$(\phi(x),\psi(x)) > 0, x \in (p,q), \ \ \phi'(p) > 0, \psi'(p) > 0, \phi'(q) < 0, \psi'(q) < 0.$$

**Proof.** Substituting the first equation in (3.4) to the second equation yields

$$-D_h\phi_{xx} + \beta\phi_x = \frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*((1-q)r_v(x)-\lambda_0)}\phi - (d_h(x)+\gamma_h(x))\phi + \lambda_0\phi. \tag{3.5}$$

Let $(R_0^{DA}, \psi^*, \phi^*)$ be the eigen-pair of problem (3.3), that is

$$\begin{cases} 0 = \dfrac{N_v^*\beta_v(x)}{N_h^* R_0^{DA}}\phi^* - (1-q)r_v(x)\psi^*, & x \in (p,q), \\ -D_h\phi^*_{xx} + \beta\phi^*_x = \dfrac{\beta_h(x)}{R_0^{DA}}\psi^* - (d_h(x)+\gamma_h(x))\phi^*, & x \in (p,q), \\ \psi^*(x) = \phi^*(x) = 0, & x = p \, \text{or} \, x = q, \end{cases} \tag{3.6}$$

which reduces to

$$-D_h\phi^*_{xx} + \beta\phi^*_x = \frac{N_v^*\beta_h(x)\beta_v(x)}{(R_0^{DA})^2 N_h^*(1-q)r_v(x)}\phi^* - (d_h(x)+\gamma_h(x))\phi^*. \tag{3.7}$$

For convenience, taking $\Psi = e^{-\frac{\beta}{2D_h}x}\phi$ in (3.5)and $\Psi^* = e^{-\frac{\beta}{2D_h}x}\phi^*$ in (3.7) yields

$$-D_h\Psi_{xx} = \frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*((1-q)r_v(x)-\lambda_0)}\Psi - (d_h(x)+\gamma_h(x))\Psi - \frac{\beta^2}{4D_h}\Psi + \lambda_0\Psi, \tag{3.8}$$

and

$$-D_h\Psi^*_{xx} = \frac{N_v^*\beta_h(x)\beta_v(x)}{(R_0^{DA})^2 N_h^*(1-q)r_v(x)}\Psi^* - (d_h(x)+\gamma_h(x))\Psi^* - \frac{\beta^2}{4D_h}\Psi^*. \tag{3.9}$$

By the multiply-multiply-subtract-integrate technique, we obtain

$$\int_p^q \frac{N_v^*\beta_h(x)\beta_v(x)}{(R_0^{DA})^2 N_h^*(1-q)r_v(x)}\Psi\Psi^* dx = \int_p^q [\frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*((1-q)r_v(x)-\lambda_0)} + \lambda_0]\Psi\Psi^* dx,$$

which means that

$$sign\,(1 - R_0^{DA}) = sign\,\lambda_0.$$

$\square$

With the above definition of the basic reproduction number of $R_0^{DA}$, we have the following properties.

**Theorem 3.1.** *The following assertions hold.*

*(i)* $R_0^{DA}$ *is a positive and monotone decreasing function of the advection intensity* $\beta$*;*

*(ii)* $R_0^{DA} \to 0$ *as* $D_h \to \infty$*;*

*(iii) If* $\Omega_1 \subsetneqq \Omega_2 \subsetneqq R^1$*, then* $R_0^{DA}(\Omega_1) \leq R_0^{DA}(\Omega_2)$*, with strict inequality if and only if* $\Omega_2 \setminus \Omega_1$ *is an open set. Moreover,*

$$\lim_{(q-p)\to\infty} R_0^{DA}((p,q), D_h, \beta) \geq \sqrt{\frac{\frac{N_v^* \beta_v^\infty \beta_h^\infty}{N_h^*(1-q) r_v^\infty}}{\frac{\beta^2}{4D_h} + d_h^\infty + \gamma_h^\infty}}$$

*provided that* $(H)$ *holds;*

*(iv) In the special case,* $\beta_v(x) \equiv \beta_v^*, \beta_h(x) \equiv \beta_h^*,\ r_v(x) \equiv r_v^*,\ d_h(x) \equiv d_h^*,$ $\gamma_h(x) \equiv \gamma_h^*$*, then*

$$R_0^{DA} = \sqrt{\frac{\frac{N_v^* \beta_v^* \beta_h^*}{N_h^*(1-q) r_v^*}}{D_h(\frac{\pi}{q-p})^2 + \frac{\beta^2}{4D_h} + d_h^* + \gamma_h^*}}.$$

**Proof.** The proof of part (i) is from the definition of $R_0^{DA}$, and part (iv) can be obtained through direct calculations, where$(\frac{\pi}{q-p})^2$ is the principal eigenvalue of $-\Delta$ operator with the null Dirichlet boundary condition in the fixed interval $(p, q)$.

For part (ii), by the definition of $R_0^{DA}$ and Poincáre's inequality, we have

$$\begin{aligned}
R_0^{DA} = R_0^{DA}((p,q), D_h, \beta) &= \sup_{\psi \in H_0^1(p,q), \psi \neq 0} \left\{ \sqrt{\frac{\int_p^q \frac{N_v^* \beta_h \beta_v}{N_h^*(1-q) r_v} \psi^2 dx}{\int_p^q (D_h \psi_x^2 + \frac{\beta^2}{4D_h}\psi^2 + (d_h + \gamma_h)\psi^2) dx}} \right\} \\
&\leq \sup_{\psi \in H_0^1(p,q), \psi \neq 0} \left\{ \sqrt{\frac{\int_p^q \frac{N_v^* \beta_h^M \beta_v^M}{N_h^*(1-q) r_v^m} \psi^2 dx}{\int_p^q (D_h \psi_x^2 + \frac{\beta^2}{4D_h}\psi^2 + (d_h^m + \gamma_h^m)\psi^2) dx}} \right\} \\
&\leq \sup_{\psi \in H_0^1(p,q), \psi \neq 0} \left\{ \sqrt{\frac{\int_p^q \frac{N_v^* \beta_h^M \beta_v^M}{N_h^*(1-q) r_v^m} \psi^2 dx}{\int_p^q (D_h (\frac{\pi}{q-p})^2 + \frac{\beta^2}{4D_h}\psi^2 + (d_h^m + \gamma_h^m)\psi^2) dx}} \right\} \\
&= \sqrt{\frac{\frac{N_v^* \beta_h^M \beta_v^M}{N_h^*(1-q) r_v^m}}{D_h(\frac{\pi}{q-p})^2 + \frac{\beta^2}{4D_h} + (d_h^m + \gamma_h^m)}} \to 0 \text{ as } D_h \to \infty,
\end{aligned}$$

where $f^M = \sup_{x \in (p,q)} \{f(x)\}, f^m = \inf_{x \in (p,q)} \{f(x)\}$ for any bounded function $f$ in the interval $(p, q)$.

The proof of the monotonicity in (iii) is similar to that of Corollary 2.3 in [10]. For the inequality, it follows from the assumption (H) that for any $\varepsilon > 0$, there exists the constant $L_0 > 0$, when $|x| > L_0$, we can obtain

$$\begin{aligned}
&\beta_v^\infty - \varepsilon \leq \beta_v(x) \leq \beta_v^\infty + \varepsilon, \quad \beta_h^\infty - \varepsilon \leq \beta_h(x) \leq \beta_h^\infty + \varepsilon, \\
&r_v^\infty - \varepsilon \leq r_v(x) \leq r_v^\infty + \varepsilon, \quad \gamma_h^\infty - \varepsilon \leq \gamma_h(x) \leq \gamma_h^\infty + \varepsilon, \quad d_h^\infty - \varepsilon \leq d_h(x) \leq d_h^\infty + \varepsilon.
\end{aligned}$$

For the case $q \geq 2L_0$, simple computations yield

$$\lim_{(q-p)\to\infty} R_0^{DA}((p,q), D_h, \beta)$$

$$\geq \sup_{\psi\in H_0^1(L_0,2L_0),\psi\neq 0}\left\{\sqrt{\frac{\int_{L_0}^{2L_0}\frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*(1-q)r_v(x)}\psi^2dx}{\int_{L_0}^{2L_0}(D_h\psi_x^2+\frac{\beta^2}{4D_h}\psi^2+(d_h(x)+\gamma_h(x))\psi^2)dx}}\right\}$$
$$\geq \sup_{\psi\in H_0^1(L_0,2L_0),\psi\neq 0}\left\{\sqrt{\frac{\int_{L_0}^{2L_0}\frac{N_v^*(\beta_h^\infty-\varepsilon)(\beta_v^\infty-\varepsilon)}{N_h^*(1-q)(r_v^\infty+\varepsilon)}\psi^2dx}{\int_{L_0}^{2L_0}(D_h\psi_x^2+\frac{\beta^2}{4D_h}\psi^2+(d_h^\infty+\gamma_h^\infty+\varepsilon)\psi^2)dx}}\right\}$$
$$\geq \sqrt{\frac{\frac{N_v^*(\beta_h^\infty-\varepsilon)(\beta_v^\infty-\varepsilon)}{N_h^*(1-q)(r_v^\infty+\varepsilon)}}{(D_h(\frac{\pi}{L_0})^2+\frac{\beta^2}{4D_h}+(d_h^\infty+\gamma_h^\infty+\varepsilon)}}.$$

Due to the arbitraries of $\varepsilon$, letting $L_0\to\infty$ yields

$$\lim_{(q-p)\to\infty} R_0^{DA}((p,q),D_h,\beta)\geq \sqrt{\frac{\frac{N_v^*\beta_h^\infty\beta_v^\infty}{N_h^*(1-q)r_v^\infty}}{\frac{\beta^2}{4D_h}+d_h^\infty+\gamma_h^\infty}}.$$

Similarly, for $p\leq -2L_0$, we obtain the same result by replacing the interval $(L_0,2L_0)$ with the interval $(-2L_0,-L_0)$. □

Noticing that the habitat $(g(t),h(t))$ is varying with $t$, so the basic reproduction number for the free boundary problem (1.4) will not be a constant and should be changing with $t$. As a result, we introduce the risk index $R_0^F(t)$ by

$$R_0^F(t):=R_0^{DA}((g(t),h(t))),D_h,\beta)$$
$$=\sup_{\psi\in H_0^1(g(t),h(t)),\psi\neq 0}\left\{\sqrt{\frac{\int_{g(t)}^{h(t)}\frac{N_v^*\beta_h(x)\beta_v(x)}{N_h^*(1-q)r_v(x)}\psi^2dx}{\int_{g(t)}^{h(t)}(D_h\psi_x^2+\frac{\beta^2}{4D_h}\psi^2+(d_h(x)+\gamma_h(x))\psi^2)dx}}\right\}.$$

Lemma 3.1 together with the above risk index definition shows that

**Lemma 3.2.** $1-R_0^F(t)$ *has the same sign as* $\lambda_0$*, where* $\lambda_0$ *is the principal eigenvalue of the problem*

$$\begin{cases} 0=\dfrac{N_v^*}{N_h^*}\beta_v(x)\phi-(1-q)r_v(x)\psi+\lambda\psi, & x\in(g(t),h(t)),\\ -D_h\phi_{xx}+\beta\phi_x=\beta_h(x)\psi-(d_h(x)+\gamma_h(x))\phi+\lambda\phi, & x\in(g(t),h(t)),\\ \psi(x)=\phi(x)=0, & x=g(t)\ or\, x=h(t). \end{cases}\tag{3.10}$$

It follows from Theorems 2.2 and 3.1

**Theorem 3.2.** *(i)* $R_0^F(t)$ *is a positive and monotonically decreasing function of* $\beta$*, and* $R_0^F(t)\to 0$ *as* $\beta\to\infty$.
*(ii)*$R_0^F(t)$ *is a strictly monotone increasing function of* $t$*, which means that if* $t_1<t_2$*, then* $R_0^F(t_1)<R_0^F(t_2)$*. Moreover,* $\lim_{t\to\infty} R_0^F(t)\geq\sqrt{\frac{\frac{N_v^*\beta_h^\infty\beta_v^\infty}{N_h^*(1-q)r_v^\infty}}{\frac{\beta^2}{4D_h}+d_h^\infty+\gamma_h^\infty}}$ *if the hypothesis* $(H)$ *holds and* $h(t)-g(t)\to\infty$ *as* $t\to\infty$.

**Remark 3.1.** When $\beta=0$, we suppose that the habitat at far distance is in high risk, that is, $R_0^F(\infty)>1$, which is equivalent to $\frac{\beta_v^\infty\beta_h^\infty N_v^*}{N_h^*r_v^\infty(1-q)}-d_h^\infty-\gamma_h^\infty>0$.

**Remark 3.2.** There exists a threshold $\beta^* = 2\sqrt{D_h(\frac{\beta_v^\infty \beta_h^\infty N_v^*}{N_h^* r_v^\infty (1-q)} - d_h^\infty - \gamma_h^\infty)}$. Recalling the monotonicity of $R_0^F(t)$ with respect to $\beta$, we derive that when $|\beta| < \beta^*$, there exists $t_0$ such that

$$R_0^F(t_0) = R_0^{DA}((g(t_0), h(t_0)), D_h, \beta) > 1$$

and

$$R_0^F(\infty) = R_0^{DA}((g_\infty, h_\infty), D_h, \beta) > 1$$

if $h_\infty - g_\infty = \infty$. When $|\beta| > \beta^*$, we derive $R_0^F(t) = R_0^{DA}((g(t), h(t)), D_h, \beta) < 1$ for any $t > 0$.

Combining the above arguments about high-risk habitat at far distance and small advection intensity, we give the assumption $(H)$.

## 4. Virus extinction

It follows from Theorem 2.2 that $x = g(t)$ is monotonically decreasing and $x = h(t)$ is monotonically increasing, therefore there exist $h_\infty, -g_\infty \in (0, +\infty]$ such that $\lim\limits_{t\to+\infty} g(t) = g_\infty$ and $\lim\limits_{t\to+\infty} h(t) = h_\infty$. The following lemma shows that both $g_\infty$ and $h_\infty$ are finite or infinite simultaneously, that is, if $h_\infty < \infty$, then $-g_\infty < \infty$, vice versa.

First, we will give the definitions of extinction and persistence of the virus.

**Definition 4.1.** We called the virus extinction eventually if

$$h_\infty - g_\infty < \infty \ \text{ and } \lim_{t\to+\infty} (||H_i(t,\cdot)||_{C([g(t),h(t)])} + ||V_i(t,\cdot)||_{C([g(t),h(t)])}) = 0,$$

and the virus persistence continuously if

$$h_\infty - g_\infty = \infty \,\text{and}\, \limsup_{t\to+\infty} (||H_i(t,\cdot)||_{C([g(t),h(t)])} + ||V_i(t,\cdot)||_{C([g(t),h(t)])}) > 0.$$

Then we give the following lemmas.

**Lemma 4.1.** *Let $(H_i, V_i; g, h)$ be the solution of* (1.4)*. If $h_\infty - g_\infty < \infty$, then there exists a constant $C > 0$ such that*

$$\|H_i(t,.)\|_{C^1([g(t),h(t)])} \le C, \ t \ge 1$$

*and*

$$\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0.$$

**Proof.** The first inequality can be proved by using the similar method as Theorem 2.1 in [39]. In the following, we only prove $\lim\limits_{t\to\infty} h'(t) = 0$, and the proof of $\lim\limits_{t\to\infty} g'(t) = 0$ is similar.

Factually,the transformation $y = \frac{x}{h(t)} h_0$, $w(t,y) = H_i(t, \frac{h(t)}{h_0} y) = H_i(t,x)$ turns problem (1.4) for $H_i(t,x)$ in $[0,+\infty) \times [0,h(t)]$ into a new problem for $w(t,y)$ in $[0,+\infty) \times [0,h_0]$. Let $\chi$ be the function in $C^3([0,h_0])$ satisfying

$$\begin{cases} \chi(y) = 1, \ \frac{h_0}{2} \le y \le h_0, \\ \chi(y) = 0, \ 0 \le y \le \frac{h_0}{8}. \end{cases}$$

Letting $z(t,y) = w(t,y) \times \chi(y)$, where $(t,y) \in [0,+\infty) \times [0,h_0]$, it follows from the $L^p$ theory of parabolic equations and the Sobolev imbedding theory that there exists $M_1 > 0$ such that

$$\|z\|_{C^{\frac{1+\alpha}{2},1+\alpha}([0,+\infty)\times[0,h_0])} \le M_1,$$

therefore, there exists $M_2 > 0$ such that $\|H_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}([0,+\infty)\times[0,h(t)])} \le M_2$. The comparison principle together with the free boundary condition yields that there exists $M_3 > 0$ such that

$$\|h\|_{C^{1+\alpha}([0,+\infty))} \le M_3,$$

which together with the assumption $h_\infty < \infty$ implies that $\lim\limits_{t\to\infty} h'(t) = 0$. □

**Lemma 4.2.** *If $h_\infty < \infty$ or $g_\infty > -\infty$, then both $h_\infty$ and $g_\infty$ are finite and*

$$R_0^{DA}((g_\infty,h_\infty),D_h,\beta)\le 1 \text{ and } \lim_{t\to\infty}(\|V_i(t,\cdot)\|_{C([g(t),\,h(t)])}+\|H_i(t,\cdot)\|_{C([g(t),\,h(t)])})=0.$$

**Proof.** Firstly, without loss of generality, we assume that $h_\infty < \infty$, and prove that $R_0^{DA} \le 1$, which implies that $g_\infty > -\infty$ by Remark 3.2. On the contrary, we assume that $R_0^{DA}((g_\infty,h_\infty),D_h,\beta) > 1$ by contradiction. Similarly as Lemma 3.1 in [16], we know that there is $\varepsilon_0 > 0$ such that $h'(t) > \varepsilon_0$. This contradicts the fact $\lim\limits_{t\to\infty} h'(t) = 0$ in Lemma 4.1.

Next, let $(\overline{V}_i(t,x), \overline{H}_i(t,x))$ be the unique solution of the problem l

$$\begin{cases} \dfrac{\partial \overline{V}_i}{\partial t} = \dfrac{\beta_v(x)(N_v^* - \overline{V}_i)\overline{H}_i}{N_h^*} - r_v(x)(1-q)\overline{V}_i, & t>0,\ g_\infty < x < h_\infty, \\ \dfrac{\partial \overline{H}_i}{\partial t} - D_h \dfrac{\partial^2 \overline{H}_i}{\partial x^2} + \beta \dfrac{\partial \overline{H}_i}{\partial x} = \dfrac{\beta_h \overline{V}_i(N_h^* - \overline{H}_i)}{N_h^*} - (d_h+\gamma_h)\overline{H}_i, & t>0,\ g_\infty < x < h_\infty, \\ \overline{V}_i(0,g_\infty) = \overline{H}_i(0,g_\infty) = \overline{V}_i(0,h_\infty) = \overline{H}_i(0,h_\infty) = 0, & t>0, \\ (\overline{V}_i(0,x), \overline{H}_i(0,x)) = (\tilde{V}_{i0}(x), \tilde{H}_{i0}(x)), & g_\infty \le x \le h_\infty, \end{cases} \tag{4.1}$$

with

$$(\tilde{V}_{i0}(x), \tilde{H}_{i0}(x)) = \begin{cases} (V_{i0}(x), H_{i0}(x)),\ g_0 \le x \le h_0, \\ (0,0), \qquad \text{otherwise.} \end{cases}$$

It follows from the comparison principle that

$$(0,0) \le (V_i(t,x), H_i(t,x)) \le (\overline{V}_i(t,x), \overline{H}_i(t,x))$$

where $t > 0,\ x \in [g(t),h(t)]$.

Recalling the fact $R_0^{DA}((g_\infty,h_\infty),D_h,\beta) \le 1$, we find that $(0,0)$ is the unique nonnegative steady-state solution of problem (4.1). Choosing the lower solution as $(0,0)$ and upper solution as $(N_v^*, N_h^*)$, it is easy to see, by the method of upper and lower solutions and its associated monotone iterations, that the time-dependent solution converges to the unique nonnegative steady-state solution. Therefore,

$$(\overline{V}_i(t,x), \overline{H}_i(t,x)) \to (0,0) \text{ uniformly } \quad x \in [g_\infty, h_\infty] \text{ as } t \to \infty,$$

and then

$$\lim_{t\to+\infty} ||V_i(t,\cdot)||_{C([g(t),\,h(t)])} = \lim_{t\to+\infty} ||H_i(t,\cdot)||_{C([g(t),\,h(t)])} = 0.$$

□

The next result shows that if $h_\infty - g_\infty < \infty$, then virus extinction will occur.

**Lemma 4.3.** *If $h_\infty - g_\infty < \infty$, then we have*

$$\lim_{t\to+\infty} (||V_i(t,\cdot)||_{C([g(t),h(t)])} + ||H_i(t,\cdot)||_{C([g(t),h(t)])}) = 0.$$

**Proof.** On the contrary, we assume that

$$\limsup_{t\to+\infty} ||H_i(t,\cdot)||_{C([g(t),h(t)])} = \delta > 0.$$

Then there exists a sequence $(t_k, x_k) \in (0,\infty)\times(g(t),h(t))$ such that $H_i(t_k,x_k) \geq \delta/2$ for all $k\in\mathbb{N}$, and $t_k\to\infty$ as $k\to\infty$.

Subsequently, we will show that

$$\|H_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}([1,+\infty)\times[g(t),h(t)])} \leq C, \tag{4.2}$$

$$||h'||_{C^{\frac{\alpha}{2}}([1,+\infty))} \leq C,\ ||g'||_{C^{\frac{\alpha}{2}}([1,+\infty))} \leq C \tag{4.3}$$

where $\alpha\in(0,1)$ and the constant $C>0$.

In fact, straighten the double free boundaries by the following transformation

$$y = \frac{2x}{h(t)-g(t)} - \frac{(h(t)+g(t))}{h(t)-g(t)},$$

let $w(t,y) = H_i(t,x)$, then the free boundary problem (1.4) is transformed into the initial boundary problem (2.2) in the fixed interval $(-h_0,h_0)$.

Combining (4.3) and the properties that $-g(t)$ and $h(t)$ are increasing and bounded, it follows from standard $L^p$ theory and the Sobolev imbedding theorem [27] that for $0<\alpha<1$, there exists a constant $C_1$ depending on $\alpha, h_0, \|H_{i0}\|_{C^2([-h_0,h_0])}$, $\|V_{i0}\|_{C^2([-h_0,h_0])}$, $g_\infty, h_\infty$ such that

$$\|w\|_{C^{\frac{1+\alpha}{2},1+\alpha}([\tau,\tau+1]\times[-h_0,h_0])} \leq C_1 \tag{4.4}$$

for any $\tau\geq 1$. Note that $C_1$ is independent of $\tau$, by using the free boundary conditions in (1.4), it is easy to see that (4.2), (4.3) hold. Using (4.3) and the assumption that $h_\infty - g_\infty < \infty$ yields

$$h'(t)\to 0 \text{ and } g'(t)\to 0 \text{ as}\, t\to+\infty.$$

It follows from the free boundary condition that $\frac{\partial H_i}{\partial x}(t_k, h(t_k)) \to 0$ as $t_k\to\infty$.

On the other hand, since $-\infty < g_\infty < g(t) < x_k < h(t) < h_\infty < \infty$, there exists a subsequence $\{x_{kn}\}$ which converges to $x_0\in[g_\infty,h_\infty]$ as $n\to\infty$. For convenience, we still denote $\{x_{kn}\}$ as $\{x_k\}$, it follows that $x_k\to x_0\in[g_\infty,h_\infty]$ as $k\to\infty$. Thanks to the uniform boundedness in (4.2), we can derive that $x_0\in(g_\infty,h_\infty)$.

Define $Z_k(t,x) = V_i(t_k+t,x)$ and $W_k(t,x) = H_i(t_k+t,x)$ for $x \in (g(t_k+t), h(t_k+t)), t \in (-t_k,\infty)$. According to the parabolic regularity, $\{(Z_k, W_k)\}$ has a subsequence $\{(Z_{k_i}, W_{k_i})\}$ which converges to $(\tilde{Z},\tilde{W})$ as $i \to \infty$, and $(\tilde{Z},\tilde{W})$ satisfies

$$\begin{cases} \tilde{Z}_t = \beta_v(x)\dfrac{N_v^* - \tilde{Z}}{N_h^*}\tilde{W} - (1-q)r_v(x)\tilde{Z} \\ \tilde{W}_t - D_h\tilde{W}_{xx} + \beta\tilde{W}_x = \beta_h(x)(1-\dfrac{\tilde{W}}{N_h^*})\tilde{Z} - (d_h(x)+\gamma_h(x))\tilde{W}, \end{cases}$$

where $t \in (-\infty,\infty)$, $x \in (g_\infty, h_\infty)$.

Since $\tilde{W}(t,x_0) \geq \delta/2$, we derive $\tilde{W} > 0$ in $(-\infty,\infty)\times(g_\infty,h_\infty)$ by the strong maximal principle. Using the Hopf lemma at the point $(0,h_\infty)$ yields $\tilde{W}_x(0,h_\infty) \leq -\sigma^*$ for some $\sigma^* > 0$.

Furthermore, the fact $\|H_i\|_{C^{\frac{1+\alpha}{2},1+\alpha}([1,+\infty)\times[g(t),h(t)])} \leq C$ implies that

$$\frac{\partial H_i}{\partial x}(t_k+0, h(t_k)) = (W_k)_x(0,h(t_k)) \to \tilde{W}_x(0,h_\infty),\ \ k\to\infty,$$

and then $\tilde{W}_x(0,h_\infty) = 0$, which is a contradiction to $\tilde{W}_x(0,h_\infty) \leq -\sigma^* < 0$; therefore

$$\lim_{t\to+\infty} ||H_i(t,\cdot)||_{C([g(t),h(t)])} = 0.$$

Note that $V_i(t,x)$ satisfies

$$\frac{\partial V_i}{\partial t} = \frac{\beta_v(x)(N_v^*-V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i,\ t>0,\ g(t)<x<h(t),$$

and $\dfrac{\beta_v(x)(N_v^*-V_i)H_i}{N_h^*} \to 0$ uniformly for $x \in [g(t),h(t)]$ as $t\to\infty$, we then deduce

$$\lim_{t\to+\infty} ||V_i(t,\cdot)||_{C([g(t),h(t)])} = 0.$$

□

Now we give sufficient conditions so that the virus will extinct.

**Lemma 4.4.** *If $R_0^F(\infty) \leq 1$, then $h_\infty - g_\infty < \infty$ and*

$$\lim_{t\to+\infty} (||V_i(t,\cdot)||_{C([g(t),h(t)])} + ||H_i(t,\cdot)||_{C([g(t),h(t)])}) = 0.$$

In this paper we assume that the far site is high-risk and consider small advection intensity, as a consequence, if $h_\infty - g_\infty = \infty$, then $R_0^F(\infty) > 1$. Therefore $R_0^F(\infty) \leq 1$ means that virus extinction happens. The following result shows that if $R_0^F(0) < 1$, the virus will extinct eventually for small initial values.

**Theorem 4.1.** *If $R_0^F(0) < 1$, then $h_\infty - g_\infty < \infty$ and*

$$\lim_{t\to+\infty} (||V_i(t,\cdot)||_{C([g(t),h(t)])} + ||H_i(t,\cdot)||_{C([g(t),h(t)])}) = 0$$

*provided that $||V_{i0}(\cdot)||_{C([-h_0,h_0])}$, $||H_{i0}(\cdot)||_{C([-h_0,h_0])}$ are sufficiently small.*

**Proof.** In order to complete the theorem, we need to construct a suitable upper solution to problem (1.4).

Since $R_0^F(0) < 1$, we can deduce from Lemma 3.3 that there exist the constant $\lambda_0 > 0$ and the functions satisfying $0 < \psi(x), \phi(x) \le 1$, $x \in (-h_0, h_0)$ such that

$$\begin{cases} 0 = \beta_v(x)\dfrac{N_v^*}{N_h^*}\phi - (1-q)r_v(x)\psi + \lambda_0\psi, & -h_0 < x < h_0, \\ -D_h\phi_{xx} + \beta\phi_x = \beta_h(x)\psi - (d_h(x) + d_h(x))\phi + \lambda_0\phi, & -h_0 < x < h_0, \\ \psi(x) = \phi(x) = 0, & x = \pm h_0. \end{cases} \tag{4.5}$$

Recalling that $\psi'(h_0), \phi'(h_0) < 0$ and $\psi'(-h_0), \phi'(-h_0) > 0$, we can derive that there exist some positive constants $C_1$ and $C_2$ such that

$$x\psi' \le C_1\psi,\ x\phi' \le C_2\phi,\ x \in (-h_0, h_0).$$

Also, we can easily derive that there exists the constant $L > 0$ such that

$$\frac{1}{L} \le \frac{\phi(x)}{\psi(x)} \le L,\ \ x \in (-h_0, h_0). \tag{4.6}$$

Similarly as statements in [14], we set

$$\sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\delta t}),\ t \ge 0,$$

and

$$\overline{V}_i = \varepsilon e^{-\delta t}\psi(xh_0/\sigma(t))e^{\frac{\beta}{2D_h}(1-\frac{h_0}{\sigma(t)})x}, t \ge 0,\ -\sigma(t) \le x \le \sigma(t).$$

$$\overline{H}_i = \varepsilon e^{-\delta t}\phi(xh_0/\sigma(t))e^{\frac{\beta}{2D_h}(1-\frac{h_0}{\sigma(t)})x}, t \ge 0,\ -\sigma(t) \le x \le \sigma(t).$$

Combining $\lambda_0 > 0$ and the continuity of the function $\beta_v(x)$, $\beta_h(x)$, $r_v(x)$, $d_h(x)$ and $\gamma_h(x)$ in $[-2h_0, 2h_0]$, we can derive from (4.6) that there exists a small $\delta > 0$ such that

$$\begin{aligned} &-\delta - \frac{\beta h_0}{4D_h}(1+\delta)\frac{{h_0}^2}{\sigma^2(t)}\delta^2 - \frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}C_1 + \lambda_0 - L\frac{N_v^*}{N_h^*}|\beta_v(x) - \beta_v(y)| \\ &+ (1-q)(r_v(x) - r_v(y)) \ge 0, \end{aligned}$$

and

$$\begin{aligned} &-\delta - \frac{\beta h_0}{4D_h}(1+\delta)\frac{{h_0}^2}{\sigma^2(t)}\delta^2 - \frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}C_2 + \frac{\beta^2}{4D_h}(1 - \frac{{h_0}^2}{\sigma^2(t)}) + \frac{{h_0}^2}{\sigma^2(t)}\lambda_0 \\ &- L|\frac{{h_0}^2}{\sigma^2(t)}\beta_h(y) - \beta_h(x)| + (d_h(x) + \gamma_h(x) - \frac{{h_0}^2}{\sigma^2(t)}(d_h(y) + \gamma_h(y))) \ge 0, \end{aligned}$$

where $y = \frac{xh_0}{\sigma(t)}$.

Direct computations yield

$$\begin{aligned} &\frac{\partial \overline{V}_i}{\partial t} - \frac{\beta_v(x)(N_v^* - \overline{V}_i)\overline{H}_i}{N_h^*} + r_v(x)(1-q)\overline{V}_i \\ \ge &\frac{\partial \overline{V}_i}{\partial t} - \beta_v(x)\overline{H}_i + r_v(x)(1-q)\overline{V}_i \end{aligned}$$

$$
\begin{aligned}
&= -\delta\overline{V}_i + \frac{\beta x}{2D_h}\frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}e^{-\delta t}\overline{V}_i - \frac{x{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}e^{-\delta t}\overline{V}_i\psi^{-1}\psi' + (\lambda_0\overline{V}_i + \frac{N_v^*}{N_h^*}\beta_v(y)\overline{H}_i \\
&\quad - (1-q)r_v(y)\overline{V}_i) - \frac{N_v^*}{N_h^*}\beta_v(x)\overline{H}_i + (1-q)r_v(x)\overline{V}_i \\
&\geq \overline{V}_i(-\delta - \frac{\beta h_0}{4D_h}(1+\delta)\frac{{h_0}^2}{\sigma^2(t)}\delta^2 - \frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}C_1 + \lambda_0 - L\frac{N_v^*}{N_h^*}|\beta_v(y) - \beta_v(x)| \\
&\quad + (1-q)(r_v(x) - r_v(y))) \\
&\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial\overline{H}_i}{\partial t} - D_h\frac{\partial^2\overline{H}_i}{\partial x^2} + \beta\frac{\partial\overline{H}_i}{\partial x} - \frac{\beta_h(x)\overline{V}_i(N_h^* - \overline{H}_i)}{N_h^*} + (d_h(x) + \gamma_h(x))\overline{H}_i, \\
\geq\ & \frac{\partial\overline{H}_i}{\partial t} - D_h\frac{\partial^2\overline{H}_i}{\partial x^2} + \beta\frac{\partial\overline{H}_i}{\partial x} - \beta_h(x)\overline{V}_i + (d_h(x) + \gamma_h(x))\overline{H}_i, \\
=\ & -\delta\overline{H}_i - \frac{\beta x}{2D_h}\frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}e^{-\delta t}\overline{H}_i - \frac{x{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}e^{-\delta t}\overline{H}_i\phi^{-1}\phi' + \frac{\beta^2}{4D_h}(1 - \frac{{h_0}^2}{\sigma^2(t)})\overline{H}_i \\
&+ \lambda_0\frac{{h_0}^2}{\sigma^2(t)}\overline{H}_i + \varepsilon e^{-\delta t}e^{\frac{\beta}{2D_h}(1-\frac{h_0}{\sigma(t)})x}\psi(\frac{{h_0}^2}{\sigma^2(t)}\beta_h(y) - \beta_h(x)) \\
&+ \varepsilon e^{-\delta t}e^{\frac{\beta}{2D_h}(1-\frac{h_0}{\sigma(t)})x}\phi(d_h(x) + \gamma_h(x) - \frac{{h_0}^2}{\sigma^2(t)}(d_h(y) + \gamma_h(y))) \\
\geq\ & \overline{H}_i(-\delta - \frac{\beta h_0}{4D_h}(1+\delta)\frac{{h_0}^2}{\sigma^2(t)}\delta^2 - \frac{{h_0}^2}{\sigma^2(t)}\frac{\delta^2}{2}C_2 + \frac{\beta^2}{4D_h}(1 - \frac{{h_0}^2}{\sigma^2(t)}) \\
&+ \frac{{h_0}^2}{\sigma^2(t)}\lambda_0 - L|\frac{{h_0}^2}{\sigma^2(t)}\beta_h(y) - \beta_h(x)| + (d_h(x) + \gamma_h(x) - \frac{{h_0}^2}{\sigma^2(t)}(d_h(y) + \gamma_h(y)))) \\
\geq\ & 0,
\end{aligned}
$$

where $t > 0$ and $x \in (-\sigma(t), \sigma(t))$.

On the other hand, we can choose $\varepsilon = \frac{\delta^2 h_0}{2\beta e^{\frac{\beta}{2D_h}h_0\delta}}\min\{\frac{-1}{\phi'(h_0)}, \frac{1}{\phi'(-h_0)}\}$ such that

$$
\begin{cases}
\dfrac{\partial\overline{V}_i}{\partial t} \geq \dfrac{\beta_v(x)(N_v^* - \overline{V}_i)\overline{H}_i}{N_h^*} - r_v(x)(1-q)\overline{V}_i, & t > 0,\ x \in (-\sigma(t), \sigma(t)), \\
\dfrac{\partial\overline{H}_i}{\partial t} \geq D_h\dfrac{\partial^2\overline{H}_i}{\partial x^2} - \beta\dfrac{\partial\overline{H}_i}{\partial x} + \dfrac{\beta_h(x)\overline{V}_i(N_h^* - \overline{H}_i)}{N_h^*} & \\
\qquad - (d_h(x) + \gamma_h(x))\overline{H}_i, & t > 0,\ x \in (-\sigma(t), \sigma(t)), \\
\overline{V}_i(t,x) = \overline{H}_i(t,x) = 0, & t > 0,\ x = \pm\sigma(t), \\
-\sigma(0) < -h_0,\ -\sigma'(t) \leq -\mu\dfrac{\partial\overline{H}_i}{\partial x}(t, -\sigma(t)), & t > 0, \\
\sigma(0) > h_0,\ \sigma'(t) \geq -\mu\frac{\partial\overline{H}_i}{\partial x}(t, \sigma(t)), & t > 0.
\end{cases}
$$

If

$$||V_{i0}||_{L^\infty((-h_0,h_0))} \leq \varepsilon \min_{[-h_0,h_0]}\psi(\frac{h_0}{1+\delta/2})e^{\frac{\beta}{2D_h}\frac{\delta}{2+\delta}(-h_0)},$$

and

$$||H_{i0}||_{L^\infty((-h_0,h_0))} \leq \varepsilon \min_{[-h_0,h_0]}\phi(\frac{h_0}{1+\delta/2})e^{\frac{\beta}{2D_h}\frac{\delta}{2+\delta}(-h_0)},$$

then we deduce that

$$V_{i0}(x) \le \varepsilon\psi(\frac{x}{1+\delta/2})e^{\frac{\beta}{2D_h}\frac{\delta}{2+\delta}x} = \overline{V}_i(0,x),\ \ x\in[-h_0,h_0]$$

and

$$H_{i0}(x) \le \varepsilon\phi(\frac{x}{1+\delta/2})e^{\frac{\beta}{2D_h}\frac{\delta}{2+\delta}x} = \overline{H}_i(0,x),\ \ x\in[-h_0,h_0],$$

that is, $(\overline{V}_i(t,x),\overline{H}_i(t,x),-\sigma(t),\sigma(t))$ is an upper solution to problem (1.4).

Applying the comparison principle, we conclude that

$$g(t)\ge -\sigma(t),\ \ h(t)\le\sigma(t),\ \ t>0.$$

It follows from Lemma 4.3 that

$$h_\infty - g_\infty \le \lim_{t\to\infty} 2\sigma(t) = 2h_0(1+\delta) < \infty$$

and

$$\lim_{t\to+\infty}(||H_i(t,\cdot)||_{C([g(t),h(t)])} + ||V_i(t,\cdot)||_{C([g(t),h(t)])}) = 0.$$

□

From the proof above, we can deduce the following result, see Lemma 3.8 in [14] for details.

**Theorem 4.2.** *Suppose $R_0^F(0)(:= R_0^{DA}((-h_0,h_0),D_h,\beta)) < 1$. Then $h_\infty - g_\infty < \infty$ and*

$$\lim_{t\to+\infty}(||V_i(t,\cdot)||_{C([g(t),h(t)])} + ||H_i(t,\cdot)||_{C([g(t),h(t)])}) = 0$$

*if $\mu$ is sufficiently small.*

# 5. Virus persistence

In this section, we are going to give the sufficient conditions that the virus will persist continuously. We first prove that if $R_0^F(0)\ge 1$, the virus are spreading continuously.

**Theorem 5.1.** *If $R_0^F(0)\ge 1$, then $h_\infty - g_\infty = \infty$ and $\liminf_{t\to+\infty}||H_i(t,\cdot)||_{C([0,h(t)])} > 0$, that is, the virus will persist continuously.*

**Proof.** We first consider the case that $R_0^F(0) := R_0^D((-h_0,h_0)) > 1$. In this case, the linear eigenvalue problem

$$\begin{cases} 0 = \dfrac{N_v^*}{N_h^*}\beta_v(x)\phi - (1-q)r_v(x)\psi + \lambda_0\psi, & -h_0<x<h_0, \\ -D_h\phi_{xx} + \beta\phi_x = \beta_h(x)\psi - (d_h(x)+\gamma_h(x))\phi + \lambda_0\phi, & -h_0<x<h_0, \\ \psi(x)=\phi(x)=0, & x=\pm h_0. \end{cases} \tag{5.1}$$

admits a positive solution $(\psi(x),\phi(x))$ with $||\psi||_{L^\infty} + ||\phi||_{L^\infty} = 1$, where $\lambda_0$ is the principal eigenvalue. It follows from Lemma 3.3 that $\lambda_0 < 0$.

We construct a suitable lower solution to problem (1.4) by define

$$\underline{H}_i(t,x) = \delta\phi(x), \quad \underline{V}_i(t,x) = \delta\psi(x),\ t \geq 0,\ -h_0 \leq x \leq h_0,$$

choose $\delta$ sufficiently small such that

$$\lambda_0 + ||\max\{\beta_v(x), \beta_h(x)\}||_{L^\infty}\frac{\delta}{K} < 0,$$

where $K = \min\{N_v^*, N_h^*\}$. Direct computations yield

$$\begin{aligned}
&\frac{\partial \underline{V}_i}{\partial t} - \frac{\beta_v(x)(N_v^* - \underline{V}_i)\underline{H}_i}{N_h^*} + (1-q)r_v(x)\underline{V}_i \\
&= -\frac{N_v^*}{N_h^*}\beta_v(x)\delta\phi + \beta_v(x)\frac{\delta\phi}{N_h^*}\delta\phi + (1-q)r_v(x)\delta\psi \\
&= \delta\psi(\lambda_0 + \beta_v(x)\frac{\delta\phi}{N_h^*}) \\
&\leq 0, \\
&\frac{\partial \underline{H}_i}{\partial t} - D_h\frac{\partial^2 \underline{H}}{\partial x^2} + \beta\frac{\partial \underline{H}_i}{\partial x} - \frac{\beta_h(x)\underline{V}_i(N_h^* - \underline{H}_i)}{N_h^*} + (d_h(x) + \gamma_h(x))\underline{H}_i \\
&= -D_h\delta\phi_{xx} + \beta\delta\phi_x - \beta_h(x)\delta\psi(1 - \frac{\delta\phi}{N_h^*}) + (d_h(x) + \gamma_h(x))\delta\phi \\
&= \delta\psi[\lambda_0 + \frac{\delta\beta_h(x)\phi}{N_h^*}], \\
&\leq 0,
\end{aligned}$$

where $t > 0$ and $x \in (-h_0, h_0)$.

Recalling $\lambda_0 < 0$, we can choose $\delta$ sufficiently small such that

$$\begin{cases}
\frac{\partial \underline{V}_i}{\partial t} \leq \frac{\beta_v(x)(N_v^* - \underline{V}_i)\underline{H}_i}{N_h^*} - (1-q)r_v(x)\underline{V}_i, & t > 0,\ -h_0 < x < h_0, \\
\quad \frac{\partial \underline{H}_i}{\partial t} - D_h\frac{\partial^2 \underline{H}}{\partial x^2} + \beta\frac{\partial \underline{H}_i}{\partial x} & \\
\leq \frac{\beta_h(x)\underline{V}_i(N_h^* - \underline{H}_i)}{N_h^*} - (d_h(x) + \gamma_h(x))\underline{H}_i, & t > 0,\ -h_0 < x < h_0, \\
\underline{V}_i(x,t) = \underline{H}_i(x,t) = 0, & t > 0,\ x = \pm h_0, \\
0 = -h_0' \geq -\mu D_h\frac{\partial \underline{H}_i}{\partial x}(-h_0, t), & t > 0, \\
0 = h_0' \leq -\mu D_h\frac{\partial \underline{H}_i}{\partial x}(h_0, t), & t > 0, \\
\underline{V_i}(x,0) \leq V_{i0}(x),\ \underline{H_i}(x,0) \leq H_{i0}(x), & -h_0 \leq x \leq h_0.
\end{cases}$$

Hence, applying the comparison principle yields that

$$H_i(t,x) \geq \underline{H}_i(t,x),\ V_i(t,x) \geq \underline{V}_i(t,x),\ (t,x) \in [0,\infty) \times [-h_0, h_0].$$

It follows that $\liminf\limits_{t\to+\infty} ||H_i(t,\cdot)||_{C([g(t),h(t)])} \geq \delta\phi(0) > 0$, therefore $h_\infty - g_\infty = +\infty$ by Lemma 4.2.

When $R_0^F(0) = 1$, then for any positive time $t_0$, we deduce $g(t_0) < -h_0$ and $h(t_0) > h_0$; therefore, $R_0^F(t_0) > R_0^F(0) = 1$ by the monotonicity in Theorem 3.4. We then have $h_\infty - g_\infty = +\infty$ as above by replacing the initial time 0 with the positive time $t_0$. □

**Remark 5.1.** It follows from the above proof that the virus persist, if and only if there exists $t_0 \geq 0$ such that $R_0^F(t_0) \geq 1$.

Epidemiologically, Theorems 4.4 and 4.5 show that if $R_0^F(0) < 1$, the virus will extinct for small initial scale of mosquitoes or small expanding capability $\mu$, and Lemma 4.3 implies that if $R_0^F(\infty) \leq 1$, the virus will persist eventually for any initial values. The next result shows that the virus will persist for large expanding capability $\mu$, see similar results and the proofs in [14].

**Theorem 5.2.** *Suppose that $R_0^F(0) < 1$. Then $h_\infty - g_\infty = \infty$ if $\mu$ is sufficiently large.*

**Theorem 5.3** (Sharp threshold)**.** *Fixed $h_0$, $V_{i0}$ and $H_{i0}$. There exists $\mu^* \in [0, \infty)$ such that the virus will persist when $\mu > \mu^*$, and the virus will extinct when $0 < \mu \leq \mu^*$.*

**Proof.** If $R_0^F(0) \geq 1$, we have $\mu^* = 0$, since in this case spreading always happens for $\mu > 0$ from Theorem 5.1.

For the remaining case $R_0^F(0) < 1$. We define

$$\mu^* := \sup\{\sigma_0 : h_\infty(\mu) - g_\infty(\mu) < \infty \text{ for } \mu \in (0, \sigma_0]\}.$$

Theorem 4.5 implies that the virus extinction happens for all small $\mu > 0$, therefore, $\mu^* \in (0, \infty]$. On the other hand, by Theorem 5.2, it is easy to derive that the virus persistence happens for all big $\mu$. Thus we have $\mu^* \in (0, \infty)$, and virus persistence happens when $\mu > \mu^*$, the virus extinction occurs when $0 < \mu < \mu^*$ by Corollary 2.4.

We now claim that the virus extinction happens when $\mu = \mu^*$. Otherwise $h_\infty - g_\infty = \infty$ for $\mu = \mu^*$. Since $\lim_{t\to\infty} R_0^F(t) \geq \sqrt{\dfrac{\frac{N_v^* \beta_h^\infty \beta_v^\infty}{N_h^*(1-q) r_v^\infty}}{\frac{\beta^2}{4D_h} + d_h^\infty + \gamma_h^\infty}} > 1$, there exists $T_0 > 0$ such that $R_0^F(T_0) := R_0^{DA}((g(T_0), h(T_0), D_h, \beta) > 1$. By the continuous dependence of $(V_i, H_i, g, h)$ on its expanding capability $\mu$, we can find small $\epsilon > 0$ such that the solution of (1.4) with $\mu = \mu^* - \epsilon$, denoted by $(V_{i\epsilon}, H_{i\epsilon}, g_\epsilon, h_\epsilon)$, satisfies $R_0^{DA}((g_\epsilon(T_0), h_\epsilon(T_0)), D_h, \beta) > 1$. This implies that spreading happens for the solution $(V_{i\epsilon}, H_{i\epsilon}, g_\epsilon, h_\epsilon)$, which contradicts the definition of $\mu^*$. The proof is complete. □

Next, we consider the long-time asymptotical behavior of the solution to (1.4) when the virus persist.

**Theorem 5.4.** *Suppose that $h_\infty = -g_\infty = \infty$, then the solution to the free boundary problem (1.4) satisfies $\lim_{t\to+\infty} (V_i(t,x), H_i(t,x)) = (V_i^*(x), H_i^*(x))$ uniformly in any bounded subset of $(-\infty, \infty)$, where $(V_i^*(x), H_i^*(x))$ is the unique bounded positive solution of the following problem*

$$\begin{cases} 0 = \dfrac{\beta_v(x)(N_v^* - V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i, & -\infty < x < \infty, \\ -D_h \dfrac{\partial^2 H_i}{\partial x^2} + \beta \dfrac{\partial H_i}{\partial x} = \dfrac{\beta_h(x) V_i (N_h^* - H_i)}{N_h^*} - (d_h(x) + \gamma_h(x))H_i, & -\infty < x < \infty, \end{cases} \tag{5.2}$$

**Proof.** Our proof will divide into several parts.

**Step 1** The existence and uniqueness of the stationary solution

It is easy to see that problem (5.2) is equivalent to

$$-D_h\frac{\partial^2 H_i}{\partial x^2}+\beta\frac{\partial H_i}{\partial x}=\frac{\beta_h(x)\beta_v(x)N_v^*H_i}{\beta_v(x)H_i+(1-q)N_h^*r_v(x)}\frac{N_h^*-H_i}{N_h^*}-(d_h(x)+\gamma_h(x))H_i \tag{5.3}$$

for $-\infty<x<\infty$. Since $h_\infty=-g_\infty=+\infty$, it follows from Remark 3.1 that there exists $t_0>0$ such that $R_0^F(t_0)=R_0^{DA}((g(t_0),h(t_0)),D_h,\beta)>1$, therefore, denoting $L_0:=\max\{-g(t_0),h(t_0)\}$, for any $L$ satisfying $L\geq L_0$, we consider the problem

$$\begin{cases}-D_h\dfrac{\partial^2 H_i^L}{\partial x^2}+\beta\dfrac{\partial H_i^L}{\partial x}=\dfrac{\beta_h\beta_vN_v^*H_i^L}{\beta_vH_i^L+(1-q)N_h^*r_v}\dfrac{N_h^*-H_i^L}{N_h^*}-(d_h+\gamma_h)H_i^L,\ -L<x<L,\\ H_i^L(\pm L)=0.\end{cases} \tag{5.4}$$

Setting $\tilde{H_i^L}=N_h^*,\hat{H_i^L}=\delta\phi(x)$, where $(\psi(x),\phi(x))$ is the corresponding eigenfunction to the principal eigenvalue $\lambda_0$ of the following problem

$$\begin{cases}-D_h\phi_{xx}+\beta\phi_x=\beta_h(x)\psi-(d_h(x)+\gamma_h(x))\phi+\lambda_0\phi, & x\in(-L,L),\\ 0=\dfrac{N_v^*}{N_h^*}\beta_v(x)\phi-(1-q)r_v(x)\psi+\lambda_0\psi, & x\in(-L,L),\\ \psi(x)=\phi(x)=0, & x=\pm L.\end{cases} \tag{5.5}$$

We can choose $\delta$ sufficiently small such that $\tilde{H_i^L}$ and $\hat{H_i^L}$ are upper and lower solutions to problem (5.4). As a result, there exists $H_i^L$ that solves problem (5.4).

Moreover, for the first equation in (5.4), taking $H_i^L=e^{\frac{\beta}{2D_h}x}u$ derives that

$$-D_hu_{xx}=-\frac{\beta^2}{4D_h}u+\frac{N_v^*\beta_v\beta_h(x)u}{\beta_ve^{\frac{\beta}{2D_h}x}u+(1-q)N_h^*r_v}(1-\frac{e^{\frac{\beta}{2D_h}x}u}{N_h^*})-(d_h+\gamma_h)u:=f(u)u.$$

It is easy to see that $f(u)$ is decreasing, therefore the positive solution is unique.

Using the comparison principle yields that as $L$ increases to infinity, $H_i^L$ increases to a positive solution $H_i^*$ to problem (5.3). The uniqueness of positive solution to problem (5.3) follows from the similar technique in [16].

**Step 2** The limit superior of the solution

We recall that the comparison principle derives

$$(V_i(t,x),H_i(t,x))\leq(\overline{V}_i(t,x),\overline{H}_i(t,x)),\ \ (t,x)\in(0,\infty)\times(-\infty,\infty),$$

where $(\overline{V}_i(t,x),\overline{H}_i(t,x))$ is the solution to the following problem

$$\begin{cases}\dfrac{\partial\overline{V}_i}{\partial t}=\dfrac{\beta_v(x)(N_v^*-\overline{V}_i)\overline{H}_i}{N_h^*}-(1-q)r_v(x)\overline{V}_i,\ \ t>0,\ -\infty<x<\infty,\\ \quad\dfrac{\partial\overline{H}_i}{\partial t}-D_h\dfrac{\partial^2\overline{H}_i}{\partial x^2}+\beta\dfrac{\partial\overline{H}_i}{\partial x}\\ =\dfrac{\beta_h(x)\overline{V}_i(N_h^*-\overline{H}_i)}{N_h^*}-(d_h(x)+\gamma_h(x))\overline{H}_i,\qquad t>0,\ -\infty<x<\infty,\\ \overline{V}_i(0,x)=N_v^*,\ \overline{H}_i(0,x)=N_h^*.\end{cases} \tag{5.6}$$

We can easily see that $(\overline{V}_i(t,x), \overline{H}_i(t,x)) \leq (\overline{V}_i(0,x), \overline{H}_i(0,x))$, therefore we deduce

$$(\overline{V}_i(t+\delta,x), \overline{H}_i(t+\delta,x)) \leq (\overline{V}_i(t,x), \overline{H}_i(t,x))$$

by comparing the initial conditions, that is, $(\overline{V}_i, \overline{H}_i)$ is monotone decreasing with respect to $t$ and $\lim\limits_{t\to\infty}(\overline{V}_i, \overline{H}_i) = (V_i^*(x), H_i^*(x))$ uniformly in any bounded subset of $(-\infty,\infty)$; therefore we deduce

$$\limsup_{t\to+\infty}\, (V_i(t,x), H_i(t,x)) \leq (V_i^*(x), H_i^*(x)) \tag{5.7}$$

uniformly in any bounded subset of $(-\infty,\infty)$.

**Step 3** The lower bound of the solution for a large time

From step 1, we can deduce that the principal eigenvalue $\lambda_0$ of

$$\begin{cases} 0 = \dfrac{N_v^*}{N_h^*}\beta_v(x)\phi - (1-q)r_v(x)\psi + \lambda_0\psi, & x \in (-L_0, L_0), \\ -D_h\phi_{xx} + \beta\phi_x = \beta_h(x)\psi - (d_h(x)+\gamma_h(x))\phi + \lambda_0\phi, & x \in (-L_0, L_0), \\ \psi(x) = \phi(x) = 0, & x = \pm L_0 \end{cases} \tag{5.8}$$

satisfies

$$\lambda_0 < 0.$$

Since $h_\infty = \infty = -g_\infty$, for any $L \geq L_0$, there exists $t_L > 0$ such that $g(t) \leq -L$ and $h(t) \geq L$ for $t \geq t_L$.

Letting $\underline{V}_i = \delta\psi$ and $\underline{H}_i = \delta\phi$, we can choose $\delta$ sufficiently small such that $(\underline{V}_i, \underline{H}_i)$ satisfies

$$\begin{cases} \dfrac{\partial \underline{V}_i}{\partial t} \leq \dfrac{\beta_v(x)(N_v^* - \underline{V}_i)\underline{H}_i}{N_h^*} - (1-q)r_v(x)\underline{V}_i, & t > t_{L_0},\ -L_0 < x < L_0, \\ \dfrac{\partial \underline{H}_i}{\partial t} - D_h\dfrac{\partial^2 \underline{H}}{\partial x^2} + \beta\dfrac{\partial \underline{H}_i}{\partial x} \leq \dfrac{\beta_h\underline{V}_i(N_h^* - \underline{H}_i)}{N_h^*} - (d_h+\gamma_h)\underline{H}_i, & t > t_{L_0},\ -L_0 < x < L_0, \\ \underline{V}_i(t,x) = \underline{H}_i(t,x) = 0, & t > t_{L_0},\ -L_0 < x < L_0, \\ \underline{V}_i(t_{L_0},x) \leq V_i(t_{L_0},x),\ \underline{H}_i(t_{L_0},x) \leq H_i(t_{L_0},x), & -L_0 \leq x \leq L_0, \end{cases}$$

which means that $(\underline{V}_i, \underline{H}_i)$ is a lower solution of $(V_i, H_i)$ in $[t_{L_0}, \infty) \times [-L_0, L_0]$. We then deduce

$$(V_i, H_i) \geq (\delta\psi, \delta\phi),\ (t,x) \in [t_{L_0}, \infty) \times [-L_0, L_0],$$

which implies that the solution can not decay to zero.

**Step 4** The limit inferior of the solution

Firstly, We extend $\phi(x)$ to $\phi_{l_0}(x)$, and extend $\psi(x)$ to $\psi_{l_0}(x)$ as follow

$$\phi_{l_0}(x) = \begin{cases} \phi(x), & -l_0 \leq x \leq l_0, \\ 0, & x < -l_0 \text{ or } x > l_0, \end{cases}$$

$$\psi_{l_0}(x) = \begin{cases} \psi(x), & -l_0 \leq x \leq l_0, \\ 0, & x < -l_0 \text{ or } x > l_0. \end{cases}$$

Now for $L \geq L_0$, $(V_i, H_i)$ satisfies

$$\begin{cases} \dfrac{\partial V_i}{\partial t} = \dfrac{\beta_v(x)(N_v^* - V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i, & t > t_L,\ g(t) < x < h(t), \\ \dfrac{\partial H_i}{\partial t} - D_h \dfrac{\partial^2 H_i}{\partial x^2} + \beta \dfrac{\partial H_i}{\partial x} = \dfrac{\beta_h V_i (N_h^* - H_i)}{N_h^*} - (d_h + \gamma_h)H_i, & t > t_L,\ g(t) < x < h(t), \\ V_i(t,x) = H_i(t,x) = 0, & t > t_L,\ x = g(t) \text{ or } x = h(t), \\ H_i(t_L, x) \geq \delta\phi_{L_0},\ V_i(t_L, x) \geq \delta\psi_{L_0}, & -L \leq x \leq L, \end{cases} \tag{5.9}$$

therefore, we have $(V_i, H_i) \geq (z, w)$ in $[t_L, \infty) \times [-L, L]$, where $(z, w)$ satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{\beta_v(x)(N_v^* - z)w}{N_h^*} - (1-q)r_v(x)z, & t > t_L,\ -L < x < L, \\ \dfrac{\partial w}{\partial t} - D_h \dfrac{\partial^2 w}{\partial x^2} + \beta \dfrac{\partial w}{\partial x} = \dfrac{\beta_h(x) z (N_h^* - w)}{N_h^*} - (d_h(x) + \gamma_h(x))w, & t > t_L,\ -L < x < L, \\ z(t,x) = w(t,x) = 0, & t > t_L,\ x = \pm L, \\ z(t_L, x) = \delta\psi_{L_0}, w(t_L, x) = \delta\phi_{L_0}, & -L \leq x \leq L. \end{cases} \tag{5.10}$$

System (5.10) is quasimonotone increasing; therefore, it follows from the upper and lower solution method and the theory of monotone dynamical systems( [34] Corollary 3.6) that

$$\lim_{t \to +\infty} (z(t,x), w(t,x)) = (V_i^L(x), H_i^L(x)) \geq (\delta\psi_{L_0}, \delta\phi_{L_0})$$

uniformly on $[-L, L]$, where $(V_i^L(x), H_i^L(x))$ satisfies

$$\begin{cases} 0 = \dfrac{\beta_v(x)(N_v^* - V_i)H_i}{N_h^*} - r_v(x)(1-q)V_i, & -L < x < L, \\ -D_h \dfrac{\partial^2 H_i}{\partial x^2} + \beta \dfrac{\partial H_i}{\partial x} = \dfrac{\beta_h(x) V_i (N_h^* - H_i)}{N_h^*} - (d_h(x) + \gamma_h(x))H_i, & -L < x < L. \\ H_i(\pm L) = 0. \end{cases} \tag{5.11}$$

Now we claim the monotonicity and show that if $0 < L_1 < L_2$, then $(V_i^{L_1}(x), H_i^{L_1}(x)) \leq (V_i^{L_2}(x), H_i^{L_2}(x))$ on $[-L_1, L_1]$. The result is derived by comparing the initial conditions and boundary conditions in (5.10) for $L = L_1$ and $L = L_2$.

Letting $L \to \infty$, by classical elliptic regularity theory and a diagonal procedure, we derive that $(V_i^L(x), H_i^L(x))$ converges uniformly on any compact subset of $(-\infty, \infty)$ to $(V_i^\infty(x), H_i^\infty(x))$, which is continuous on $(-\infty, \infty)$ and satisfies

$$\begin{cases} 0 = \dfrac{\beta_v(x)(N_v^* - V_i^\infty)H_i^\infty}{N_h^*} - r_v(x)(1-q)V_i^\infty, & -\infty < x < \infty, \\ \quad - D_h \dfrac{\partial^2 H_i^\infty}{\partial x^2} + \beta \dfrac{\partial H_i^\infty}{\partial x} & \\ = \frac{\beta_h(x) V_i^\infty (N_h^* - H_i^\infty)}{N_h^*} - (d_h(x) + \gamma_h(x))H_i^\infty, & -\infty < x < \infty, \\ V_i^\infty(x) \geq \delta\psi_{L_0}, H_i^\infty(x) \geq \delta\phi_{L_0}, & -\infty < x < \infty. \end{cases} \tag{5.12}$$

It follows from step 1 that $V_i^{\infty}(x) = V_i^*(x)$ and $H_i^{\infty}(x) = H_i^*(x)$.

Now for any given interval $[-X, X]$ with $X \geq L_0$, since $(V_i^L(x), H_i^L(x)) \to (V_i^*(x), H_i^*(x))$ uniformly in $[-X, X]$, which is the compact subset of $(-\infty, \infty)$, as $L \to \infty$, we deduce that for any $\varepsilon > 0$, there exists $L^* > L_0$ such that $(V_i^{L^*}(x), H_i L^*(x)) \geq (V_i^*(x) - \varepsilon, H_i^*(x) - \varepsilon)$ in $[-X, X]$. As above, there is $t_{L^*}$ such that $[-L^*, L^*] \subseteq [g(t), h(t)]$ for $t \geq t_{L^*}$. Therefore,

$$(V_i(t,x), H_i(t,x)) \geq (z(t,x), w(t,x)), \ \ (t,x) \in [t_{L^*}, \infty) \times [-L^*, L^*],$$

and

$$\lim_{t \to +\infty} (z(t,x), w(t,x)) = (V_i^{L^*}(x), H_i^{L^*}(x)), \ \ x \in [-L^*, L^*].$$

Using the fact that $(V_i^{L^*}(x), H_i^{L^*}(x)) \geq (V_i^*(x) - \varepsilon, H_i^*(x) - \varepsilon)$ in $[-X, X]$ derives

$$\liminf_{t \to +\infty} (V_i(t,x), H_i(t,x)) \geq (V_i^*(x) - \varepsilon, H_i^*(x) - \varepsilon), \ \ x \in [-X, X].$$

Since $\varepsilon > 0$ is arbitrary, we deduce that

$$\liminf_{t \to +\infty} V_i(t,x) \geq V_i^*(x), \ \liminf_{t \to +\infty} H_i(t,x) \geq H_i^*(x) \ \text{ uniformly on } [-X, X],$$

which together with (5.7) imply that

$$\lim_{t \to +\infty} V_i(t,x) = V_i^*(x), \ \ \lim_{t \to +\infty} H_i(t,x) = H_i^*(x)$$

uniformly in any bounded subset of $(-\infty, \infty)$. $\square$

Combining Remarks 3.1 and 5.1, Lemma 4.1 and Theorem 5.4, we immediately obtain the following spreading-vanishing dichotomy:

**Theorem 5.5.** *If the hypothesis condition* $(H)$ *holds and*

$$|\beta| < 2\sqrt{D_h\left(\frac{\beta_v^{\infty} \beta_h^{\infty} N_v^*}{N_h^* r_v^{\infty}(1-q)} - d_h^{\infty} - \gamma_h^{\infty}\right)}.$$

*Let* $(V_i(t,x), H_i(t,x); g(t), h(t))$ *be the solution of free boundary problem* (1.4). *Then, the following spreading-vanishing dichotomy holds:*

*Either*

$(i)$ Spreading: $h_\infty - g_\infty = +\infty$ *and* $\lim_{t \to +\infty} (V_i(t,x), H_i(t,x)) = (V_i^*(x), H_i^*(x))$ *uniformly in any bounded subset of* $(-\infty, \infty)$*;*

*or*

$(ii)$ Vanishing: $h_\infty - g_\infty < \infty$ *with* $R_0^{DA}((g_\infty, h_\infty), D_h, \beta) \leq 1$ *and*

$$\lim_{t \to +\infty} (||V_i(t, \cdot)||_{C([g(t), h(t)])} + ||H_i(t, \cdot)||_{C([g(t), h(t)])}) = 0.$$

# 6. Simulation

In this section, we will carry out numerical simulations to illustrate the theoretical results given above. In light of the free boundaries are to be determined, we will use an implicit scheme to simulate problem (1.4) and change it to a nonlinear system of algebraic equations, which was solved with Newton-Raphson method.

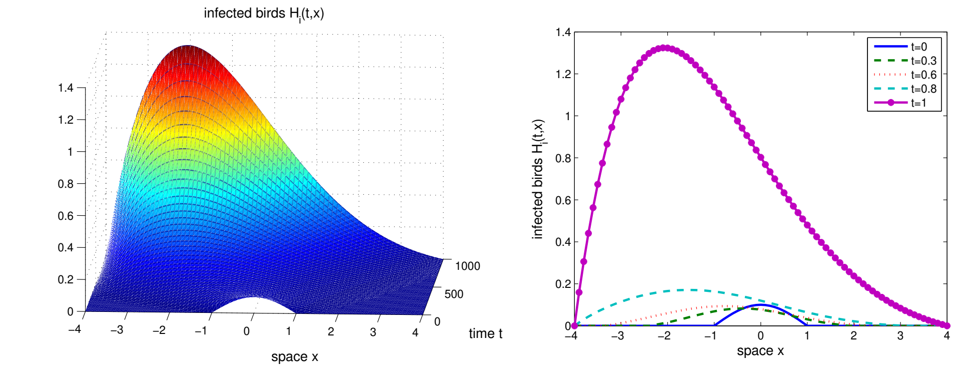


**Figure 2.** $\mu = 0.4$ and $\beta = 0.0008$. The solution $H_i(t,x)$ turns left and stabilizes to a positive equilibrium.

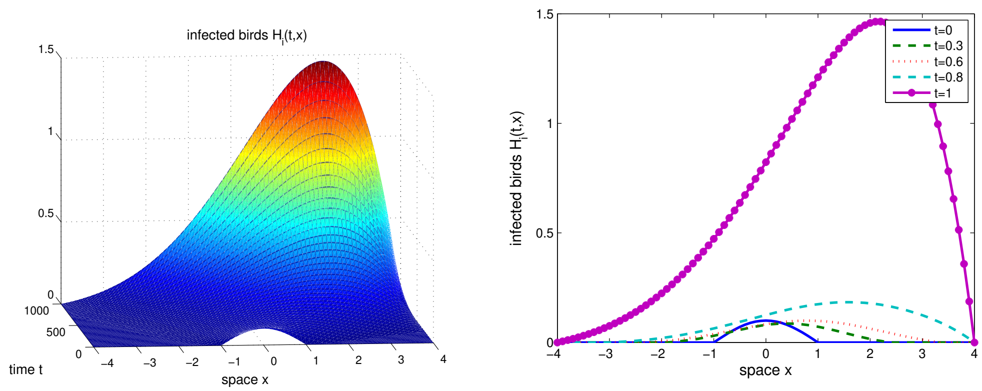


**Figure 3.** $\mu = 0.4$ and $\beta = -0.0008$. The solution $H_i(t,x)$ turns right and stabilizes to a steady state solution.

To explore the impact of the advection intensity $\beta$ and expanding capability $\mu$ on the long-time behaviors of the steady state solution to problem (1.4), we first fix some coefficients and functions as follows.

$$D_h = 0.006,\ N_h^* = 2000,\ N_v^* = 10000,\ \gamma_h(x) = 0.018\sin(\pi x),$$

$$p = 0.007,\ d_h = 0.0002\sin(\pi x),\ \beta_h(x) = 0.012\sin(\pi x), \beta_v(x) = 0.028\sin(\pi x),$$

$$h_0 = 1,\ H_i(0,x) = 0.1\cos(\frac{\pi}{2h_0}x),\ V_i(0,x) = 0.5\cos(\frac{\pi}{2h_0}x).$$

**Example 6.1.** For big expanding capability $\mu = 0.4$, we first choose the advection intensity $\beta = 0.0008$. It is easy to see from Fig. 2 that the free boundaries $x = g(t)$ and $x = h(t)$ increase fast, and the solution $H_i(t,x)$ stabilizes to a positive equilibrium gradually. Moreover, the left boundary moves faster that the right one . If we choose the advection intensity $\beta = -0.0008$, we can see from Fig. 3 that the right boundary goes faster than the left one.

**Example 6.2.** Let $\beta = -0.009$ and we now choose small expanding capability $\mu = 0.05$, compared the free boundaries in Fig. 4 with those in Fig. 3, the free

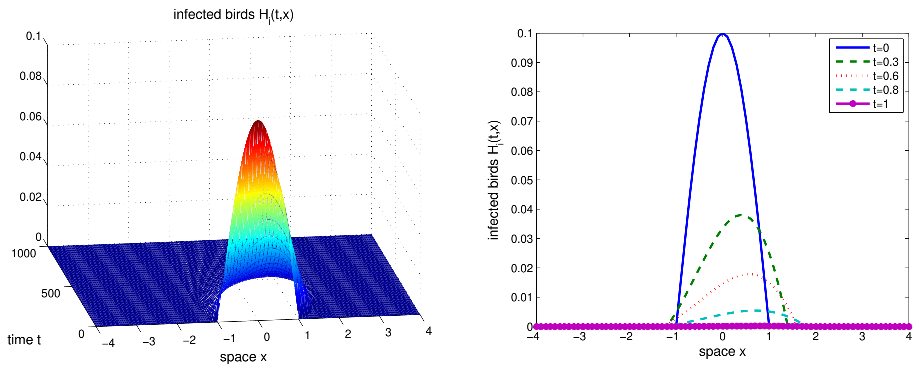


**Figure 4.** $\beta = -0.009$ and $\mu = 0.05$. The solution $H_i(t, x)$ decays to zero quickly and the free boundaries increase slowly.

boundaries $x = h(t)$ and $g(t)$ in Fig. 4 increase slower than those in Fig. 3. Moreover, the solution $H_i(t, x)$ decays to zero quickly.

## 7. Biological interpretations and discussion

The main purpose of the paper is to explore the spatial transmission mechanism of WNv where the environment are heterogeneous and advection. In order to better understand the threshold dynamics, we firstly introduce the thresholds $R_0^{DA}$ for the reaction-diffusion-advection problem with Dirichlet boundary condition in a fixed interval $(p, q)$. On this basis, we derive the index risk $R_0^F(t)$ for the problem with the free boundary by variation method and give some properties of the index risk $R_0^F(t)$. With the risk index $R_0^F(t)$ as threshold, we give some sufficient conditions for the virus to extinct or persist. Our theoretical results shown that if $R_0^F(t_0) \geq 1$ for some $t_0 \geq 0$, the virus persist eventually (Theorem 5.1, Remark 5.1). If $R_0^F(0) < 1$, the virus extinction happens provided that the initial scales of the infected mosquitoes and infected birds are small (Theorem 4.4) or the expanding capability $\mu$ is small enough (Theorem 4.5), while persistence occurs for the large expanding capability (Theorem 5.2).

The risk index $R_0^F(t)$, which change with time $t$, is similar as the threshold of the basic reproduction number in fixed region, that is an important parameter in epidemiology. Additionally, from the expression of risk index $R_0^F(t)$, we can derive that if the advection intensity is big, the risk index $R_0^F(t)$ will become small, which is beneficial to the extinction for the virus. As we know, some environmental factors, such as landuse, climate, and host community composition, can influence the abundance of WNv hosts and vectors, and subsequently, affect WNv transmission rates. Therefore, in our work we consider the environmental heterogeneity. That means the spatial-dependent rates considered in our model close more to the reality. Specifically, it is an effective way to stop WNv transmission by improving breeding grounds for infected mosquitoes and infected birds, such as keeping environment clean, getting rid of stagnant and dirty water, which can lower the transmission rates.

There are still many meaningful and challenging mathematical questions which need to be studied for the free boundary problem. For example, in this paper, we only consider small advection case and present the spreading-vanishing dichotomy. However, for large advection, we believe, it will cause more complex transmission dynamics, such as virtual spreading, virtual vanishing and transition, which deserves further study. Additionally, the model can also be generalized to the case as discussed in Madana and Yang [31], where the infected mosquitoes and infected birds are all dispersal and have the influence of advection. We leave these for future investigations.

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