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DISCONTINUOUS FRACTIONAL STURM-LIOUVILLE PROBLEMS WITH EIGEN-DEPENDENT BOUNDARY CONDITIONS

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Abstract In this paper, a fractional discontinuous Sturm-Liouville type boundary-value problem with eigenparameter-dependent boundary conditions and with two fractional transmission conditions is investigated. Using operator theory, a new inner product is defined by combining the parameters in the boundary and transmission conditions, then the boundary value transmission problem is transferred to an operator in a new Hilbert space such that the eigenvalues and eigenfunctions of the main problem coincide with those of this operator. Moreover, the fundamental solutions are constructed, and then the characteristic function whose zeros are the eigenvalues of the problem is established.

Keywords Fractional Sturm-Liouville problem, fractional eigen-dependent boundary conditions, fractional transmission conditions.

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1. Introduction

It is well known that the Sturm-Liouville type boundary-value problems play a significant role in many areas of science, engineering and mathematics, especially about the eigenvalues and eigenfunctions of the Sturm-Liouville boundary-value problem. The Sturmian theory was founded 170 years ago, and there have been a lot of researches and works related to it. The research on the nature of the regular Sturm-Liouville problem is relatively complete [16, 17, 24]. In the classic Sturm-Liouville theory, it is required that the solution and its quasi-derivative are absolutely continuous, but in many practical applications, this perfect situation is difficult to be satisfied. For example, the heat conduction problem of plates formed by overlapping materials with different characteristics, the basic solution of the problem on a single plate can be derived from the classical heat conduction equation, and the structure of the solution on the entire laminated plate leads us to consider the continuity of the solutions at each junction. Therefore, scholars focus to investigate the discontinuity of the solution or the various derivatives of the solution of the differential equation. Generally speaking, the eigenparameter only appears in the differential equation, however, a large number of applications in

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the mathematical physics, engineering etc. require that the eigenparameter appears not only in the differential equation, but also in the boundary conditions. Specific research and various applications on such issues can be found in the literature [1–3, 8, 12–14, 18].

Fractional calculus is a well known topic since it was initiated in 17th century, and there are a lot of related literatures. Among them, Podlubny [19] and Kilbass [9] introduced the history of fractional calculus. There are many different types of definitions of fractional derivatives and integrals. The great widely used is Riemann-Liouville fractional derivatives and integrals, another derivative is the Caputo fractional derivative, introduced by Caputo in 1967. West [22] and Magin [15] showed that in many applications solutions based on fractional-order derivatives model are more accurate than those based on integer-order derivatives model. Baleanu and Uğurlu [6] construct a regular dissipative fractional operator associated with a fractional boundary value problem and extend considerably the possibility to extract new features from the dynamics of complex systems involving non-local effects.

In recent years, more and more researchers pay attention to fractional Sturm-Liouville type boundary value problem and study the numerical problems, eigenvalues and eigenfunctions of fractional Sturm-Liouville operators, see for example [4, 7, 10, 11, 20, 21, 25]. In [21], the author used different fractional operators to solve common Sturm-Liouville problem, and studied the eigenvalues and eigenfunctions related to those operators respectively. Zayernouri and Karniadakis [25] considered the eigenvalues of two types of fractional Sturm-Liouville problems on compact interval. Akdogan and Yakar [23] investigated a class of discontinuous Sturm-Liouville problem with fractional derivatives and constructed the corresponding fundamental solutions. Akdogan, Yakar and Demirici [5] considered a fractional discontinuous Sturm-Liouville problem which boundary conditions are different with [23], they defined an operator A associated the problem in Hilbert space $L_2(-1, 1)$, and established the characteristic function of the discontinuous fractional Sturm-Liouville problem.

In this paper, we generalized the results of [5,23] to a class of discontinuous fractional Sturm-Liouville problems with eigen-parameter contained in the boundary conditions. Using operator theories and analytical skills, we transfer the considered boundary value problem to a symmetric fractional Sturm-Liouville operator by introducing a new Hilbert space associated with the parameters in the boundary and transmission conditions. Note that the operator we defined in this paper is different from the previous papers. We prove the symmetry of the operator, and investigate the properties of its eigenvalues and eigenfunctions. Finally, the characteristic function of the problem is established.

The paper is organized as follows: In Section 2, some basic theoretical knowledge about Riemann-Liouville and Caputo fractional calculus are given. In Section 3, we investigate the discontinuous fractional Sturm-Liouville problems with eigenparameter dependent boundary conditions. In Section 4, we define fractional Sturm-Liouville operator associated with the boundary value transmission problem and construct the fundamental solutions, then we establish the characteristic function whose zeros are the eigenvalues of the problem.

2. Preliminaries

In this section, we shall recall some basic definitions and properties of fractional calculus which are necessary for the development of the paper. In addition, we shall introduce some lemmas and give their proofs if needed.

Definition 2.1 (c.f. [9]). (Left and right Riemann-Liouville (R-L) fractional integrals)

Let $[a,b] \subset \mathbb{R}, Re(\alpha) > 0$ and $f \in L^1[a,b]$. Then the left and right Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order $\alpha \in \mathbb{C}$ are given by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \ x \in (a,b],$$
$$I^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \ x \in [a,b),$$

respectively.

Definition 2.2 (c.f. [9]). (Left and right Riemann-Liouville (R-L) fractional derivatives)

Let $[a,b] \subset \mathbb{R}, Re(\alpha) \in (0,1)$ and $f \in L^1[a,b]$. Then the left and right Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ of function f are defined as

$$D^{\alpha}_{a+} f(x) := D I^{1-\alpha}_{a+} f(x), \ x \in (a,b],$$
$$D^{\alpha}_{b-} f(x) := -D I^{1-\alpha}_{b-} f(x), \ x \in [a,b),$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

Definition 2.3 (c.f. [9]). (Left and right Caputo fractional derivatives)

Let $[a,b] \subset \mathbb{R}, Re(\alpha) \in (0,1)$ and $f \in L^1[a,b]$. Then the left and right Caputo fractional derivatives of order $\alpha \in \mathbb{C}$ are

$${}^c D^{\alpha}_{a+} f(x) := I^{1-\alpha}_{a+} D f(x), x \in (a,b],$$
$${}^c D^{\alpha}_{b-} f(x) := -I^{1-\alpha}_{b-} D f(x), x \in [a,b),$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

Lemma 2.1 (c.f. [9]).

$$D^{\alpha}_{a+} I^{\alpha}_{a+} f(x) = f(x),$$
$$D^{\alpha}_{b-} I^{\alpha}_{b-} f(x) = f(x).$$

and

$$I^{\alpha}_{a+} D^{\alpha}_{a+} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha}_{a+} f(a),$$
$$I^{\alpha}_{b-} D^{\alpha}_{b-} f(x) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha}_{a+} f(b),$$

where $\alpha \in (0,1)$.

According to the above equations, we can see that R-L derivative is the left inverse of the R-L integral, but not the right inverse.

Lemma 2.2 (c.f. [9]).

$$
{}^{c}D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(x) = f(x),\\
{}^{c}D_{b^-}^{\alpha} I_{b^-}^{\alpha} f(x) = f(x).
$$

and

$$
I_{a^+}^{\alpha}\ {}^{c}D_{a^+}^{\alpha} f(x) = f(x) - f(a),\\
I_{b^-}^{\alpha}\ {}^{c}D_{b^-}^{\alpha} f(x) = f(x) - f(b),
$$

where $\alpha \in (0,1)$.

Now, we state the following lemmas which will be used in the later sections.

Lemma 2.3 (c.f. [5]). *Assume that* $0 < \alpha < 1, f \in AC[a,b]$ *and* $g \in L^p(a,b)(1 \leq p \leq \infty)$. *Then the following equation holds:*

$$
\int_a^b f(x) D_{a^+}^{\alpha} g(x)dx = \int_a^b g(x)^c D_{b^-}^{\alpha} f(x)dx + f(x) I_{a^+}^{1-\alpha} g(x)|_{x=a}^{x=b}.
$$

Lemma 2.4 (c.f. [23]). *Let* $f \in L^2(a,b)$ *and* $\alpha \in (0,1)$, *then:*

(i) $I_{a^+}^{\alpha}\ {}^{c}D_{b^-}^{\alpha} f(x) = M_g(x) + (-1)^{\alpha}(f(x) - f(b))$,

(ii) $I_{a^+}^{\alpha}\ {}^{c}D_{b^-}^{\alpha} f(x) = (-1)^{\alpha-1} I_{a^+}^{\alpha} N_f(x) + (-1)^{\alpha}(f(x) - f(a))$.

where $M_g(x) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-t)^{\alpha-1} g(t)dt,\ N_f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^b (x-t)^{-\alpha} f'(t)dt$, $g(x) =^c D_{b^-}^{\alpha} f(x)$.

Lemma 2.5 (c.f. [9]). *The fractional integral operators* $I_{a^+}^{\alpha}$ *and* $I_{b^-}^{\alpha}$ *with* $Re(\alpha) > 0$ *are bounded in* $L^p(a,b)(1 \leq p \leq \infty)$:

$$
\|I_{a^+}^{\alpha} f\|_p \leq k\|f\|_p\ , \|I_{b^-}^{\alpha} f\|_p \leq k\|f\|_p\ (k = \frac{(b-a)^{Re(\alpha)}}{Re(\alpha)|\Gamma(\alpha)|}). \tag{2.1}
$$

Lemma 2.6 (c.f. [9]). *Let* $Re(\alpha) \geq 0$, *then the Caputo fractional differential operators* ${}^{c}D_{a^+}^{\alpha}$ *and* ${}^{c}D_{b^-}^{\alpha}$ *are bounded from the space* $C^n[a,b]$ *to the space* $C_a[a,b]$ *and* $C_b[a,b]$,

$$
\|{}^{c}D_{a^+}^{\alpha} y\|_{C_a} \leq k_\alpha \|y\|_{C^n},\ \|{}^{c}D_{b^-}^{\alpha} y\|_{C_b} \leq k_\alpha \|y\|_{C^n} (k_\alpha = \frac{(b-a)^{n-Re(\alpha)}}{\Gamma(n-\alpha)[n - Re(\alpha) + 1]}), \tag{2.2}
$$

where $C^n[a,b], C_a[a,b]$ *and* $C_b[a,b]$ *defined by* (1.1.25) *and* (1.1.26) *in [9].*

For more details, we refer to [9].

3. Fractional Sturm-Liouville problems with eigen-dependent boundary and transmission conditions

In this section, we consider the following fractional Sturm-Liouville differential expression $\pounds_\alpha$ as follows

$$
\pounds_\alpha = \begin{cases} {}^{c}D_{0^-}^{\alpha} p(x) D_{-1^+}^{\alpha} + q(x), & x \in [-1,0), \\ {}^{c}D_{1^-}^{\alpha} p(x) D_{0^+}^{\alpha} + q(x), & x \in (0,1]. \end{cases}
$$

Then we shall consider the following fractional Sturm-Liouville problem on I , where $I = [-1, 0) \cup (0, 1]$,

$$\pounds_\alpha u + \lambda u = 0, \tag{3.1}$$

with boundary conditions:

$$\begin{aligned} L_1(u) :=& r_1 I^{1-\alpha}_{-1^+} u(-1) - r_2 p_1 D^{\alpha}_{-1^+} u(-1) \\ &- \lambda(r'_1 I^{1-\alpha}_{-1^+} u(-1) - r'_2 p_1 D^{\alpha}_{-1^+} u(-1)) = 0, \end{aligned} \tag{3.2}$$

$$L_2(u) := I^{1-\alpha}_{0^+} u(1) - \beta p_2 D^{\alpha}_{0^+} u(1) = 0 \tag{3.3}$$

and transmission conditions:

$$L_3(u) := h_1 I^{1-\alpha}_{-1^+} u(0-) - I^{1-\alpha}_{0^+} u(0+) = 0, \tag{3.4}$$

$$L_4(u) := D^{\alpha}_{-1^+} u(0-) - h_2 D^{\alpha}_{0^+} u(0+) = 0, \tag{3.5}$$

where $\frac{1}{2} \leq \alpha \leq 1, \lambda \in \mathbb{C}$ and λ is eigenparameter.

$$\rho = r'_1 r_2 - r_1 r'_2 > 0; \theta = \frac{h_1}{h_2} > 0.$$

$$p(x) = \begin{cases} p_1, & x \in [-1, 0), \\ p_2, & x \in (0, 1]. \end{cases}$$

$q(x)$ is real-valued and continuous in both [-1,0) and (0,1], $h_1, h_2 \neq 0$, and h_1, h_2 are real numbers, p_1, p_2 are all positive real numbers.

4. The operator Formulation of fractional Sturm-Liouville problem

We define the following inner product in the Hilbert space $L^2(I)$ by

$$\langle f, g \rangle_1 = \frac{\theta}{p_1} \int_{-1}^{0} f(x)\overline{g(x)}dx + \frac{1}{p_2} \int_{0}^{1} f(x)\overline{g(x)}dx \tag{4.1}$$

for arbitrary $f, g \in L^2(I)$. Obviously, $H_1 = (L^2(I), \langle \cdot, \cdot \rangle_1)$ is a Hilbert space. We define the following inner product in the Hilbert space $H := H_1 \oplus \mathbb{C}$ as

$$\langle F, G \rangle = \langle f, g \rangle_1 + \frac{\theta}{\rho} f_1 \overline{g}_1. \tag{4.2}$$

for $F = (f(x), f_1), G = (g(x), g_1) \in H, f(x), g(x) \in H_1,\ f_1, g_1 \in \mathbb{C}$.

In the Hilbert space H, consider the operator $\mathcal{A}$ which is defined by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H,$$

where the domain $\mathcal{D}(\mathcal{A})$ is defined as follows

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{ (f(x), f_1) \in H | & f(x), D^{\alpha}_{-1^+} f(x), {}^c D^{\alpha}_{1^-} f(x) \in AC([-1, 0) \cup (0, 1]; \\ & f(0\pm), D^{\alpha}_{-1^+} f(0\pm), I^{1-\alpha}_{-1^+} f(0\pm) \text{all have finite limits;} \\ & L_i f = 0, i = 2, 3, 4; \\ & f_1 = r'_1 I^{1-\alpha}_{-1^+} f(-1) - r'_2 p_1 D^{\alpha}_{-1^+} \} \end{aligned}$$

$$\mathcal{A}F = \mathcal{A}(f(x), f_1) = (\pounds_\alpha f, r_1 I_{-1^+}^{1-\alpha} f(-1) - r_2 p_1 D_{-1^+}^{\alpha} f(-1)),$$

for

$$F = (f(x), r_1' I_{-1^+}^{1-\alpha} f(-1) - r_2' p_1 D_{-1^+}^{\alpha} f(-1)).$$

For simplicity, let

$$N(f) = r_1 I_{-1^+}^{1-\alpha} f(-1) - r_2 p_1 D_{-1^+}^{\alpha} f(-1),$$
$$N'(f) = r_1' I_{-1^+}^{1-\alpha} f(-1) - r_2' p_1 D_{-1^+}^{\alpha} f(-1).$$

Now, we can rewrite the considered problem (3.1)-(3.5) in operator form

$$\mathcal{A}u = \lambda u. \tag{4.3}$$

Obviously, the following lemma holds.

Lemma 4.1. *(i) The eigenvalues of boundary value problem (3.1)-(3.5) coincide with those of operator $\mathcal{A}$.*

(ii) The eigenfunctions of boundary value problem (3.1)-(3.5) are the first component of corresponding eigen element of operator $\mathcal{A}$.

Theorem 4.1. *The operator $\mathcal{A}$ is symmetric.*

Proof. For any $F, G \in \mathcal{D}(\mathcal{A}), F = (f(x), N'(f)), G = (g(x), N'(g))$.

$$< \mathcal{A}F, G >=< \pounds_\alpha f, g >_1 + \frac{\theta}{\rho} N(f)\overline{N'(g)}.$$

$$\begin{aligned} < \pounds_\alpha f, g >_1 =& \frac{\theta}{p_1}\int_{-1}^{0} ({}^c D_{0^-}^{\alpha} p_1 D_{-1^+}^{\alpha} f(x))\overline{g(x)}dx + \frac{\theta}{p_1}\int_{-1}^{0} q(x) f(x)\overline{g(x)}dx \\ &+ \frac{1}{p_2}\int_{0}^{1} ({}^c D_{1^-}^{\alpha} p_2 D_{0^+}^{\alpha} f(x))\overline{g(x)}dx + \frac{1}{p_2}\int_{0}^{1} q(x) f(x)\overline{g(x)}dx. \end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned} < \pounds_\alpha f, g >_1 =& \theta(\int_{-1}^{0} f(x)^c D_{0^-}^{\alpha} D_{-1^+}^{\alpha}\overline{g(x)}dx + D_{-1^+}^{\alpha}\overline{g(x)} I_{-1^+}^{1-\alpha} f(x)|_{-1}^{0} \\ &- D_{-1^+}^{\alpha} f(x) I_{-1^+}^{1-\alpha}\overline{g(x)}|_{-1}^{0}) \\ &+ \int_{0}^{1} f(x)^c D_{1^-}^{\alpha} D_{0^+}^{\alpha}\overline{g(x)}dx + D_{0^+}^{\alpha}\overline{g(x)} I_{0^+}^{1-\alpha} f(x)|_{0}^{1} \\ &- D_{0^+}^{\alpha} f(x) I_{0^+}^{1-\alpha}\overline{g(x)}|_{0}^{1} \\ &+ \frac{\theta}{p_1}\int_{-1}^{0} q(x) f(x)\overline{g(x)}dx + \frac{1}{p_2}\int_{0}^{1} q(x) f(x)\overline{g(x)}dx. \end{aligned}$$

Hence, we have

$$\begin{aligned} < \mathcal{A}F, G >=& < F, \mathcal{A}G > + \theta W_1(f, \overline{g}, 0-) - \theta W_1(f, \overline{g}, -1) + W_2(f, \overline{g}, 1), \\ &- W_2(f, \overline{g}, 0+) + \frac{\theta}{\rho} N(f)\overline{N'(g)} - \frac{\theta}{\rho} N'(f)\overline{N(g)}, \end{aligned}$$

where

$$W_1(f, \overline{g}, x) = I_{-1^+}^{1-\alpha} f(x) D_{-1^+}^{\alpha}\overline{g(x)} - I_{-1^+}^{1-\alpha}\overline{g(x)} D_{-1^+}^{\alpha} f(x),$$
$$W_2(f, \overline{g}, x) = I_{0^+}^{1-\alpha} f(x) D_{0^+}^{\alpha}\overline{g(x)} - I_{0^+}^{1-\alpha}\overline{g(x)} D_{0^+}^{\alpha} f(x).$$

By transmission conditions (3.4) and (3.5), we get

$$\begin{aligned} W_1(f,\overline{g},0-) &= I^{1-\alpha}_{-1^+}f(0-)D^{\alpha}_{-1^+}\overline{g(0-)} - I^{1-\alpha}_{-1^+}\overline{g(0-)}D^{\alpha}_{-1^+}f(0-) \\ &= \frac{h_2}{h_1}[I^{1-\alpha}_{0^+}f(0+)D^{\alpha}_{0^+}\overline{g(0+)} - I^{1-\alpha}_{0^+}\overline{g(0+)}D^{\alpha}_{0^+}f(0+)] \\ &= \frac{1}{\theta}W_2(f,\overline{g},0+). \end{aligned}$$

By (3.3), we get

$$\begin{aligned} &W_2(f,\overline{g},1) = 0, \\ &\frac{\theta}{\rho}N(f)\overline{N'(g)} - \frac{\theta}{\rho}N'(f)\overline{N(g)} = \theta W_1(f,\overline{g},-1). \end{aligned}$$

Therefore,

$$< \mathcal{A}F, G >=< F, \mathcal{A}G >,$$

which implies $\mathcal{A}$ is symmetric. $\square$

Corollary 4.1. *All eigenvalues of boundary value problem (3.1)-(3.5) are all real-valued.*

Now we can assume that all eigenfunctions of problem (3.1)-(3.5) are real-valued.

Corollary 4.2 (Orthogonality)**.** *Let λ_1 and λ_2 be two different eigenvalues of the problem (3.1)-(3.5), then the corresponding eigenfunctions $f(x), g(x)$ of this problem satisfy the following equation*

$$\begin{aligned} &\frac{\theta}{p_1}\int_{-1}^{0} f(x)\overline{g(x)}dx + \frac{1}{p_2}\int_0^1 f(x)\overline{g(x)}dx \\ &+ \frac{\theta}{\rho}(r_1' I^{1-\alpha}_{-1^+}f(-1) - r_2' p_1 D^{\alpha}_{-1^+}f(-1))(r_1' I^{1-\alpha}_{-1^+}\overline{g(-1)} - r_2' p_1 D^{\alpha}_{-1^+}\overline{g(-1)}) = 0. \end{aligned}$$

Corollary 4.2 implies that the eigenfunctions of the operator A corresponding to different eigenvalues are orthogonal under the inner product (4.2) in the Hilbert space H.

Lemma 4.2. *The equivalent integral form of equation (3.1) with fractional conditions (3.4)-(3.5) is given as*

$$u(x) = u_0(x) + \frac{1}{p_2\Gamma(2\alpha)}\int_0^x [N_u(t) + (-1)^{1-\alpha}(x-t)^{2\alpha-1}(\lambda + q(t))u(t)]dt, \quad (4.4)$$

where $u_0(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}(h_1 I^{1-\alpha}_{-1^+}u(0-)) + I^{\alpha}_{0^+}(\frac{1}{h_2}D^{\alpha}_{-1^+}u(0-))$.

Proof. Let us consider the equation (3.1) : $\mathcal{L}_\alpha u + \lambda u = 0$,

$${}^cD^{\alpha}_{1^-}p_2 D^{\alpha}_{0^+}u(x) + (\lambda + q(x))u(x) = 0,\ x \in (0,1].$$

Using fractional integral operator $I^{\alpha}_{0^+}$ acting on above equation, we obtain

$$I^{\alpha}_{0^+}\ {}^cD^{\alpha}_{1^-}p_2 D^{\alpha}_{0^+}u(x) + I^{\alpha}_{0^+}(\lambda + q(x))u(x) = 0,\ x \in (0,1]. \quad (4.5)$$

Then by Lemma 2.4, we have

$$I^{\alpha}_{0^+}\ {}^cD^{\alpha}_{1^-}p_2 D^{\alpha}_{0^+}u(x) = (-1)^{\alpha-1}I^{\alpha}_{0^+}N_u(x) + (-1)^{\alpha}(p_2 D^{\alpha}_{0^+}u(x) - p_2 D^{\alpha}_{0^+}u(0)).$$

Then equation (4.5) becomes

$$p_2 D^{\alpha}_{0^+}u(x) = I^{\alpha}_{0^+}N_u(x) + p_2 D^{\alpha}_{0^+}u(0) + (-1)^{1-\alpha} I^{\alpha}_{0^+}(\lambda + q(x))u(x). \tag{4.6}$$

Applying $I^{\alpha}_{0^+}$ on both sides of (4.6), we get

$$p_2 I^{\alpha}_{0^+}D^{\alpha}_{0^+}u(x) = I^{\alpha}_{0^+}I^{\alpha}_{0^+}N_u(x) + p_2 I^{\alpha}_{0^+}D^{\alpha}_{0^+}u(0) + (-1)^{1-\alpha} I^{\alpha}_{0^+}I^{\alpha}_{0^+}(\lambda + q(x))u(x). \tag{4.7}$$

By Lemma 2.1 and transmission conditions (3.4), (3.5). Then equation (4.7) becomes

$$\begin{aligned} p_2[u(x) - \frac{x^{\alpha-1}}{\Gamma(\alpha)}(h_1 I^{1-\alpha}_{-1^+}u(0-))] =& I^{2\alpha}_{0^+}N_u(x) + p_2 I^{\alpha}_{0^+}(\frac{1}{h_2}D^{\alpha}_{-1^+}u(0-)) \\ &+ (-1)^{1-\alpha} I^{2\alpha}_{0^+}(\lambda + q(x))u(x). \end{aligned}$$

Then we reach

$$u(x) = u_0(x) + \frac{1}{p_2} I^{2\alpha}_{0^+}[N_u(x) + (-1)^{1-\alpha}(\lambda + q(x))u(x)].$$

□

Next, we use the conclusion of Lemma 4.2 to construct $u_m(x, \lambda)$, and then discuss the successive approximations.

$$u_m(x, \lambda) = u_0(x, \lambda) + \frac{\int_0^x (x-t)^{2\alpha-1}[N_{u_{m-1}}(t) + (-1)^{1-\alpha}(\lambda + q(t))u_{m-1}(t)]dt}{p_2\Gamma(2\alpha)}. \tag{4.8}$$

If $\alpha = 1$, the above problem becomes classical Sturm-Liouville problem.

Lemma 4.3. *Let* $Q := \max_{x\in(0,1]} |q(x)|$, $P_R := \max_{|\lambda|\le R} P(\lambda)$ *and* $P(\lambda) := \max_{x\in(0,1]} |u_0(x, \lambda)|$.

Then, for any m*, the following estimate*

$$\|u_m(x, \lambda) - u_{m-1}(x, \lambda)\| \le P_R\{\frac{|\lambda| + 2k_\alpha + Q}{p_2\Gamma(2\alpha + 1)}\}^m \tag{4.9}$$

holds, where $k_\alpha := \frac{1}{(2-\alpha)\Gamma(1-\alpha)}$.

Proof. The proof of this lemma can be referred to [16]. □

For the following initial value problem

$$\begin{cases} {}^cD^{\alpha}_{0^-}p_1 D^{\alpha}_{-1^+}u(x) + (q(x) + \lambda)u(x) = 0, \ x \in [-1, 0); \\ I^{1-\alpha}_{-1^+}u(-1) = (r_2 - \lambda r_2')p_1, \\ D^{\alpha}_{-1^+}u(-1) = r_1 - \lambda r_1'. \end{cases} \tag{4.10}$$

If we use a similar way in Lemma 4.2, we can get a corresponding integral equation of the problem (4.10) as follows:

$$u(x) = u_0(x) + \frac{1}{p_1} I^{2\alpha}_{-1^+}[N_u(x) + (-1)^{1-\alpha}(q(x) + \lambda)u(x)], \tag{4.11}$$

where $u_0(x) = \frac{(x+1)^{\alpha-1}}{\Gamma(\alpha)}p_1(r_2 - \lambda r_2') + \frac{(x+1)^{\alpha}}{\Gamma(\alpha+1)}(r_1 - \lambda r_1')$.

Lemma 4.4. *The initial value problem (4.10) has an unique solution on* $[-1,0)$ *provided that* $\frac{1}{p_1\Gamma(2\alpha+1)}(|\lambda|+2k_\alpha+Q)<1$.

Proof. We shall use the idea of contraction mapping to prove this lemma. From above discussion, we know

$$u(x)=u_0(x)+\frac{1}{p_1}I^{2\alpha}_{-1^+}[N_u(x)+(-1)^{1-\alpha}(q(x)+\lambda)u(x)],$$

where $u_0(x)=\frac{(x+1)^{\alpha-1}}{\Gamma(\alpha)}p_1(r_2-\lambda r_2')+\frac{(x+1)^{\alpha}}{\Gamma(\alpha+1)}(r_1-\lambda r_1')$. We construct the integral equation as

$$\phi=T\phi,$$

and action law of operator T:

$$Tf=u_0+\frac{1}{p_1}I^{2\alpha}_{-1^+}[N_f+(-1)^{1-\alpha}(q(x)+\lambda)f],$$

then we obtain

$$\|Tf-Tg\|=\|\frac{1}{p_1}I^{2\alpha}_{-1^+}[(N_f-N_g)+(-1)^{1-\alpha}(q(x)+\lambda)(f-g)]\|.$$

By Lemma 2.5,

$$\begin{aligned}\|Tf-Tg\|&\le\frac{1}{p_1\Gamma(2\alpha+1)}\|(N_f-N_g)+(-1)^{1-\alpha}(q(x)+\lambda)(f-g)\|\\&\le\frac{1}{p_1\Gamma(2\alpha+1)}\|(N_f-N_g)\|+\|(q(x)+\lambda)(f-g)\|.\end{aligned}$$

According to Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned}N_f-N_g&={}^c D^{\alpha}_{0^-}(f(x)-g(x))+(-1)^{\alpha-1}\;{}^cD^{\alpha}_{-1^+}(f(x)-g(x))\\&\le 2k_\alpha\|(f-g)\|.\ k_\alpha=\frac{1}{(2-\alpha)\Gamma(1-\alpha)}.\end{aligned}$$

Therefore,

$$\|Tf-Tg\|\le\frac{1}{p_1\Gamma(2\alpha+1)}(|\lambda|+2k_\alpha+Q)\|(f-g)\|.$$

If $\frac{1}{p_1\Gamma(2\alpha+1)}(|\lambda|+2k_\alpha+Q)<1$, we get the mapping T is a contraction on the space $\langle C[-1,0),\|\cdot\|\rangle$ by contraction mapping principle. Consequently,

$$\phi=T\phi$$

has an unique solution. $\square$

For any $\lambda\in\mathbb{C}$, let $\phi_{1,\lambda}(x):=\phi_1(x,\lambda)$ be the solution of equation (3.1) on interval $[-1,0)$, and satisfies initial conditions:

$$\begin{cases}I^{1-\alpha}_{-1^+}u(-1)=(r_2-\lambda r_2')p_1,\\D^{\alpha}_{-1^+}u(-1)=(r_1-\lambda r_1'),\end{cases}\tag{4.12}$$

$\phi_1(x,\lambda)$ is an entire function of λ for each $x\in[-1,0)$. By considering Lemma 4.4, the equation (3.1) with initial conditions (4.12) has a unique solution $\phi_1(x,\lambda)$.

Let $\phi_{2,\lambda}(x) := \phi_2(x,\lambda)$ be the solution of equation (3.1) on interval $(0,1]$, and satify

$$\begin{cases} I_{0^+}^{1-\alpha}\phi_2(0+) = h_1 I_{-1^+}^{1-\alpha}\phi_1(0-,\lambda), \\ D_{0^+}^{\alpha}\phi_2(0+) = \frac{1}{h_2} D_{-1^+}^{\alpha}\phi_1(0-,\lambda), \end{cases} \tag{4.13}$$

$\phi_2(x,\lambda)$ also is an entire function of λ for each $x \in (0,1]$.

Remark 4.1. For the following initial value problem

$$\begin{cases} {}^cD_{0^-}^{\alpha} p_1 D_{-1^+}^{\alpha} u(x) + (q(x)+\lambda)u(x) = 0, \ x \in [-1,0), \\ I_{0^+}^{1-\alpha}u(0+) = h_1 I_{-1^+}^{1-\alpha}u(0-), \\ D_{0^+}^{\alpha}u(0+) = \frac{1}{h_2} D_{-1^+}^{\alpha}u(0-), \end{cases} \tag{4.14}$$

By using similar way in Lemma 4.2 and Lemma 4.4, we can get the intitial value problem (4.14) has an unique solution $\phi_2(x,\lambda)$ on (0,1] provided that $\frac{1}{p_2\Gamma(2\alpha+1)}(|\lambda| + 2k_\alpha + Q) < 1$, in what follows, we will always assume that this condition holds.

Obviously, the function $\phi(x,\lambda)$ defined on $[-1,0) \cup (0,1]$ by

$$\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), & x \in [-1,0), \\ \phi_2(x,\lambda), & x \in (0,1], \end{cases}$$

is such a solution of equation (3.1) on the whole of $[-1,0) \cup (0,1]$, which satisfies the boundary conditions (3.2), and both transmission conditions (3.4) and (3.5).

Similarly, we see that the problem (3.1) with initial conditions:

$$\begin{cases} I_{0^+}^{1-\alpha}u(1) = \beta p_2, \\ D_{0^+}^{\alpha}u(1) = 1, \end{cases} \tag{4.15}$$

has an unique solution $\chi_2(x,\lambda)$, which is an entire function of the parameter λ for each fixed $x \in (0,1]$. As the same as above discussion, we can define the solution $\chi_1(x,\lambda)$ of equation (3.1) by initial conditions:

$$\begin{cases} I_{-1^+}^{1-\alpha}u(0-) = \frac{1}{h_1} I_{0^+}^{1-\alpha}\chi_2(0+,\lambda), \\ D_{-1^+}^{\alpha}u(0-) = h_2 D_{0^+}^{\alpha}\chi_2(0+,\lambda), \end{cases} \tag{4.16}$$

$\chi_1(x,\lambda)$ is an entire function of the parameter λ for each fixed $x \in [-1,0)$. Hence, the function $\chi(x,\lambda)$ defined on $[-1,0) \cup (0,1]$ by

$$\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), & x \in [-1,0), \\ \chi_2(x,\lambda), & x \in (0,1], \end{cases}$$

is such a solution of equation (3.1) on $[-1,0) \cup (0,1]$, and $\chi(x,\lambda)$ satisfies the boundary conditions (3.3), and both transmission conditions (3.4) and (3.5).

Let us consider fractional Wronskians

$$\omega_1(\lambda) = W_1(\phi_1(x,\lambda), \chi_1(x,\lambda),) = \begin{vmatrix} I_{-1^+}^{1-\alpha}\phi_1(x,\lambda) & I_{-1^+}^{1-\alpha}\chi_1(x,\lambda) \\ D_{-1^+}^{\alpha}\phi_1(x,\lambda) & D_{-1^+}^{\alpha}\chi_1(x,\lambda) \end{vmatrix},$$

$$\omega_2(\lambda) = W_2(\phi_2(x,\lambda), \chi_2(x,\lambda),) = \begin{vmatrix} I_{0^+}^{1-\alpha}\phi_2(x,\lambda) & I_{0^+}^{1-\alpha}\chi_2(x,\lambda) \\ D_{0^+}^{\alpha}\phi_2(x,\lambda) & D_{0^+}^{\alpha}\chi_2(x,\lambda) \end{vmatrix},$$

$\omega_1(\lambda)$ and $\omega_2(\lambda)$ are entire functions, and independent of x. According to (3.4) and (3.5), we have

$$\begin{aligned}
\omega_1(\lambda) &= I^{1-\alpha}_{-1^+}\phi_1(x,\lambda)D^{\alpha}_{-1^+}\chi_1(x,\lambda) - I^{1-\alpha}_{-1^+}\chi_1(x,\lambda)D^{\alpha}_{-1^+}\phi_1(x,\lambda) \\
&= I^{1-\alpha}_{-1^+}\phi_1(0,\lambda)D^{\alpha}_{-1^+}\chi_1(0,\lambda) - I^{1-\alpha}_{-1^+}\chi_1(0,\lambda)D^{\alpha}_{-1^+}\phi_1(0,\lambda) \\
&= \frac{1}{h_1}I^{1-\alpha}_{0^+}\phi_2(0,\lambda)h_2D^{\alpha}_{0^+}\chi_2(0,\lambda) - \frac{1}{h_1}I^{1-\alpha}_{0^+}\chi_2(0,\lambda)h_2D^{\alpha}_{0^+}\phi_2(0,\lambda) \\
&= \frac{1}{\theta}[I^{1-\alpha}_{0^+}\phi_2(0,\lambda)D^{\alpha}_{0^+}\chi_2(0,\lambda) - I^{1-\alpha}_{0^+}\chi_2(0,\lambda)D^{\alpha}_{0^+}\phi_2(0,\lambda)] \\
&= \frac{1}{\theta}\omega_2(\lambda).
\end{aligned}$$

For convenience, let $\omega(\lambda) := \omega_1(\lambda) = \frac{h_2}{h_1}\omega_2(\lambda)$.

Lemma 4.5. *For any $\lambda \in \mathbb{C}$, the fractional Wronskian W_F satisfies the following relation:*

$$W_F(x,\lambda) = -\frac{h_1^2}{h_2}\omega^3(\lambda), \tag{4.17}$$

where

$$W_F(x,\lambda) = \begin{vmatrix} L_1(\phi_1) & L_1(\chi_1) & L_1(\phi_2) & L_1(\chi_2) \\ L_2(\phi_1) & L_2(\chi_1) & L_2(\phi_2) & L_2(\chi_2) \\ L_3(\phi_1) & L_3(\chi_1) & L_3(\phi_2) & L_3(\chi_2) \\ L_4(\phi_1) & L_4(\chi_1) & L_4(\phi_2) & L_4(\chi_2) \end{vmatrix}.$$

Proof. Applying the definitions of the functions ϕ_i, χ_i $(i = 1, 2)$, we get

$$\begin{aligned}
W_F(x,\lambda) &= \begin{vmatrix} 0 & -\omega_1(\lambda) & 0 & 0 \\ 0 & 0 & \omega_2(\lambda) & 0 \\ h_1I^{1-\alpha}_{-1^+}\phi_1(0-,\lambda) & h_1I^{1-\alpha}_{-1^+}\chi_1(0-,\lambda) & I^{1-\alpha}_{0^+}\phi_2(0+,\lambda) & I^{1-\alpha}_{0^+}\chi_2(0+,\lambda) \\ D^{\alpha}_{-1^+}\phi_1(0-,\lambda) & D^{\alpha}_{-1^+}\chi_1(0-,\lambda) & h_2D^{\alpha}_{0^+}\phi_2(0+,\lambda) & h_2D^{\alpha}_{0^+}\chi_2(0+,\lambda) \end{vmatrix} \\
&= -\omega_1(\lambda)\omega_2(\lambda)\begin{vmatrix} h_1I^{1-\alpha}_{-1^+}\phi_1(0-,\lambda) & I^{1-\alpha}_{0^+}\chi_2(0+,\lambda) \\ D^{\alpha}_{-1^+}\phi_1(0-,\lambda) & h_2D^{\alpha}_{0^+}\chi_2(0+,\lambda) \end{vmatrix} \\
&= -\omega_1(\lambda)\omega_2(\lambda)\begin{vmatrix} h_1I^{1-\alpha}_{-1^+}\phi_1(0-,\lambda) & h_1I^{1-\alpha}_{-1^+}\chi_1(0-,\lambda) \\ D^{\alpha}_{-1^+}\phi_1(0-,\lambda) & D^{\alpha}_{-1^+}\chi_1(0-,\lambda) \end{vmatrix} \\
&= -h_1\omega_1^2(\lambda)\omega_2(\lambda) \\
&= -h_1\omega_1^2(\lambda)\frac{h_1}{h_2}\omega_1(\lambda) \\
&= -\frac{h_1^2}{h_2}\omega^3(\lambda).
\end{aligned}$$

□

Corollary 4.3. *The zeros of the W_F consist of the zeros of the $\omega(\lambda)$.*

Lemma 4.6. *Let $u_0(x)$ be an any eigenfunction corresponding to eigenvalue λ_0 , then the function $u_0(x)$ may be represented in the form:*

$$u_0(x) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [-1,0), \\ c_3\phi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (0,1], \end{cases}$$

where at least one of the constants c_i $(i = 1,2,3,4)$ is not zero.

Theorem 4.2. *The eigenvalues of the fractional boundary value problem (3.1)-(3.5) are coincide with the roots of the characteristic function $\omega(\lambda)$.*

Proof. $\Leftarrow$: Let λ_0 be a zero of $\omega(\lambda)$, i.e. $\omega(\lambda_0) = 0$, then, $W(\phi_1(x,\lambda_0), \chi_1(x,\lambda_0)) = 0$. We see that $\phi_1(x,\lambda_0)$ and $\chi_1(x,\lambda_0)$ are linearly dependent, that is, there exists $k_1 \neq 0$ such that

$$\chi_1(x,\lambda_0) = k_1\phi_1(x,\lambda_0).$$

By (4.12), we have

$$\begin{aligned} & r_1 I^{1-\alpha}_{-1^+}\chi_1(-1,\lambda_0) - r_2 p_1 D^{\alpha}_{-1^+}\chi_1(-1,\lambda_0) \\ & - \lambda[r_1' I^{1-\alpha}_{-1^+}\chi_1(-1,\lambda_0) - r_2' p_1) D^{\alpha}_{-1^+}\chi_1(-1,\lambda_0)] \\ =& k_1[(r_1 - \lambda r_1')(r_2 - \lambda r_2')p_1 - (r_2 - \lambda r_2')p_1(r_1 - \lambda r_1')] \\ =& 0. \end{aligned}$$

Consequently, the function $\chi(x,\lambda_0)$ satisfies the fractional boundary condition (3.3).

From above discussion, we know that $\chi(x,\lambda_0)$ satisfies (3.3)-(3.5). Hence, $\chi(x,\lambda_0)$ is an eigenfunction of the problem (3.1)-(3.5) corresponding to the eigenvalue λ_0.
$\Rightarrow$: Let λ_0 be an eigenvalue of the problem (3.1)-(3.5), $u_0(x)$ be the corresponding eigenfunction. Let us suppose that $\omega(\lambda_0) \neq 0$, then ϕ_1 and χ_1, ϕ_2 and χ_2 are linearly independent, respectively. Consequently, we can let

$$u_0(x) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [-1,0), \\ c_3\phi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (0,1], \end{cases}$$

where at least one of the constants $c_i, (i = 1,2,3,4)$ is not zero. By the definition of $u_0(x)$ and transmission condition (3.4), we obtain

$$\begin{aligned} I^{1-\alpha}_{0^+}u_0(0+) &= c_3 I^{1-\alpha}_{0^+}\phi_2(0,\lambda_0) + c_4 I^{1-\alpha}_{0^+}\chi_2(0,\lambda_0) \\ &= h_1[c_3 I^{1-\alpha}_{-1^+}\phi_1(0,\lambda_0) + c_4 I^{1-\alpha}_{-1^+}\chi_1(0,\lambda_0)]. \\ I^{1-\alpha}_{0^+}u_0(0+) &= h_1 I^{1-\alpha}_{-1^+}u_0(0-) \\ &= h_1[c_1 I^{1-\alpha}_{-1^+}\phi_1(0,\lambda_0) + c_2 I^{1-\alpha}_{-1^+}\chi_1(0,\lambda_0)]. \end{aligned}$$

Then, we get

$$h_1(c_3 - c_1)I^{1-\alpha}_{-1^+}\phi_1(0,\lambda_0) + h_1(c_4 - c_2)I^{1-\alpha}_{-1^+}\chi_1(0,\lambda_0) = 0.$$

Applying the similar way of above, we also get

$$\frac{1}{h_2}(c_3 - c_1)D^{\alpha}_{-1^+}\phi_1(0,\lambda_0) + \frac{1}{h_2}(c_4 - c_2)D^{\alpha}_{-1^+}\chi_1(0,\lambda_0) = 0.$$

Then, we find

$$\begin{vmatrix} h_1 & 0 \\ 0 & \frac{1}{h_2} \end{vmatrix} \begin{vmatrix} (c_3-c_1)I^{1-\alpha}_{-1^+}\phi_1(0,\lambda_0)+(c_4-c_2)I^{1-\alpha}_{-1^+}\chi_1(0,\lambda_0) \\ (c_3-c_1)D^{\alpha}_{-1^+}\phi_1(0,\lambda_0)+(c_4-c_2)D^{\alpha}_{-1^+}\chi_1(0,\lambda_0) \end{vmatrix} = 0.$$

Consequence, $c_3 = c_1, c_4 = c_2$.

By the boundary conditions (3.2)-(3.3) and (4.12), (4.16), we obtain

$$\begin{cases} L_1u_0(x) := (r_1-\lambda r_1')I^{1-\alpha}_{-1^+}u_0(-1)-(r_2-\lambda r_2')p_1D^{\alpha}_{-1^+}u_0(-1)=0, \\ L_2u_0(x) := I^{1-\alpha}_{0^+}u_0(1)-\beta p_2D^{\alpha}_{0^+}u_0(1)=0, \end{cases}$$

$$\begin{aligned} L_1u_0(x) =& c_1L_1(\phi_1(x,\lambda_0))+c_2L_1(\chi_1(x,\lambda_0)) \\ =& c_1[(r_1-\lambda r_1')I^{1-\alpha}_{-1^+}\phi_1(-1,\lambda_0)-(r_2-\lambda r_2')p_1D^{\alpha}_{-1^+}\phi_1(-1,\lambda_0)] \\ &+ c_2[(r_1-\lambda r_1')I^{1-\alpha}_{-1^+}\chi_1(-1,\lambda_0)-(r_2-\lambda r_2')p_1D^{\alpha}_{-1^+}\chi_1(-1,\lambda_0)] \\ =& -c_2\omega_1(\lambda_0) \\ =& 0. \end{aligned}$$

Due to $\omega_1(\lambda_0)\neq 0$, so, $c_2 = c_4 = 0$. In a similar way, we get

$$\begin{aligned} L_2u_0(x) =& c_3L_2(\phi_2(x,\lambda_0))+c_4L_2(\chi_2(x,\lambda_0)) \\ =& c_3(I^{1-\alpha}_{0^+}\phi_2(1,\lambda_0)-\beta p_2D^{\alpha}_{0^+}\phi_2(1,\lambda_0)) \\ &+ c_4(I^{1-\alpha}_{0^+}\chi_2(1,\lambda_0)-\beta p_2D^{\alpha}_{0^+}\chi_2(1,\lambda_0)) \\ =& c_3\omega_2(\lambda_0) \\ =& 0. \end{aligned}$$

Due to $\omega_2(\lambda_0)\neq 0$, so, $c_1 = c_3 = 0$. So, we obtain $c_1 = c_2 = c_3 = c_4 = 0$. This conclusion contradicts the assumption, i.e. assumption fails. As a result, $\omega(\lambda_0) = 0$, which completes the proof. □

5. Conclusions

In this paper, we use the Riemann-Liouville fractional and Caputo fractional operator to research a class of discontinuous Sturm-Liouville type boundary-value problem, and the boundary conditions of this problem contain an eigen-parameter. We studied the eigenvalues and eigenfunctions of fractional Sturm-Liouville problem and proved that fractional operator $\mathcal{A}$ is symmetric, as well as the eigenfunctions corresponding to different eigenvalues of the problem are orthogonal and the eigenvalues are real. We give the expression of fractional Wronskian W_F and obtain that the zeros of the W_F consist of the zeros of w_λ. This conclusion provides a basis for getting the asymptotic formula of eigenvalues of fractional S-L problems in the forthcoming work.

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