Journal of Applied Analysis and Computation
Volume 11, Number 4, August 2021, 2052–2069
Website:http://www.jaac-online.com
DOI:10.11948/20200314

ABUNDANT NEW NON-TRAVELING WAVE SOLUTIONS FOR THE (3+1)-DIMENSIONAL BOITI-LEON-MANNA-PEMPINELLI EQUATION*

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Abstract Seeking exact solutions of higher-dimensional nonlinear partial differential equations has recently received tremendous attention in mathematics and physics. In this paper, we investigate exact solutions of (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation which describes nonlinear wave propagation in incompressible fluid. Firstly, by means of extended homoclinic test approach, we get eight kinds of non-traveling wave solutions of (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. Then, combining the improved tanh function method and new ansatz solutions, we obtain abundant new exact non-traveling wave solutions of (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. These results include not only many results obtained in other literatures, but also some new exact non-traveling wave solutions. Moreover, the exact kink wave solutions, periodic solitary wave solutions and singular solitary wave solutions are given when arbitrary functions contained in these solutions are taken as some special functions.

Keywords Extended homoclinic test approach, improved tanh function method, generalized Riccati equation, exact solutions.

MSC(2010) 35Q53, 37K10, 68W30.

1. Introduction

Many significant phenomena in physics and engineering are represented by nonlinear partial differential equations (NLPDEs). Compared with the low-dimensional systems, the higher-dimensional nonlinear partial differential systems have more complex behaviors. The investigation of the explicit solutions for NLPDEs plays an important role in the study of nonlinear phenomena. Recently, in order to explore exact solutions of NLPDEs and figure out these phenomena in nature, a mass of methods have been established, for instance, bilinear method, Darboux transformations, symmetry reductions, and so on. Using (G'/G)-expansion method, the ansatz (positive quadratic and exponential functions) technique, the generalized uni-

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*The authors were supported by National Natural Science Foundation of China (No. 11371183) and Natural Science Foundation of Shandong (Nos. ZR2018BA016, ZR2020MA013).

fied method and Lie symmetry technique, Refs. [5, 9, 13] obtained nonautonomous complex wave solutions, exact solutions, double-wave solutions, non-traveling wave solutions et al. for (2+1)-dimensional and (3+1)-dimensional nonlinear partial differential equations.

As we known, traveling wave solution, which describes the evolution of physical quantities, is a very special solution of partial differential equations. More solutions of NLPDEs are non-traveling wave solutions. Seeking exact non-traveling wave solutions of nonlinear partial differential equations has recently received tremendous attention in mathematics and physics. Using (G'/G)-expansion method, Guo [1] and Liu [6] studied non-traveling wave solutions of (2+1)-dimensional Painlevé integrable Burgers equation, (2+1)-dimensional breaking soliton equation and (3+1)-dimensional generalized shallow water equation. Lin [3] got non-traveling wave solutions for (2+1)-dimensional Burgers equation by means of the generalized direct ansatz method and different test functions. Shang [11, 15] have studied the non-traveling wave solutions of (3+1)-dimensional potential-YTSF equation and Calogero equation by combining the extended homoclinic test approach with the method of separation of variables. Sheng [17] and Zhang [21] obtained exact non-traveling wave solutions of (2+1)-dimensional KD equation and (3+1)-dimensional KP equation with the aid of symbolic computation. Therefore, it is meaningful to consider the non-traveling wave solutions of higher-dimensional nonlinear partial differential equations.

As one of the most important higher-dimensional NLPDEs, the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation

$$u_{yt} + u_{zt} + u_{xxxy} + u_{xxxz} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z) = 0 \tag{1.1}$$

describes nonlinear wave propagation in incompressible fluid, where u is a function depending on spatial variables (x, y, z) and temporal variable t. The researchers have obtained some types of traveling wave solutions to Eq.(1.1) by many methods, such as bilinear method [2, 14, 16], extended three-wave approach [7], extended homoclinic test technique [4, 18], extended three-soliton method [12], Painlevé analysis [19], modified exponential function method [20], respectively. Liu [8] and Luo [10] obtained non-traveling wave solutions of Eq.(1.1) by using (G'/G)-expansion method and Bäcklund transformation, respectively. Although many researchers investigated exact traveling wave solutions of Eq.(1.1), there are few results on the non-traveling wave solutions. In this paper, we will investigate non-traveling wave solutions of Eq.(1.1) by utilizing the extended homoclinic test approach [22] and the improved tanh function method [21].

The rest of this paper is organized as follows: In section 2, eight kinds of non-traveling wave solutions are constructed by using the extended homoclinic test approach. When arbitrary functions in these solutions are taken as some special functions, we will get kink solutions, periodic solitary wave solutions, singular solitary wave solutions and single solitary wave solutions, which can be seen in [12, 18, 19]. In section 3, by constructing new ansatz solutions of equation (1.1), we employ the improved tanh method to get twenty-seven types of non-traveling wave solutions for Eq.(1.1). Furthermore, we can get one hundred and eight types of non-traveling wave solutions for Eq.(1.1) based on the four forms of arbitrary functions η and b. Moreover, we give the graphic analyses of solutions obtained in sections 2. At last, some conclusions and discussions are given.

2. Extended homoclinic test approach

In this section, we will apply the extended homoclinic test approach to get eight kinds of non-traveling wave solutions of Eq.(1.1).

We assume that the solutions of equation (1.1) in form:

$$u(x,y,z,t) = \varphi(\xi,t) + q(z), \tag{2.1}$$

where $\xi = x + my + nt + \theta(z)$, m, n are two nonzero constants, $\varphi(\xi,t), q(z)$ and $\theta(z)$ are three functions to be determined later. Substituting (2.1) into (1.1), we obtain

$$\begin{aligned}&(m+\theta'(z))\varphi_{\xi\xi\xi\xi} + (mn + n\theta'(z) - 3q'(z))\varphi_{\xi\xi} + (-6m - 6\theta'(z))\varphi_\xi\varphi_{\xi\xi}\\ &+(m+\theta'(z))\varphi_{\xi t} = 0.\end{aligned} \tag{2.2}$$

To simplify equation (2.2), we let

$$mn + n\theta'(z) - 3q'(z) = 0. \tag{2.3}$$

From (2.3), we get

$$q(z) = \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \tag{2.4}$$

where c is an integral constant. Therefore, in the condition of $m+\theta'(z) \neq 0$, Eq.(2.2) reduces to

$$\varphi_{\xi\xi\xi\xi} - 6\varphi_\xi\varphi_{\xi\xi} + \varphi_{\xi t} = 0. \tag{2.5}$$

Integrating (2.5) once with respect to ξ and taking the integral constant to be zero, we get

$$\varphi_{\xi\xi\xi} - 3\varphi_\xi^2 + \varphi_t = 0. \tag{2.6}$$

In order to solving (2.6), we introduce a nonlinear function transformation of dependent variable

$$\varphi = -2(\ln\phi)_\xi, \tag{2.7}$$

where $\phi(\xi,t)$ is an undetermined function. Substituting (2.7) into (2.6), we can get a bilinear equation

$$(D_\xi D_t + D_\xi^4)\phi\cdot\phi = 0, \tag{2.8}$$

where the bilinear operator D is defined as

$$D_\xi^m D_t^n f\cdot g = (\partial\xi - \partial\xi')^m(\partial t - \partial t')^n f(\xi,t)g(\xi',t')|_{(\xi',t')=(\xi,t)}.$$

In this section, we seek for the solution in the following form

$$\phi = k_1\cos(\zeta_1) + k_2\exp(\zeta_2) + \exp(-\zeta_2), \tag{2.9}$$

where $\zeta_i = a_i\xi + b_i t$, $i = 1,2$, $k_1, k_2 \in R$ and $a_1, a_2, b_1, b_2 \in C$ are undetermined constants. Substituting (2.9) into (2.8) and setting coefficients of $\cos^2(\zeta_1)$, $\cos(\zeta_1)\exp(\zeta_2)$, $\cos(\zeta_1)\exp(-\zeta_2)$, $\sin^2(\zeta_1)$, $\sin(\zeta_1)\exp(\zeta_2)$, $\sin(\zeta_1)\exp(-\zeta_2)$ and the constant term to zero, we obtain a set of nonlinear algebraic equations with respect to a_i, b_i and k_i, $i = 1,2$

$$\begin{cases} k_1^2(4a_1^4 - a_1b_1) = 0,\\ k_1k_2(a_1^4 + a_2^4 - 6a_1^2a_2^2 + a_2b_2 - a_1b_1) = 0,\\ k_1(a_1^4 + a_2^4 - 6a_1^2a_2^2 + a_2b_2 - a_1b_1) = 0,\\ k_2(16a_2^4 + 4a_2b_2) = 0,\\ k_1k_2(4a_1a_2^3 - 4a_1^3a_2 + a_1b_2 + a_2b_1) = 0,\\ k_1(-4a_1a_2^3 + 4a_1^3a_2 - a_1b_2 - a_2b_1) = 0.\end{cases} \tag{2.10}$$

Solving (2.10) with the aid of Maple, we have the following results.

Case 1:

$$\begin{cases} a_1 = a_1, & b_1 = b_1, & k_1 = 0, \\ a_2 = a_2, & b_2 = -4a_2^3, & k_2 = k_2. \end{cases} \tag{2.11}$$

Collecting (2.11), (2.9), (2.7), (2.4) with (2.1), one obtains the solution

$$u(x,y,z,t) = -2a_2 \frac{k_2 \exp(\zeta_2) - \exp(-\zeta_2)}{k_2 \exp(\zeta_2) + \exp(-\zeta_2)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \tag{2.12}$$

where $\zeta_2 = a_2(x + my + (n - 4a_2^2)t + \theta(z))$.

In particular, solution (2.12) can be written as follows:

$$u_1(x,y,z,t) = -2a_2 \tanh(\zeta_2 + \frac{1}{2}\ln k_2) + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad k_2 > 0, \tag{2.13}$$

$$u_2(x,y,z,t) = -2a_2 \coth(\zeta_2 + \frac{1}{2}\ln(-k_2)) + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad k_2 < 0, \tag{2.14}$$

where $\zeta_2 = a_2(x + my + (n - 4a_2^2)t + \theta(z))$.

Case 2:

$$\begin{cases} a_1 = a_1, & b_1 = 4a_1^3, & k_1 = k_1, \\ a_2 = \pm i a_1, & b_2 = -4a_2^3, & k_2 = k_2. \end{cases} \tag{2.15}$$

Collecting (2.15), (2.9), (2.7), (2.4) with (2.1), we obtain the solution

$$u = 2a_1 \frac{k_1 \sin(\zeta_1) \mp i k_2 \exp(\zeta_2) \pm i \exp(-\zeta_2)}{k_1 cos(\zeta_1) + k_2 \exp(\zeta_2) + \exp(-\zeta_2)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \tag{2.16}$$

where $\zeta_1 = a_1(x + my + (n + 4a_1^2)t + \theta(z))$ and $\zeta_2 = \pm i\zeta_1$.

In particular, solution (2.16) becomes

$$\begin{aligned} u_3(x,y,z,t) &= 2a_1 \frac{[K^2 - (1-k_2)^2]\sin(\zeta_1)cos(\zeta_1) \pm K(1-k_2)i}{K^2\cos^2(\zeta_1) + (1-k_2)^2\sin^2(\zeta_1)} \\ &\quad + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad K = k_1 + k_2 + 1, \quad a_1 \in R, \end{aligned} \tag{2.17}$$

$$\begin{aligned} u_4(x,y,z,t) &= 2k_3 \frac{k_1 \sinh(\zeta_1^*) \pm 2\sqrt{k_2}\sinh(\pm\zeta_1^* + \frac{1}{2}\ln(k_2))}{k_1\cosh(\zeta_1^*) + 2\sqrt{k_2}\cosh(\pm\zeta_1^* + \frac{1}{2}\ln(k_2))} \\ &\quad + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad k_2 > 0,\ a_1 = k_3 i,\ k_3 \in R, \end{aligned} \tag{2.18}$$

$$\begin{aligned} u_5(x,y,z,t) &= 2k_3 \frac{k_1 \sinh(\zeta_1^*) \mp 2\sqrt{-k_2}\cosh(\pm\zeta_1^* + \frac{1}{2}\ln(-k_2))}{k_1\cosh(\zeta_1^*) - 2\sqrt{-k_2}\sinh(\pm\zeta_1^* + \frac{1}{2}\ln(-k_2))} \\ &\quad + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad k_2 < 0,\ a_1 = k_3 i,\ k_3 \in R, \end{aligned} \tag{2.19}$$

where $\zeta_1 = a_1(x + my + (n + 4a_1^2)t + \theta(z))$ and $\zeta_1^* = i\zeta_1$.

Case 3:

$$\begin{cases} a_1 = a_1, & b_1 = 4a_1^3, & k_1 = k_1, \\ a_2 = \pm i a_1, & b_2 = -4a_2^3, & k_2 = 0. \end{cases} \tag{2.20}$$

Collecting (2.20), (2.9), (2.7), (2.4) with (2.1), one gets

$$u = 2a_1 \frac{k_1 sin(\zeta_1) \pm i \exp(-\zeta_2)}{k_1 cos(\zeta_1) + \exp(-\zeta_2)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \tag{2.21}$$

where $\zeta_1 = a_1(x + my + (n + 4a_1^2)t + \theta(z))$ and $\zeta_2 = \pm i\zeta_1$.

In particular, solution (2.21) becomes

$$u_6(x,y,z,t) = 2a_1 \frac{[(k_1+1)^2 - 1]sin(\zeta_1)\cos(\zeta_1) \pm i(k_1+1)}{(k_1+1)^2\cos^2(\zeta_1) + \sin^2(\zeta_1)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad a_1 \in R, \tag{2.22}$$

$$u_7(x,y,z,t) = 2k_3 \frac{(k_1+1)\sinh(\zeta_1^*) \mp cosh(\zeta_1^*)}{(k_1+1)\cosh(\zeta_1^*) \mp \sinh(\zeta_1^*)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \quad a_1 = k_3 i, \ k_3 \in R, \tag{2.23}$$

where $\zeta_1 = a_1(x + my + (n + 4a_1^2)t + \theta(z))$ and $\zeta_1^* = i\zeta_1$.

Case 4:

$$\begin{cases} a_1 = 0, & b_1 = 0, & k_1 = k_1, \\ a_2 = a_2, & b_2 = -a_2^3, & k_2 = 0. \end{cases} \tag{2.24}$$

Collecting (2.24), (2.9), (2.7), (2.4) with (2.1), one obtains

$$u_8(x,y,z,t) = 2a_2 \frac{\exp(-\zeta_2)}{k_1 + \exp(-\zeta_2)} + \frac{n}{3}\theta(z) + \frac{mn}{3}z + c, \tag{2.25}$$

where $\zeta_2 = a_2(x + my + (n - a_2^2)t + \theta(z))$. In $u_1 - u_8$, $\theta(z)$ is an arbitrary first order derivable function. Moreover, when $a_1 = k_4 + ik_3$, we can get many other type solutions from (2.16) and (2.21), where k_3, k_4 are nonzero real numbers. Here, we omit the detail expressions of these solutions.

3. Improved tanh function method

In this section, we will obtain many non-traveling wave solutions of Eq.(1.1) by combining the improved tanh function method with a generalized Riccati equation.

Balancing the u_{xxxy} and $u_x u_{xy}$, we construct the solution of the following new form

$$u(x,y,z,t) = a(y,z,t)\phi^{-1}(kx+\eta(y,z,t)) + b(y,z,t) + c(y,z,t)\phi(kx+\eta(y,z,t)) \tag{3.1}$$

with

$$\phi' = r + p\phi + q\phi^2, \tag{3.2}$$

where k, r, p and q are all real constants, $a(y,z,t)$, $b(y,z,t)$, $c(y,z,t)$ and $\eta(y,z,t)$ are all differentiable functions. As we known, Eq.(3.2) is the generalized Riccati equation and has 27 solutions which were given by Zhang et al. [21](see Appendix 1). Substituting (3.1) and (3.2) into (1.1), combining similar terms of $\phi(kx + \eta(y,z,t))$, and setting each coefficient to zero, we have the following system of over-determined partial differential equations for $a(y,z,t)$, $b(y,z,t)$, $c(y,z,t)$ and $\eta(y,z,t)$

$$\begin{cases}(24k^3r^4a+12k^2r^3a^2)(\eta_y+\eta_z)=0,\\ (24k^3q^4c+12k^2q^3c^2)(\eta_y+\eta_z)=0,\\ -4kr^3\eta_t(\eta_y+\eta_z)+(64k^4r^4q-4k^4r^3p^2)(\eta_y+\eta_z)+12k^3r^3(b_y+b_z)=0,\\ 4kq^3\eta_t(\eta_y+\eta_z)+(-64k^4q^4r+4k^4q^3p^2)(\eta_y+\eta_z)-12k^3q^3(b_y+b_z)=0,\\ 2kr^2(\eta_{yt}+\eta_{zt})-6kr^2p\eta_t(\eta_y+\eta_z)+(120k^4r^3pq-6k^4r^2p^3)(\eta_y+\eta_z)\\ +18k^3r^2p(b_y+b_z)=0,\\ 2kq^2(\eta_{yt}+\eta_{zt})+6kq^2p\eta_t(\eta_y+\eta_z)+(-120k^4q^3pr+6k^4q^2p^3)(\eta_y+\eta_z)\\ -18k^3q^2p(b_y+b_z)=0,\\ (2arq+ap^2)\eta_t(\eta_y+\eta_z)-ap(\eta_{yt}+\eta_{zt})+(-2k^4p^4r+64k^4r^3q^2+52k^4r^2p^2q)\\ (\eta_y+\eta_z)+(3k^2p^2a-6k^2rqa)(b_y+b_z)=0,\\ (2crq+cp^2)\eta_t(\eta_y+\eta_z)+cp(\eta_{yt}+\eta_{zt})+(2k^4p^4q-64k^4q^3r^2-52k^4q^2p^2r)\\ (\eta_y+\eta_z)+(-3k^2p^2c-6k^2rqc)(b_y+b_z)=0,\\ (apq+cpr)\eta_t(\eta_y+\eta_z)+(cr-aq)(\eta_{yt}+\eta_{zt})+24k^4q^2r^2p(\eta_y+\eta_z)\\ -(3k^2pqa+3k^2prc)(b_y+b_z)+(b_{yt}+b_{zt})=0.\end{cases} \tag{3.3}$$

Solving the system of (3.3), we can get the following non-trivial results.

Case 1:

$$\begin{cases}\eta(y,z,t)=F(t)G(y,z),\\ a(y,z,t)=-2kr,\quad b(y,z,t)=b_1(t)b_2(y,z),\quad c(y,z,t)=2kq,\end{cases}$$

where $F(t)$, $G(y,z)$, $b_1(t)$ and $b_2(y,z)$ are all arbitrary differentiable functions satisfying

$$b_{2y}(y,z)+b_{2z}(y,z)=0,\qquad G_y(y,z)+G_z(y,z)=0.$$

Case 2:

$$\begin{cases}\eta(y,z,t)=F(t)G(y,z),\\ a(y,z,t)=-2kr,\quad b(y,z,t)=b_1(t)+b_2(y,z),\quad c(y,z,t)=2kq,\end{cases}$$

where $F(t)$, $G(y,z)$, $b_1(t)$ and $b_2(y,z)$ are all arbitrary differentiable functions satisfying

$$b_{2y}(y,z)+b_{2z}(y,z)=0,\qquad G_y(y,z)+G_z(y,z)=0.$$

Case 3:

$$\begin{cases}\eta(y,z,t)=F(t)+G(y,z),\\ a(y,z,t)=-2kr,\quad b(y,z,t)=b_1(t)b_2(y,z),\quad c(y,z,t)=2kq,\end{cases}$$

where $F(t)$, $G(y,z)$, $b_1(t)$ and $b_2(y,z)$ are all arbitrary differentiable functions satisfying

$$b_{2y}(y,z)+b_{2z}(y,z)=0,\qquad G_y(y,z)+G_z(y,z)=0.$$

Case 4:

$$\begin{cases}\eta(y,z,t)=F(t)+G(y,z),\\ a(y,z,t)=-2kr,\quad b(y,z,t)=b_1(t)+b_2(y,z),\quad c(y,z,t)=2kq,\end{cases}$$

where $F(t)$, $G(y,z)$, $b_1(t)$ and $b_2(y,z)$ are all arbitrary differentiable functions satisfying

$$b_{2y}(y,z) + b_{2z}(y,z) = 0, \qquad G_y(y,z) + G_z(y,z) = 0.$$

For simplification, in the rest of this paper, we take

$$M = \frac{\sqrt{p^2 - 4qr}}{2}, \quad N = \frac{\sqrt{4pr - q^2}}{2}, \quad w = kx + \eta(y,z,t).$$

From *case 1*, (3.1) and Types 1-4 of Appendix 1 in [21], we have 27 solutions of equation (1.1) as follows.

Family 1: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), we get twelve solutions

$$u_1 = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(Mw)]^2}{[p + 2M\tanh(Mw)]},$$

$$u_2 = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\coth(Mw)]^2}{[p + 2M\coth(Mw)]},$$

$$u_3 = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(2Mw) \pm isech(2Mw)]^2}{[p + 2M\tanh(2Mw) \pm isech(2Mw)]},$$

$$u_4 = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\coth(2Mw) \pm csch(2Mw)]^2}{[p + 2M\coth(2Mw) \pm csch(2Mw)]},$$

$$u_5 = b_1(t)b_2(y,z) + \frac{8kqr - 2k[p + M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]^2}{[2p + 2M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]},$$

$$u_6 = b_1(t)b_2(y,z) + \frac{-k[2M\sinh(Mw) - p\cosh(Mw)]^2 + 4kqr[\cosh(Mw)]^2}{\cosh(Mw)[2M\sinh(Mw) - p\cosh(Mw)]},$$

$$u_7 = b_1(t)b_2(y,z) + \frac{k[p\sinh(Mw) - 2M\cosh(Mw)]^2 - 4kqr[\sinh(Mw)]^2}{\sinh(Mw)[p\sinh(Mw) - 2M\cosh(Mw)]},$$

$$u_8 = b_1(t)b_2(y,z) - \frac{k[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]^2 - 4kqr[\cosh(2Mw)]^2}{\cosh(2Mw)[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]},$$

$$u_9 = b_1(t)b_2(y,z) - \frac{k[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]^2 - 4kqr[\sinh(2Mw)]^2}{\sinh(2Mw)[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]},$$

$$u_{10} = b_1(t)b_2(y,z) + \frac{8kqr[\Phi(w)]^2 - 2k[-p\Phi(w) + 2M\cosh^2(\frac{Mw}{2}) - M]^2}{\Phi(w)[-2p\Phi(w) + 4M\cosh^2(\frac{Mw}{2}) - 2M]},$$

$$u_{11} = b_1(t)b_2(y,z) + F_1(\sinh, \cosh),$$

$$u_{12} = b_1(t)b_2(y,z) + G_1(\cosh, \sinh),$$

where

$$\Phi(w) = \sinh(\frac{Mw}{2})\cosh(\frac{Mw}{2}), \tag{3.4}$$

$$F_1(f,g) = \frac{-4kqr[\Psi(w)]^2 + k[-p\Psi(w) + 2M\sqrt{A^2+B^2} - 2AMg(2Mw)]^2}{\Psi(w)[-p\Psi(w) + 2M\sqrt{A^2+B^2} - 2AMg(2Mw)]}, \tag{3.5}$$

$$G_1(f,g) = \frac{-4kqr[\Psi(w)]^2 + k[-p\Psi(w) - 2M\sqrt{A^2+B^2} - 2AMg(2Mw)]^2}{\Psi(w)[-p\Psi(w) - 2M\sqrt{A^2+B^2} - 2AMg(2Mw)]}, \tag{3.6}$$

$$\Psi(w) = Af(2Mw) + B,$$

A and B are two nonzero real constants and satisfy $B^2 - A^2 > 0$.

Family 2: When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), one obtains twelve solutions

$$u_{13} = b_1(t)b_2(y,z) + \frac{-4kqr + k[-p + 2N\tan(Nw)]^2}{-p + 2N\tan(Nw)},$$

$$u_{14} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2N\cot(Nw)]^2}{p + 2N\cot(Nw)},$$

$$u_{15} = b_1(t)b_2(y,z) + \frac{-4kqr + k[-p + 2N(\tan(2Nw) \pm \sec(2Nw))]^2}{-p + 2N(\tan(2Nw) \pm \sec(2Nw))},$$

$$u_{16} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2N(\cot(2Nw) \pm \csc(2Nw))]^2}{p + 2N(\cot(2Nw) \pm \csc(2Nw))},$$

$$u_{17} = b_1(t)b_2(y,z) + \frac{-8kqr + 2k[-p + N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))]^2}{-2p + 2N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))},$$

$$u_{18} = b_1(t)b_2(y,z) + \frac{k[2N\sin(Nw) + p\cos(Nw)]^2 - 4kqr[\cos(Nw)]^2}{\cos(Nw)[2N\sin(Nw) + p\cos(Nw)]},$$

$$u_{19} = b_1(t)b_2(y,z) + \frac{-k[2N\cos(Nw) - p\sin(Nw)]^2 + 4kqr[\sin(Nw)]^2}{\sin(Nw)[2N\cos(Nw) - p\sin(Nw)]},$$

$$u_{20} = b_1(t)b_2(y,z) + \frac{k[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]^2 - 4kqr[\cos(2Nw)]^2}{\cos(2Nw)[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]},$$

$$u_{21} = b_1(t)b_2(y,z) + \frac{-k[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]^2 + 4kqr[\sin(2Nw)]^2}{\sin(2Nw)[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]},$$

$$u_{22} = b_1(t)b_2(y,z) + \frac{8kqr[\Phi_1(w)]^2 - 2k[-p\Phi_1(w) + 2N\cos^2(\frac{Nw}{2}) - N]^2}{\Phi_1(w)[-2p\Phi_1(w) + 4N\cos^2(\frac{Nw}{2}) - 2N]},$$

$$u_{23} = b_1(t)b_2(y,z) + F_2(\sin,\cos),$$

$$u_{24} = b_1(t)b_2(y,z) + G_2(\cos,\sin),$$

where

$$\Phi_1(w) = \sin(\frac{Nw}{2})\cos(\frac{Nw}{2}), \tag{3.7}$$

$$F_2(f,g) = \frac{-4kqr[\Psi_1(w)]^2 + k[-p\Psi_1(w) \pm 2N\sqrt{A^2 - B^2} - 2ANg(2Nw)]^2}{\Psi_1(w)[-p\Psi_1(w) \pm 2N\sqrt{A^2 - B^2} - 2ANg(2Nw)]}, \tag{3.8}$$

$$G_2(f,g) = \frac{-4kqr[\Psi_1(w)]^2 + k[-p\Psi_1(w) \pm 2N\sqrt{A^2 - B^2} + 2ANg(2Nw)]^2}{\Psi_1(w)[-p\Psi_1(w) \pm 2N\sqrt{A^2 - B^2} + 2ANg(2Nw)]}, \tag{3.9}$$

$$\Psi_1(w) = Af(2Nw) + B,$$

A and B are two nonzero real constants and satisfy $A^2 - B^2 > 0$.

Family 3: When $r = 0$ and $pq \neq 0$, one gets two solutions

$$u_{25} = b_1(t)b_2(y,z) + \frac{2krq[d + \cosh(pw) - \sinh(pw)]^2 - 2kp^2d^2}{pd[d + \cosh(pw) - \sinh(pw)]},$$

$$u_{26} = b_1(t)b_2(y,z)$$

$$+\frac{2kqr[d+\cosh(pw)+\sinh(pw)]^2-2kp^2[\cosh(pw)+\sinh(pw)]^2}{p[\cosh(pw)+\sinh(pw)][d+\cosh(pw)+\sinh(pw)]},$$

where d is an arbitrary constant.

Family 4: When $q \neq 0$, $r = p = 0$, we obtain the solution

$$u_{27} = b_1(t)b_2(y,z) + \frac{2kr(qw+d)^2-2kq}{qw+d},$$

where d is an arbitrary constant.

From *case 2*, (3.1) and Types 1-4 of Appendix 1 in [21], we have 27 solutions of equation (1.1) as follows.

Family 5: When $p^2-4qr>0$ and $pq \neq 0$ (or $qr \neq 0$), we obtain twelve solutions

$$u_{28} = b_1(t)+b_2(y,z)+\frac{4kqr-k[p+2M\tanh(Mw)]^2}{[p+2M\tanh(Mw)]},$$
$$u_{29} = b_1(t)+b_2(y,z)+\frac{4kqr-k[p+2M\coth(Mw)]^2}{[p+2M\coth(Mw)]},$$
$$u_{30} = b_1(t)+b_2(y,z)+\frac{4kqr-k[p+2M\tanh(2Mw)\pm isech(2Mw)]^2}{[p+2M\tanh(2Mw)\pm isech(2Mw)]},$$
$$u_{31} = b_1(t)+b_2(y,z)+\frac{4kqr-k[p+2M\coth(2Mw)\pm csch(2Mw)]^2}{[p+2M\coth(2Mw)\pm csch(2Mw)]},$$
$$u_{32} = b_1(t)+b_2(y,z)+\frac{8kqr-2k[p+M(\tanh(\frac{Mw}{2})+\coth(\frac{Mw}{2}))]^2}{[2p+2M(\tanh(\frac{Mw}{2})+\coth(\frac{Mw}{2}))]},$$
$$u_{33} = b_1(t)+b_2(y,z)+\frac{-k[2M\sinh(Mw)-p\cosh(Mw)]^2+4kqr[\cosh(Mw)]^2}{\cosh(Mw)[2M\sinh(Mw)-p\cosh(Mw)]},$$
$$u_{34} = b_1(t)+b_2(y,z)+\frac{k[p\sinh(Mw)-2M\cosh(Mw)]^2-4kqr[\sinh(Mw)]^2}{\sinh(Mw)[p\sinh(Mw)-2M\cosh(Mw)]},$$
$$u_{35} = b_1(t)+b_2(y,z) - \frac{k[2M\sinh(2Mw)-p\cosh(2Mw)\pm 2iM]^2-4kqr[\cosh(2Mw)]^2}{\cosh(2Mw)[2M\sinh(2Mw)-p\cosh(2Mw)\pm 2iM]},$$
$$u_{36} = b_1(t)+b_2(y,z) - \frac{k[2M\cosh(2Mw)-p\sinh(2Mw)\pm 2M]^2-4kqr[\sinh(2Mw)]^2}{\sinh(2Mw)[2M\cosh(2Mw)-p\sinh(2Mw)\pm 2M]},$$
$$u_{37} = b_1(t)+b_2(y,z)+\frac{8kqr[\Phi(w)]^2-2k[-p\Phi(w)+2M\cosh^2(\frac{Mw}{2})-M]^2}{\Phi(w)[-2p\Phi(w)+4M\cosh^2(\frac{Mw}{2})-2M]},$$
$$u_{38} = b_1(t)+b_2(y,z)+F_1(\sinh,\cosh),$$
$$u_{39} = b_1(t)+b_2(y,z)+G_1(\cosh,\sinh),$$

where $\Phi(w)$, $F_1(\sinh,\cosh)$, $G_1(\cosh,\sinh)$ are defined as (3.4), (3.5) and (3.6), respectively, A and B are two nonzero real constants and satisfy $B^2-A^2>0$.

Family 6: When $p^2-4qr<0$ and $pq \neq 0$ (or $qr \neq 0$), one gets twelve soluitons

$$u_{40} = b_1(t)+b_2(y,z)+\frac{-4kqr+k[-p+2N\tan(Nw)]^2}{-p+2N\tan(Nw)},$$

$$u_{41} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2N\cot(Nw)]^2}{p + 2N\cot(Nw)},$$
$$u_{42} = b_1(t) + b_2(y,z) + \frac{-4kqr + k[-p + 2N(\tan(2Nw) \pm \sec(2Nw))]^2}{-p + 2N(\tan(2Nw) \pm \sec(2Nw))},$$
$$u_{43} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2N(\cot(2Nw) \pm \csc(2Nw))]^2}{p + 2N(\cot(2Nw) \pm \csc(2Nw))},$$
$$u_{44} = b_1(t) + b_2(y,z) + \frac{-8kqr + 2k[-p + N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))]^2}{-2p + 2N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))},$$
$$u_{45} = b_1(t) + b_2(y,z) + \frac{k[2N\sin(Nw) + p\cos(Nw)]^2 - 4kqr[\cos(Nw)]^2}{\cos(Nw)[2N\sin(Nw) + p\cos(Nw)]},$$
$$u_{46} = b_1(t) + b_2(y,z) + \frac{-k[2N\cos(Nw) - p\sin(Nw)]^2 + 4kqr[\sin(Nw)]^2}{\sin(Nw)[2N\cos(Nw) - p\sin(Nw)]},$$
$$u_{47} = b_1(t) + b_2(y,z) + \frac{k[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]^2 - 4kqr[\cos(2Nw)]^2}{\cos(2Nw)[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]},$$
$$u_{48} = b_1(t) + b_2(y,z) + \frac{-k[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]^2 + 4kqr[\sin(2Nw)]^2}{\sin(2Nw)[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]},$$
$$u_{49} = b_1(t) + b_2(y,z) + \frac{8kqr[\Phi_1(w)]^2 - 2k[-p\Phi_1(w) + 2N\cos^2(\frac{Nw}{2}) - N]^2}{\Phi_1(w)[-2p\Phi_1(w) + 4N\cos^2(\frac{Nw}{2}) - 2N]},$$
$$u_{50} = b_1(t) + b_2(y,z) + F_2(\sin,\cos),$$
$$u_{51} = b_1(t) + b_2(y,z) + G_2(\cos,\sin),$$

where $\Phi_1(w)$, $F_2(\sin,\cos)$, $G_2(\cos,\sin)$ are defined as (3.7), (3.8) and (3.9) respectively, A and B are two nonzero real constants and satisfy $A^2 - B^2 > 0$.

Family 7: When $r = 0$ and $pq \neq 0$, one obtains two solutions

$$u_{52} = b_1(t) + b_2(y,z) + \frac{2krq[d + \cosh(pw) - \sinh(pw)]^2 - 2kp^2d^2}{pd[d + \cosh(pw) - \sinh(pw)]},$$
$$u_{53} = b_1(t) + b_2(y,z) + \frac{2kqr[d + \cosh(pw) + \sinh(pw)]^2 - 2kp^2[\cosh(pw) + \sinh(pw)]^2}{p[\cosh(pw) + \sinh(pw)][d + \cosh(pw) + \sinh(pw)]},$$

where d is an arbitrary constant.

Family 8: When $q \neq 0$, $r = p = 0$, we get the solution

$$u_{54} = b_1(t) + b_2(y,z) + \frac{2kr(qw + d)^2 - 2kq}{qw + d},$$

where d is an arbitrary constant.

From *case 3*, (3.1) and Types 1-4 of Appendix 1 in [21], we have 27 solutions of equation (1.1) as follows.

Family 9: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), one gets twelve solutions

$$u_{55} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(Mw)]^2}{[p + 2M\tanh(Mw)]},$$
$$u_{56} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\coth(Mw)]^2}{[p + 2M\coth(Mw)]},$$

$$u_{57} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(2Mw) \pm isech(2Mw)]^2}{[p + 2M\tanh(2Mw) \pm isech(2Mw)]},$$

$$u_{58} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2M\coth(2Mw) \pm csch(2Mw)]^2}{[p + 2M\coth(2Mw) \pm csch(2Mw)]},$$

$$u_{59} = b_1(t)b_2(y,z) + \frac{8kqr - 2k[p + M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]^2}{[2p + 2M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]},$$

$$u_{60} = b_1(t)b_2(y,z) + \frac{-k[2M\sinh(Mw) - p\cosh(Mw)]^2 + 4kqr[\cosh(Mw)]^2}{\cosh(Mw)[2M\sinh(Mw) - p\cosh(Mw)]},$$

$$u_{61} = b_1(t)b_2(y,z) + \frac{k[p\sinh(Mw) - 2M\cosh(Mw)]^2 - 4kqr[\sinh(Mw)]^2}{\sinh(Mw)[p\sinh(Mw) - 2M\cosh(Mw)]},$$

$$u_{62} = b_1(t)b_2(y,z) - \frac{k[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]^2 - 4kqr[\cosh(2Mw)]^2}{\cosh(2Mw)[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]},$$

$$u_{63} = b_1(t)b_2(y,z) - \frac{k[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]^2 - 4kqr[\sinh(2Mw)]^2}{\sinh(2Mw)[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]},$$

$$u_{64} = b_1(t)b_2(y,z) + \frac{8kqr[\Phi(w)]^2 - 2k[-p\Phi(w) + 2M\cosh^2(\frac{Mw}{2}) - M]^2}{\Phi(w)[-2p\Phi(w) + 4M\cosh^2(\frac{Mw}{2}) - 2M]},$$

$$u_{65} = b_1(t)b_2(y,z) + F_1(\sinh,\cosh),$$

$$u_{66} = b_1(t)b_2(y,z) + G_1(\cosh,\sinh),$$

where $\Phi(w)$, $F_1(\sinh,\cosh)$, $G_1(\cosh,\sinh)$ are defined as (3.4), (3.5) and (3.6) respectively, A and B are two nonzero real constants and satisfy $B^2 - A^2 > 0$.

Family 10: When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), we obtain twelve solutions

$$u_{67} = b_1(t)b_2(y,z) + \frac{-4kqr + k[-p + 2N\tan(Nw)]^2}{-p + 2N\tan(Nw)},$$

$$u_{68} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2N\cot(Nw)]^2}{p + 2N\cot(Nw)},$$

$$u_{69} = b_1(t)b_2(y,z) + \frac{-4kqr + k[-p + 2N(\tan(2Nw) \pm \sec(2Nw))]^2}{-p + 2N(\tan(2Nw) \pm \sec(2Nw))},$$

$$u_{70} = b_1(t)b_2(y,z) + \frac{4kqr - k[p + 2N(\cot(2Nw) \pm \csc(2Nw))]^2}{p + 2N(\cot(2Nw) \pm \csc(2Nw))},$$

$$u_{71} = b_1(t)b_2(y,z) + \frac{-8kqr + 2k[-p + N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))]^2}{-2p + 2N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))},$$

$$u_{72} = b_1(t)b_2(y,z) + \frac{k[2N\sin(Nw) + p\cos(Nw)]^2 - 4kqr[\cos(Nw)]^2}{\cos(Nw)[2N\sin(Nw) + p\cos(Nw)]},$$

$$u_{73} = b_1(t)b_2(y,z) + \frac{-k[2N\cos(Nw) - p\sin(Nw)]^2 + 4kqr[\sin(Nw)]^2}{\sin(Nw)[2N\cos(Nw) - p\sin(Nw)]},$$

$$u_{74} = b_1(t)b_2(y,z) + \frac{k[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]^2 - 4kqr[\cos(2Nw)]^2}{\cos(2Nw)[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]},$$

$$u_{75} = b_1(t)b_2(y,z) + \frac{-k[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]^2 + 4kqr[\sin(2Nw)]^2}{\sin(2Nw)[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]},$$

$$u_{76} = b_1(t)b_2(y,z) + \frac{8kqr[\Phi_1(w)]^2 - 2k[-p\Phi_1(w) + 2N\cos^2(\frac{Nw}{2}) - N]^2}{\Phi_1(w)[-2p\Phi_1(w) + 4N\cos^2(\frac{Nw}{2}) - 2N]},$$

$$u_{77} = b_1(t)b_2(y,z) + F_2(\sin,\cos),$$

$$u_{78} = b_1(t)b_2(y,z) + G_2(\cos,\sin),$$

where $\Phi_1(w)$, $F_2(\sin,\cos)$, $G_2(\cos,\sin)$ are defined as (3.7), (3.8) and (3.9) respectively, A and B are two nonzero real constants and satisfy $A^2 - B^2 > 0$.

Family 11: When $r = 0$ and $pq \neq 0$, one obtains two solutions

$$u_{79} = b_1(t)b_2(y,z) + \frac{2krq[d + \cosh(pw) - \sinh(pw)]^2 - 2kp^2d^2}{pd[d + \cosh(pw) - \sinh(pw)]},$$

$$u_{80} = b_1(t)b_2(y,z) + \frac{2kqr[d + \cosh(pw) + \sinh(pw)]^2 - 2kp^2[\cosh(pw) + \sinh(pw)]^2}{p[\cosh(pw) + \sinh(pw)][d + \cosh(pw) + \sinh(pw)]},$$

where d is an arbitrary constant.

Family 12: When $q \neq 0$, $r = p = 0$, we obtain the solution

$$u_{81} = b_1(t)b_2(y,z) + \frac{2kr(qw + d)^2 - 2kq}{qw + d},$$

where d is an arbitrary constant.

From *case 4*, (3.1) and Types 1-4 of Appendix 1 in [21], we have 27 solutions of equation (1.1) as follows.

Family 13: When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $qr \neq 0$), we obtain twelve solutions

$$u_{82} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(Mw)]^2}{[p + 2M\tanh(Mw)]},$$

$$u_{83} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2M\coth(Mw)]^2}{[p + 2M\coth(Mw)]},$$

$$u_{84} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2M\tanh(2Mw) \pm isech(2Mw)]^2}{[p + 2M\tanh(2Mw) \pm isech(2Mw)]},$$

$$u_{85} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2M\coth(2Mw) \pm csch(2Mw)]^2}{[p + 2M\coth(2Mw) \pm csch(2Mw)]},$$

$$u_{86} = b_1(t) + b_2(y,z) + \frac{8kqr - 2k[p + M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]^2}{[2p + 2M(\tanh(\frac{Mw}{2}) + \coth(\frac{Mw}{2}))]},$$

$$u_{87} = b_1(t) + b_2(y,z) + \frac{-k[2M\sinh(Mw) - p\cosh(Mw)]^2 + 4kqr[\cosh(Mw)]^2}{\cosh(Mw)[2M\sinh(Mw) - p\cosh(Mw)]},$$

$$u_{88} = b_1(t) + b_2(y,z) + \frac{k[p\sinh(Mw) - 2M\cosh(Mw)]^2 - 4kqr[\sinh(Mw)]^2}{\sinh(Mw)[p\sinh(Mw) - 2M\cosh(Mw)]},$$

$$u_{89} = b_1(t) + b_2(y,z) - \frac{k[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]^2 - 4kqr[\cosh(2Mw)]^2}{\cosh(2Mw)[2M\sinh(2Mw) - p\cosh(2Mw) \pm 2iM]},$$

$$u_{90} = b_1(t) + b_2(y,z) - \frac{k[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]^2 - 4kqr[\sinh(2Mw)]^2}{\sinh(2Mw)[2M\cosh(2Mw) - p\sinh(2Mw) \pm 2M]},$$

$$u_{91} = b_1(t) + b_2(y,z) + \frac{8kqr[\Phi(w)]^2 - 2k[-p\Phi(w) + 2M\cosh^2(\frac{Mw}{2}) - M]^2}{\Phi(w)[-2p\Phi(w) + 4M\cosh^2(\frac{Mw}{2}) - 2M]},$$
$$u_{92} = b_1(t) + b_2(y,z) + F_1(\sinh,\cosh),$$
$$u_{93} = b_1(t) + b_2(y,z) + G_1(\cosh,\sinh),$$

where $\Phi(w)$, $F_1(\sinh,\cosh)$, $G_1(\cosh,\sinh)$ are defined as (3.4), (3.5) and (3.6) respectively, A and B are two nonzero real constants and satisfy $B^2 - A^2 > 0$.

Family 14: When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $qr \neq 0$), one gets twelve solutions

$$u_{94} = b_1(t) + b_2(y,z) + \frac{-4kqr + k[-p + 2N\tan(Nw)]^2}{-p + 2N\tan(Nw)},$$
$$u_{95} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2N\cot(Nw)]^2}{p + 2N\cot(Nw)},$$
$$u_{96} = b_1(t) + b_2(y,z) + \frac{-4kqr + k[-p + 2N(\tan(2Nw) \pm \sec(2Nw))]^2}{-p + 2N(\tan(2Nw) \pm \sec(2Nw))},$$
$$u_{97} = b_1(t) + b_2(y,z) + \frac{4kqr - k[p + 2N(\cot(2Nw) \pm \csc(2Nw))]^2}{p + 2N(\cot(2Nw) \pm \csc(2Nw))},$$
$$u_{98} = b_1(t) + b_2(y,z) + \frac{-8kqr + 2k[-p + N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))]^2}{-2p + 2N(\tan(\frac{Nw}{2}) - \cot(\frac{Nw}{2}))},$$
$$u_{99} = b_1(t) + b_2(y,z) + \frac{k[2N\sin(Nw) + p\cos(Nw)]^2 - 4kqr[\cos(Nw)]^2}{\cos(Nw)[2N\sin(Nw) + p\cos(Nw)]},$$
$$u_{100} = b_1(t) + b_2(y,z) + \frac{-k[2N\cos(Nw) - p\sin(Nw)]^2 + 4kqr[\sin(Nw)]^2}{\sin(Nw)[2N\cos(Nw) - p\sin(Nw)]},$$
$$u_{101} = b_1(t) + b_2(y,z) + \frac{k[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]^2 - 4kqr[\cos(2Nw)]^2}{\cos(2Nw)[2N\sin(2Nw) + p\cos(2Nw) \pm 2N]},$$
$$u_{102} = b_1(t) + b_2(y,z) + \frac{-k[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]^2 + 4kqr[\sin(2Nw)]^2}{\sin(2Nw)[2N\cos(2Nw) - p\sin(2Nw) \pm 2N]},$$
$$u_{103} = b_1(t) + b_2(y,z) + \frac{8kqr[\Phi_1(w)]^2 - 2k[-p\Phi_1(w) + 2N\cos^2(\frac{Nw}{2}) - N]^2}{\Phi_1(w)[-2p\Phi_1(w) + 4N\cos^2(\frac{Nw}{2}) - 2N]},$$
$$u_{104} = b_1(t) + b_2(y,z) + F_2(\sin,\cos),$$
$$u_{105} = b_1(t) + b_2(y,z) + G_2(\cos,\sin),$$

where $\Phi_1(w)$, $F_2(\sin,\cos)$, $G_2(\cos,\sin)$ are defined as (3.7), (3.8) and (3.9) respectively, A and B are two nonzero real constants and satisfy $A^2 - B^2 > 0$.

Family 15: When $r = 0$ and $pq \neq 0$, we obtain two solutions

$$u_{106} = b_1(t) + b_2(y,z) + \frac{2krq[d + \cosh(pw) - \sinh(pw)]^2 - 2kp^2d^2}{pd[d + \cosh(pw) - \sinh(pw)]},$$
$$u_{107} = b_1(t) + b_2(y,z) + \frac{2kqr[d + \cosh(pw) + \sinh(pw)]^2 - 2kp^2[\cosh(pw) + \sinh(pw)]^2}{p[\cosh(pw) + \sinh(pw)][d + \cosh(pw) + \sinh(pw)]},$$

where d is an arbitrary constant.

Family 16: When $q \neq 0$, $r = p = 0$, one gets the solution

$$u_{108} = b_1(t) + b_2(y,z) + \frac{2kr(qw+d)^2 - 2kq}{qw+d},$$

where d is an arbitrary constant.

4. Graphic analyses

In section 2, we employ the extended homoclinic test approach to get eight types of solutions. These solutions have a tail. The tails in these solutions maybe give a prediction of physical phenomenon and the free parameters in these solutions of Eq.(1.1) have rich mathematical structures, which may be important for explaining some physical phenomena in variety of branches. According to the expressions of solutions, the non-traveling wave solutions u_1, u_4 with $k_1 + 2k_2 \neq 0$, u_5 with $k_1 - 2k_2 \leq 0$ or $k_1 + 2k_2 \geq 0$ and u_7 with $k_1 \geq 0$ or $k_1 \leq -2$ can be seen as kink-like types. u_2, u_5 with $-2k_2 > k_1 > 2k_2$ and u_7 with $-2 < k_1 < 0$ can be regarded as singular solitary wave-like types. The non-traveling wave solutions u_3 and u_6 can be seen as periodic solitary wave-like types. u_8 can be regarded as single solitary wave-like type. Especially, when $\theta(z) = z^2$, the solutions u_1-u_8 have a parabolic tail. As $\theta(z) = z$, the solutions u_1-u_8 have a linear tail. These results reveal the complex structures of the solutions for (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation (1.1). Some cross sections of these solutions are solitary wave forms. Here, through 3D graphic, we draw the cross sections of some solutions with a linear tail or a parabolic tail.

Figure 1 and Figure 2 show the kink type solitary solution u_1 with a linear tail and kink-like type solution u_1 with a parabolic tail, respectively. Figure 1 and Figure 2 are kinking, but Figure 2 has a parabolic slot. Figure 3 and Figure 4 express the singular solitary wave solution u_2 with a linear tail and singular solitary wave-like solution u_2 with a parabolic tail. The white parts of Figure 3 and Figure 4 represent singularities. Figure 5 shows the periodic solitary wave-like solution u_3 with a parabolic tail. Figure 6 expresses the singular solitary wave-like solution u_7 with a parabolic tail.

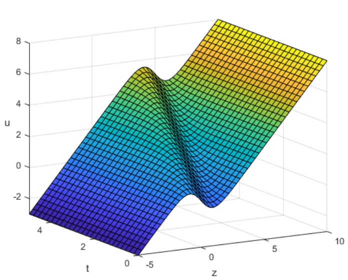

Figure 1. Exact kink type solitary solution for u_1 as $a_2 = -1, k_2 = 1, n = 3, m = c = 0, \theta(z) = z, x = y = 0$.

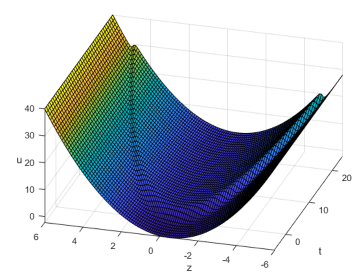

Figure 2. Kink-like solution with a parabolic tail for u_1 as $a_2 = -1, k_2 = m = 1, n = 3, c = 0, \theta(z) = z^2, x = y = 0$.

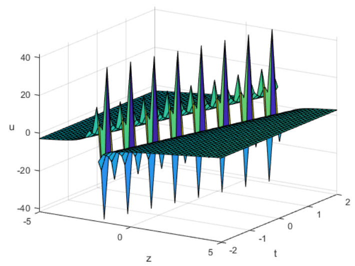

Figure 3. Exact singular solitary wave solution for u_2 as $a_2 = k_2 = -1, n = 3, m = c = 0, \theta(z) = z, x = y = 0$.

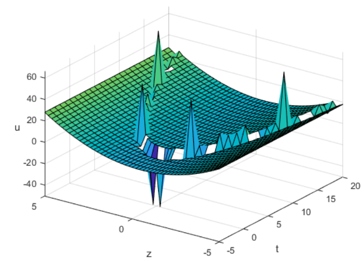

Figure 4. Singular solitary wave-like solution with a parabolic tail for u_2 as $a_2 = k_2 = -1, m = 1, n = 3, c = 0, \theta(z) = z^2, x = y = 0$.

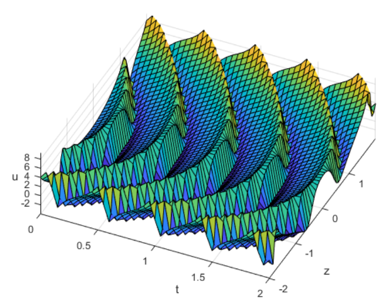

Figure 5. Periodic solitary wave-like solution for u_3 with a parabolic tail as $a_1 = k_1 = m = 1, k_2 = 2, n = 3, c = 0, \theta(z) = z^2, x = y = 0$.

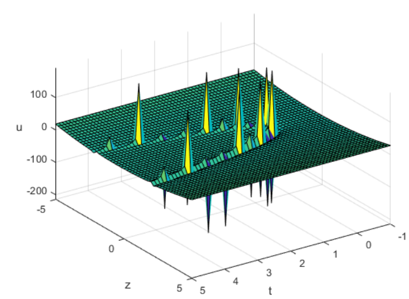

Figure 6. Singular solitary wave-like solution with a parabolic tail for u_7 as $a_1 = i, a_2 = ia_1 = -1, k_1 = -0.5, k_3 = m = 1, n = 3, c = 0, \theta(z) = z^2, x = y = 0$.

In a word, the figures of solutions will change greatly under the influence of a tail, such as u_1, u_2 and u_7, which have parabolic characteristics under the action of parabolic tail.

5. Conclusion and discussion

In conclusion, though there are various effective methods to solve the solutions of nonlinear partial differential equations during the past several decades, such as homogeneous balance method, tanh-function method, Hirota's bilinear operators method, F-expansion method, auxiliary equation method, Exp-function method and so on, these methods may only obtain one form or several forms of solutions. The extended homoclinic test approach (EHTA) used in this manuscript, which is based on the bilinear form of the nonlinear partial differential equations, is a fairly effective method to seek solutions. Moreover, applying extended homoclinic test approach, some new types of special solutions including breather type of soliton and two soliton, and periodic type of soliton can be obtained.

In this paper, we firstly employ the multi-linear variable separation approach to reduce (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation (1.1) to (1+1)-dimensional nonlinear equation with variable coefficients (2.2). Then, we introduce an appropriate transformation to simplify (2.2) to a constant coefficients equation. Furthermore, by the extended homoclinic test approach, we solve the simplified equation and obtain eight kinds of non-traveling wave solutions of (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation in conditions of $a_1 \in R$ or $a_1 = k_3 i, k_3 \in R$. Finally, combining the improved tanh function method with a generalized Riccati equation, we obtain abundant new exact non-traveling wave solutions of (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. These solutions include kink-like type solutions, periodic solitary-wave-like type solutions, singular solitary wave-like type solutions, and so on. As arbitrary functions included in these solutions are taken as some special functions, kink-type solitons, singular solitary wave solutions, periodic solitary wave solutions and so on are presented. Also, the arbitrary functions in these solutions imply that these solutions have rich local structures. Therefore, the results obtained in this paper are the supplement and extension of the results of the existing literature [2, 4, 7, 8, 10, 12, 14, 16, 18–20].

From our abundant results obtained in this paper, the methods employed here have been proved to be fairly effective methods for seeking non-traveling wave solutions of higher-dimensional nonlinear partial differential equations. It is expected that our results are helpful for theoretical study of the associated higher-dimensional nonlinear partial differential equations in mathematics and physics. Moreover, in practical applications, most of real nonlinear partial differential equations possess variable coefficients. The exact solutions of the variable coefficients nonlinear partial differential equations have a greater application value. As we known, there is few research on solutions for (3+1)-dimensional variable coefficients Boiti-Leon-Manna-Pempinelli equation. Next, we will set out to study the traveling wave solutions and non-traveling wave solutions of (3+1)-dimensional variable coefficients Boiti-Leon-Manna-Pempinelli(VC-BLMP) equation. As the special cases of (3+1)-VC-BLMP equation, several generalized BLMP equations with different form and the corresponding results of them will be given.

Acknowledgements

The authors would like to express their thanks to the anonymous referee for their valuable remarks and helpful suggestions on the earlier version of the paper.

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