Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 2017–2036
Website:http://www.jaac-online.com
DOI:10.11948/20200307

GROUND STATE SIGN–CHANGING SOLUTIONS FOR FRACTIONAL KIRCHHOFF TYPE EQUATIONS IN $\mathbb{R}^3$

Guofeng Che[1,†] and Haibo Chen[2]

Abstract In this paper, we investigate the existence of ground state sign–changing solutions for the following fractional Kirchhoff equation

$$\left(a+b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(x)u=K(x)f(u)\quad \text{in } \mathbb{R}^3,$$

where $\alpha\in(0,1)$, a,b are positive parameters, $V(x)$, $K(x)$ are nonnegative continuous functions and f is a continuous function with quasicritical growth. By establishing a new inequality, we prove the above system possesses a ground state sign–changing solutions u_b with precisely two nodal domains, and its energy is strictly larger than twice that of the ground state solutions of Nehari–type. Moreover, we obtain the convergence property of u_b as the parameter $b\to 0$. Our conditions weaken the usual increasing condition on $f(t)/|t|^3$.

Keywords Fractional Kirchhoff equations, ground state energy sign–changing solutions, non–Nehari manifold method, variational methods.

MSC(2010) 35B38, 35G20.

1. Introduction

In this paper, we consider the following fractional Kirchhoff equation:

$$\left(a+b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(x)u=K(x)f(u)\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\alpha\in(0,1)$, a,b are positive parameters and $V(x)$, $K(x)$ are nonnegative continuous functions.

Eq.(1.1) is a nonlocal problem because of the appearance of the terms $(-\triangle)^{\alpha}u$ and $\int_{\mathbb{R}^3}|(-\triangle)^{\alpha}u|^2\mathrm{d}x$, which provoke some mathematical difficulties. This also makes the study of Eq.(1.1) particularly interesting.

When $\alpha=1$, Eq.(1.1) reduces to the well–known Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\triangle u+V(x)u=K(x)f(u)\quad \text{in } \mathbb{R}^3, \tag{1.2}$$

[†]The corresponding author. Email: cheguofeng222@163.com (G. Che)
[1]School of Applied Mathematics, Guangdong University of Technology, Waihuan Xi Road, 510006 Guangzhou, China
[2]School of Mathematics and Statistics, Central South University, Yuelu Street, 410083 Changsha, China

which was related to the stationary analogue of the following equation

$$\rho\frac{\partial^2 u}{\partial^2 t}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\right)\frac{\partial^2 u}{\partial^2 x}=0, \tag{1.3}$$

where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Eq.(1.3) was presented by Kirchhoff [25] as an extension of the classical D'Alembert wave equation for free vibrations of clastic strings. Recently, many mathematicians have devoted to the study of Eq.(1.2), especially on the existence of positive solutions, ground state solutions, sign–changing solutions, multiple solutions and bound state solutions, see [13–15, 18, 20, 21, 30, 35–37, 45, 46] and the references therein. For instance, by using the Nehari manifold and the concentration compactness principle, Lü [30] established the existence of ground state solutions for Eq.(1.2) with $V_\lambda(x)=1+\lambda g(x)$ and $f(x,u)=\left(\frac{1}{|x|^\alpha}*|u|^p\right)|u|^{p-2}u$. Moreover, the concentration behaviors of these solutions were obtained as $\lambda\to\infty$. By introducing a new constraint of the Nehari manifold, Sun and Wu [37] obtained multiple positive solutions for Eq.(1.2) when $f(x,u)=f(x)|u|^{p-2}u, 2<p<4$, and $V(x)$ satisfies the steep potential well condition.

When $b=0$, Eq.(1.1) reduces to the following fractional Schrödinger equation:

$$a(-\Delta)^\alpha u+V(x)u=K(x)f(u)\quad\text{in }\mathbb{R}^3, \tag{1.4}$$

which was proposed by Laskin [27] in fractional quantum mechanics as a result of extending the Feynman integrals from the brownian like to the Lévy like quantum mechanicals paths. In the past several decades, with the aid of variational methods, the existence and multiplicity of nontrivial solutions for the fractional Schrödinger equation have been extensively investigated in the literature, see [4, 6, 9, 16, 17, 22, 24, 26, 31, 32, 40] and the references therein. In [4], when $V(x), K(x)$ and f satisfy some suitable conditions, Ambrosio *et al.* studied the existence of a sign–changing solution for Eq.(1.4) with $a=1$, $\mathbb{R}^3$ being replaced by $\mathbb{R}^N, N>2\alpha$. Moreover, the existence of infinitely many weak solutions is obtained when f is odd. In [31], using the Mountain Pass Theorem, Secchi proved that Eq.(1.4) had at least a nontrivial solution when f has subcritical growth and satisfies the famous Ambrosetti–Rabinowitz condition. By virtue of the harmonic extension techniques of Caffarelli and Silvestre [12], Teng and He [40] proved the existence of ground state solutions by using the concentration–compactness principle and methods of Brezis and Nirenberg.

To the best of our knowledge, there are few papers in the literature that considered Eq.(1.1). In [7], Ambrosio and Isernia studied the following fractional Kirchhoff equation:

$$\left(a+b\int_{\mathbb{R}^N}|(-\triangle)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\right)(-\triangle)^\alpha u+V(x)u=f(u)\quad\text{in }\mathbb{R}^N, \tag{1.5}$$

where f is an odd subcritical nonlinearity satisfying the well–known Berestycki–Lions assumptions [11]. By minimax arguments, the authors obtained a multiplicity result in the radial space $H^\alpha_{rad}(\mathbb{R}^N)$ when the parameter b is sufficiently small. In [5], by using penalization techniques and Ljusternik–Schnirelmann theory, Ambrosio and Isernia studied the existence and multiplicity of positive solutions for a class of more general fractional Kirchhoff equation. Furthermore, the relation between the number of positive solutions with the topology of the set where the potential attains its minimum was also obtained by them. Recently, when f satisfied the Berestycki–Lions type conditions of critical type [47], Liu *et al.* [28] obtained the existence of positive ground state solutions for Eq.(1.5) by using the monotonicity trick and the profile decomposition. Moreover, the nonlinearity does not satisfy the Ambrosetti–Rabinowitz type condition or monotonicity assumptions. In [34], By using the Moser iteration scheme, Su and Chen considered the existence, nonexistence and multiplicity of nontrivial solution for Eq.(1.5) with critical Hardy–Littlewood–Sobolev critical exponent. By using the Nehari manifold technique, Isernia [23] obtained the existence of the least energy solution for Eq.(1.5). Moreover, the multiplicity result was also obtained by the author.

Inspired by the above works, more precisely by [28], our goal is to deal with Eq.(1.1) and study the existence of ground state sign–changing solutions for Eq.(1.1) without the variant Nehari–type condition. Moreover, we prove that the energy of any sign–changing solutions for Eq.(1.1) is strictly larger than twice that of the ground state solutions for Eq.(1.1) and obtain the convergence of the least sign–changing solutions for Eq.(1.1) as $b \to 0$.

We denote the fractional Sobolev space $H^{\alpha}(\mathbb{R}^3)$ with the product

$$(u,v)=\int_{\mathbb{R}^3}\left(a(-\triangle)^{\frac{\alpha}{2}}u(-\triangle)^{\frac{\alpha}{2}}v+uv\right)\mathrm{d}x$$

and the norm

$$||u||=\left(\int_{\mathbb{R}^3}(a|(-\Delta)^{\frac{\alpha}{2}}u|^2+u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $D^{\alpha,2}(\mathbb{R}^3)$ be the completion of $C_0^{\infty}(\mathbb{R}^3)$ with respect to the Gagliardo norm

$$||u||_{D^{\alpha,2}}=\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

In this paper, we consider the space

$$E=\begin{cases} H_r^{\alpha}(\mathbb{R}^3)=\{u\in H^{\alpha}(\mathbb{R}^3):u(x)=u(|x|)\}, & \text{if } V(x) \text{ is a constant},\\ \left\{u\in D^{\alpha,2}(\mathbb{R}^3)|\int_{\mathbb{R}^3}V(x)u^2\mathrm{d}x<+\infty\right\} & \text{if } V(x) \text{ is not a constant},\end{cases}$$

with the norm

$$\|u\|=\left(\int_{\mathbb{R}^3}(a|(-\Delta)^{\frac{\alpha}{2}}u|^2+V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

To avoid involving too much details for checking the compactness, for $V(x)$ not being a constant, similar to the arguments of [19, 42], we may assume that:
(V) $V\in C(\mathbb{R}^3,\mathbb{R}^+)$ such that $E\subset H^{\alpha}(\mathbb{R}^3)$ and the embedding $E\to L^r(\mathbb{R}^3)$, $r\in(2,2_\alpha^*)$ is compact.

Define the energy functional $J_b: E\to\mathbb{R}$ by

$$J_b(u)=\frac{1}{2}||u||^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^3}K(x)F(u)\mathrm{d}x. \tag{1.6}$$

Then J_b is well defined on E and $J_b \in C^1(E,\mathbb{R})$. Furthermore, for any $u,\ v \in E$, we have

$$\begin{aligned}\langle J_b'(u), v\rangle = &\int_{\mathbb{R}^3}\Big(a(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v + V(x)uv\Big)\mathrm{d}x - \int_{\mathbb{R}^3}K(x)f(u)v\mathrm{d}x \\ &+ b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v\mathrm{d}x.\end{aligned} \tag{1.7}$$

Hence, if $u \in E$ is a critical point of J_b, then u is a solution of Eq.(1.1). Moreover, if $u \in E$ is a solution of Eq.(1.1) with $u^{\pm} \neq 0$, then u is a sign–changing solution of Eq.(1.1), where

$$u^+(x) := \max\big\{u(x), 0\big\} \quad \text{and} \quad u^-(x) := \min\big\{u(x), 0\big\}.$$

Here, a solution is called a ground state (or least energy) sign–changing solution if it possesses the least energy among all sign–changing solutions. By a simple calculation, (1.6) and (1.7) imply that

$$J_b(u) = J_b(u^+) + J_b(u^-) + \frac{b}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x, \tag{1.8}$$

$$\langle J_b'(u), u^+\rangle = \langle J_b'(u^+), u^+\rangle + b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x, \tag{1.9}$$

$$\langle J_b'(u), u^-\rangle = \langle J_b'(u^-), u^-\rangle + b\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x. \tag{1.10}$$

When $b = 0$, Eq.(1.1) does not depend on the nonlocal term $\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u|^2\mathrm{d}x$ any more, i.e., it reduces to Eq.(1.4), which corresponds to the energy functional $J_0 : E \to \mathbb{R}$ by

$$J_0(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^3}K(x)F(u)\mathrm{d}x. \tag{1.11}$$

Analogously, J_0 is well defined and $J_0 \in C^1(E,\mathbb{R})$. Furthermore

$$\langle J_0'(u), v\rangle = a\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v\mathrm{d}x - \int_{\mathbb{R}^3}K(x)f(u)v\mathrm{d}x. \tag{1.12}$$

From (1.8)-(1.10), it is easy to see that there are some essential differences in studying the sign-changing solutions for Eq.(1.1) between $b > 0$ and $b = 0$, the existence of sign–changing solutions for Eq.(1.4) has been extensively studied, for instance, see [29,41,43] and the references therein. However, the methods of looking for sign-changing solutions heavily rely on the decompositions of (1.8)-(1.10) with $b = 0$, which seems to be not applicable to Eq.(1.1). Motivated by the above works, we will consider the following minimization problems:

$$m_b = \inf_{\mathcal{M}_b} J_b(u) \quad \text{and} \quad m_0 = \inf_{\mathcal{M}_0} J_0(u), \tag{1.13}$$

where

$$\mathcal{M}_b = \big\{u \in E : u^{\pm} \neq 0, \langle J_b'(u), u^+\rangle = \langle J_b'(u), u^-\rangle = 0\big\}, \tag{1.14}$$

and

$$\mathcal{M}_0 = \big\{u \in E : u^{\pm} \neq 0, \langle J_0'(u), u^+\rangle = \langle J_0'(u), u^-\rangle = 0\big\}, \tag{1.15}$$

whose minimizers are the sign–changing solutions for Eq.(1.1) and Eq.(1.4), respectively.

In order to show the energy of any sign–changing solutions of Eq.(1.1) is larger than twice that of the ground state solutions of Eq.(1.1) and obtain the convergence of least energy sign–changing solution for Eq.(1.1) as $b \to 0$. As usual, we first get the ground state solutions of Nehari type to Eq.(1.1) and Eq.(1.4) by seeking the minimizers of corresponding energy functionals J_b and J_0 on the following Nehari manifolds:

$$\mathcal{N}_b := \big\{u \in E \setminus \{0\}, \langle J_b'(u), u\rangle = 0\big\}, \tag{1.16}$$

and

$$\mathcal{N}_0 := \big\{u \in E \setminus \{0\}, \langle J_0'(u), u\rangle = 0\big\}. \tag{1.17}$$

Furthermore, we suppose more general conditions involving the functions $V(x)$ and $K(x)$, such that (VK) in [3, 8] can be seen as a particular case. Throughout this paper, we say that $(V, K) \in \mathcal{K}$ if the following conditions hold:
(H_1) $V(x),\ K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$;
(H_2) if $\{A_n\} \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure of A_n is less than R, for all n and some $R > 0$, then

$$\lim_{r\to\infty} \int_{A_n \bigcap B_r^c(0)} K(x)\mathrm{d}x = 0, \quad \text{uniformly in } n \in \mathbb{N};$$

(H_3) $\frac{K}{V} \in L^\infty(\mathbb{R}^3)$;
or
(H_4) there exists $p \in (2, 2_\alpha^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{2_\alpha^*-p}{2_\alpha^*-2}}} \to 0 \qquad \text{as} \quad |x| \to \infty.$$

This kind of conditions were firstly introduced by Alves and Souto [1] to get a positive ground state solution of (1.4) with $\alpha = 1$. Similar to Proposition 2.1 in [1], we can prove the space X given by

$$X = \left\{u \in D^{\alpha,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x < \infty\right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|(-\triangle)^{\frac{\alpha}{2}}u|^2 + V(x)u^2)\mathrm{d}x\right)^{\frac{1}{2}}$$

is compactly embedded into the weighted Lebesgue space

$$L_K^r(\mathbb{R}^3) = \left\{u : \mathbb{R}^3 \to \mathbb{R} :\ \text{u is measurable and} \int_{\mathbb{R}^3} K(x)|u|^r\mathrm{d}x < \infty\right\}$$

for some $r \in (2, 2_\alpha^*)$. Moreover, we also have $\mathcal{M}_b \neq \emptyset$.

To state our results, we introduce the following conditions:
(K) $K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$;
(F_1) $\lim\limits_{t\to 0} \frac{f(t)}{t} = 0$;
(F_2) f has a “quasicritical growth”, that is, $\lim\limits_{|t|\to\infty} \frac{f(t)}{t^{2_\alpha^*-1}} = 0$;

(F_3) $\lim\limits_{|t|\to\infty} \frac{f(t)}{t^3} = \infty$;

(F_4) there exists a $\theta_0 \in (0,1)$ such that for any $x \in \mathbb{R}^3$, $t > 0$ and $\tau \neq 0$,

$$K(x)\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right]\text{sign}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0.$$

Now, we are ready to state our main results.

Theorem 1.1. *Suppose that conditions (V), (K) and $(F_1) - (F_4)$ hold. Then Eq.(1.1) has a sign–changing solution $u_b \in \mathcal{M}_b$ such that $J_b(u_b) = \inf\limits_{\mathcal{M}_b} J_b(u) > 0$, which has precisely two nodal domains.*

Theorem 1.2. *Suppose that conditions (V), (K) and $(F_1) - (F_4)$ hold. Then Eq.(1.1) has a sign–changing solution $\bar{u} \in \mathcal{N}_b$ such that $J_b(\bar{u}) = \inf\limits_{\mathcal{N}_b} J_b(u) > 0$. Furthermore, $m_b > 2c_b$, where $c_b = \inf\limits_{u\in\mathcal{N}_b} J_b(u)$.*

Theorem 1.3. *Suppose that conditions (V), (K) and $(F_1) - (F_4)$ hold. Then Eq.(1.4) has a sign–changing solution $v_0 \in \mathcal{M}_0$ such that $J_0(v_0) = \inf\limits_{\mathcal{N}_0} J_0 > 0$, which has precisely two nodal domains. Moreover, for any sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$, there exists a subsequence which we label in the same way such that $u_{b_n} \to u_0$ in E, where $u_0 \in \mathcal{M}_0$ is a sign–changing solution of Eq.(1.6) with $J_0(u_0) = \inf\limits_{\mathcal{M}_0} J_0(u)$.*

Corollary 1.1. Suppose that $(V,K) \in \mathcal{K}$ and f verifies $(F_1) - (F_4)$. Then all the conclusions in Theorems 1.1, 1.2 and 1.3 hold in X.

Remark 1.1. It is worthy stressing that the condition (F_4) is weaker than (F_4') as follows:

(F_4') The map $t \mapsto \frac{f(t)}{|t|^3}$ is nondecreasing for all $t \in \mathbb{R} \setminus \{0\}$.

In fact, it is not difficult to find some functions satisfying assumptions (F_1)–(F_4), but not (F_4'). For instance, let

$$f(t) = \begin{cases} |t|^3 t, & |t| \leq \rho, \\ \alpha|t|^3 t + \frac{1}{3M}t, & |t| > \rho, \end{cases}$$

where $M > 0$, $\alpha,\ \rho > 0$. We can easily verify that f satisfies $(F_1) - (F_4)$, but not (F_4').

Remark 1.2. By using Non–Nehari manifold method introduced in [38] to seek ground state solutions for Eq.(1.1), we can prove the existence of a sign–changing solution for Eq.(1.1) directly instead of using Proposition 3.1 in [10].

Remark 1.3. We also give an affirmative answer to an open question that the energy of any sign–changing solutions of Eq.(1.1) is strictly larger than twice that of the ground state solutions of Eq.(1.1).

Notation 1.1. Throughout this paper, C denotes various positive generic constants, which may vary from line to line. $2^*_\alpha = \frac{6}{3-2\alpha}$ is the critical Sobolev exponent. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

The remainder of this paper is as follows. In Section 2, some preliminary lemmas and corollaries are presented. In Section 3, we prove the existence of a ground state sign–changing solution with precisely two nodal domains. In Section 4, we first investigate the ground state solutions of Nehari type and then prove Theorem 1.2. The proofs of Theorem 1.3 and Corlllary 1.1 are given in Sects. 5 and 6, respectively.

2. Variational setting and preliminaries

In this section, we give some preliminary lemmas and corollaries, which will play crucial roles in proving our results.

Lemma 2.1. *Assume conditions* (V), (K) *and* $(F_1)-(F_4)$ *hold. Then*

$$\begin{aligned} J_b(u) \geq J_b(su^+ + tu^-) &+ \frac{1-s^4}{4}\langle J_b'(u), u^+\rangle + \frac{1-t^4}{4}\langle J_b'(u), u^-\rangle \\ &+ \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 \\ &+ \frac{b(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x, \end{aligned} \tag{2.1}$$

for any $u = u^+ + u^- \in E, \quad s,\ t \geq 0.$

Proof. For any $x \in \mathbb{R}^3$, $t \geq 0$, $\tau \in \mathbb{R}$, it follows from (F_4) that

$$\begin{aligned} &K(x)\big[\frac{1-t^4}{4}\tau f(\tau) + F(t\tau) - F(\tau)\big] + \frac{\theta_0 V(x)}{4}(1-t^2)^2\tau^2 \\ &= \int_t^1 \big[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} + \frac{\theta_0 V(x)(1-s^2)}{(s\tau)^2}\big]s^3\tau^4\mathrm{d}s \geq 0. \end{aligned} \tag{2.2}$$

It follows from (1.6), (1.7) and (2.2) that

$$\begin{aligned} &J_b(u) - J_b(su^+ + tu^-) \\ &= \frac{1}{2}(\|u\|^2 - \|su^+ + tu^-\|^2) + \frac{b}{4}\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u|^2\mathrm{d}x\big)^2 \\ &\quad + \frac{b}{4}\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}(su^+ + tu^-)|^2\mathrm{d}x\big)^2 \\ &\quad - \int_{\mathbb{R}^3}K(x)[F(su^+ + tu^-) - F(u)]\mathrm{d}x \\ &= \frac{1-s^4}{4}\Big(\|u^+\|^2 + b\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\big)^2\Big) \\ &\quad + \frac{1-t^4}{4}\Big(\|u^-\|^2 + b\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x\big)^2\Big) \\ &\quad + \frac{(1-s^2)^2}{4}\|u^+\|^2 + \frac{(1-t^2)^2}{4}\|u^-\|^2 \\ &\quad + \frac{b(1-s^2t^2)}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}K(x)[F(su^+) + F(tu^-) - F(u^+) - F(u^-)]\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-s^4}{4}\langle J_b'(u), u^+\rangle + \frac{1-t^4}{4}\langle J_b'(u), u^-\rangle + \frac{(1-s^2)^2}{4}\|u^+\|^2 \\
&\quad + \frac{(1-t^2)^2}{4}\|u^-\|^2 + \frac{b(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}K(x)[\frac{1-s^4}{4}f(u^+)u^+ + F(su^+) - F(u^+)]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}K(x)[\frac{1-t^4}{4}f(u^-)u^- + F(tu^-) - F(u^-)]\mathrm{d}x \\
&\geq \frac{1-s^4}{4}\langle J_b'(u), u^+\rangle + \frac{1-t^4}{4}\langle J_b'(u), u^-\rangle + \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 \\
&\quad + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 + \frac{b(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\left\{K(x)[\frac{1-s^4}{4}f(u^+)u^+ + F(su^+) - F(u^+)] + \frac{\theta_0 V(x)}{4}(1-s^2)^2|u^+|^2\right\}\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\left\{K(x)[\frac{1-t^4}{4}f(u^-)u^- + F(su^-) - F(u^-)] + \frac{\theta_0 V(x)}{4}(1-t^2)^2|u^-|^2\right\}\mathrm{d}x \\
&\geq \frac{1-s^4}{4}\langle J_b'(u), u^+\rangle + \frac{1-t^4}{4}\langle J_b'(u), u^-\rangle + \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 \\
&\quad + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 + \frac{b(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x,
\end{aligned}
$$

for any $s,\ t \geq 0$. This shows that (2.1) holds. □

Corollary 2.1. *Assume* (V), (K) *and* $(F_1) - (F_4)$ *hold. If* $u = u^+ + u^- \in \mathcal{M}_b$, *then*

$$
\begin{aligned}
J_b(u) \geq J_b(su^+ + tu^-) &+ \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 \\
&+ \frac{b(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x, \quad \forall\ s,\ t \geq 0.
\end{aligned}
\tag{2.3}
$$

Corollary 2.2. *Assume* (V), (K) *and* $(F_1) - (F_4)$ *hold. If* $u = u^+ + u^- \in \mathcal{M}_b$, *then*

$$
J_b(u) = \max_{s,t\geq 0} J_b(su^+ + tu^-). \tag{2.4}
$$

Lemma 2.2. *Assume* (V), (K) *and* $(F_1) - (F_4)$ *hold. If* $u = u^+ + u^- \in E$ *with* $u^\pm \neq 0$, *then there exists a unique pair* (s_u, t_u) *of positive numbers such that* $s_u u^+ + t_u u^- \in \mathcal{M}_b$.

Proof. For any $u \in E$ with $u^\pm \neq 0$, we first adopt the idea used in [2] to prove the existence of (s_u, t_u). Let

$$
\begin{aligned}
g_1(s,t) = s^2\|u^+\|^2 &+ bs^4(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x)^2 - \int_{\mathbb{R}^3}K(x)f(su^+)su^+\mathrm{d}x \\
&+ bs^2t^2\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u^-|^2\mathrm{d}x
\end{aligned}
\tag{2.5}
$$

and

$$g_2(s,t) = t^2\|u^-\|^2 + bt^4(\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}}u^-|^2 \mathrm{d}x)^2 - \int_{\mathbb{R}^3} K(x)f(tu^-)tu^- \mathrm{d}x \\ + bs^2t^2\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}}u^+|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}}u^-|^2 \mathrm{d}x. \tag{2.6}$$

For any fixed $t > 0$, using (F_1) and (F_3), it is easy to verify that $g_1(0,t) = 0$, $g_1(s,t) > 0$ for $s > 0$ small and $g_1(s,t) < 0$ for $s > 0$ large. From the continuity of $g_1(s,t)$ on s, there exists a $s_t > 0$ such that $g_1(s_t,t) = 0$ for $t \geq 0$. We claim that $s_t > 0$ is unique for any $t \geq 0$. In fact, for any fixed $t_0 \geq 0$, let $\tilde{s}_1,\ \tilde{s}_2 > 0$ such that

$$g_1(\tilde{s}_1,t_0) = g_2(\tilde{s}_2,t_0) = 0. \tag{2.7}$$

Then it follows from (1.7), (2.5) and (2.7) that

$$\langle J_b'(\tilde{s}_1u^+ + t_0u^-), \tilde{s}_1u^+\rangle = \langle J_b'(\tilde{s}_2u^+ + t_0u^-), \tilde{s}_2u^+\rangle = 0. \tag{2.8}$$

Then it follows from (2.2) and (2.8) that

$$\begin{aligned} J_b(\tilde{s}_1u^+ + t_0u^-) &\geq J_b(\tilde{s}_2u^+ + t_0u^-) + \frac{\tilde{s}_1^4 - \tilde{s}_2^4}{4\tilde{s}_1^4}\langle J_b'(\tilde{s}_1u^+ + t_0u^-), \tilde{s}_1u^+\rangle \\ &\quad + \frac{(1-\theta_0)(\tilde{s}_1^2 - \tilde{s}_2^2)^2}{4\tilde{s}_1^2}\|u^+\|^2 \\ &= J_b(\tilde{s}_2u^+ + t_0u^-) + \frac{(1-\theta_0)(\tilde{s}_1^2 - \tilde{s}_2^2)^2}{4\tilde{s}_1^2}\|u^+\|^2 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} J_b(\tilde{s}_2u^+ + t_0u^-) &\geq J_b(\tilde{s}_2u^+ + t_0u^-) + \frac{\tilde{s}_2^4 - \tilde{s}_1^4}{4\tilde{s}_2^4}\langle J_b'(\tilde{s}_2u^+ + t_0u^-), \tilde{s}_2u^+\rangle \\ &\quad + \frac{(1-\theta_0)(\tilde{s}_2^2 - \tilde{s}_1^2)^2}{4\tilde{s}_2^2}\|u^+\|^2 \\ &= J_b(\tilde{s}_2u^+ + t_0u^-) + \frac{(1-\theta_0)(\tilde{s}_2^2 - \tilde{s}_1^2)^2}{4\tilde{s}_2^2}\|u^+\|^2. \end{aligned} \tag{2.10}$$

(2.9) and (2.10) imply that $\tilde{s}_1 = \tilde{s}_2$. Hence, $s_t = \tilde{s}(t) > 0$ is unique for all $t \geq 0$. i.e., $g_1(s,t) = 0$ defines an implicit function $s = \tilde{s}(t)$ for all $t \geq 0$. Since $s_0 = \tilde{s}(0)$ and for every $t \geq 0$, $g_1(s,t) > 0$ for small $s > 0$ and $g_1(s,t) < 0$ for large $s > 0$. Then one has

$$g_1(s_t,t) = 0,\ \forall\ t \geq 0;\quad s_t > t \quad \text{for small } t \geq 0,\ s_t < t \quad \text{for large } t \geq 0. \tag{2.11}$$

Similarly, $g_2(s,t) = 0$ defines an implicit function $t = t_s = \tilde{t}(s)$ such that

$$g_2(s,t_s) = 0,\ \forall\ s \geq 0;\quad t_s > s \quad \text{for small } s \geq 0,\ t_s < s \quad \text{for large } s \geq 0. \tag{2.12}$$

(2.11) and (2.12) imply that the planar curves $s = \tilde{s}(t)$ and $t = \tilde{t}(s)$ intersect at some point (s_u,t_u) with $s_u,\ t_u > 0$. Hence, $s_uu^+ + t_uu^- \in \mathcal{M}_b$.

Next, we prove the uniqueness. Choosing (s_1,t_1) and (s_2,t_2) such that $s_iu^+ + t_iu^- \in \mathcal{M}_b,\ i=1,\ 2$. Then it follows from Corollary 2.1 that

$$\begin{aligned} J_b(\tilde{s}_1u^+ + t_1u^-) \geq J_b(\tilde{s}_2u^+ + t_2u^-) + \frac{(1-\theta_0)(\tilde{s}_1^2-\tilde{s}_2^2)^2}{4\tilde{s}_1^2}\|u^+\|^2 \\ + \frac{(1-\theta_0)(\tilde{t}_1^2-\tilde{t}_2^2)^2}{4\tilde{t}_1^2}\|u^-\|^2 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} J_b(\tilde{s}_2u^+ + t_2u^-) \geq J_b(\tilde{s}_1u^+ + t_1u^-) + \frac{(1-\theta_0)(\tilde{s}_2^2-\tilde{s}_1^2)^2}{4\tilde{s}_2^2}\|u^+\|^2 \\ + \frac{(1-\theta_0)(\tilde{t}_1^2-\tilde{t}_2^2)^2}{4\tilde{t}_2^2}\|u^-\|^2. \end{aligned} \tag{2.14}$$

Both (2.13) and (2.14) imply that $(s_1,t_1)=(s_2,t_2)$. The proof is complete. □

Lemma 2.3. *Assume* (V), (K) *and* $(F_1)-(F_4)$ *hold. Then*

$$\inf_{\mathcal{M}_b} J_b(u) = m_b = \inf_{u\in E, u^\pm\neq 0}\max_{s,t\geq 0} J_b(su^+ + tu^-).$$

Proof. Both Corollary 2.2 and Lemma 2.2 imply the above Lemma. □

Lemma 2.4. *Assume* (V), (K) *and* $(F_1)-(F_4)$ *hold. Then* $m_b>0$ *is achieved.*

Proof. Similar as Lemma 2.7 in [39], it follows from $(F_1)-(F_3)$ that there exists a constant $\beta>0$ such that $\|u^\pm\|>\beta$ for all $u\in\mathcal{M}_b$. Let $\{u_n\}\subset\mathcal{M}_b$ be such that $J_b(u_n)\to m_b$. Observe that (2.2) with $t=0$ yields

$$K(x)[\frac{1}{4}f(\tau)\tau - F(\tau)] + \frac{\theta_0 V(x)}{4}\tau^2 \geq 0,\quad \forall\ x\in\mathbb{R}^3,\ \tau\in\mathbb{R}. \tag{2.15}$$

Then it follows from (1.6), (1.7) and (2.15) that for large $n\in N$, we derive

$$\begin{aligned} m_b+1 &\geq J_b'(u_n) - \frac{1}{4}\langle J_b'(u_n),u_n\rangle \\ &\geq \frac{1-\theta_0}{4}\|u_n\|^2 + \int_{\mathbb{R}^3}\{K(x)[\frac{1}{4}f(u_n)u_n - F(u_n)] + \frac{\theta_0 V(x)}{4}|u_n|^2\}\mathrm{d}x \\ &\geq \frac{1-\theta_0}{4}\|u_n\|^2, \end{aligned} \tag{2.16}$$

which implies that $\{u_n\}$ is bounded in E. Then there exists $u_b\in E$ such that $u_n^\pm \rightharpoonup u_b^\pm$ in E. Thus, from (V), (K), $(F_1)-(F_4)$, (2.2) and Lemma A.1 in [44], we obtain

$$\begin{aligned} 0<\beta &\leq \|u_n^\pm\|^2 + b\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2\mathrm{d}x\big)^2 \\ &= \int_{\mathbb{R}^3}K(x)f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^3}K(x)f(u)u\mathrm{d}x + o(1), \end{aligned} \tag{2.17}$$

showing that $u_b^\pm\neq 0$. Therefore, by (2.7), Fatou's Lemma and the weak semicontinuity of the norm, we derive

$$\begin{aligned} \|u_b^\pm\|^2 + b\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u_b|^2\mathrm{d}x\big)^2 &= \liminf_{n\to\infty}\big[\|u_n^\pm\|^2 + b\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u_n|^2\mathrm{d}x\big)^2\big] \\ &= \int_{\mathbb{R}^3}K(x)f(u)u\mathrm{d}x, \end{aligned} \tag{2.18}$$

showing that

$$\langle J_b'(u_b), u_b^{\pm} \rangle \leq 0. \tag{2.19}$$

Then by (1.6), (1.7), (2.1), (2.15), (2.19), Fatou's Lemma, the weak semicontinuity and Lemma 2.3, we derive

$$\begin{aligned}
m_b &= \lim_{n\to\infty} \left[J_b'(u_n) - \frac{1}{4}\langle J_b'(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty} \left\{\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)[\frac{1}{4}f(u_n)u_n - F(u_n)]\mathrm{d}x\right\} \\
&\geq \frac{1}{4}\liminf_{n\to\infty}\left[a\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3}V(x)|u_n|^2\mathrm{d}x\right] \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^3}\left\{K(x)[\frac{1}{4}f(u_n)u_n - F(u_n)]\mathrm{d}x + \frac{\theta_0}{4}V(x)|u_n|^2\right\}\mathrm{d}x \\
&\geq \frac{1}{4}\liminf_{n\to\infty}\left[a\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u_b|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3}V(x)|u_b|^2\mathrm{d}x\right] \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^3}\left\{K(x)[\frac{1}{4}f(u_b)u_b - F(u_b)]\mathrm{d}x + \frac{\theta_0}{4}V(x)|u_b|^2\right\}\mathrm{d}x \\
&= \frac{1}{4}\|u_b\|^2 + \int_{\mathbb{R}^3}K(x)[\frac{1}{4}f(u_b)u_b - F(u_b)]\mathrm{d}x \\
&= J_b(u_b) - \frac{1}{4}\langle J_b'(u_b), u_b\rangle \\
&\geq \sup_{s,t\geq 0}\left[J_b(su_b^+ + tu_b^-) + \frac{1-s^4}{4}\langle J_b'(u_b), u_b^+\rangle + \frac{1-t^4}{4}\langle J_b'(u_b), u_b^-\rangle\right] \\
&\quad - \frac{1}{4}\langle J_b'(u_b), u_b\rangle \\
&\geq \sup_{s,t\geq 0} J_b(su_b^+ + tu_b^-) \\
&\geq m_b,
\end{aligned} \tag{2.20}$$

which implies that

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u_n|^2\mathrm{d}x = \int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u_b|^2\mathrm{d}x,\ \lim_{n\to\infty}\int_{\mathbb{R}^3}V(x)|u_n|^2\mathrm{d}x \\
= \int_{\mathbb{R}^3}V(x)|u_b|^2\mathrm{d}x.
\end{aligned} \tag{2.21}$$

Hence, $u_n \to u_b$ in E, then $J_b(u_b) = m_b$ and $u_b \in \mathcal{M}_b$. □

Lemma 2.5. *Assume* (V), (K) *and* $(F_1)-(F_4)$ *hold. If* $u_0 \in \mathcal{M}_b$ *and* $J_b(u_0) = m_b$, *then* u_0 *is a critical point of* J_b.

Proof. Assume that $u_0 = u_0^+ + u_0^- \in \mathcal{M}_b$, $J_b'(u_0) \neq 0$. Then there exist $\omega,\ \delta > 0$ such that

$$u \in E,\ \|u - u_0\| \leq 3\delta \Rightarrow \|J_b'(u)\| \geq \omega. \tag{2.22}$$

Then for all $s,\ t \geq 0$, it follows from Corollary 2.1 that

$$\begin{aligned} J_b(su^+ + tu^-) &\leq J_b(u) - \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 \\ &= m_b - \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2. \end{aligned} \tag{2.23}$$

Let $D = (0.5, 1.5) \times (0.5, 1.5)$, Then by (2.23), we obtain

$$\kappa := \max_{\partial D} J_b(su_0^+ + tu_0^-) < m_b. \tag{2.24}$$

For $\varepsilon := \min\{(m_b - \kappa)/3, \delta\omega/8\}$ and $S := B(u_0, \delta)$, Lemma 2.3 in [44] yields a deformation $\eta \in C([0,1] \times E, E)$ such that

(i) $\eta(1, u) = u$ if $u \notin J_b^-([m_b - 2\varepsilon, m_b + 2\varepsilon]) \cap S_{2\delta}$.

(ii) $\eta(1, J_b^{m_b+\varepsilon} \cap B(u_0, \delta)) \subset J_b^{m_b-\varepsilon}$.

(iii)$J_b(\eta(1, u)) \leq J_b(u),\ \forall\ u \in E$.

From Corollary 2.2, $J_b(su_0^+ + tu_0^-) \leq J_b(u_0) = m_b$ for all $s, t \geq 0$. Then it follows from (ii) that

$$J_b(\eta(1, su_0^+ + tu_0^-)) \leq m_b - \varepsilon, \quad \forall\ s, t \geq 0,\ |s-1|^2 + |t-1|^2 < \delta^2/\|u_0\|^2. \tag{2.25}$$

On the other hand, by (iii) and (2.23), for any $s, t \geq 0$, $|s-1|^2 + |t-1|^2 \geq \delta^2/\|u_0\|^2$, there holds

$$\begin{aligned} J_b(\eta(1, su_0^+ + tu_0^-)) &\leq J_b(su_0^+ + tu_0^-) \\ &= m_b - \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 \\ &\leq m_b - \frac{(1-\theta_0)\delta^2}{8\|u_0\|^2}\min\{\|u_0^+\|^2, \|u_0^-\|^2\}. \end{aligned} \tag{2.26}$$

Then it follows from (2.25) and (2.26) that

$$\max_{\bar{D}} J_b(\eta(1, su_0^+ + tu_0^-)) < m_b. \tag{2.27}$$

Define $g(s,t) := su_0^+ + tu_0^-$. By a similar argument as [33], we get $\eta(1, g(D)) \cap \mathcal{M}_b \neq \emptyset$, which contradicts to the definition of m_b. The proof is complete. □

3. Sign–changing solutions

Proof of Theorem 1.1. It follows from Lemma 2.4 and Lemma 2.5 that there exists a $u_b \in \mathcal{M}_b$ such that $J_b(u_b) = m_b$ and $J_b'(u_b) = 0$. Thus, u_b is a sign–changing solution of (1.1).

Next, we prove that u_b has exactly two nodal domains. Let $u_b = u_1 + u_2 + u_3$, where

$$u_1 \geq 0,\ u_2 \leq 0,\ \Omega_1 \cap \Omega_2 = \emptyset,\ u_1|_{\Omega_2 \cup \Omega_3} = u_2|_{\Omega_1 \cup \Omega_3} = u_3|_{\Omega_1 \cup \Omega_2} = 0, \tag{3.1}$$

$$\Omega_1 = \{x \in \mathbb{R}^3 : u_1(x) > 0\},\ \Omega_2 = \{x \in \mathbb{R}^3 : u_2(x) < 0\},\ \Omega_3 = \mathbb{R}^3 \setminus \{(\Omega_1 \cup \Omega_2)\}, \tag{3.2}$$

and Ω_1, Ω_2 are connected open subsets of $\mathbb{R}^3$.

Setting $v = u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e. $v^\pm \neq 0$. Then it follows from (1.6), (1.7), (2.1), (2.15) and (3.1) that

$$\begin{aligned}
m_b &= J_b(u_b) = J_b(u_b) - \frac{1}{4}\langle J_b'(u_b), u_b\rangle \\
&= J_b(v) + J_b(u_3) + \frac{b}{2}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} v|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} u_3|^2 \mathrm{d}x \\
&\quad - \frac{1}{4}\big[\langle J_b'(v), v\rangle + \langle J_b'(u_3), u_3\rangle + 2b\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} v|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} u_3|^2 \mathrm{d}x\big] \\
&\geq \sup_{s,t\geq 0}\left[J_b(sv^+ + tv^-) + \frac{1-s^4}{4}\langle J_b'(v), v^+\rangle + \frac{1-t^4}{4}\langle J_b'(v), v^-\rangle\right] \\
&\quad - \frac{1}{4}\langle J_b'(v), v\rangle + J_b(u_3) - \frac{1}{4}\langle J_b'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0}\left[J_b(sv^+ + tv^-) + \frac{b}{4}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} v^+|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} u_3|^2 \mathrm{d}x\right. \\
&\quad \left. + \frac{b}{4}\int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} v^-|^2 \mathrm{d}x \int_{\mathbb{R}^3} |(-\triangle)^{\frac{\alpha}{2}} u_3|^2 \mathrm{d}x\right] + \frac{1}{4}\|u_3\|^2 \\
&\quad + \int_{\mathbb{R}^3} K(x)\big[\frac{1}{4}f(u_3)u_3 - F(u_3)\big]\mathrm{d}x \\
&\geq \sup_{s,t\geq 0} J_b(sv^+ + tv^-) + \frac{1-\theta_0}{4}\|u_3\|^2 \\
&\geq m_b + \frac{1-\theta_0}{4}\|u_3\|^2,
\end{aligned}$$

which implies that $u_3 = 0$. Hence, u_b has exactly two nodal domains. The proof is complete. □

4. Ground state solutions of Nehari type

In this section, we will use Non–Nehari manifold's method to seek the ground state solutions of Nehari type for Eq.(1.1). First, we can prove the following lemmas and corollaries as in Section 2.

Lemma 4.1. *Assume (V), (K) and $(F_1) - (F_4)$ hold. Then*

$$J_b(u) \geq J_b(tu) + \frac{1-t^4}{4}\langle J_b'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2 \tag{4.1}$$

for any $u \in E$, $t \geq 0$.

Corollary 4.1. *Assume (V), (K) and $(F_1) - (F_4)$ hold. Then for $u \in \mathcal{N}_b$,*

$$J_b(u) \geq J_b(tu) + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall\ u \in E, \quad t \geq 0. \tag{4.2}$$

Corollary 4.2. *Assume (V), (K) and $(F_1) - (F_4)$ hold. Then for $u \in \mathcal{N}_b$,*

$$J_b(u) = \max_{t\geq 0} J_b(tu). \tag{4.3}$$

Lemma 4.2. *Assume (V), (K) and $(F_1)-(F_4)$ hold. If $u \in E \setminus \{0\}$, then there exists a unique pair $t_u > 0$ such that $t_u u \in \mathcal{N}_b$.*

Lemma 4.3. *Assume (V), (K) and $(F_1)-(F_4)$ hold. Then*

$$\inf_{\mathcal{N}_b} J_b(u) = c_b = \inf_{u\in E, u^{\pm}\neq 0} \max_{t\geq 0} J_b(tu).$$

Lemma 4.4. *Assume (V), (K) and $(F_1)-(F_4)$ hold. Then there exist a constant $c_* \in (0, c_b]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$J_b(u_n) \to c_*, \quad (1 + \|u_n\|)\|J_b'(u_n)\| \to 0. \tag{4.4}$$

Proof. It follows from (2.1), (F_1) and (F_2) that there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$J_b(u) \geq \rho_0, \quad \|u\| = \delta_0. \tag{4.5}$$

Choosing $v_k \in \mathcal{N}_b$ such that

$$c_b \leq J_b(v_k) < c_b + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.6}$$

Since $J_b(0) = 0$ and $J_b(tv_k) < 0$ for large $t > 0$, then it follows from Mountain Pass lemma that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ such that

$$J_b(u_{k,n}) \to c_k, \quad (1 + \|u_{k,n}\|)\|J_b'(u_{k,n})\| \to 0, \quad k \in \mathbb{N}, \tag{4.7}$$

where $c_k \in [\rho_0, \sup_{t\geq 0} J_b(tv_k)]$. In view of Corollary 4.1, we derive

$$J_b(v_k) \geq J_b(tv_k), \quad \forall\ t \geq 0,$$

showing that $J_b(v_k) = \sup_{t\geq 0} J_b(tv_k)$. Therefore, it follows from (4.5) and (4.7) that

$$J_b(u_{k,n}) \to c_k \in [\rho_0, c_b + \frac{1}{k}], \quad (1 + \|u_{k,n}\|)\|J_b'(u_{k,n})\| \to 0, \quad k \in \mathbb{N}. \tag{4.8}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$J_b(u_{k,n_k}) \in [\rho_0, c_b + \frac{1}{k}], \quad (1 + \|u_{k,n_k}\|)\|J_b'(u_{k,n_k})\| < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.9}$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then passing to a subsequence if necessary, we obtain

$$J_b(u_n) \to c_* \in [\rho_0, c_b], \qquad (1 + \|u_n\|)\|J_b'(u_n)\| \to 0.$$

The proof is complete. □

Proof of Theorem 1.2. It follows from Lemma 4.4 that there exists a sequence $\{u_n\} \subset E$ satisfying (4.4), showing that

$$J_b(u_n) \to c_*, \quad \langle J_b'(u_n), u_n\rangle \to 0. \tag{4.10}$$

For large $n \in \mathbb{N}$, it follows from (1.6), (1.7), (2.15) and (4.10) that

$$c_* + 1 \geq J_b(u_n) - \langle J_b'(u_n, u_n\rangle \geq \frac{1-\theta}{4}\|u_n\|^2,$$

which implies that $\{u_n\}$ is bounded in E. By a standard argument, we can prove that there exists a $u_0 \in E \setminus \{0\}$ such that $J_b'(u_0) = 0$. This shows that $u_0 \in \mathcal{N}_b$ is a nontrivial solution of problem (1.1) and $J_b(u_0) \geq c_b$. On the other hand, it follows from (1.6), (1.7), (2.15), Fatou's lemma and the weak semicontinuity of the norm that

$$\begin{aligned}
c_b \geq c_* &= \lim_{n\to\infty} \big[J_b'(u_n) - \frac{1}{4}\langle J_b'(u_n), u_n\rangle\big] \\
&= \lim_{n\to\infty} \Big\{\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)[\frac{1}{4}f(u_n)u_n - F(u_n)]\mathrm{d}x\Big\} \\
&\geq \frac{1}{4}\liminf_{n\to\infty} \big[\int_{\mathbb{R}^3} a|(-\Delta)^{\frac{\alpha}{2}}u_n|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|u_n|^2\mathrm{d}x\big] \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^3}\Big\{K(x)\big[\frac{1}{4}f(u_n)u_n - F(u_n)]\mathrm{d}x + \frac{\theta_0}{4}V(x)|u_n|^2\Big\}\mathrm{d}x \\
&\geq \frac{1}{4}\liminf_{n\to\infty} \big[a\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}u_0|^2\mathrm{d}x + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|u_0|^2\mathrm{d}x\big] \\
&\quad + \lim_{n\to\infty}\int_{\mathbb{R}^3}\Big\{K(x)\big[\frac{1}{4}f(u_0)u_0 - F(u_0)]\mathrm{d}x + \frac{\theta_0}{4}V(x)|u_0|^2\Big\}\mathrm{d}x \\
&= \frac{1}{4}\|u_0\|^2 + \int_{\mathbb{R}^3} K(x)\big[\frac{1}{4}f(u_0)u_0 - F(u_0)]\mathrm{d}x \\
&= J_b(u_0) - \frac{1}{4}\langle J_b'(u_0), u_0\rangle \\
&= J_b(u_0),
\end{aligned}$$

which implies that $J_b(u_0) \leq c_*$, then $J_b(u_0) = c_b = \inf\limits_{\mathcal{N}_b} J_b > 0$.

It follows from Theorem 1.1 that there exists a $u_b \in \mathcal{M}_b$ such that $J_b(u_b) = m_b$. Then by (2.1), Lemma 2.1, Corollary 2.2 and Lemma 4.3, we obtain

$$\begin{aligned}
m_b = J_b(u_b) &= \sup_{s,t\geq 0} J_b(su_b^+ + tu_b^-) \\
&= \sup_{s,t\geq 0}\Big\{J_b(su_b^+) + J_b(tu_b^-) + \frac{bs^2t^2}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u_b^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}u_b^-|^2\mathrm{d}x\Big\} \\
&\geq \sup_{s\geq 0} J_b(su_b^+) + \sup_{t\geq 0} J_b(tu_b^-) \geq 2c_b.
\end{aligned}$$

The proof is complete. □

5. The convergence property

In this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. In the arguments of Section 2, $b = 0$ is allowed. Hence, it follows from the assumptions of Theorem 1.3 that there exists a $v_0 \in \mathcal{M}_0$ such that $J_0'(v_0) = 0$ and $J_0(v_0) = m_0 = \inf\limits_{u\in\mathcal{M}_0} J_0(u)$, i.e., Eq.(1.6) has the least energy sign–changing solution, which changes sign only once.

Choosing $\varphi_0 \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi_0^\pm \neq 0$. It follows from (H_1) and $(F_1)-(F_3)$ that there exist $\gamma_1 > 0$ and

$$\gamma_2 \geq \max\big\{\big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^+|^2\mathrm{d}x\big)^2, \big(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^-|^2\mathrm{d}x\big)^2\big\}$$

such that

$$\int_{\mathbb{R}^3} K(x)F(s\varphi_0^+)\mathrm{d}x \geq \gamma_2|s|^4 - \gamma_1, \quad \int_{\mathbb{R}^3} K(x)F(s\varphi_0^-)\mathrm{d}x \geq \gamma_2|t|^4 - \gamma_1, \tag{5.1}$$

for any $s,\ t \in \mathbb{R}$. Then for any $b \in [0,1]$, it follows from (1.6), (1.13), (5.1) and Lemma 2.2 that

$$\begin{aligned}
J_b(u_b) &= m_b \leq \max_{s,t\geq 0} J_b(s\varphi_0^+ + t\varphi_0^-)\\
&= \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|\varphi_0^+\|^2 + \frac{bs^4}{4}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^+|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3} K(x)F(s\varphi_0^+)\mathrm{d}x\\
&\quad + \frac{t^2}{2}\|\varphi_0^-\|^2 + \frac{bt^4}{4}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^-|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3} K(x)F(t\varphi_0^-)\mathrm{d}x\\
&\quad + \frac{bs^2t^2}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^-|^2\mathrm{d}x\Bigg]\\
&= \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|\varphi_0^+\|^2 + \frac{bs^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^+|^2\mathrm{d}x\big)^2 + 2\gamma_1 - \gamma_2 s^4\\
&\quad + \frac{t^2}{2}\|\varphi_0^-\|^2 + \frac{bt^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^-|^2\mathrm{d}x\big)^2 - \gamma_2 t^4\Bigg]\\
&\leq \max_{s,t\geq 0}\Bigg[\frac{s^2}{2}\|\varphi_0^+\|^2 - \frac{s^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^+|^2\mathrm{d}x\big)^2 + 2\gamma_1 + \frac{t^2}{2}\|\varphi_0^-\|^2\\
&\quad - \frac{t^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}\varphi_0^-|^2\mathrm{d}x\big)^2\Bigg]\\
&:= \Lambda_0 \in (0,+\infty).
\end{aligned} \tag{5.2}$$

For any sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$. For large $n \in \mathbb{N}$, it follows from (1.6), (1.7), (2.15) and (5.2) that

$$\Lambda_0 + 1 \geq J_{b_n}(u_{b_n}) - \frac{1}{4}\langle J'_{b_n}(u_{b_n}), u_{b_n}\rangle \geq \frac{1-\theta_0}{4}\|u_{b_n}\|^2,$$

which implies that $\{u_{b_n}\}$ is bounded in E due to $\theta_0 \in (0,1)$. Therefore, there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$ and $u_{b_n} \to u_0$ in E. By a standard argument (see [33]), we can prove that $u_{b_n}^{\pm} \to u_0^{\pm}$ in E. Note that

$$\begin{aligned}
\langle J_0'(u_0), \varphi\rangle &= \int_{\mathbb{R}^3}(a(-\triangle)^{\frac{\alpha}{2}}u_0(-\triangle)^{\frac{\alpha}{2}}\varphi + V(x)u_0\varphi)\mathrm{d}x - \int_{\mathbb{R}^3} K(x)f(u_0)\varphi\mathrm{d}x\\
&= \lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^3} a((-\triangle)^{\frac{\alpha}{2}}u_{b_n}(-\triangle)^{\frac{\alpha}{2}}\varphi + V(x)u_{b_n}\varphi)\mathrm{d}x\\
&\quad + b_n(\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u_{b_n}\mathrm{d}x)^2\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}u_{b_n}(-\Delta)^{\frac{\alpha}{2}}\varphi\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^3} K(x)f(u_{b_n})\varphi\mathrm{d}x\Bigg]\\
&= \lim_{n\to\infty}\langle J'_{b_n}(u_{b_n}), \varphi\rangle, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^3),
\end{aligned}$$

which implies that $J_0'(u_0) = 0$, and so $u_0 \in \mathcal{M}_0$ and $J_0(u_0) \geq m_0$. Next, we show that $J_0(u_0) = m_0$. Let $b_n \in [0,1]$. Then it follows from (F_3) that there exists a number $N_0 > 0$ such that

$$
\begin{aligned}
J_{b_n}(sv_0^+ + tv_0^-) = & \frac{s^2}{2}\|v_0^+\|^2 + \frac{b_n s^4}{4}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}v_0^+|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3}K(x)F(sv_0^+)\mathrm{d}x \\
& + \frac{t^2}{2}\|v_0^-\|^2 + \frac{b_n t^4}{4}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}v_0^-|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3}K(x)F(tv_0^-)\mathrm{d}x \\
& + \frac{b_n s^2 t^2}{2}\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0^+|^2\mathrm{d}x\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0^-|^2\mathrm{d}x\Big] \\
\le & \frac{s^2}{2}\|v_0^+\|^2 + \frac{s^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}v_0^+|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3}K(x)F(sv_0^+)\mathrm{d}x \\
& + \frac{t^2}{2}\|v_0^-\|^2 + \frac{t^4}{2}\int_{\mathbb{R}^3}\big(|(-\triangle)^{\frac{\alpha}{2}}v_0^-|^2\mathrm{d}x\big)^2 - \int_{\mathbb{R}^3}K(x)F(tv_0^-)\mathrm{d}x \\
< & \, 0, \quad \forall\ s+t \ge N_0.
\end{aligned} \tag{5.3}
$$

In view of Lemma 2.2, there exists (s_n, t_n) such that $s_n v_0^+ + t_n v_0^- \in \mathcal{M}_{b_n}$, which, together with (5.2), implies that $0 < s_n, t_n < N_0$. Therefore, it follows from (1.6), (1.7), (1.11) and (2.1) that

$$
\begin{aligned}
m_0 &= J_0(v_0) \\
&= J_{b_n}(v_0) - \frac{b_n}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0|^2\mathrm{d}x)^2 \\
&\ge J_{b_n}(s_n v_0^+ + t_n v_0^-) + \frac{1-s_n^4}{4}\langle J'_{b_n}(v_0), v_0^+\rangle + \frac{1-t_n^4}{4}\langle J'_{b_n}(v_0), v_0^-\rangle \\
&\quad - \frac{b_n}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0|^2\mathrm{d}x)^2 \\
&\ge m_{b_n} - \frac{1+N_0^4}{4}|\langle J'_{b_n}(v_0), v_0^+\rangle| - \frac{1+N_0^4}{4}|\langle J'_{b_n}(v_0), v_0^-\rangle| \\
&\quad - \frac{b_n}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0|^2\mathrm{d}x)^2 \\
&= m_{b_n} - \frac{1+N_0^4}{4}|\langle J'_{b_n}(v_0), v_0^+\rangle| - \frac{1+N_0^4}{4}|\langle J'_{b_n}(v_0), v_0^-\rangle| \\
&\quad - \frac{b_n}{4}(\int_{\mathbb{R}^3}|(-\triangle)^{\frac{\alpha}{2}}v_0|^2\mathrm{d}x)^2,
\end{aligned}
$$

showing that

$$\limsup_{n\to\infty} m_{b_n} \le m_0. \tag{5.4}$$

It follows from (1.4), (1.6) and (5.4) that

$$m_0 = J_0(u_0) = \limsup_{n\to\infty} J_{b_n}(u_{b_n}) = \limsup_{n\to\infty} m_{b_n} \le m_0,$$

which implies that $J_0(u_0) = m_0$. The proof is complete. □

6. Proof of Corollary 1.1

Similar to Proposition 2.1 in [1], we have the following Lemma.

Lemma 6.1. *Assume $(V, K) \in \mathcal{K}$. If (H_3) or (H_4) holds, then the embedding $X \hookrightarrow L^r_K(\mathbb{R}^3)$ is compact for $2 \le r < 2^*_\alpha$.*

Proof. The proof is analogous to Proposition 2.1 in [1], we omit it here. □

Proof of Corollary 1.1. From Lemma 6.1 and the assumptions of Corollary 1.1, we can easily verify that J_b satisfies the similar geometry structure as the case where (V) and (K) hold. Therefore, Corollary 1.1 follows by slightly modification of Sets. 2–5. □

Acknowledgements

The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

References

[1] C. O. Alves and M. A. S. Souto, *Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity,* J. Differential Equations, 2013, 254, 1977–1991.

[2] C. O. Alves and M. A. S. Souto, *Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains,* Z. Angew. Math. Phys., 2014, 65, 1153–1166.

[3] A. Ambrosetti, V. Felli and A. Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity,* J. Eur. Math. Soc., 2005, 7, 117–144.

[4] V. Ambrosio, G. M. Figueiredo, T. Isernia and G. Molica Bisci, *Sign–changing solutions for a class of zero mass nonlocal Schrödinger equations,* Adv. Nonlinear Stud., 2019, 19, 113–132.

[5] V. Ambrosio and T. Isernia, *Concentration phenomena for a fractional Schrödinger-Kirchhoff type equation,* Math. Meth. Appl. Sci., 2018, 41, 615–645.

[6] V. Ambrosio and T. Isernia, *Sign–changing solutions for a class of Schrödinger equations with vanishing potentials,* Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 2018, 29, 127–152.

[7] V. Ambrosion and T. Isernia, *A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity,* Commum. Contemp. Math., 2018, 20, 1750054.

[8] A. Ambrosetti and Z. Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials,* Differ. Integral Equ., 2005, 18, 1321–1332.

[9] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$,* J. Differential Equations, 2013, 255, 2340–2362.

[10] T. Bartsch, T. Weth and M. Willem, *Partial symmetry of least energy nodal solutions to some variational problems,* J. Anal. Math, 2005, 96, 1–18.

[11] H. Berestycki and P. Lions, *Nonlinear scalar field equations. I. Existence of a ground state state,* Arch. Ration. Mech. Anal., 1983, 82, 313–345.

[12] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian,* Commun. Part. Diff. Equat., 2007, 32, 1245–1260.

[13] D. Cassani, Z. Liu, C. Tarsi and J. Zhang, *Multiplicity of sign–changing solutions for Kirchhoff–type equations,* Nonlinear Anal., 2019, 186, 145–161.

[14] G. Che and H. Chen, *Existence and multiplicity of positive solutions for Kirchhoff–Schrödinger–Poisson system with critical growth,* Rev. Real Acad. Cienc. Exactas F., 2020, 114, 78.

[15] G. Che and H. Chen, *Existence and concentration result for Kirchhoff equations with critical exponent and Hartree nonlinearity,* J. Appl. Anal. Comput., 2020, 10, 2121–2144.

[16] G. Che, H. Chen, H. Shi and Z. Wang, *Existence of nontrivial solutions for fractional Schrödinger–Poisson system with sign–changing potentials,* Math. Meth. Appl. Sci., 2018, 41, 5050–5064.

[17] G. Che, H. Chen and T.F. Wu, *Existence and multiplicity of positive solutions for fractional Laplacian systems with nonlinear coupling,* J. Math. Phys., 2019, 60, 081511.

[18] G. Che and T.F. Wu, *Three positive solutions for Kirchhoff problems with steep potential well and concave–convex nonlinearities,* Appl. Math. Lett., 2021, 121, 107348.

[19] S. Chen and X. Tang, *Ground state sign–changing solutions for a class of Schrödinger–Poisson type problems in* $\mathbb{R}^3$*,* Z. Angew. Math. Phys., 2016, 67, 1–18.

[20] Y. Deng, S. Peng and W. Shuai, *Existence and asymptotic behavior of nodal solutions for the Kirchhoff type problema in* $\mathbb{R}^3$*,* J. Funct. Anal., 2015, 269, 3500–3527.

[21] G. M. Figueiredo, M. B. Guimar$\widetilde{a}$es and R. d. S. Rodrigues, *Solutions for a Kirchhoff equation with weight and nonlinearity with subcritical and critical Caffarelli–Kohn–Nirenberg growth,* Proc. Edinburgh Math. Soc., 2016, 59, 925–944.

[22] R. L. Frank, E. Lenzmann and L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian,* Commun. Pure Appl. Math., 2016, 69, 1671–1726.

[23] T. Isernia, *Sign–changing solutions for a fractional Kirchhoff equation,* Nonlinear Anal., 2020, 190, 111623.

[24] T. Isernia, *Fractional* p & q*–Laplacian problems with potentials vanishing at infinity,* Opuscula Math., 2020, 40, 93–110.

[25] G. Kirchhoff, *Mechanik,* Teubner, 1883.

[26] S. Khoutir and H. Chen, *Existence of infinitely many high energy solutions for a fractional Schrödinger equation in* $\mathbb{R}^N$*,* Appl. Math. Lett., 2016, 61, 156–162.

[27] N. Laskin, *Fractional Schrödinger equation,* Phy. Rev. E., 2002, 66, 05618.

[28] Z. Liu, M. Squassina and J. Zhang, *Ground states for fractional Kirchhoff equations with critical nonlinearity in low dimensions,* NODEA-Nonlinear Differ. Equ. Ap., 2017, 24, 1–32.

[29] W. Long, S. Peng and J. Yang, *Infinitely positive and sign–changing solutions for nonlinear fractional scalar field equations,* Discrete Contin. Dyn. Syst., 2015, 36, 917–939.

[30] D. Lü, *A note on Kirchhoff–type equations with Hartree–type nonlinearities,* Nonlinear Anal., 2014, 99, 35–48.

[31] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$*,* J. Math. Phys., 2013, 54, 031501.

[32] H. Shi and H. Chen, *Multiple solutions for fractional Schrödinger equation,* Electron. J. Differ. Equ., 25 (2015) 1–11.

[33] W. Shuai, *Sign–changing solutions for a class of Kirchhoff-type problem in bounded domains,* J. Differential Equations, 2015, 259, 1256–1274.

[34] Y. Su and H. Chen, *Fractional Kirchhoff–type equation with Hardy–Littlewood–Sobolev critical exponent,* Comput. Math. Appl., 2019, 78, 2063–2082.

[35] J. Sun and T. Wu, *Ground state solutions for an indefinite Kirchhoff type problem with steep potential well,* J. Differential Equations, 2014, 256, 1771–1792 .

[36] J. Sun, Y. Cheng, T. Wu and Z. Feng, *Positive solutions of a superlinear Kirchhoff type equation in* $\mathbb{R}^N (N \geq 4)$*,* Commun. Nonlinear Sci. Numer. Simulat., 2019, 71, 141–160.

[37] J. Sun and T. Wu, *Steep potential well may help Kirchhoff type equations to generate multiple solutions,* Nonlinear Anal., 2020, 190, 111609.

[38] X. Tang, *Non–Nehari manifold method for superlinear Schrödinger equation,* Taiwanese J. Math., 2014, 18, 1950–1972.

[39] X. Tang and B. Cheng, *Ground state sign–changing solutions for Kirchhoff type problems in bounded domains,* J. Differential Equations, 2016, 261, 2384–2402.

[40] K. Teng and X. He, *Ground state solutions for fractional Schrödinger equations with critical Sobolev exponent,* Commun. Pure Appl. Anal., 2016, 15, 991–1008.

[41] K. Teng, K. Wang and R. Wang, *A sign–changing solution for nonlinear problems involving the fractional laplacian,* Electron. J. Differ. Equ., 2015, 2015, 1–12.

[42] Z. Wang and H. Zhou, *Sign–changing solutions for the nonlinear Schrödinger–Poisson system in* $\mathbb{R}^3$*,* Calc. Var. Partial Diff. Equ., 2015, 52, 927–943.

[43] Z. Wang and H. Zhou, *Radial sign–changing solution for fractional Schrödinger equation,* Discrete Contin. Dyn. Syst., 2016, 36, 499–508.

[44] M. Willem, *Minimax Theorems,* Birkhäuser, Berlin, 1996.

[45] Q. Xie, *Bounded state solution of degenerate Kirchhoff type problem with a critical exponent,* J. Math. Anal. Appl., 2019, 479, 1–24.

[46] J. Zhang, Z. Liu and M. Squassina, *Modulational stability of ground states to nonlinear Kirchhoff equations,* J. Math. Anal. Appl., 2019, 477, 844–859.

[47] J. Zhang and W. Zou, *A Berestycki–Lions theorem revisted,* Commun. Contemp. Math., 2012, 14, 1250033.