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SUPERCLOSENESS ANALYSIS OF STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION EQUATIONS

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Abstract Recently, a stabilizer free weak Galerkin (SFWG) method is proposed in [14], which is easier to implement and more efficient. In this paper, we developed an SFWG scheme for solving the general second-order elliptic problem on triangular meshes in 2D. This new SFWG method will dramatically reduce the error between the L^2-projection of the exact solution and the numerical solution.

Keywords Stabilizer free, weak Galerkin finite element methods, second-order elliptic problems, weak gradient, error estimates, supercloseness.

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1. Introduction

In this paper, we are concerned with the development of an SFWG finite element method using the following convection-diffusion equation

$$-\nabla\cdot(\alpha\nabla u)+\nabla\cdot(\boldsymbol{\beta}u)+cu=f \quad \text{in } \Omega, \tag{1.1}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{1.2}$$

as the model problem, where Ω is a polygonal domain in $\mathbb{R}^2$, $\alpha=\alpha(x)$ is the diffusion coefficient matrix, $\boldsymbol{\beta}=\boldsymbol{\beta}(x)$ is the convection coefficient, and $c=c(x)$ is the reaction coefficient. We suppose that $\alpha=(\alpha_{ij})_{2\times 2}\in[W^{1,\infty}(\Omega)]^{2\times 2}, c\in W^{1,\infty}(\Omega), 0\le c(x)\le M, \boldsymbol{\beta}\in[W^{1,\infty}(\Omega)]^2$, and $c-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\ge c_0>0$ for some constant c_0 (cf., e.g., [12]), and there exists positive constants $\alpha_m\le\alpha_M$ such that

$$\alpha_m\xi^T\xi\le\xi^T\alpha(x)\xi\le\alpha_M\xi^T\xi, \quad \forall\xi\in\mathbb{R}^2, x\in\Omega.$$

Using integration by parts, we derive a variational formulation for the problem (1.1)-(1.2) as follows: seek $u\in H_0^1(\Omega)$ such that

$$(\alpha\nabla u,\nabla v)+(\nabla\cdot(\boldsymbol{\beta}u),v)+(cu,v)=(f,v), \quad \forall v\in H_0^1(\Omega). \tag{1.3}$$

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The weak Galerkin (WG) finite element method for the model problem (1.1)-(1.2) has the following formulation

$$(\alpha\nabla_w u_h, \nabla_w v) + (\nabla_w \cdot (\boldsymbol{\beta} u_h), v) + (cu_h, v) + s(u_h, v) = (f, v), \tag{1.4}$$

where ∇_w is the weak gradient operator, $\nabla_w \cdot (\boldsymbol{\beta} u_h)$ is the weak divergence, and $s(u_h, v)$ in (1.4) is a stabilizer term that enforces a sufficient weak continuity for the numerical solution. Recently, the weak Galerkin method has been developed to solve the elliptic equations [16], the nonlinear convection–diffusion problems in 1D and 2D [6,7], convection-dominated diffusion equations [9], Crank-Nicolson scheme for Parabolic interface problems [8], and singularly perturbed convection–diffusion problems [10,11].

A stabilizer free weak Galerkin finite element method is proposed by Ye and Zhang in [14] for the solution of Poisson equation on polytopal meshes in 2D or 3D, where $\big(P_k(T), P_k(e), [P_j(T)]^d\big)$ elements are used. It is shown that there is a $j_0 > 0$ so that the SFWG method converges with optimal order for any $j \geq j_0$. However, when j is too large, the magnitude of the weak gradient can be extremely large, causing numerical instability. In [1], the optimal j_0 is given to improve the efficiency and to avoid unnecessary numerical difficulties. In this setting, if $\big(P_k(T), P_k(e), [P_j(T)]^2\big)$ elements are used for a triangular mesh, $j_0 = k + 1$, where $k \geq 1$. Recently, the SFWG finite element method has been developed to solve the parabolic equations [3], Stokes equations [15] and the second order elliptic problem [4]. In [2], the authors proposed a scheme using $\big(P_0(T), P_1(e), [P_1(T)]^2\big)$ elements for triangular meshes with the optimal order of convergence. In [5], a SFWG finite element $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ is investigated for second order elliptic problems.

In this paper, we will develop the theoretical foundation of the SFWG scheme using the general $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements for solving the convection-diffusion equation (1.1)-(1.2) on a triangle mesh in $2D$. As one of the main contributions of this paper, it is shown that by replacing $\big(P_k(T), P_k(e), [P_{k+1}(T)]^2\big)$ elements with $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements, the error between L^2-projection of the exact solution and the numerical solution will be dramatically reduced.

The rest of this paper is organized as follows. In Section 2, the notations and finite element spaces are introduced. Section 3 is devoted to investigating the error equations and several other required inequalities. The error analysis for the SFWG solutions in an energy norm is studied in Section 4. In Section 5, we will derive the L^2 error estimates for the SFWG finite element method for solving (1.1)-(1.2). Numerical test results are presented in Section 6. Concluding remarks are given in Section 7.

2. Notations

In this section, we shall introduce some notations and definitions. Suppose $\mathcal{T}_h$ is a quasi uniform triangular partition of Ω. For every element $T \in \mathcal{T}_h$, denote by h_T its diameter and $h = \max_{T\in\mathcal{T}_h} h_T$. Let $\mathcal{E}_h$ be the set of all the edges in $\mathcal{T}_h$. For $k \geq 0$, the weak Galerkin finite element space is defined as follows:

$$V_h = \{\{v_0, v_b\} : v_0 \in P_k(T), \forall T \in \mathcal{T}_h, \text{ and } v_b \in P_{k+1}(e), \forall e \in \mathcal{E}_h\}, \tag{2.1}$$

where the component v_0 symbolizes the interior value of v, and the component v_b symbolizes the edge value of v on each T and e, respectively. Let V_h^0 be the subspace of V_h defined as:

$$V_h^0 = \{v : v \in V_h, v_b = 0 \text{ on } \partial\Omega\}. \tag{2.2}$$

For each element $T \in \mathcal{T}_h$, let $Q_i, i = 0, 1, k+1$, be the L^2-projection onto $P_{t_i}(T)$ and let $\mathbb{Q}_h$ be the L^2-projection onto $[P_{k+1}(T)]^2$, where $t_0 = k, t_1 = 1$, and $t_{k+1} = k+1$, respectively. Also, let $\hat{Q}_h$ and $\hat{Q}_0$ be the L^2-projection onto $[P_1(T)]^2$ and $P_0(T)$, respectively, for each $T \in \mathcal{T}_h$. On each edge e, denote by Q_b the L^2-projection operator onto $P_{k+1}(e)$. Combining Q_0 and Q_b, denote by $Q_h = \{Q_0, Q_b\}$ the L^2-projection operator onto V_h.

For any $v = \{v_0, v_b\} \in V_h$, the weak gradient $\nabla_w v \in [P_{k+1}(T)]^2$ is defined on T as the unique polynomial satisfying

$$(\nabla_w v, \mathbf{q})_T = -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{q} \in [P_{k+1}(T)]^2, \tag{2.3}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂T.

Definition 2.1. For any $v = \{v_0, v_b\} \in V_h$, on each element $T \in \mathcal{T}_h$, the weak divergence of $\boldsymbol{\beta} v$ is defined as the polynomial $\nabla_w \cdot (\boldsymbol{\beta} v) \in P_k(T)$ satisfying

$$(\nabla_w \cdot (\boldsymbol{\beta} v), w)_T = -(\boldsymbol{\beta} v_0, \nabla w)_T + \langle \boldsymbol{\beta} \cdot \mathbf{n} v_b, w \rangle_{\partial T}, \quad \forall w \in P_k(T). \tag{2.4}$$

For simplicity, we adopt the following notations:

$$(v, w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw dx,$$

$$\langle v, w \rangle_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw ds.$$

We are ready to introduce an SFWG finite element scheme for the problems (1.1)-(1.2).

Algorithm 1 Stabilizer Free Weak Galerkin Algorithm

A numerical approximation for (1.1)-(1.2) can be obtained by finding $u_h = \{u_0, u_b\} \in V_h^0$, such that the following equation holds

$$(\alpha \nabla_w u_h, \nabla_w v)_{\mathcal{T}_h} + (\nabla_w \cdot (\boldsymbol{\beta} u_h), v_0)_{\mathcal{T}_h} + (c u_0, v_0) = (f, v_0), \tag{2.5}$$

for all $v = \{v_0, v_b\} \in V_h^0$.

We define the following energy norm $\|\!|\cdot|\!\|$ on V_h:

$$\|\!|v|\!\|^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w v\|_T^2 + \sum_{T \in \mathcal{T}_h} \|v_0\|_T^2. \tag{2.6}$$

An H^1 semi norm on V_h is defined as:

$$\|v\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \left(\|\nabla v_0\|_T^2 + h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right).$$

It is easy to show that $\|v\|_{1,h}$ defines a norm on V_h^0.

The following lemmas will be needed.

Lemma 2.1 (Lemma 3.2, [5]). *There exist $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 \|v\|_{1,h}^2 \leq \sum_{T \in \mathcal{T}_h} \|\nabla_w v\|_T^2 \leq C_2 \|v\|_{1,h}^2, \quad \forall v \in V_h. \tag{2.7}$$

3. Error equation

In this section, we derive an error estimate of Algorithm 1. For simplicity, we will confine our attention to the case where α in (1.1) is a piecewise constant matrix with respect to the finite element partition $\mathcal{T}_h$.

Lemma 3.1 (see [13]). *For any function $\psi \in H^1(T)$, the following trace inequality holds true:*

$$\|\psi\|_e^2 \leq C\left(h_T^{-1}\|\psi\|_T^2 + h_T\|\nabla\psi\|_T^2\right). \tag{3.1}$$

Lemma 3.2 (Inverse Inequality see [13]). *There exists a constants C such that for any piecewise polynomial $\psi|_T \in P_k(T)$,*

$$\|\nabla\psi\|_T \leq Ch_T^{-1}\|\psi\|_T, \qquad \forall T \in \mathcal{T}_h. \tag{3.2}$$

Lemma 3.3. *Let $u \in H_0^{k+2}(\Omega)$, and $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumptions. Then, the L^2 projections Q_0 and $\mathbb{Q}_h$ satisfy*

$$\sum_{T\in\mathcal{T}_h}\left(\|u - Q_0u\|_T^2 + h_T^2\|\nabla(u - Q_0u)\|_T^2\right) \leq Ch^{2(k+1)}\|u\|_{k+1}^2, \tag{3.3}$$

$$\sum_{T\in\mathcal{T}_h}\left(\|\nabla u - \mathbb{Q}_h\nabla u\|_T^2 + h_T^2\|\nabla u - \mathbb{Q}_h\nabla u)\|_{1,T}^2\right) \leq Ch^{2(k+2)}\|u\|_{k+2}^2. \tag{3.4}$$

Lemma 3.4. *Let $\phi \in H^1(\Omega)$. Then for each element $T \in \mathcal{T}_h$, we have*

$$\mathbb{Q}_h(\nabla\phi) = \nabla_w Q_h\phi. \tag{3.5}$$

Proof. By definition (2.3) and integration by parts, for each $\mathbf{q} \in [P_{k+1}(T)]^2$ we have

$$\begin{aligned}(\mathbb{Q}_h(\nabla\phi), \mathbf{q})_T &= -(\phi, \nabla\cdot\mathbf{q})_T + \langle\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(Q_0\phi, \nabla\cdot\mathbf{q})_T + \langle Q_b\phi, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla_w Q_h\phi, \mathbf{q})_T,\end{aligned}$$

which implies (3.5). □

Lemma 3.5. *Let u be the solution of the convection-diffusion problem (1.1)-(1.2) and then for $v \in V_h^0$,*

$$(\nabla\cdot(\boldsymbol{\beta}u), v_0)_{\mathcal{T}_h} = (\nabla_w\cdot(\boldsymbol{\beta}Q_hu), v_0)_{\mathcal{T}_h} - \ell_{\boldsymbol{\beta_1}}(u, v) + \ell_{\boldsymbol{\beta_2}}(u, v), \tag{3.6}$$

where

$$\begin{aligned}\ell_{\boldsymbol{\beta_1}}(u, v) &= (u - Q_0u, \boldsymbol{\beta}\cdot\nabla v_0)_{\mathcal{T}_h},\\ \ell_{\boldsymbol{\beta_2}}(u, v) &= \langle u - Q_bu, \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b)\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Proof. From definition 2.1 we get

$$\begin{aligned}(\nabla\cdot(\boldsymbol{\beta}u), v_0)_{\mathcal{T}_h} &= -(\boldsymbol{\beta}u, \nabla v_0)_{\mathcal{T}_h} + \langle\boldsymbol{\beta}\cdot\mathbf{n}u, v_0\rangle_{\partial\mathcal{T}_h}\\ &= -(\boldsymbol{\beta}Q_0u, \nabla v_0)_{\mathcal{T}_h} + (\boldsymbol{\beta}Q_0u - \boldsymbol{\beta}u, \nabla v_0)_{\mathcal{T}_h}\\ &\quad + \langle\boldsymbol{\beta}\cdot\mathbf{n}Q_bu, v_0\rangle_{\partial\mathcal{T}_h} + \langle\boldsymbol{\beta}\cdot\mathbf{n}(u - Q_bu), v_0\rangle_{\partial\mathcal{T}_h}\\ &= (\nabla_w\cdot(\boldsymbol{\beta}Q_hu), v_0)_{\mathcal{T}_h} - \ell_{\boldsymbol{\beta_1}}(u, v) + \ell_{\boldsymbol{\beta_2}}(u, v)\end{aligned}$$

where in the last equality we have used the fact that $\langle\boldsymbol{\beta}\cdot\mathbf{n}(u - Q_bu), v_b\rangle_{\partial\mathcal{T}_h} = 0$, and thus completes the proof. □

Lemma 3.6. *For any $T \in \mathcal{T}_h$ and $\mathbf{q} \in [P_{k+1}(T)]^2$,*

$$(\nabla_w v, \mathbf{q})_T = (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}. \tag{3.7}$$

Proof. (3.7) follows directly from the definition of $\nabla_w v$ and integration by parts:

$$\begin{aligned}(\nabla_w v, \mathbf{q})_T &= -(v_0, \nabla \cdot \mathbf{q})_T + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.7. *Let $\phi \in H^1(\Omega)$. Then for all $v \in V_h^0$, we have*

$$(\alpha \nabla \phi, \nabla v_0)_T = (\alpha \nabla_w Q_h \phi, \nabla_w v)_T + \langle (\mathbb{Q}_h(\alpha \nabla \phi)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}. \tag{3.8}$$

Proof. Let $\mathbf{q} = \mathbb{Q}_h(\alpha \nabla \phi)$. It follows from Lemma 3.6 that

$$\begin{aligned}(\nabla_w v, \mathbf{q})_T &= (\nabla v_0, \mathbf{q})_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla v_0, \alpha \nabla \phi)_T + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}.\end{aligned} \tag{3.9}$$

Then (3.8) follows from Lemma 3.4 and (3.9). □

Lemma 3.8. *Let $e_h = Q_h u - u_h \in V_h^0$. Then for any $v \in V_h^0$, we have*

$$\begin{aligned}&(\alpha \nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} + (\nabla_w \cdot (\boldsymbol{\beta} e_h), v_0)_{\mathcal{T}_h} + (c e_0, v_0) \\ =&\ell_\alpha(u, v) + \ell_{\boldsymbol{\beta}}(u, v) + \ell_c(u, v),\end{aligned} \tag{3.10}$$

where $\ell_\alpha(u, v), \ell_{\boldsymbol{\beta}}(u, v)$ and $\ell_c(u, v)$ are defined as follows:

$$\begin{aligned}\ell_\alpha(u, v) &= \sum_{T \in \mathcal{T}_h} \langle (\alpha \nabla u - \mathbb{Q}_h(\alpha \nabla u)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}, \\ \ell_{\boldsymbol{\beta}}(u, v) &= \ell_{\boldsymbol{\beta_1}}(u, v) - \ell_{\boldsymbol{\beta_2}}(u, v), \\ \ell_c(u, v) &= -(cu - cQ_0 u, v_0).\end{aligned}$$

Proof. Testing (1.1) by $v = \{v_0, v_b\} \in V_h^0$, we obtain

$$-(\nabla \cdot (\alpha \nabla u), v_0) + (\nabla \cdot (\boldsymbol{\beta} u), v_0) + (cu, v_0) = (f, v_0). \tag{3.11}$$

Using integration by part and the fact that $\sum_{T \in \mathcal{T}_h} \langle \alpha \nabla u \cdot \mathbf{n}, v_b \rangle_{\partial T} = 0$, we get

$$-(\nabla \cdot (\alpha \nabla u), v_0) = (\alpha \nabla u, \nabla v_0)_T - \langle \alpha \nabla u \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}. \tag{3.12}$$

It follows from Lemma 3.7 that

$$(\alpha \nabla u, \nabla v_0)_T = (\alpha \nabla_w (Q_h u), \nabla_w v)_T + \langle (\mathbb{Q}_h(\alpha \nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}. \tag{3.13}$$

Combining (3.12) and (3.13) gives

$$-(\nabla \cdot (\alpha \nabla u), v_0) = (\alpha \nabla_w (Q_h u), \nabla_w v)_T - \ell_\alpha(u, v). \tag{3.14}$$

Using (3.6) and (3.14), (3.11) becomes

$$\begin{aligned}&(\alpha \nabla_w (Q_h u), \nabla_w v)_{\mathcal{T}_h} + (\nabla_w \cdot (\boldsymbol{\beta} Q_h u), v_0)_{\mathcal{T}_h} + (cQ_0 u, v_0) \\ =& (f, v_0) + \ell_\alpha(u, v) + \ell_{\boldsymbol{\beta}}(u, v) + \ell_c(u, v).\end{aligned} \tag{3.15}$$

Subtracting (2.5) from (3.15) yields the error equation (3.10), and thus completes the proof. □

4. Error Estimates

We will derive error estimates in this section.

Lemma 4.1. *Let $u \in H_0^{k+2}(\Omega)$ be the solution of the problem (1.1)-(1.2). Then for $v \in V_h^0$,*

$$|\ell_\alpha(u,v)| \le Ch^{k+2}\|u\|_{k+3}|||v|||, \tag{4.1}$$
$$|\ell_{\boldsymbol{\beta}}(u,v)| \le Ch^{k+2}\|u\|_{k+2}|||v|||, \tag{4.2}$$
$$|\ell_c(u,v)| \le Ch^{k+2}\|u\|_{k+1}|||v|||. \tag{4.3}$$

Proof. By using Cauchy-Schwarz inequality, the trace inequality (3.1), Lemmas 3.3 and 2.1, we obtain

$$\begin{aligned}
|\ell_\alpha(u,v)| &\le \sum_{T\in\mathcal{T}_h} |\langle \alpha(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}| \\
&\le C\sum_{T\in\mathcal{T}_h} \|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}\|v_0 - v_b\|_{\partial T} \\
&\le C\Big(\sum_{T\in\mathcal{T}_h} h_T\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\Big)^{1/2} \\
&\le Ch^{k+2}\|u\|_{k+3}|||v|||.
\end{aligned}$$

To prove (4.2), we need to estimate $\ell_{\boldsymbol{\beta_1}}(u,v)$ and $\ell_{\boldsymbol{\beta_2}}(u,v)$ first. It follows from Cauchy-Schwarz inequality and 3.3 that

$$\begin{aligned}
|\ell_{\boldsymbol{\beta_1}}(u,v)| &= |(u - Q_0u, \boldsymbol{\beta}\cdot\nabla v_0)_{\mathcal{T}_h}| \\
&= \Big|(u - Q_0u, (\hat{Q}_h\boldsymbol{\beta})\cdot\nabla v_0)_{\mathcal{T}_h} + (u - Q_0u, (\boldsymbol{\beta} - \hat{Q}_h\boldsymbol{\beta})\cdot\nabla v_0)_{\mathcal{T}_h}\Big| \\
&= \Big|\sum_{T\in\mathcal{T}_h}(u - Q_0u, (\boldsymbol{\beta} - \hat{Q}_h\boldsymbol{\beta})\cdot\nabla v_0)_T\Big| \\
&\le Ch\sum_{T\in\mathcal{T}_h}\|u - Q_0u\|_T\|\boldsymbol{\beta}\|_{W^{1,\infty}(T)}\|\nabla v_0\|_T \\
&\le Ch\sum_{T\in\mathcal{T}_h}\|u - Q_0u\|_T\|\nabla v_0\|_T \\
&\le Ch\sum_{T\in\mathcal{T}_h}\|u - Q_0u\|_T\|v\|_{1,h,T} \\
&\le Ch^{k+2}\|u\|_{k+1}|||v|||.
\end{aligned} \tag{4.4}$$

It follows from the definition of Q_b and the Cauchy-Schwarz inequality that

$$\begin{aligned}
|\ell_{\boldsymbol{\beta_2}}(u,v)| &= |\,\langle u - Q_bu, \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b)\rangle_{\partial\mathcal{T}_h}\,| \\
&\le C\sum_{T\in\mathcal{T}_h}\|u - Q_bu\|_{\partial T}\|\boldsymbol{\beta}\|_{L^\infty(T)}\|v_0 - v_b\|_{\partial T} \\
&\le C\left(\sum_{T\in\mathcal{T}_h}\sum_{e\in\partial T} h_T\|u - Q_{k+1}u\|_e^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|v_0 - v_b\|_{\partial T}^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq Ch^{k+2}\|u\|_{k+2}|||v|||. \tag{4.5}$$

(4.2) follows from combining (4.4) and (4.5) and where we used the fact $\|u - Q_{k+1}u\|_e \geq \|u - Q_b u\|_e$ in the inequality (4.5).

The last estimate (4.3) is resulting from the Cauchy-Schwarz inequality and Lemma 3.3:

$$\begin{aligned}
|\ell_c(u,v)| &= |(cu - cQ_0u, v_0)| \\
&= \left|(u - Q_0u, (\hat{Q}_0c)v_0) + (u - Q_0u, (c - \hat{Q}_0c)v_0)\right| \\
&\leq \left|(Q_0u - u, (c - \hat{Q}_0c)v_0)\right| \\
&\leq Ch^{k+2}\|u\|_{k+1}|||v|||\|c\|_{1,\infty}.
\end{aligned}$$

This completes the proof. □

Remark 4.1. If we drop the requirement $c \in C^1(\bar{\Omega})$, then

$$|\ell_c(u,v)| \leq Ch^{k+1}\|u\|_{k+1}|||v|||. \tag{4.6}$$

Lemma 4.2. *For any $v \in V_h^0$,*

$$(\nabla_w \cdot (\boldsymbol{\beta}v), v_0)_{\mathcal{T}_h} = -\frac{1}{2}(\nabla \cdot \boldsymbol{\beta}v_0, v_0)_{\mathcal{T}_h} - \frac{1}{2}\left\langle \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b), v_0 - v_b\right\rangle_{\partial\mathcal{T}_h}. \tag{4.7}$$

Proof. Let $v, w \in V_h^0$. From the definition 2.1 and integration by parts, we obtain

$$\begin{aligned}
(\nabla_w \cdot (\boldsymbol{\beta}v), w_0)_{\mathcal{T}_h} &= -(\boldsymbol{\beta}v_0, \nabla w_0)_{\mathcal{T}_h} + \langle \boldsymbol{\beta}\cdot\mathbf{n}v_b, w_0\rangle_{\partial\mathcal{T}_h} \\
&= (\nabla\cdot\boldsymbol{\beta}v_0 + \boldsymbol{\beta}\cdot\nabla v_0, w_0)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b), w_0\rangle_{\partial\mathcal{T}_h},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
(\nabla_w \cdot (\boldsymbol{\beta}w), v_0)_{\mathcal{T}_h} &= -(\boldsymbol{\beta}w_0, \nabla v_0)_{\mathcal{T}_h} + \langle \boldsymbol{\beta}\cdot\mathbf{n}w_b, v_0\rangle_{\partial\mathcal{T}_h} \\
&= -(\boldsymbol{\beta}w_0, \nabla v_0)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b), w_b\rangle_{\partial\mathcal{T}_h},
\end{aligned} \tag{4.9}$$

here we have used that the fact that $\langle \boldsymbol{\beta}\cdot\mathbf{n}v_b, w_b\rangle_{\partial\mathcal{T}_h} = 0$ in the last equality. By summing (4.8) and (4.9), and setting $v = w$, we have

$$(\nabla_w \cdot (\boldsymbol{\beta}v), v_0)_{\mathcal{T}_h} = -\frac{1}{2}(\nabla \cdot \boldsymbol{\beta}v_0, v_0)_{\mathcal{T}_h} - \frac{1}{2}\left\langle \boldsymbol{\beta}\cdot\mathbf{n}(v_0 - v_b), v_0 - v_b\right\rangle_{\partial\mathcal{T}_h}.$$

□

Lemma 4.3. *Let $e_h = \{e_0, e_b\} = \{Q_0u - u_0, Q_bu - u_b\}$. Then, there exists a constant $\gamma > 0$, such that*

$$(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w \cdot (\boldsymbol{\beta}e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0) > \gamma|||e_h|||^2, \tag{4.10}$$

if $0 < h \ll 1$.

Proof. Let $\gamma = \frac{1}{2}\min\{\alpha_m, c_0\}$ and $v = e_h$. From (4.7) and the definition of $|||\cdot|||$ in (2.6), we get

$$(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w \cdot (\boldsymbol{\beta}e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0)$$

$$
\begin{aligned}
&= (\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} - \frac{1}{2}(\nabla\cdot(\boldsymbol{\beta} e_0), e_0)_{\mathcal{T}_h} - \frac{1}{2}\langle \boldsymbol{\beta}\cdot\mathbf{n}(e_0-e_b), e_0-e_b\rangle_{\partial\mathcal{T}_h} + (ce_0, e_0)\\
&\geq (\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} - \frac{1}{2}\sum_{T\in\mathcal{T}_h}(e_0, \nabla\cdot(\boldsymbol{\beta} e_0)) + (ce_0, e_0) - \left|\frac{1}{2}\langle \boldsymbol{\beta}\cdot(e_0-e_b), e_0-e_b\rangle_{\partial\mathcal{T}_h}\right|\\
&\geq (\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (ce_0, e_0) - Ch\sum_{T\in\mathcal{T}_h}\frac{1}{h_T}\langle e_0-e_b, e_0-e_b\rangle_{\partial T}\\
&\geq \alpha_m(\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + c_0(e_0, e_0) - Ch|\!|\!|e_h|\!|\!|^2\\
&> \gamma|\!|\!|e_h|\!|\!|^2,
\end{aligned}
$$

if $Ch < \gamma$, which completes the proof. □

Lemma 4.4. *The weak Galerkin scheme 2.5 has one and only one solution when h is small enough.*

Proof. It suffices to verify the uniqueness for the homogeneous equation. Assume that $u_h^{(1)}$ and $u_h^{(2)}$ are two solutions of (2.5). Then $e_h = u_h^{(1)} - u_h^{(2)}$ would satisfy the forthcoming equation

$$(\alpha\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} + (\nabla_w\cdot(\boldsymbol{\beta} e_h), v_0)_{\mathcal{T}_h} + (ce_0, v_0) = 0, \quad \forall v\in V_h^0. \tag{4.11}$$

Note that $e_h \in V_h^0$. Suppose that $v = e_h$, in the equation (4.11) we obtain

$$(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w\cdot(\boldsymbol{\beta} e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0) = 0.$$

From Lemma 4.3, we have

$$\gamma|\!|\!|e_h|\!|\!| < (\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w\cdot(\boldsymbol{\beta} e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0) = 0.$$

It follows from (2.7) that $\|e_h\|_{1,h} = 0$. Since $\|\cdot\|_{1,h}$ is a norm in V_h^0 which implies $e_h = 0$. Consequently, $u_h^{(1)} \equiv u_h^{(2)}$ and thus completes the proof. □

Theorem 4.1. *Let $u_h \in V_h$ be the SFWG finite element solution of (2.5). In addition, assuming the regularity of exact solution $u \in H_0^{k+2}(\Omega)$, then there exists a constant C such that*

$$|\!|\!|Q_h u - u_h|\!|\!| \leq Ch^{k+2}\|u\|_{k+3}. \tag{4.12}$$

Proof. It follows from (4.10) that

$$|\!|\!|Q_h u - u_h|\!|\!|^2 \leq C\left((\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w\cdot(\boldsymbol{\beta} e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0)\right).$$

Letting $v = e_h$ in (3.10), yields

$$(\alpha\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} + (\nabla_w\cdot(\boldsymbol{\beta} e_h), e_0)_{\mathcal{T}_h} + (ce_0, e_0) = \ell_\alpha(u, e_h) + \ell_{\boldsymbol{\beta}}(u, e_h) + \ell_c(u, e_h).$$

Then (4.12) follows from Lemma 4.1. □

5. Error Estimates in L^2 norm

A duality argument is utilized to get L^2 error estimate. Let $e_h = \{e_0, e_b\} = Q_h u - u_h = \{Q_0 u - u_0, Q_b u - u_b\}$. Consider the dual problem of seeking $\Phi \in H_0^2(\Omega)$ satisfying

$$-\nabla\cdot(\alpha\nabla\Phi) - \boldsymbol{\beta}\cdot\nabla\Phi + c\Phi = e_0, \quad \text{in } \Omega \tag{5.1}$$

$$\Phi = 0, \quad \text{on } \partial\Omega.$$

Suppose that the following H^2-regularity condition holds true

$$\|\Phi\|_2 \le C\|e_0\|. \tag{5.2}$$

Theorem 5.1. *Let $u_h = \{u_0, u_b\} \in V_h$ be the SFWG finite element solution of (2.5). Assume that the exact solution $u \in H_0^{k+2}(\Omega)$ and (5.2) holds true. Then, there exists a constant C such that when $0 < h \ll 1$,*

$$\|Q_0 u - u_0\|_0 \le Ch^{k+3}\|u\|_{k+3}, \tag{5.3}$$

$$\|u - u_0\|_0 \le Ch^{k+1}\|u\|_{k+1}. \tag{5.4}$$

Proof. Testing (5.1) with e_0, we obtain

$$\|e_0\|^2 = -(\nabla\cdot(\alpha\nabla\Phi), e_0) - (\boldsymbol{\beta}\cdot\nabla\Phi, e_0) + (c\Phi, e_0). \tag{5.5}$$

Using integration by part and the fact that $\sum_{T\in\mathcal{T}_h}\langle\alpha\nabla\Phi\cdot\mathbf{n}, e_b\rangle_{\partial T} = 0$, we have

$$-(\nabla\cdot(\alpha\nabla\Phi), e_0) = (\alpha\nabla\Phi, \nabla e_0)_{\mathcal{T}_h} - \langle\alpha\nabla\Phi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h}. \tag{5.6}$$

Substituting (5.6) into (5.5) yields

$$\begin{aligned}
\|e_0\|^2 &= (\alpha\nabla\Phi, \nabla e_0)_{\mathcal{T}_h} - \langle\alpha\nabla\Phi\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{\beta}e_0, \nabla\Phi) + (c\Phi, e_0)\\
&= (\alpha\nabla_w e_h, \nabla_w Q_h\Phi) - \langle\alpha(\nabla\Phi - Q_h\nabla\Phi)\cdot\mathbf{n}, e_0 - e_b\rangle + (\nabla_w\cdot(\boldsymbol{\beta}e_h), Q_0\Phi)\\
&\quad - \langle\boldsymbol{\beta}\cdot\mathbf{n}e_b, Q_0\Phi\rangle - (\boldsymbol{\beta}e_0, \nabla(\Phi - Q_0\Phi)) + (ce_0, Q_0\Phi) + (ce_0, \Phi - Q_0\Phi)\\
&= \ell_\alpha(u, Q_h\Phi) + \ell_{\boldsymbol{\beta}}(u, Q_h\Phi) + \ell_c(u, Q_h\Phi) - \ell_\alpha(\Phi, e_h) - \langle\boldsymbol{\beta}\cdot\mathbf{n}e_b, Q_0\Phi\rangle\\
&\quad - (\boldsymbol{\beta}e_0, \nabla(\Phi - Q_0\Phi)) + (ce_0, \Phi - Q_0\Phi)\\
&= \ell_\alpha(u, Q_h\Phi) + \ell_{\boldsymbol{\beta}}(u, Q_h\Phi) + \ell_c(u, Q_h\Phi) - \ell_\alpha(\Phi, e_h) - I_1 - I_2 + I_3.
\end{aligned} \tag{5.7}$$

To estimate the terms on the right-hand side of the above equation, we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|\ell_\alpha(u, Q_h\Phi)| &= \left|\sum_{T\in\mathcal{T}_h}\langle\alpha(\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, Q_0\Phi - Q_b\Phi\rangle_{\partial T}\right|\\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\right)^{\frac12}\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b\Phi\|_{\partial T}^2\right)^{\frac12}.
\end{aligned} \tag{5.8}$$

By using the trace inequality (3.1) and the definition of Q_b, we get

$$\begin{aligned}
\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - Q_b\Phi\|_{\partial T}^2\right)^{\frac12} &\le \left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - \Phi\|_{\partial T}^2 + \|\Phi - Q_b\Phi\|_{\partial T}^2\right)^{\frac12}\\
&\le \left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi - \Phi\|_{\partial T}^2\right)^{\frac12} \le Ch^{3/2}\|\Phi\|_2,
\end{aligned} \tag{5.9}$$

and

$$\left(\sum_{T\in\mathcal{T}_h}\|\nabla u - \mathbb{Q}_h\nabla u\|_{\partial T}^2\right)^{\frac12} \le Ch^{k+3/2}\|u\|_{k+3}. \tag{5.10}$$

By combining (5.9) and (5.10) with (5.8), we obtain

$$|\ell_\alpha(u, Q_h\Phi)| \le Ch^{k+3}\|u\|_{k+3}\|\Phi\|_2. \tag{5.11}$$

It follows from the Cauchy-Schwarz inequality, the trace inequality (3.1), (2.7), (4.1) and (4.12) that

$$\begin{aligned}
|\ell_\alpha(\Phi, e_h)| &= \left|\sum_{T\in\mathcal{T}_h}\langle\alpha(\nabla\Phi-\mathbb{Q}_h\nabla\Phi)\cdot\mathbf{n}, e_0-e_b\rangle_{\partial T}\right| \\
&\le C\left(\sum_{T\in\mathcal{T}_h} h_T\|\nabla\Phi-\mathbb{Q}_h\nabla\Phi\|_{\partial T}^2\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|e_0-e_b\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch\|\Phi\|_2{|\!|\!|}e_h{|\!|\!|} \\
&\le Ch^{k+3}\|u\|_{k+3}\|\Phi\|_2.
\end{aligned} \tag{5.12}$$

The estimate (4.2) and Lemma 3.3 give

$$\begin{aligned}
|\ell_{\boldsymbol\beta}(u, Q_h\Phi)| &= \left|(u-Q_0u, \boldsymbol\beta\cdot\nabla\Phi_0)_{\mathcal{T}_h} - \langle u-Q_bu, \boldsymbol\beta\cdot\mathbf{n}(\Phi_0-\Phi_b)\rangle_{\partial\mathcal{T}_h}\right| \\
&\le Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2 + Ch^{k+2}\|u\|_{k+2}\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\Phi-Q_b\Phi\|_{\partial T}^2\right)^{\frac{1}{2}} \\
&\le Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2.
\end{aligned} \tag{5.13}$$

It follows from the Cauchy-Schwarz inequality and the trace inequality (3.1 that

$$\begin{aligned}
|I_1| &= |\langle\boldsymbol\beta\cdot\mathbf{n}e_b, Q_0\Phi\rangle| \\
&\le |\langle\boldsymbol\beta\cdot\mathbf{n}e_b, Q_0\Phi-\Phi\rangle| \\
&\le |\langle\boldsymbol\beta\cdot\mathbf{n}(e_b-e_0), Q_0\Phi-\Phi\rangle| + |\langle\boldsymbol\beta\cdot\mathbf{n}e_0, Q_0\Phi-\Phi\rangle| \\
&\le C\sum_{T\in\mathcal{T}_h}\|\boldsymbol\beta\|_{L^\infty(\partial T)}\|Q_0\Phi-\Phi\|_{\partial T}\left(\|e_0-e_b\|_{\partial T}+\|e_0\|_{\partial T}\right) \\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|Q_0\Phi-\Phi\|_{\partial T}^2\right)^{\frac{1}{2}}\left[\left(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|e_0-e_b\|_{\partial T}^2\right)^{\frac{1}{2}} + \frac{\|e_0\|}{h^{\frac{1}{2}}}\right] \\
&\le Ch\|\Phi\|_2\left({|\!|\!|}e_h{|\!|\!|} + h^{\frac{-1}{2}}\|e_0\|\right) \\
&\le Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2 + Ch^{\frac{1}{2}}\|e_0\|^2.
\end{aligned} \tag{5.14}$$

The estimates (4.2), (4.12), and Lemma 3.3 give

$$\begin{aligned}
|I_2| = |(\boldsymbol\beta e_0, \nabla(\Phi-Q_0\Phi))| &\le Ch\sum_{T\in\mathcal{T}_h}\|\nabla(\Phi-Q_0\Phi)\|_T\|\boldsymbol\beta\|_{W^{1,\infty}(T)}\|e_0\|_T \\
&\le Ch\sum_{T\in\mathcal{T}_h}\|\nabla(\Phi-Q_0\Phi)\|_T\|e_0\|_T \\
&\le Ch\left(\sum_{T\in\mathcal{T}_h}\|\nabla(\Phi-Q_0\Phi)\|_T^2\right)^{\frac{1}{2}}{|\!|\!|}e_h{|\!|\!|} \\
&\le Ch^{k+2}\|\Phi\|_2\|u\|_{k+2}
\end{aligned}$$

$$\leq Ch^{k+3}\|\Phi\|_2\|u\|_{k+2}. \tag{5.15}$$

It follows from the estimate (4.3) that

$$|I_3| = |(ce_0, \Phi - Q_0\Phi)| \leq Ch^{k+2}\|\Phi\|_2 |||e_h||| \leq Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2. \tag{5.16}$$

Similarly,

$$|\ell_c(u, Q_h\Phi)| \leq Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2. \tag{5.17}$$

Now combining (5.7) with the estimates (5.11)-(5.17), we have

$$\|e_0\|^2 \leq Ch^{k+3}\|u\|_{k+2}\|\Phi\|_2 + Ch^{\frac{1}{2}}\|e_0\|^2, \tag{5.18}$$

which combined with (5.2) and the triangle inequality, provides the required error estimate (5.3). (5.4) follows from

$$\|u - u_0\|_0 \leq \|u - Q_0u\|_0 + \|Q_0u - u_0\|_0 \leq Ch^{k+1}\|u\|_{k+1}.$$

□

6. Numerical Experiments

In this section, various numerical examples in $2D$ uniform triangular meshes are presented to validate the theoretical results derived in previous sections. We will compare our SFWG method (2.5) with the older version of the SFWG method proposed in [1, 14] and the WG method in [10].

6.1. Example 1 (Constant diffusion α, convection $\boldsymbol{\beta}$ and reaction c)

In this example, we use the SFWG scheme (2.5) to solve the convection-diffusion equations (1.1)-(1.2) posed on the unit square $\Omega = (0,1) \times (0,1)$ with the following data: $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{\beta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $c = 1$, and the analytic solution is

$$u(x, y) = \cos(x)\cos(\pi y).$$

The source term $f(x, y)$ and the boundary conditions are computed accordingly. We applied the SFWG algorithm 1 with $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements and the older version of the SFWG algorithm with $(P_k(T), P_k(e), [P_{k+1}(T)]^2)$ elements in the computation. As we can see in Table 1 that the error between u_0 and Q_0u, the numerical solution obtained by using the SFWG method (2.5) and the L^2-projection of u, respectively, is $\|Q_0u - u_0\| = \mathcal{O}(h^{k+3})$. If the older version of the SFWG method is used, $\|Q_0u - u_0\| = \mathcal{O}(h^{k+1})$. If the WG is used with $\big(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\big)$ elements, $\|Q_0u - u_0\| = \mathcal{O}(h^{k+1})$, as can be see from Figure 1. Thus our new SFWG method is much more accurate. Since we have increased the complexity, the new SFWG method is slower than the older version of the SFWG method. However, in comparison with the WG method, the new SFWG method is faster as can be seen from Figure 2.

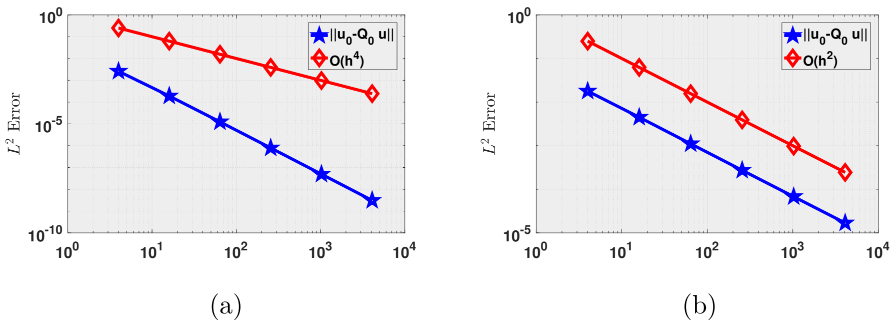


Figure 1. Example 1: Plot of the L^2-error and convergence rate for $\big(P_1(T), P_2(e), [P_2(T)]^2\big)$ and $h = 1/64$: (a) SFWG method (2.5); (b) WG method.

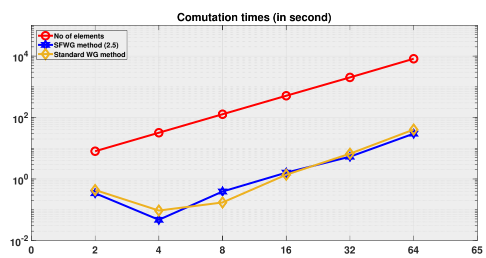


Figure 2. Example 1: Comparison of computation times for the SFWG algorithm 1 and the WG algorithm using $\big(P_1(T), P_2(e), [P_2(T)]^2\big)$ elements.

Table 1. Example 1:Errors and convergence results.

		New SFWG				**Old SFWG**			
k	$1/h$	$\\|Q_h u - u_h\\|$	**Rate**	$\\|Q_0 u - u_0\\|$	**Rate**	$\\|Q_h u - u_h\\|$	**Rate**	$\\|Q_0 u - u_0\\|$	**Rate**
	2	1.4996E-01	-	6.7207E-03	-	-	-	-	-
	4	4.7127E-02	1.67	3.1720E-03	1.08	-	-	-	-
0	8	1.2476E-02	1.92	9.5774E-40	1.73	-	-	-	-
	16	3.1744E-03	1.97	2.5094E-04	1.93	-	-	-	-
	32	7.9838E-04	1.99	6.3476E-05	1.98	-	-	-	-
	64	2.0006E-04	2.00	1.5916-E-05	2.00	-	-	-	-
	2	3.0030E-02	-	2.5934E-03	-	4.7984E-01	-	3.5246E-02	-
	4	4.1036E-03	2.87	1.9037E-04	3.77	2.3818E-01	1.01	9.5301E-03	1.89
1	8	5.3169E-04	2.95	1.2436E-05	3.94	1.1868E-01	1.00	2.4462E-03	1.96
	16	6.7463E-05	2.98	7.8713E-07	3.98	5.9286E-02	1.00	6.1618E-04	1.99
	32	8.4889E-06	2.99	4.9375E-08	3.99	2.9636E-02	1.00	1.5435E-04	2.00
	64	1.0644E-06	3.00	3.0892E-09	4.00	1.4817E-02	1.00	3.8609E-05	2.00
	2	3.2735E-03	-	1.4234E-04	-	8.6482E-02	-	3.4577E-03	-
	4	2.1578E-04	3.92	4.7017E-06	4.92	2.1811E-02	1.99	4.2976E-04	3.01
2	8	1.3693E-05	3.98	1.4986E-07	4.97	5.4417E-03	2.00	5.3423E-05	3.01
	16	8.6072E-07	3.99	4.7291E-09	4.99	1.3588E-03	2.00	6.6681E-06	3.02
	32	5.3930E-08	4.00	1.4851E-10	4.99	3.3960E-04	2.00	8.3318E-07	3.00
	64	3.3747E-09	4.00	4.6531E-12	5.00	8.4893E-05	2.00	1.0413E-07	3.00

6.2. Example 2 (L-shaped domain)

In this example, we consider the problem (1.1)-(1.2) posed on an L-shaped domain $\Omega = [-1,1]^2 \backslash (0,1)\times(-1,0)$ with the following data: $\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\boldsymbol{\beta} = (1,-1)^\top, c = \sin(xy)$, and $f(x,y)$ is given such that the exact solution is

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Table 2 lists errors and convergence rates in $\|\!|\cdot|\!\|$-norm and L^2-norm. It can be observed from Table 2 that the numerical solutions obtained by our SFWG algorithm 2.5 converge at rate of $k+2$ and $k+3$ in H^1-norm and L^2-norm, respectively, while the corresponding rates are k and $k+1$, respectively, by using the older version of SFWG algorithm. If the WG method is used with $\left(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2\right)$ elements, converge at rate of k in H^1-norm and k+1 in L^2-norm, as can be see from Figure 3. The numerical solutions for the SFWG (2.5) are plotted in Figure 4.

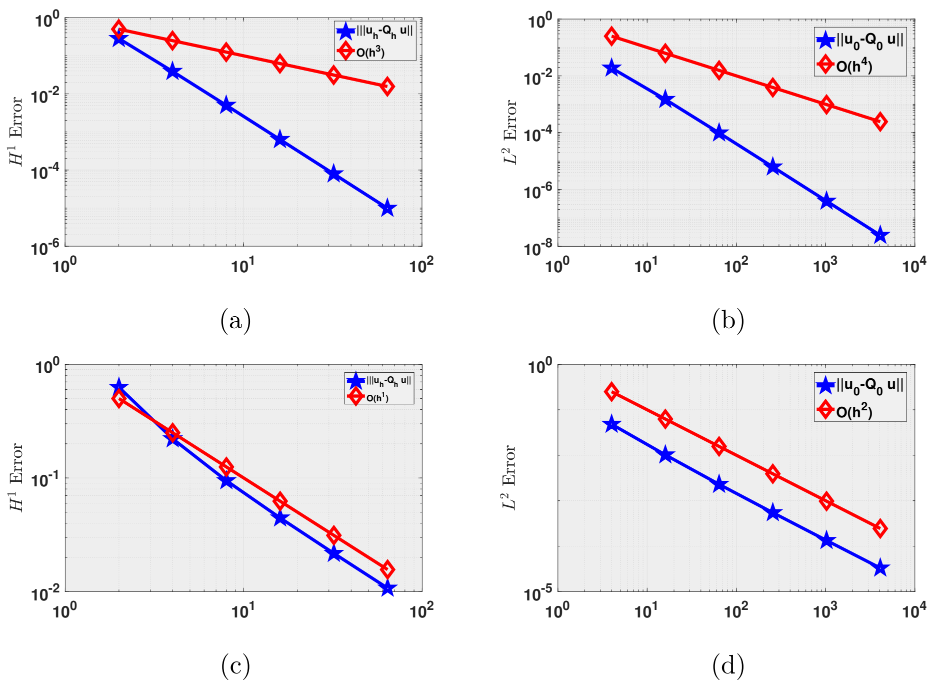
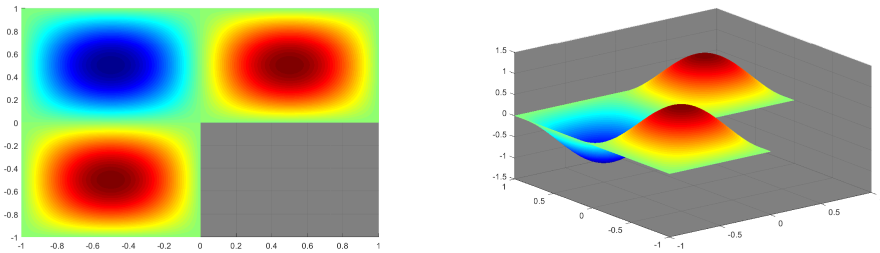


Figure 3. Example 2: Plot of the errors and convergence rate for $\left(P_1(T), P_2(e), [P_2(T)]^2\right)$ and $h = 1/64$, for errors measured by $\|Q_0u - u\|$ and $\|\!|Q_hu - u|\!\|$: (a), (b) SFWG method (2.5); (c), (d) WG method.

6.3. Example 3

Interior layer-continuous boundary condition. This example is adopted from [10]. Let $\Omega = (0,1)\times(0,1)$ with the following data: $\boldsymbol{\beta} = (1,0)^\top, c = 1$, and the exact solution is given by

$$u(x,y) = 0.5x(1-x)y(1-y)\left(1 - \tanh\frac{\eta - x}{\gamma}\right),$$



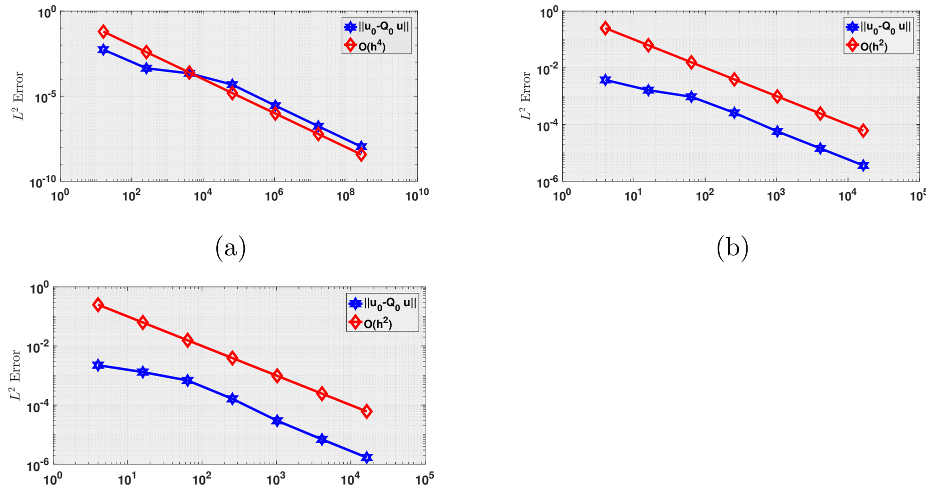
(a) (b)

Figure 4. Example 2: Plot of numerical solutions for $(P_1(T), P_2(e), [P_2(T)]^2)$ element using SFWG method (2.5)and $h = 1/64$: (a) 2D plot; (b) 3D plot.

Table 2. Example 2: Errors profile for the SFWG method using $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements and the older version of SFWG method using $(P_k(T), P_k(e), [P_{k+1}(T)]^2)$ elements.

		New SFWG				**Old SFWG**			
k	$1/h$	$\|\|\|Q_hu - u_h\|\|\|$	**Rate**	$\|\|Q_0u - u_0\|\|$	**Rate**	$\|\|\|Q_hu - u_h\|\|\|$	**Rate**	$\|\|Q_0u - u_0\|\|$	**Rate**
0	2	1.7121E-00	-	1.5180E-01	-	-	-	-	-
	4	4.9968E-01	1.77	5.3307E-02	1.51	-	-	-	-
	8	1.3033E-01	1.94	1.4578E-02	1.88	-	-	-	-
	16	3.2954E-02	1.98	3.7287E-03	1.97	-	-	-	-
	32	8.2639E-03	2.00	9.3754E-04	1.99	-	-	-	-
	64	2.0678E-03	2.00	2.3472E-04	2.00	-	-	-	-
1	2	2.8477E-01	-	1.8883E-02	-	2.4382E-00	-	1.3594E-01	-
	4	3.8981E-02	2.87	1.4809E-03	3.67	1.2058E-00	1.02	4.3569E-02	1.64
	8	5.0153E-03	2.96	9.8282E-05	3.91	5.9758E-01	1.019	1.1610E-02	1.90
	16	6.3272E-04	2.99	6.2388E-06	3.98	2.9794E-01	1.00	2.9807E-03	1.97
	32	7.9338E-05	3.00	3.9147E-07	3.99	1.4886E-01	1.00	7.4879E-04	1.99
	64	9.9290E-06	3.00	2.4491E-08	4.00	7.4414E-02	1.00	1.8743E-04	2.00
2	2	4.7787E-02	-	1.4200E-03	-	7.2082E-01	-	1.9190E-02	-
	4	3.1684E-03	3.91	4.3315E-05	5.03	1.8136E-01	1.99	2.2813E-03	3.07
	8	2.0035E-04	3.98	1.3446E-06	5.00	4.5056E-02	2.00	2.7751E-04	3.04
	16	1.2560E-05	4.00	4.2024E-08	5.00	1.1235E-02	2.00	3.4353E-05	3.01
	32	7.8588E-07	4.00	1.3142E-09	5.00	2.8069E-03	2.00	4.2782E-06	3.00
	64	4.9140E-08	4.00	4.1084E-11	5.00	7.0165E-04	2.00	5.3392E-07	3.00

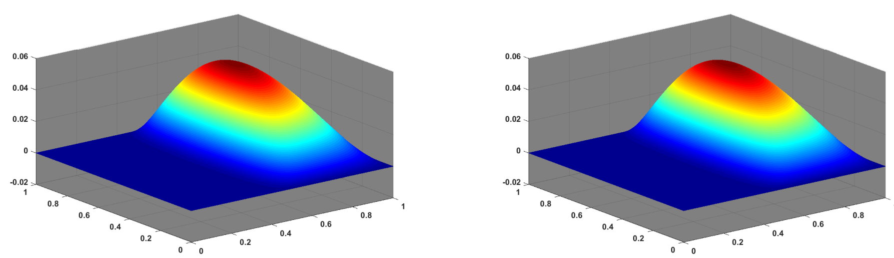
where the parameters η and γ control the location and thickness of the interior layer. Figures 5 shows that the error between u_0 and Q_0u, the numerical results obtained by using the SFWG method (2.5) and the L^2-projection of u, respectively, is $\|Q_0u - u_0\| = \mathcal{O}(h^{k+3})$. If the older version of the SFWG method is used, $\|Q_0u - u_0\| = \mathcal{O}(h^{k+1})$. If the WG is used with $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements, $\|Q_0u - u_0\| = \mathcal{O}(h^{k+1})$. Table 3 shows that the SFWG scheme (2.5) with $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$, $k = 0, 1, 2$ elements has convergence rate of $\mathcal{O}(h^{k+3})$ in $\|Q_0u - u_0\|$ and $\mathcal{O}(h^{k+2})$ in $\|\|\|Q_hu - u_h\|\|\|$ for the convection-diffusion problem with the diffusion coefficient matrix $\alpha = 0.1I_2$ and $\alpha = 0.01I_2, \eta = 0.5$, and $\gamma = 0.05$. We can capture the interior layer accurately. The numerical solutions and exact solution for the SFWG (2.5) with $\alpha = 0.01I_2$ and $(P_1(T), P_2(e), [P_2(T)]^2)$ elements are plotted in Figure 6.



(a) (b)

(c)

Figure 5. Example 3: Plot of the errors and convergence rate for errors measured by $\|Q_0u - u\|$ and $h = 1/128$: (a) SFWG method (2.5); (b) old version of SFWG method with $(P_1(T), P_2(e), [P_2(T)]^2)$ elements, (c) WG method.



(a) (b)

Figure 6. Example 3: Plot of: (a) SFWG solutions (2.5); (b) Exact solution, for $\left(P_1(T), P_2(e), [P_2(T)]^2\right)$ and $h = 1/128$.

6.4. Example 4

In this example, we use the SFWG scheme (2.5) to solve the convection-diffusion equations (1.1)-(1.2) posed on the unit square $\Omega = (0, 1)^2$ with the following data: $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{\beta} = [0.5 - y, x - 0.5]$, $c = 1$, and the exact solution is

$$u(x, y) = \exp(x)\cos(y).$$

Table 3. Example 3: Error profiles and convergence rate.

		When $\alpha = 0.1I_2$				**When** $\alpha = 0.01I_2$			
k	$1/h$	$\lvert\lvert\lvert Q_hu - u_h\rvert\rvert\rvert$	**Rate**	$\lVert Q_0u - u_0\rVert$	**Rate**	$\lvert\lvert\lvert Q_hu - u_h\rvert\rvert\rvert$	**Rate**	$\lVert Q_0u - u_0\rVert$	**Rate**
	2	2.5742E-02	-	6.7135E-03	-	8.4620E-03	-	4.4487E-03	-
	4	1.6868E-02	0.61	3.3898E-03	0.99	5.5596E-03	0.61	1.4750E-03	1.59
0	8	6.6918E-03	1.33	9.1608E-04	1.89	2.2516E-03	1.30	4.0512E-04	1.86
	16	2.7046E-03	1.31	2.4865E-04	1.88	5.6001E-04	2.01	8.9084E-05	2.19
	32	1.3364E-03	1.02	1.0143E-04	1.29	2.9797E-04	0.91	3.5546E-05	1.33
	64	3.4969E-04	1.93	2.7012E-05	1.91	7.1323E-05	2.06	9.0668E-06	1.97
	128	8.8184E-05	1.99	6.8605E-06	1.98	1.7589E-05	2.02	2.2879E-06	1.99
	2	2.0623E-02	-	5.4186E-03	-	1.1963E-02	-	5.3159E-03	-
	4	6.1873E-03	1.74	4.3667E-04	3.63	2.0356E-03	2.56	3.6794E-04	3.85
1	8	3.1236E-03	0.99	2.1944E-044	0.99	7.2972E-04	1.48	1.1031E-04	1.74
	16	1.1438E-03	1.45	4.7538E-05	2.21	3.0271E-04	1.27	2.5223E-05	2.13
	32	1.3086E-04	3.13	2.7709E-06	4.10	3.9110E-05	2.95	1.7315E-06	3.86
	64	1.6017E-05	3.03	1.7072E-07	4.02	5.0209E-06	2.96	9.5886E-08	4.17
	128	2.0202E-06	2.99	1.0794E-08	3.98	6.3771E-07	2.98	5.7702E-09	4.05
	2	1.0848E-02	-	2.4768E-03	-	6.0108E-03	-	2.6754E-03	-
	4	3.8843E-03	1.48	3.3173E-04	2.90	1.1415E-03	2.40	3.1075E-04	3.11
2	8	1.7795E-03	1.13	6.9767E-05	2.25	5.0803E-04	1.17	5.0676E-05	2.62
	16	1.2276E-04	3.86	2.1735E-06	5.00	3.7420E-05	3.76	1.7881E-06	4.82
	32	1.0665E-05	3.52	9.9126E-08	4.45	3.3317E-06	3.49	8.5683E-08	4.38
	64	8.5554E-07	3.64	4.0046E-09	4.63	2.6905E-07	3.63	3.8464E-09	4.48
	128	5.3825E-08	3.99	1.2619E-10	4.99	1.6994E-08	3.98	1.2499E-10	4.94

Table 4 shows that the numerical performance of the $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements on the uniform triangular partition. It can be observed in Table 4 that the numerical solutions obtained by our SFWG algorithm 2.5 converge at rate of $k+2$ in H^1-norm and $k+3$ L^2-norm. We observe from Table 4 that the numerical performance is the same as those in Tables 1-2 and 3, two orders of superconvergence in both L^2-norm and three-bar norm.

6.5. Example 5(Continuous convection $\boldsymbol{\beta}$)

In this example, we perform the SFWG scheme (2.5) to solve the convection-diffusion equations (1.1)-(1.2) on a square domain $\Omega = (0,1)^2$ and the following data: $\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{\beta} = (\exp(1-x), \exp(xy))^\top$, $c = 1$, and $f(x,y)$ is chosen such that the exact solution is

$$u(x,y) = \sin(x)\cos(y).$$

Table 5 shows that the numerical solutions obtained by our SFWG algorithm 2.5 converge at rate of $k+2$ in H^1-norm and $k+3$ L^2-norm. We can see from Table 5 that we do have two orders of superconvergence in both three-bar norm and L^2 norms.

Table 4. Example 4:Errors and numerical rates of convergence for the SFWG (2.5).

k	$1/h$	$\vert\vert\vert Q_hu - u_h\vert\vert\vert$	**Rate**	$\Vert Q_0u - u_0\Vert$	**Rate**
0	2	1.4514E-02	-	7.1372E-04	-
	4	4.2583E-03	1.77	9.8371E-05	2.86
	8	1.1341E-03	1.91	1.2700E-05	2.95
	16	2.9178E-04	1.96	1.6118E-06	2.98
	32	7.3951E-05	1.98	2.0303E-07	2.99
1	2	1.0015E-03	-	4.1487E-05	-
	4	1.3438E-04	2.90	2.6963E-06	3.94
	8	1.7343E-05	2.95	1.7282E-07	3.96
	16	2.2012E-06	2.98	1.0954E-08	3.98
	32	2.7721E-07	2.99	6.8966E-10	3.99
2	2	5.2444E-05	-	1.6300E-06	-
	4	3.3573E-06	3.97	4.8725E-08	5.06
	8	2.1187E-07	3.99	1.4821E-09	5.04
	16	1.3298E-08	3.99	4.5626E-11	5.02
	32	8.3235E-10	4.00	1.4157E-12	5.01

Table 5. Example 5:Errors and numerical rates of convergence for the SFWG (2.5).

k	$1/h$	$\vert\vert\vert Q_hu - u_h\vert\vert\vert$	**Rate**	$\Vert Q_0u - u_0\Vert$	**Rate**
0	2	1.2635E-02	-	8.5356E-04	-
	4	3.6334E-03	1.80	2.9683E-04	1.52
	8	9.5726E-04	1.92	8.0842E-05	1.88
	16	2.4457E-04	1.97	2.0658E-05	1.97
	32	6.1740E-05	1.99	5.1931E-06	1.99
1	2	1.5277E-03	-	1.0329E-04	-
	4	2.0418E-04	2.90	7.3227E-06	3.82
	8	2.6333E-05	2.95	4.7690E-07	3.94
	16	3.3403E-06	2.98	3.0231E-08	3.98
	32	4.2051E-07	2.99	1.8992E-09	3.99
2	2	6.1133E-05	-	2.3140E-06	-
	4	3.9230E-06	3.96	7.2691E-08	4.99
	8	2.4765E-07	3.99	2.2949E-09	4.99
	16	1.5553E-08	3.99	7.2316E-11	4.99
	32	9.7450E-10	4.00	2.2715E-12	4.99

6.6. Example 6(L-shaped domain)

In this example, we perform the SFWG scheme (2.5) with $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements to solve the convection-diffusion equations (1.1)-(1.2) on an L-shaped domain $\Omega = [-1,1]^2 \setminus (0,1) \times (-1,0)$ with the following data: $\alpha = I_2$, $\boldsymbol{\beta} = (x,y)^\top$, $c = x + y + 1$, and $f(x,y)$ is chosen such that the exact solution is

$$u(x,y) = x^5y^2.$$

Table 6. Example 6:Errors and numerical rates of convergence for the SFWG (2.5).

k	$1/h$	$\|\!\|Q_h u - u_h\|\!\|$	**Rate**	$\|Q_0 u - u_0\|$	**Rate**
0	2	2.7089E-01	-	3.3577E-02	-
	4	48.5930E-02	1.66	1.0564E-02	1.66
	8	2.3611E-02	1.86	2.8116E-03	1.91
	16	6.1484E-03	1.94	7.1460E-04	1.98
	32	1.5660E-03	1.97	1.7940E-04	1.99
1	2	5.8131E-02	-	3.5443E-03	-
	4	8.2445E-03	2.82	2.6175E-04	3.76
	8	1.0838E-03	2.93	1.7429E-05	3.91
	16	1.3861E-04	2.97	1.1153E-06	3.97
	32	1.7517E-05	2.98	7.0321E-08	3.99
2	2	7.0599E-03	-	2.8088E-04	-
	4	4.6786E-04	3.92	8.9436E-06	4.97
	8	2.9788E-05	3.97	2.8126E-07	4.99
	16	1.8750E-06	3.99	8.8160E-09	5.00
	32	1.1755E-07	4.00	2.7590E-10	5.00

Table 6 shows that the numerical solution obtained by our SFWG algorithm (2.5) converge at rate of $\mathcal{O}(h^{k+2})$ in H^1-norm and $\mathcal{O}(h^{k+3})$ in L^2-norm. As one can observe from Table 6 that we can capture two order of superconvergence in both L^2-norm and H^1-norm by using SFWG algorithm (2.5).

7. Concluding remarks

In this paper, we presented an SFWG finite element method for solving the general second-order elliptic problem on triangular meshes in 2D. We have shown both theoretically and numerically that on a triangular mesh, using $(P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)$ elements instead of $(P_k(T), P_k(e), [P_{k+1}(T)]^2)$ elements, the accuracy of the approximation to the L^2-projection of the exact solution can be greatly improved.

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