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A NOVEL WAY CONSTRUCTING SYMPLECTIC STOCHASTIC PARTITIONED RUNGE-KUTTA METHODS FOR STOCHASTIC HAMILTONIAN SYSTEMS*

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Abstract In this paper, a novel way of constructing symplectic stochastic partitioned Runge-Kutta methods for stochastic Hamiltonian systems is presented. First, a new class of continuous-stage stochastic partitioned Runge-Kutta methods for partitioned stochastic differential equations are proposed. The order conditions of the continuous-stage stochastic partitioned Runge-Kutta methods are derived via the stochastic B-series theory. The symplectic conditions of the continuous-stage stochastic partitioned Runge-Kutta methods when applied to stochastic Hamiltonian systems are analyzed. Then we prove applying any quadrature formula to a symplectic continuous-stage stochastic partitioned Runge-Kutta method will result in a classical symplectic stochastic partitioned Runge-Kutta method. In this way, various symplectic stochastic partitioned Runge-Kutta methods can be easily constructed by using different quadrature formulas. A concrete symplectic continuous-stage stochastic partitioned Runge-Kutta method of order 1 is constructed and two retrieved stochastic partitioned Runge-Kutta methods are obtained. Numerical experiments are presented to verify the theoretical results and show the effectiveness of the derived methods.

Keywords Stochastic Hamiltonian systems, symplectic, stochastic partitioned Runge-Kutta methods, continuous-stage, stochastic B-series.

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1. Introduction

Stochastic differential equations (SDEs) are widely used to model many dynamical and social systems [18]. Since most SDEs cannot be solved analytically, the development of numerical methods for the approximation of SDEs has become a field of increasing interest in recent years (see [2,13,14,19,24] and references therein).

Since many systems possess some important geometrical or physical properties, numerical integrators that can preserve the intrinsic properties of the original systems have received much attention recently. In this paper we are concerned with the

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stochastic Hamiltonian system determined by two sufficiently smooth real-valued functions $H(p,q)$ and $\tilde{H}(p,q)$ over the phase space $\mathbb{R}^{2d}$, which is represented by the following SDEs in the Stratonovich sense

$$\begin{cases} dp = -\frac{\partial H(p,q)}{\partial q}dt - \frac{\partial \tilde{H}(p,q)}{\partial q} \circ dW(t), & p(0) = p_0 \in \mathbb{R}^d, \\ dq = \frac{\partial H(p,q)}{\partial p}dt + \frac{\partial \tilde{H}(p,q)}{\partial p} \circ dW(t), & q(0) = q_0 \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $p,\ q,\ \frac{\partial H(p,q)}{\partial p},\ \frac{\partial H(p,q)}{\partial q},\ \frac{\partial \tilde{H}(p,q)}{\partial p},\ \frac{\partial \tilde{H}(p,q)}{\partial q},\ p_0$ and q_0 are d-dimensional column vectors with components $p^i,\ q^i,\ \frac{\partial H(p,q)}{\partial p^i},\ \frac{\partial H(p,q)}{\partial q^i},\ \frac{\partial \tilde{H}(p,q)}{\partial p^i},\ \frac{\partial \tilde{H}(p,q)}{\partial q^i},\ p_0^i$ and q_0^i respectively, $i = 1, \ldots, d$. $W(t)$ is a standard 1-dimensional Wiener process. Stochastic Hamiltonian systems are one of the most important classes of dynamical systems. It is known that the phase flows of (1.1) have the property of preserving symplectic structure, i.e.,

$$dp(t) \wedge dq(t) = dp_0 \wedge dq_0, \quad \forall t \geq 0, \tag{1.2}$$

where the differential two-form $dp \wedge dq$ is defined by

$$dp \wedge dq = dp^1 \wedge dq^1 + dp^2 \wedge dq^2 + \ldots + dp^d \wedge dq^d. \tag{1.3}$$

Naturally, a numerical method is called symplectic if the condition

$$dp_{n+1} \wedge dq_{n+1} = dp_n \wedge dq_n, \quad n = 0, 1, \ldots, \tag{1.4}$$

holds, where (p_n, q_n) is the approximation of $(p(t), q(t))$ at $t = nh$.

Many excellent results on deterministic symplectic numerical methods have been derived during the last four decades (e.g., [7–9, 11, 26–30, 34]). Recently, stochastic symplectic methods have also received some attention. Milstein [20] constructs some symplectic methods for stochastic Hamiltonian system with multiplicative noise by using the technique of truncated Brownian motion increments. Misawa [21] derives symplectic integrators from composition methods for stochastic Hamiltonian systems. Wang [32] proposes symplectic methods by computing an approximate stochastic generating function. Ma [16, 17] studies symplectic stochastic Runge-Kutta methods and symplectic stochastic partitioned Runge-Kutta methods for stochastic Hamiltonian systems with multiplicative noise. Wang constructs symplectic Runge-Kutta methods for three types of stochastic Hamiltonian systems in [33]. Han [12] derives high-order stochastic symplectic partitioned Runge-Kutta methods for stochastic Hamiltonian systems with additive noise. Continuous-stage Runge-Kutta methods were firstly presented by Butcher in 1970s [5], and then they have been considered e.g. in [10, 22, 31] and more recently also for stochastic differential equations [35].

In this work, we aim to find a new and efficient way to construct symplectic stochastic partitioned Runge-Kutta methods for solving (1.1). The rest of the paper is organized as follows. In Section 2, we present the continuous-stage stochastic partitioned Runge-Kutta (CSSPRK) methods for general partitioned SDEs with R partitions and M noises. Based on the stochastic B-series theory, we derive the order conditions. In Section 3, we apply the CSSPRK methods to the stochastic Hamiltonian system (1.1) to obtain the symplectic conditions, furthermore, we prove the classical stochastic partitioned Runge-Kutta (SPRK) methods retrieved from the CSSPRK methods with any quadrature formula are symplectic provided the

CSSPRK methods are symplectic. In Section 4, we construct a concrete symplectic CSSPRK method of order 1 and derive the corresponding classical SPRK methods by means of some quadrature formulas. Numerical experiments are presented in Section 5 to report our theoretical results.

2. CSSPRK methods and order conditions

For the partitioned SDEs with R partitions and M noises in the Stratonovich sense

$$\begin{cases} dX^{(r)}(t) = \sum\limits_{m=0}^{M} g_m^{(r)}(X^{(1)}(t), X^{(2)}(t), \dots, X^{(R)}(t)) * dW_m(t), \ t \in [0,T], \\ X^{(r)}(0) = x_0^{(r)}, \ r = 1, \dots, R, \end{cases} \tag{2.1}$$

where the deterministic terms are represented by $m = 0$ and the stochastic terms are represented by $m = 1, \dots, M$. $*dW_0(t) = dt$ and $*dW_m(t) = \circ dW_m(t)$ for $m = 1, \dots, M$. $W_m(t)$ $(m = 1, \dots, M)$ are 1-dimensional and pairwise independent Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{t \geqslant 0})$ fulfilling the usual conditions. The coefficients $g_m^{(r)} : \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_R} \to \mathbb{R}^{d_r}$ are sufficiently smooth and satisfy the conditions such that the solution of (2.1) uniquely exists. Denote the initial values vector by $x_0 = (x_0^{(1)}, \dots, x_0^{(R)})$, which is independent of the σ-algebra generated by the underlying Wiener process with $E\|x_0\|^2 < \infty$.

For an equidistant discretization of the interval $[0,T]$ with a fixed step size $h > 0$, we denote the numerical approximation of $X^{(r)}$ at $t_n = nh$ by $Y_n^{(r)}$. Now we define the CSSPRK method for solving (2.1) as following

$$\begin{aligned} Y_\varepsilon^{(r)} &= Y_n^{(r)} + \sum_{m=0}^{M} \int_0^1 Z_{\varepsilon,\sigma}^{(r,m)} g_m^{(r)}(Y_\sigma^{(1)}, \dots, Y_\sigma^{(R)}) d\sigma, \ r = 1, \dots, R, \\ Y_{n+1}^{(r)} &= Y_n^{(r)} + \sum_{m=0}^{M} \int_0^1 z_\varepsilon^{(r,m)} g_m^{(r)}(Y_\varepsilon^{(1)}, \dots, Y_\varepsilon^{(R)}) d\varepsilon, \ r = 1, \dots, R, \end{aligned} \tag{2.2}$$

where $Z_{\varepsilon,\sigma}^{(r,m)}$ is a function of two variables $\varepsilon, \sigma \in [0,1]$, $z_\varepsilon^{(r,m)}$ is a function of $\varepsilon \in [0,1]$, the random variables are included in the coefficients $Z_{\varepsilon,\sigma}^{(r,m)}$ and $z_{\varepsilon,\sigma}^{(r,m)}$.

Before discussing order conditions of the proposed CSSPRK method (2.2), we first recall some theories about stochastic B-series and rooted trees. The B-series theory for ordinary differential equations was introduced in [4], and B-series for stochastic case were developed in [1–3, 6, 15, 25].

Definition 2.1 ([1]). The set of shaped, rooted trees

$$T = T_1 \cup T_2 \cup \cdots \cup T_R$$

where

$$T_r = \{\emptyset_r\} \cup T_{r,0} \cup T_{r,1} \cup \cdots \cup T_{r,M}$$

for $r = 1, \dots, R$ is recursively defined by
(I) $\bullet_{r,m}$ belongs to $T_{r,m}$ representing one vertex of shape r and color m;
(II) If $\tau_1, \dots, \tau_\kappa \ \in T \backslash \{\emptyset_0, \dots, \emptyset_R\}$, then $[\tau_1, \dots, \tau_\kappa]_{r,m} \in T_{r,m}$ denotes the tree by grafting the roots of $\tau_1, \dots, \tau_\kappa$ to a new vertex of shape r and color m.

Definition 2.2 ([1]). For a tree $\tau \in T$, the elementary differential is a mapping $F(\tau) : \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_R} \to \mathbb{R}^{d_r}$ defined recursively by
(I) $F(\emptyset_r)(x_0) = x_0^{(r)},\ \emptyset_r \in T_r,$
(II) $F(\bullet_{r,m})(x_0) = g_m^{(r)}(x_0),$
(III) If $\tau = [\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m}$, then

$$F(\tau)(x_0) = (D_{r_1 \ldots r_\kappa} g_m^{(r)})(x_0)(F(\tau_1)(x_0), \ldots, F(\tau_\kappa)(x_0)),$$

where r_k is the shape of $\tau_k,\ k = 1, \ldots, \kappa$, and $D_{r_1 \ldots r_\kappa} = \frac{\partial^\kappa}{\partial x^{(r_1)} \ldots \partial x^{(r_\kappa)}}$ is the derivative operator of order κ.

Lemma 2.1 ([1]). *If $Y^{(r)}(h)$ is a B-series $B^{(r)}(\varphi, x_0; h)$ as*

$$Y^{(r)}(h) = B^{(r)}(\varphi, x_0; h) = \sum_{\tau \in T_r} \alpha(\tau) \cdot \varphi(\tau)(h) \cdot F(\tau)(x_0),$$

for $r = 1, \ldots, R$, where $\alpha(\tau)$ is defined by

$$\alpha(\emptyset_r) = 1,\ \alpha(\bullet_{r,m}) = 1,\ \alpha([\tau_1, \ldots, \tau_\kappa]_{r,m}) = \frac{1}{\mu_1! \mu_2! \cdots} \prod_{k=1}^{\kappa} \alpha(\tau_k),$$

here $\mu_1, \mu_2, \ldots$ count equal trees among $\tau_1, \ldots, \tau_\kappa$. Then $g_m^{(r)}(Y^{(1)}(h), \ldots, Y^{(R)}(h))$ can be represented as a formal series of the form

$$g_m^{(r)}(Y^{(1)}(h), \ldots, Y^{(R)}(h)) = \sum_{\tau \in T_{r,m}} \alpha(\tau) \cdot \varphi'_{r,m}(\tau)(h) \cdot F(\tau)(x_0),$$

for $r = 1, \ldots, R,\ m = 0, \ldots, M$, where

$$\varphi'_{r,m}(\tau)(h) = \begin{cases} 1, & if\ \tau = \bullet_{r,m}, \\ \prod_{k=1}^{\kappa} \varphi(\tau_k)(h), & if\ \tau = [\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m}. \end{cases}$$

Lemma 2.2 ([1]). *The exact solution $X^{(r)}(t_0 + h)$ of (2.1) can be written as a B-series $B^{(r)}(\phi, x_0; h)$*

$$X^{(r)}(t_0 + h) = B^{(r)}(\phi, x_0; h) = \sum_{\tau \in T_r} \alpha(\tau) \cdot \phi(\tau)(h) \cdot F(\tau)(x_0),$$

with

$$\phi(\emptyset_r)(h) = 1,\ \phi(\bullet_{r,m})(h) = W_m(h),$$
$$\phi([\tau_1, \ldots, \tau_\kappa]_{r,m})(h) = \int_0^h \prod_{k=1}^{\kappa} \phi(\tau_k)(s) * dW_m(s),$$

for all $[\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m},\ r = 1, \ldots, R,\ m = 0, \ldots, M$.

Now we put forward to the first main result of our work.

Theorem 2.1. *The continuous-stage values $Y_\varepsilon^{(r)}$ and the numerical solution $Y_1^{(r)}$ can be written in the form of B-series*

$$Y_\varepsilon^{(r)} = B^{(r)}(\Psi_\varepsilon, x_0; h) = \sum_{\tau \in T_r} \alpha(\tau) \cdot \Psi_\varepsilon(\tau)(h) \cdot F(\tau)(x_0),$$
$$Y_1^{(r)} = B^{(r)}(\Phi, x_0; h) = \sum_{\tau \in T_r} \alpha(\tau) \cdot \Phi(\tau)(h) \cdot F(\tau)(x_0),$$

for $r = 1, \ldots, R$, with

$$\begin{aligned} &\Psi_\varepsilon(\emptyset_r)(h) = 1,\ \Psi_\varepsilon(\bullet_{r,m})(h) = \int_0^1 Z_{\varepsilon,\sigma}^{(r,m)} d\sigma, \\ &\Psi_\varepsilon([\tau_1, \ldots, \tau_\kappa]_{r,m})(h) = \int_0^1 Z_{\varepsilon,\sigma}^{(r,m)} \prod_{k=1}^{\kappa} \Psi_\sigma(\tau_k)(h) d\sigma, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\Phi(\emptyset_r)(h) = 1,\ \Phi(\bullet_{r,m})(h) = \int_0^1 z_\varepsilon^{(r,m)} d\varepsilon, \\ &\Phi([\tau_1, \ldots, \tau_\kappa]_{r,m})(h) = \int_0^1 z_\varepsilon^{(r,m)} \prod_{k=1}^{\kappa} \Psi_\varepsilon(\tau_k)(h) d\varepsilon, \end{aligned} \tag{2.4}$$

for all $[\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m}$, $r = 1, \ldots, R$, $m = 0, \ldots, M$.

Proof. The proof follows the way proving Theorem 2.2 in [1]. Write $Y_\varepsilon^{(r)}$ as B-series

$$Y_\varepsilon^{(r)} = \sum_{\tau \in T_r} \alpha(\tau) \cdot \Psi_\varepsilon(\tau)(h) \cdot F(\tau)(x_0),\ r = 1, \ldots, R. \tag{2.5}$$

From Lemma 2.1 and the first equality of (2.2) we get

$$\begin{aligned} Y_\varepsilon^{(r)} &= x_0^{(r)} + \sum_{m=0}^{M} \int_0^1 Z_{\varepsilon,\sigma}^{(r,m)} \sum_{\tau \in T_{r,m}} \alpha(\tau) \cdot \Psi'_{\sigma,r,m}(\tau)(h) \cdot F(\tau)(x_0) d\sigma \\ &= x_0^{(r)} + \sum_{m=0}^{M} \sum_{\tau \in T_{r,m}} \alpha(\tau) \cdot \int_0^1 Z_{\varepsilon,\sigma}^{(r,m)} \Psi'_{\sigma,r,m}(\tau)(h) d\sigma \cdot F(\tau)(x_0), \end{aligned} \tag{2.6}$$

where

$$\Psi'_{\sigma,r,m}(\tau)(h) = \begin{cases} 1,\ \text{if } \tau = \bullet_{r,m}, \\ \prod_{k=1}^{\kappa} \Psi_\sigma(\tau_k)(h),\ \text{if } \tau = [\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m}. \end{cases} \tag{2.7}$$

Comparing (2.5) with (2.6) term by term, we obtain (2.3).

Similarly, suppose

$$Y_1^{(r)} = \sum_{\tau \in T_r} \alpha(\tau) \cdot \Phi(\tau)(h) \cdot F(\tau)(x_0),\ r = 1, \ldots, R. \tag{2.8}$$

By the second equality of (2.2), Lemma 2.1 and (2.5) we have

$$\begin{aligned} Y_1^{(r)} &= x_0^{(r)} + \sum_{m=0}^{M} \int_0^1 z_\varepsilon^{(r,m)} \sum_{\tau \in T_{r,m}} \alpha(\tau) \cdot \Psi'_{\varepsilon,r,m}(\tau)(h) \cdot F(\tau)(x_0) d\varepsilon \\ &= x_0^{(r)} + \sum_{m=0}^{M} \sum_{\tau \in T_{r,m}} \alpha(\tau) \cdot \int_0^1 z_\varepsilon^{(r,m)} \Psi'_{\varepsilon,r,m}(\tau)(h) d\varepsilon \cdot F(\tau)(x_0), \end{aligned} \tag{2.9}$$

where

$$\Psi'_{\varepsilon,r,m}(\tau)(h) = \begin{cases} 1,\ \text{if } \tau = \bullet_{r,m}, \\ \prod_{k=1}^{\kappa} \Psi_\varepsilon(\tau_k)(h),\ \text{if } \tau = [\tau_1, \ldots, \tau_\kappa]_{r,m} \in T_{r,m}, \end{cases} \tag{2.10}$$

thus (2.4) follows from comparing (2.8) with (2.9) term by term. $\square$

Definition 2.3. [1,3] The order of a tree $\tau \in T$ is defined by

$$\rho(\emptyset_r) = 0,\ \rho(\tau = [\tau_1, \ldots, \tau_\kappa]_{r,m}) = \sum_{k=1}^{\kappa} \rho(\tau_k) + \begin{cases} 1, \text{ for } m = 0, \\ \frac{1}{2}, \text{ otherwise.} \end{cases}$$

By comparing the derived B-series of the exact solution and the numerical solution, now we can obtain the order conditions of the proposed CSSPRK method.

Theorem 2.2. *The CSSPRK method (2.2) has mean square convergence order P if*

$$\begin{aligned} &\Phi(\tau)(h) = \phi(\tau)(h),\ \forall \tau \in T \textit{ with } \rho(\tau) \le P, \\ &E\Phi(\tau)(h) = E\phi(\tau)(h),\ \forall \tau \in T \textit{ with } \rho(\tau) = P + \tfrac{1}{2}. \end{aligned} \tag{2.11}$$

The result (2.11) follows from Lemma 2.2 and Theorem 2.1, see [3] for details.

3. Symplectic CSSPRK methods

3.1. Symplectic conditions

The stochastic Hamiltonian system (1.1) is of the form (2.1) with $R = 2,\ M = 1$. In this section, we will apply the CSSPRK method (2.2) to (1.1) to study the symplectic conditions. For convenience, we denote $f_1(p,q) = \frac{\partial H(p,q)}{\partial q}$, $f_2(p,q) = \frac{\partial H(p,q)}{\partial p}$, $g_1(p,q) = \frac{\partial \tilde{H}(p,q)}{\partial q}$ and $g_2(p,q) = \frac{\partial \tilde{H}(p,q)}{\partial p}$ with the components $f_1^i(p,q) = \frac{\partial H(p,q)}{\partial q^i}$, $f_2^i(p,q) = \frac{\partial H(p,q)}{\partial p^i}$, $g_1^i(p,q) = \frac{\partial \tilde{H}(p,q)}{\partial q^i}$ and $g_2^i(p,q) = \frac{\partial \tilde{H}(p,q)}{\partial p^i}$, respectively, $i = 1, \ldots, d$. For a given step size $h > 0$, with the initial values $(P_0, Q_0) = (p_0, q_0)$, we apply the CSSPRK method (2.2) with $Z_{\varepsilon,\sigma}^{(1,0)} = hA_{\varepsilon,\sigma}$, $Z_{\varepsilon,\sigma}^{(1,1)} = \Delta W(h) B_{\varepsilon,\sigma}$, $Z_{\varepsilon,\sigma}^{(2,0)} = h\tilde{A}_{\varepsilon,\sigma}$, $Z_{\varepsilon,\sigma}^{(2,1)} = \Delta W(h)\tilde{B}_{\varepsilon,\sigma}$, $z_\varepsilon^{(1,0)} = hC_\varepsilon$, $z_\varepsilon^{(1,1)} = \Delta W(h) D_\varepsilon$, $z_\varepsilon^{(2,0)} = h\tilde{C}_\varepsilon$ and $z_\varepsilon^{(2,1)} = \Delta W(h)\tilde{D}_\varepsilon$ to (1.1) to derive the following iterative scheme

$$p_\varepsilon = P_n - h\int_0^1 A_{\varepsilon,\sigma} f_1(p_\sigma, q_\sigma) d\sigma - \Delta W(h)\int_0^1 B_{\varepsilon,\sigma} g_1(p_\sigma, q_\sigma) d\sigma, \tag{3.1a}$$

$$q_\varepsilon = Q_n + h\int_0^1 \tilde{A}_{\varepsilon,\sigma} f_2(p_\sigma, q_\sigma) d\sigma + \Delta W(h)\int_0^1 \tilde{B}_{\varepsilon,\sigma} g_2(p_\sigma, q_\sigma) d\sigma, \tag{3.1b}$$

$$P_{n+1} = P_n - h\int_0^1 C_\varepsilon f_1(p_\varepsilon, q_\varepsilon) d\varepsilon - \Delta W(h)\int_0^1 D_\varepsilon g_1(p_\varepsilon, q_\varepsilon) d\varepsilon, \tag{3.1c}$$

$$Q_{n+1} = Q_n + h\int_0^1 \tilde{C}_\varepsilon f_2(p_\varepsilon, q_\varepsilon) d\varepsilon + \Delta W(h)\int_0^1 \tilde{D}_\varepsilon g_2(p_\varepsilon, q_\varepsilon) d\varepsilon, \tag{3.1d}$$

where $\Delta W(h) = W(t_{n+1}) - W(t_n)$ are independent Gaussian random variables with $N(0,h)$ distribution. Now we propose the symplectic conditions results.

Theorem 3.1. *The scheme (3.1) preserves the symplectic structure of (1.1), i.e.,*

$dP_{n+1} \wedge dQ_{n+1} = dP_n \wedge dQ_n$, *if the coefficients satisfy the following conditions*

$$\begin{aligned}
&C_\varepsilon \tilde{A}_{\varepsilon,\sigma} + \tilde{C}_\sigma A_{\sigma,\varepsilon} = C_\varepsilon \tilde{C}_\sigma,\\
&D_\varepsilon \tilde{A}_{\varepsilon,\sigma} + \tilde{C}_\sigma B_{\sigma,\varepsilon} = D_\varepsilon \tilde{C}_\sigma,\\
&C_\varepsilon \tilde{B}_{\varepsilon,\sigma} + \tilde{D}_\sigma A_{\sigma,\varepsilon} = C_\varepsilon \tilde{D}_\sigma,\\
&D_\varepsilon \tilde{B}_{\varepsilon,\sigma} + \tilde{D}_\sigma B_{\sigma,\varepsilon} = D_\varepsilon \tilde{D}_\sigma,\\
&C_\varepsilon = \tilde{C}_\varepsilon,\ D_\varepsilon = \tilde{D}_\varepsilon.
\end{aligned} \tag{3.2}$$

Proof. For simplifying notations, we denote $f_1(p_\varepsilon, q_\varepsilon) = f_{1,\varepsilon},\ f_1(p_\sigma, q_\sigma) = f_{1,\sigma}$, $f_2(p_\varepsilon, q_\varepsilon) = f_{2,\varepsilon},\ f_2(p_\sigma, q_\sigma) = f_{2,\sigma},\ g_1(p_\varepsilon, q_\varepsilon) = g_{1,\varepsilon},\ g_1(p_\sigma, q_\sigma) = g_{1,\sigma},\ g_2(p_\varepsilon, q_\varepsilon) = g_{2,\varepsilon},\ g_2(p_\sigma, q_\sigma) = g_{2,\sigma}$. Differentiating on both sides of (3.1c) and (3.1d) with respect to P_n and Q_n yields

$$\begin{aligned}
&dP_{n+1} = dP_n - h\int_0^1 C_\varepsilon df_{1,\varepsilon} d\varepsilon - \Delta W(h)\int_0^1 D_\varepsilon dg_{1,\varepsilon} d\varepsilon,\\
&dQ_{n+1} = dQ_n + h\int_0^1 \tilde{C}_\varepsilon df_{2,\varepsilon} d\varepsilon + \Delta W(h)\int_0^1 \tilde{D}_\varepsilon dg_{2,\varepsilon} d\varepsilon,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
df_{1,\varepsilon} &= \frac{\partial f_{1,\varepsilon}}{\partial P_n} dP_n + \frac{\partial f_{1,\varepsilon}}{\partial Q_n} dQ_n,\ dg_{1,\varepsilon} = \frac{\partial g_{1,\varepsilon}}{\partial P_n} dP_n + \frac{\partial g_{1,\varepsilon}}{\partial Q_n} dQ_n,\\
df_{2,\varepsilon} &= \frac{\partial f_{2,\varepsilon}}{\partial P_n} dP_n + \frac{\partial f_{2,\varepsilon}}{\partial Q_n} dQ_n,\ dg_{2,\varepsilon} = \frac{\partial g_{2,\varepsilon}}{\partial P_n} dP_n + \frac{\partial g_{2,\varepsilon}}{\partial Q_n} dQ_n.
\end{aligned}$$

Then according to the properties of wedge product [23], by a straightforward computation, we obtain

$$\begin{aligned}
dP_{n+1} \wedge dQ_{n+1} =& dP_n \wedge dQ_n + h\int_0^1 \tilde{C}_\varepsilon dP_n \wedge df_{2,\varepsilon} d\varepsilon\\
&+ \Delta W(h)\int_0^1 \tilde{D}_\varepsilon dP_n \wedge dg_{2,\varepsilon} d\varepsilon - h\int_0^1 C_\varepsilon df_{1,\varepsilon} \wedge dQ_n d\varepsilon\\
&- h^2 \int_0^1\int_0^1 C_\varepsilon \tilde{C}_\sigma df_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon\\
&- \Delta W(h) h\int_0^1\int_0^1 C_\varepsilon \tilde{D}_\sigma df_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon\\
&- \Delta W(h)\int_0^1 D_\varepsilon dg_{1,\varepsilon} \wedge dQ_n d\varepsilon\\
&- \Delta W(h) h\int_0^1\int_0^1 D_\varepsilon \tilde{C}_\sigma dg_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon\\
&- \Delta W^2(h)\int_0^1\int_0^1 D_\varepsilon \tilde{D}_\sigma dg_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon.
\end{aligned} \tag{3.4}$$

Differentiating on both sides of (3.1a) and (3.1b) with respect to P_n and Q_n yields

$$\begin{aligned}
&dp_\varepsilon = dP_n - h\int_0^1 A_{\varepsilon,\sigma} df_{1,\sigma} d\sigma - \Delta W(h)\int_0^1 B_{\varepsilon,\sigma} dg_{1,\sigma} d\sigma,\\
&dq_\varepsilon = dQ_n + h\int_0^1 \tilde{A}_{\varepsilon,\sigma} df_{2,\sigma} d\sigma + \Delta W(h)\int_0^1 \tilde{B}_{\varepsilon,\sigma} dg_{2,\sigma} d\sigma,
\end{aligned}$$

then there are

$$
\begin{aligned}
dP_n \wedge df_{2,\varepsilon} = &\ dp_\varepsilon \wedge df_{2,\varepsilon} + h\int_0^1 A_{\varepsilon,\sigma} df_{1,\sigma} \wedge df_{2,\varepsilon} d\sigma \\
&+ \Delta W(h)\int_0^1 B_{\varepsilon,\sigma} dg_{1,\sigma} \wedge df_{2,\varepsilon} d\sigma, \\
dP_n \wedge dg_{2,\varepsilon} = &\ dp_\varepsilon \wedge dg_{2,\varepsilon} + h\int_0^1 A_{\varepsilon,\sigma} df_{1,\sigma} \wedge dg_{2,\varepsilon} d\sigma \\
&+ \Delta W(h)\int_0^1 B_{\varepsilon,\sigma} dg_{1,\sigma} \wedge dg_{2,\varepsilon} d\sigma, \\
df_{1,\varepsilon} \wedge dQ_n = &\ df_{1,\varepsilon} \wedge dq_\varepsilon - h\int_0^1 \tilde{A}_{\varepsilon,\sigma} df_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma \\
&- \Delta W(h)\int_0^1 \tilde{B}_{\varepsilon,\sigma} df_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma, \\
dg_{1,\varepsilon} \wedge dQ_n = &\ dg_{1,\varepsilon} \wedge dq_\varepsilon - h\int_0^1 \tilde{A}_{\varepsilon,\sigma} dg_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma \\
&- \Delta W(h)\int_0^1 \tilde{B}_{\varepsilon,\sigma} dg_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma.
\end{aligned}
\tag{3.5}
$$

Inserting (3.5) into (3.4), we get

$$
\begin{aligned}
& dP_{n+1} \wedge dQ_{n+1} \\
=& dP_n \wedge dQ_n + h\int_0^1 \tilde{C}_\varepsilon dp_\varepsilon \wedge df_{2,\varepsilon} d\varepsilon + h^2 \int_0^1 \tilde{C}_\varepsilon \int_0^1 A_{\varepsilon,\sigma} df_{1,\sigma} \wedge df_{2,\varepsilon} d\sigma d\varepsilon \\
&+ \Delta W(h) h \int_0^1 \tilde{C}_\varepsilon \int_0^1 B_{\varepsilon,\sigma} dg_{1,\sigma} \wedge df_{2,\varepsilon} d\sigma d\varepsilon + \Delta W(h) \int_0^1 \tilde{D}_\varepsilon dp_\varepsilon \wedge dg_{2,\varepsilon} d\varepsilon \\
&+ \Delta W(h) h \int_0^1 \tilde{D}_\varepsilon \int_0^1 A_{\varepsilon,\sigma} df_{1,\sigma} \wedge dg_{2,\varepsilon} d\sigma d\varepsilon \\
&+ \Delta W^2(h) \int_0^1 \tilde{D}_\varepsilon \int_0^1 B_{\varepsilon,\sigma} dg_{1,\sigma} \wedge dg_{2,\varepsilon} d\sigma d\varepsilon \\
&- h\int_0^1 C_\varepsilon df_{1,\varepsilon} \wedge dq_\varepsilon d\varepsilon + h^2 \int_0^1 C_\varepsilon \int_0^1 \tilde{A}_{\varepsilon,\sigma} df_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon \\
&+ \Delta W(h) h \int_0^1 C_\varepsilon \int_0^1 \tilde{B}_{\varepsilon,\sigma} df_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon - h^2 \int_0^1 \int_0^1 C_\varepsilon \tilde{C}_\sigma df_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon \\
&- \Delta W(h) h \int_0^1 \int_0^1 C_\varepsilon \tilde{D}_\sigma df_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon - \Delta W(h) \int_0^1 D_\varepsilon dg_{1,\varepsilon} \wedge dq_\varepsilon d\varepsilon \\
&+ \Delta W(h) h \int_0^1 D_\varepsilon \int_0^1 \tilde{A}_{\varepsilon,\sigma} dg_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon \\
&+ \Delta W^2(h) \int_0^1 D_\varepsilon \int_0^1 \tilde{B}_{\varepsilon,\sigma} dg_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon \\
&- \Delta W(h) h \int_0^1 \int_0^1 D_\varepsilon \tilde{C}_\sigma dg_{1,\varepsilon} \wedge df_{2,\sigma} d\sigma d\varepsilon \\
&- \Delta W^2(h) \int_0^1 \int_0^1 D_\varepsilon \tilde{D}_\sigma dg_{1,\varepsilon} \wedge dg_{2,\sigma} d\sigma d\varepsilon.
\end{aligned}
\tag{3.6}
$$

Substituting the conditions $C_\varepsilon = \tilde{C}_\varepsilon,\ D_\varepsilon = \tilde{D}_\varepsilon$ in (3.2) into (3.6), we deduce that

$$
\begin{aligned}
& dP_{n+1} \wedge dQ_{n+1} \\
=& dP_n \wedge dQ_n + h\int_0^1 C_\varepsilon (dp_\varepsilon \wedge df_{2,\varepsilon} - df_{1,\varepsilon} \wedge dq_\varepsilon) d\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta W(h)\int_0^1 D_\varepsilon(dp_\varepsilon\wedge dg_{2,\varepsilon}-dg_{1,\varepsilon}\wedge dq_\varepsilon)d\varepsilon\\
&+h^2\int_0^1\int_0^1(C_\varepsilon\tilde{A}_{\varepsilon,\sigma}+\tilde{C}_\sigma A_{\sigma,\varepsilon}-C_\varepsilon\tilde{C}_\sigma)df_{1,\varepsilon}\wedge df_{2,\sigma}d\sigma d\varepsilon\\
&+\Delta W(h)h\int_0^1\int_0^1(D_\varepsilon\tilde{A}_{\varepsilon,\sigma}+\tilde{C}_\sigma B_{\sigma,\varepsilon}-D_\varepsilon\tilde{C}_\sigma)dg_{1,\varepsilon}\wedge df_{2,\sigma}d\sigma d\varepsilon\\
&+\Delta W(h)h\int_0^1\int_0^1(C_\varepsilon\tilde{B}_{\varepsilon,\sigma}+\tilde{D}_\sigma A_{\sigma,\varepsilon}-C_\varepsilon\tilde{D}_\sigma)df_{1,\varepsilon}\wedge dg_{2,\sigma}d\sigma d\varepsilon\\
&+\Delta W^2(h)\int_0^1\int_0^1(D_\varepsilon\tilde{B}_{\varepsilon,\sigma}+\tilde{D}_\sigma B_{\sigma,\varepsilon}-D_\varepsilon\tilde{D}_\sigma)dg_{1,\varepsilon}\wedge dg_{2,\sigma}d\sigma d\varepsilon,
\end{aligned}
\tag{3.7}
$$

where we have used the equality

$$\int_0^1\tilde{C}_\varepsilon\int_0^1 A_{\varepsilon,\sigma}df_{1,\sigma}\wedge df_{2,\varepsilon}d\sigma d\varepsilon=\int_0^1\int_0^1\tilde{C}_\sigma A_{\sigma,\varepsilon}df_{1,\varepsilon}\wedge df_{2,\sigma}d\sigma d\varepsilon,$$

and other similar equalities in the derivation of (3.7).

By (1.3) we have

$$
\begin{aligned}
&dp_\varepsilon\wedge df_{2,\varepsilon}-df_{1,\varepsilon}\wedge dq_\varepsilon\\
=&\sum_{k=1}^d(dp_\varepsilon^k\wedge df_{2,\varepsilon}^k-df_{1,\varepsilon}^k\wedge dq_\varepsilon^k)\\
=&\sum_{k=1}^d[dp_\varepsilon^k\wedge(\sum_{j=1}^d\frac{\partial f_{2,\varepsilon}^k}{\partial p_\varepsilon^j}dp_\varepsilon^j+\sum_{j=1}^d\frac{\partial f_{2,\varepsilon}^k}{\partial q_\varepsilon^j}dq_\varepsilon^j)\\
&-(\sum_{j=1}^d\frac{\partial f_{1,\varepsilon}^k}{\partial p_\varepsilon^j}dp_\varepsilon^j+\sum_{j=1}^d\frac{\partial f_{1,\varepsilon}^k}{\partial q_\varepsilon^j}dq_\varepsilon^j)\wedge dq_\varepsilon^k]\\
=&\sum_{k,j=1}^d\frac{\partial f_{2,\varepsilon}^k}{\partial p_\varepsilon^j}dp_\varepsilon^k\wedge dp_\varepsilon^j+\sum_{k,j=1}^d(\frac{\partial f_{2,\varepsilon}^k}{\partial q_\varepsilon^j}dp_\varepsilon^k\wedge dq_\varepsilon^j-\frac{\partial f_{1,\varepsilon}^k}{\partial p_\varepsilon^j}dp_\varepsilon^j\wedge dq_\varepsilon^k)\\
&-\sum_{k,j=1}^d\frac{\partial f_{1,\varepsilon}^k}{\partial q_\varepsilon^j}dq_\varepsilon^j\wedge dq_\varepsilon^k\\
=&\sum_{k<j}(\frac{\partial f_{2,\varepsilon}^k}{\partial p_\varepsilon^j}-\frac{\partial f_{2,\varepsilon}^j}{\partial p_\varepsilon^k})dp_\varepsilon^k\wedge dp_\varepsilon^j+\sum_{k,j=1}^d(\frac{\partial f_{2,\varepsilon}^k}{\partial q_\varepsilon^j}-\frac{\partial f_{1,\varepsilon}^j}{\partial p_\varepsilon^k})dp_\varepsilon^k\wedge dq_\varepsilon^j\\
&-\sum_{j<k}(\frac{\partial f_{1,\varepsilon}^k}{\partial q_\varepsilon^j}-\frac{\partial f_{1,\varepsilon}^j}{\partial q_\varepsilon^k})dq_\varepsilon^j\wedge dq_\varepsilon^k.
\end{aligned}
\tag{3.8}
$$

It follows from the sufficiently smooth property of $H(p,q)$ that

$$
\begin{aligned}
\frac{\partial f_{2,\varepsilon}^k}{\partial q_\varepsilon^j}-\frac{\partial f_{1,\varepsilon}^j}{\partial p_\varepsilon^k}&=\frac{\partial}{\partial q_\varepsilon^j}\left(\frac{\partial H(p_\varepsilon,q_\varepsilon)}{\partial p_\varepsilon^k}\right)-\frac{\partial}{\partial p_\varepsilon^k}\left(\frac{\partial H(p_\varepsilon,q_\varepsilon)}{\partial q_\varepsilon^j}\right)\\
&=\frac{\partial^2 H(p_\varepsilon,q_\varepsilon)}{\partial q_\varepsilon^j\partial p_\varepsilon^k}-\frac{\partial^2 H(p_\varepsilon,q_\varepsilon)}{\partial p_\varepsilon^k\partial q_\varepsilon^j}=0,\quad k,\ j=1,\dots,d.
\end{aligned}
\tag{3.9}
$$

Similarly, we can easily check

$$\frac{\partial f_{2,\varepsilon}^k}{\partial p_\varepsilon^j} - \frac{\partial f_{2,\varepsilon}^j}{\partial p_\varepsilon^k} = 0, \ \frac{\partial f_{1,\varepsilon}^k}{\partial q_\varepsilon^j} - \frac{\partial f_{1,\varepsilon}^j}{\partial q_\varepsilon^k} = 0, \quad k, \ j = 1, \ldots, d. \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.8), we obtain

$$dp_\varepsilon \wedge df_{2,\varepsilon} - df_{1,\varepsilon} \wedge dq_\varepsilon = 0, \tag{3.11}$$

then in the same way, we can deduce that

$$dp_\varepsilon \wedge dg_{2,\varepsilon} - dg_{1,\varepsilon} \wedge dq_\varepsilon = 0. \tag{3.12}$$

Finally, substituting (3.11), (3.12) and the first four conditions in (3.2) into (3.7), we derive

$$dP_{n+1} \wedge dQ_{n+1} = dP_n \wedge dQ_n,$$

which completes the proof. □

3.2. SPRK methods retrieved from CSSPRK methods

So far, we have constructed the CSSPRK methods, studied the order conditions, and analyzed the symplectic conditions of the CSSPRK methods when applied to the stochastic Hamiltonian system (1.1). Notice there are integrals in the proposed CSSPRK methods, hence the use of numerical quadrature formulas is necessary in practical implementation.

Applying a quadrature formula denoted by $(b_i, c_i)_{i=1}^s$ to (3.1), we get a classical SPRK method of s-stage for solving (1.1) by

$$\begin{aligned}
p_i &= P_n - h\sum_{j=1}^s b_j A_{c_i,c_j} f_1(p_j,q_j) - \Delta W(h)\sum_{j=1}^s b_j B_{c_i,c_j} g_1(p_j,q_j), \ i = 1,\ldots,s,\\
q_i &= Q_n + h\sum_{j=1}^s b_j \tilde{A}_{c_i,c_j} f_2(p_j,q_j) + \Delta W(h)\sum_{j=1}^s b_j \tilde{B}_{c_i,c_j} g_2(p_j,q_j), \ i = 1,\ldots,s,\\
p_{n+1} &= P_n - h\sum_{i=1}^s b_i C_{c_i} f_1(p_i,q_i) - \Delta W(h)\sum_{i=1}^s b_i D_{c_i} g_1(p_i,q_i),\\
q_{n+1} &= Q_n + h\sum_{i=1}^s b_i \tilde{C}_{c_i} f_2(p_i,q_i) + \Delta W(h)\sum_{i=1}^s b_i \tilde{D}_{c_i} g_2(p_i,q_i),
\end{aligned} \tag{3.13}$$

where $A_{c_i,c_j} = A_{\varepsilon,\sigma}|_{\varepsilon=c_i,\sigma=c_j}$, $B_{c_i,c_j} = B_{\varepsilon,\sigma}|_{\varepsilon=c_i,\sigma=c_j}$, $\tilde{A}_{c_i,c_j} = \tilde{A}_{\varepsilon,\sigma}|_{\varepsilon=c_i,\sigma=c_j}$, $\tilde{B}_{c_i,c_j} = \tilde{B}_{\varepsilon,\sigma}|_{\varepsilon=c_i,\sigma=c_j}$, $C_{c_i} = C_\varepsilon|_{\varepsilon=c_i}$, $D_{c_i} = D_\varepsilon|_{\varepsilon=c_i}$, $\tilde{C}_{c_i} = \tilde{C}_\varepsilon|_{\varepsilon=c_i}$, $\tilde{D}_{c_i} = \tilde{D}_\varepsilon|_{\varepsilon=c_i}$.

(3.13) is a classical SPRK method applied to (1.1), which can be denoted by the following Butcher tableau

$$\begin{array}{c|cccc}
 & (b_j A_{c_i,c_j})_{s\times s} & (b_j B_{c_i,c_j})_{s\times s} & (b_j \tilde{A}_{c_i,c_j})_{s\times s} & (b_j \tilde{B}_{c_i,c_j})_{s\times s} \\
\hline
 & u & v & \tilde{u} & \tilde{v}
\end{array}, \tag{3.14}$$

where $u = (b_1 C_{c_1}, \ldots, b_s C_{c_s})$, $v = (b_1 D_{c_1}, \ldots, b_s D_{c_s})$, $\tilde{u} = (b_1 \tilde{C}_{c_1}, \ldots, b_s \tilde{C}_{c_s})$, $\tilde{v} = (b_1 \tilde{D}_{c_1}, \ldots, b_s \tilde{D}_{c_s})$.

Lemma 3.1 ([16]). *For a SPRK method denoted by the Butcher tableau*

$$\begin{array}{c|cccc} & (a_{ij})_{s\times s} & (b_{ij})_{s\times s} & (\tilde{a}_{ij})_{s\times s} & (\tilde{b}_{ij})_{s\times s} \\ \hline & \alpha & \beta & \tilde{\alpha} & \tilde{\beta} \end{array}, \tag{3.15}$$

where $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\beta = (\beta_1, \ldots, \beta_s)$, $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_s)$, $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_s)$, *when applied to the stochastic Hamiltonian system (1.1), (3.15) is symplectic if the coefficients satisfy*

$$\begin{aligned} &\alpha_i \tilde{a}_{ij} + \tilde{\alpha}_j a_{ji} = \alpha_i \tilde{\alpha}_j, \ i, \ j = 1, \ldots, s, \\ &\beta_i \tilde{a}_{ij} + \tilde{\alpha}_j b_{ji} = \beta_i \tilde{\alpha}_j, \ i, \ j = 1, \ldots, s, \\ &\alpha_i \tilde{b}_{ij} + \tilde{\beta}_j a_{ji} = \alpha_i \tilde{\beta}_j, \ i, \ j = 1, \ldots, s, \\ &\beta_i \tilde{b}_{ij} + \tilde{\beta}_j b_{ji} = \beta_i \tilde{\beta}_j, \ i, \ j = 1, \ldots, s, \\ &\alpha_i = \tilde{\alpha}_i, \ \beta_i = \tilde{\beta}_i, \ i = 1, \ldots, s. \end{aligned} \tag{3.16}$$

Next we will show the symplectic conditions of (3.13).

Theorem 3.2. *If the coefficients of the CSSPRK method (3.1) satisfy the symplectic conditions (3.2), then the SPRK method (3.13) retrieved from (3.1) and the quadrature formula* $(b_i, c_i)_{i=1}^s$ *is symplectic for solving the stochastic Hamiltonian system (1.1).*

Proof. Let $\varepsilon = c_i, \ \sigma = c_j$ in (3.2), then we have

$$\begin{aligned} &C_{c_i} \tilde{A}_{c_i,c_j} + \tilde{C}_{c_j} A_{c_j,c_i} = C_{c_i} \tilde{C}_{c_j}, \\ &D_{c_i} \tilde{A}_{c_i,c_j} + \tilde{C}_{c_j} B_{c_j,c_i} = D_{c_i} \tilde{C}_{c_j}, \\ &C_{c_i} \tilde{B}_{c_i,c_j} + \tilde{D}_{c_j} A_{c_j,c_i} = C_{c_i} \tilde{D}_{c_j}, \\ &D_{c_i} \tilde{B}_{c_i,c_j} + \tilde{D}_{c_j} B_{c_j,c_i} = D_{c_i} \tilde{D}_{c_j}, \\ &C_{c_i} = \tilde{C}_{c_i}, \ D_{c_i} = \tilde{D}_{c_i}. \end{aligned} \tag{3.17}$$

Multiplying $b_i b_j$ on both sides of the first four equalities in (3.17) and multiplying b_i on both sides of the last two equalities in (3.17) yield

$$\begin{aligned} &(b_i C_{c_i})(b_j \tilde{A}_{c_i,c_j}) + (b_j \tilde{C}_{c_j})(b_i A_{c_j,c_i}) = (b_i C_{c_i})(b_j \tilde{C}_{c_j}), \ i, \ j = 1, \ldots, s, \\ &(b_i D_{c_i})(b_j \tilde{A}_{c_i,c_j}) + (b_j \tilde{C}_{c_j})(b_i B_{c_j,c_i}) = (b_i D_{c_i})(b_j \tilde{C}_{c_j}), \ i, \ j = 1, \ldots, s, \\ &(b_i C_{c_i})(b_j \tilde{B}_{c_i,c_j}) + (b_j \tilde{D}_{c_j})(b_i A_{c_j,c_i}) = (b_i C_{c_i})(b_j \tilde{D}_{c_j}), \ i, \ j = 1, \ldots, s, \\ &(b_i D_{c_i})(b_j \tilde{B}_{c_i,c_j}) + (b_j \tilde{D}_{c_j})(b_i B_{c_j,c_i}) = (b_i D_{c_i})(b_j \tilde{D}_{c_j}), \ i, \ j = 1, \ldots, s, \\ &b_i C_{c_i} = b_i \tilde{C}_{c_i}, \ b_i D_{c_i} = b_i \tilde{D}_{c_i}, \ i = 1, \ldots, s, \end{aligned} \tag{3.18}$$

which are exactly the symplectic conditions of the SPRK method (3.13) according to Lemma 3.1. The proof is completed. $\square$

Theorem 3.2 reveals the SPRK methods retrieved from the CSSPRK methods will always be symplectic if the CSSPRK methods are symplectic, no matter what quadrature formulas are chosen. It implies that we can construct various symplectic SPRK methods by choosing different quadrature formulas based on one CSSPRK method, which is a new and cheap way to derive symplectic SPRK methods.

4. Construction of a symplectic CSSPRK method and two symplectic SPRK methods

In this section, we will construct a concrete symplectic CSSPRK method (3.1) for solving (1.1) according to the symplectic conditions derived in Section 3, then get the corresponding SPRK methods by using some quadrature formulas. Since the convergence order 1.5 for Runge-Kutta types schemes can not be surpassed if just the increment $\Delta W(h)$ of the Wiener process is used [2], in view of the form of (3.1), here we only focus on constructing a symplectic CSSPRK method of order 1.

By Theorem 2.2, (3.1) has order 1 if the two conditions

$$\begin{aligned} &\Phi(\tau)(h) = \phi(\tau)(h),\ \forall \tau \in T \text{ with } \rho(\tau) \le 1, \\ &E\Phi(\tau)(h) = E\phi(\tau)(h),\ \forall \tau \in T \text{ with } \rho(\tau) = 1.5, \end{aligned} \tag{4.1}$$

hold. Notice the second condition in (4.1) is always true, since τ with $\rho(\tau) = 1.5$ must have an odd number of stochastic nodes so that the expectations will be 0. We list all the trees with $\rho(\tau) \le 1$ in Table 1, where we use $\bullet_{r,0}$ to denote a deterministic node in partition r, and $\bullet_{r,1}$ to denote a stochastic node in partition r for $r = 1, 2$. According to the first condition in (4.1), we get the following equations

$$\begin{aligned} &\int_0^1 D_\varepsilon d\varepsilon = 1, \int_0^1 \tilde{D}_\varepsilon d\varepsilon = 1, \\ &\int_0^1 C_\varepsilon d\varepsilon = 1, \int_0^1 \tilde{C}_\varepsilon d\varepsilon = 1, \\ &\int_0^1 D_\varepsilon (\int_0^1 \tilde{B}_{\varepsilon,\sigma} d\sigma) d\varepsilon = \frac{1}{2}, \\ &\int_0^1 \tilde{D}_\varepsilon (\int_0^1 B_{\varepsilon,\sigma} d\sigma) d\varepsilon = \frac{1}{2}, \\ &\int_0^1 D_\varepsilon (\int_0^1 B_{\varepsilon,\sigma} d\sigma) d\varepsilon = \frac{1}{2}, \\ &\int_0^1 \tilde{D}_\varepsilon (\int_0^1 \tilde{B}_{\varepsilon,\sigma} d\sigma) d\varepsilon = \frac{1}{2}. \end{aligned} \tag{4.2}$$

Table 1. Trees with order $\rho(\tau) \le 1$ and corresponding functions

No.	τ	$\rho(\tau)$	$\phi(\tau)$	$\Phi(\tau)$
1	$\bullet\, 1,1$	0.5	$\Delta W(h)$	$\Delta W(h) \int_0^1 D_\varepsilon d\varepsilon$
2	$\bullet\, 2,1$	0.5	$\Delta W(h)$	$\Delta W(h) \int_0^1 \tilde{D}_\varepsilon d\varepsilon$
3	$\bullet\, 1,0$	1	h	$h \int_0^1 C_\varepsilon d\varepsilon$
4	$\bullet\, 2,0$	1	h	$h \int_0^1 \tilde{C}_\varepsilon d\varepsilon$
5	$\begin{smallmatrix}\bullet\ 2,1\\ \mid\\ \bullet\ 1,1\end{smallmatrix}$	1	$\dfrac{\Delta W^2(h)}{2}$	$\Delta W^2(h) \int_0^1 D_\varepsilon \int_0^1 \tilde{B}_{\varepsilon,\sigma} d\sigma d\varepsilon$
6	$\begin{smallmatrix}\bullet\ 1,1\\ \mid\\ \bullet\ 2,1\end{smallmatrix}$	1	$\dfrac{\Delta W^2(h)}{2}$	$\Delta W^2(h) \int_0^1 \tilde{D}_\varepsilon \int_0^1 B_{\varepsilon,\sigma} d\sigma d\varepsilon$
7	$\begin{smallmatrix}\bullet\ 1,1\\ \mid\\ \bullet\ 1,1\end{smallmatrix}$	1	$\dfrac{\Delta W^2(h)}{2}$	$\Delta W^2(h) \int_0^1 D_\varepsilon \int_0^1 B_{\varepsilon,\sigma} d\sigma d\varepsilon$
8	$\begin{smallmatrix}\bullet\ 2,1\\ \mid\\ \bullet\ 2,1\end{smallmatrix}$	1	$\dfrac{\Delta W^2(h)}{2}$	$\Delta W^2(h) \int_0^1 \tilde{D}_\varepsilon \int_0^1 \tilde{B}_{\varepsilon,\sigma} d\sigma d\varepsilon$

The symplectic conditions for (3.1) are derived in Theorem 3.1 as

$$
\begin{aligned}
&C_\varepsilon \tilde{A}_{\varepsilon,\sigma} + \tilde{C}_\sigma A_{\sigma,\varepsilon} = C_\varepsilon \tilde{C}_\sigma,\\
&D_\varepsilon \tilde{A}_{\varepsilon,\sigma} + \tilde{C}_\sigma B_{\sigma,\varepsilon} = D_\varepsilon \tilde{C}_\sigma,\\
&C_\varepsilon \tilde{B}_{\varepsilon,\sigma} + \tilde{D}_\sigma A_{\sigma,\varepsilon} = C_\varepsilon \tilde{D}_\sigma,\\
&D_\varepsilon \tilde{B}_{\varepsilon,\sigma} + \tilde{D}_\sigma B_{\sigma,\varepsilon} = D_\varepsilon \tilde{D}_\sigma,\\
&C_\varepsilon = \tilde{C}_\varepsilon,\ D_\varepsilon = \tilde{D}_\varepsilon.
\end{aligned} \tag{4.3}
$$

Now we solve the equations (4.2)-(4.3). For simplifying conditions, we set

$$D_\varepsilon = \tilde{D}_\varepsilon = 1,\ C_\varepsilon = \tilde{C}_\varepsilon = 2\varepsilon, \tag{4.4}$$

then the equations (4.2)-(4.3) reduce to

$$
\begin{aligned}
&\int_0^1 \int_0^1 B_{\varepsilon,\sigma} d\sigma d\varepsilon = \frac{1}{2},\\
&2\varepsilon \tilde{A}_{\varepsilon,\sigma} + 2\sigma A_{\sigma,\varepsilon} = 4\varepsilon\sigma,\\
&\tilde{A}_{\varepsilon,\sigma} + 2\sigma B_{\sigma,\varepsilon} = 2\sigma,\\
&2\varepsilon \tilde{B}_{\varepsilon,\sigma} + A_{\sigma,\varepsilon} = 2\varepsilon,\\
&\tilde{B}_{\varepsilon,\sigma} + B_{\sigma,\varepsilon} = 1.
\end{aligned} \tag{4.5}
$$

By solving (4.5), we get a solution

$$A_{\varepsilon,\sigma} = 4\varepsilon\sigma^2,\ B_{\varepsilon,\sigma} = 2\varepsilon\sigma,\ \tilde{A}_{\varepsilon,\sigma} = 2\sigma(1 - 2\varepsilon\sigma),\ \tilde{B}_{\varepsilon,\sigma} = 1 - 2\varepsilon\sigma. \tag{4.6}$$

Notice there are many solutions satisfying (4.2)-(4.3), while here we just take the solution (4.4) with (4.6) as an example. Inserting (4.4) and (4.6) into (3.1), we obtain a symplectic CSSPRK method with order 1 for solving (1.1) as following

$$
\begin{aligned}
&p_\varepsilon = P_n - h\int_0^1 4\varepsilon\sigma^2 f_1(p_\sigma, q_\sigma)d\sigma - \Delta W(h)\int_0^1 2\varepsilon\sigma g_1(p_\sigma, q_\sigma)d\sigma,\\
&q_\varepsilon = Q_n + h\int_0^1 2\sigma(1-2\varepsilon\sigma) f_2(p_\sigma, q_\sigma)d\sigma + \Delta W(h)\int_0^1 (1-2\varepsilon\sigma) g_2(p_\sigma, q_\sigma)d\sigma,\\
&P_{n+1} = P_n - h\int_0^1 2\varepsilon f_1(p_\varepsilon, q_\varepsilon)d\varepsilon - \Delta W(h)\int_0^1 g_1(p_\varepsilon, q_\varepsilon)d\varepsilon,\\
&Q_{n+1} = Q_n + h\int_0^1 2\varepsilon f_2(p_\varepsilon, q_\varepsilon)d\varepsilon + \Delta W(h)\int_0^1 g_2(p_\varepsilon, q_\varepsilon)d\varepsilon.
\end{aligned}
\tag{4.7}
$$

By applying different quadrature formulas to (4.7), we will derive different symplectic SPRK methods. Next we demonstrate the order results of the retrieved SPRK methods.

Theorem 4.1. *Assume the coefficients of the CSSPRK method (3.1) satisfy (4.4) and (4.6), then the SPRK method (3.13) retrieved from (3.1) with the quadrature formula $(b_i, c_i)_{i=1}^s$ has convergence order 1 as long as the quadrature formula has order $\hat{p} \geq 2$.*

Proof. Since $D_\varepsilon = \tilde{D}_\varepsilon = 1$ are constants, $C_\varepsilon = \tilde{C}_\varepsilon = 2\varepsilon$ are polynomials of degree 1, $B_{\varepsilon,\sigma} = 2\varepsilon\sigma$ and $\tilde{B}_{\varepsilon,\sigma} = 1 - 2\varepsilon\sigma$ are bivariate polynomials of degree 1 in ε and degree 1 in σ in (4.4) and (4.6), applying the quadrature formula $(b_i, c_i)_{i=1}^s$ of order $\hat{p} \geq 2$ to the integrals in (4.2) yields

$$
\begin{aligned}
&\sum_{i=1}^s b_i D_{c_i} = \int_0^1 D_\varepsilon d\varepsilon = 1,\ \ \sum_{i=1}^s b_i \tilde{D}_{c_i} = \int_0^1 \tilde{D}_\varepsilon d\varepsilon = 1,\\
&\sum_{i=1}^s b_i C_{c_i} = \int_0^1 C_\varepsilon d\varepsilon = 1,\ \ \sum_{i=1}^s b_i \tilde{C}_{c_i} = \int_0^1 \tilde{C}_\varepsilon d\varepsilon = 1,\\
&\sum_{i=1}^s\sum_{j=1}^s (b_i D_{c_i})(b_j \tilde{B}_{c_i,c_j}) = \int_0^1 D_\varepsilon\left(\int_0^1 \tilde{B}_{\varepsilon,\sigma}d\sigma\right)d\varepsilon = \frac{1}{2},\\
&\sum_{i=1}^s\sum_{j=1}^s (b_i \tilde{D}_{c_i})(b_j B_{c_i,c_j}) = \int_0^1 \tilde{D}_\varepsilon\left(\int_0^1 B_{\varepsilon,\sigma}d\sigma\right)d\varepsilon = \frac{1}{2},\\
&\sum_{i=1}^s\sum_{j=1}^s (b_i D_{c_i})(b_j B_{c_i,c_j}) = \int_0^1 D_\varepsilon\left(\int_0^1 B_{\varepsilon,\sigma}d\sigma\right)d\varepsilon = \frac{1}{2},\\
&\sum_{i=1}^s\sum_{j=1}^s (b_i \tilde{D}_{c_i})(b_j \tilde{B}_{c_i,c_j}) = \int_0^1 \tilde{D}_\varepsilon\left(\int_0^1 \tilde{B}_{\varepsilon,\sigma}d\sigma\right)d\varepsilon = \frac{1}{2},
\end{aligned}
\tag{4.8}
$$

which are the conditions of convergence order 1 for (3.13) according to [16]. The proof is completed. $\square$

Now we construct two SPRK methods by applying two different quadrature formulas of order $\hat{p} \geq 2$ to (4.7) respectively. First, we use Gauss quadrature formula $(b_i, c_i)_{i=1}^2$ with $(b_1, b_2) = (\frac{1}{2}, \frac{1}{2})$ and $(c_1, c_2) = (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$ to get a 2-stage SPRK method denoted by the following Butcher tableau

$$
\begin{array}{c|cc|cc|cc|cc}
 & \frac{1}{2}-\frac{5\sqrt{3}}{18} & \frac{1}{6}+\frac{\sqrt{3}}{18} & \frac{1}{3}-\frac{\sqrt{3}}{6} & \frac{1}{6} & \frac{\sqrt{3}}{9} & \frac{1}{3}+\frac{\sqrt{3}}{9} & \frac{1}{6}+\frac{\sqrt{3}}{6} & \frac{1}{3}\\
 & \frac{1}{6}-\frac{\sqrt{3}}{18} & \frac{1}{2}+\frac{5\sqrt{3}}{18} & \frac{1}{6} & \frac{1}{3}+\frac{\sqrt{3}}{6} & \frac{1}{3}-\frac{\sqrt{3}}{9} & -\frac{\sqrt{3}}{9} & \frac{1}{3} & \frac{1}{6}-\frac{\sqrt{3}}{6}\\
\hline
 & \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{1}{2}
\end{array}
\tag{4.9}
$$

Second, we choose a quadrature formula $(b_i, c_i)_{i=1}^3$ with $(b_1, b_2, b_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$ and $(c_1, c_2, c_3) = (0, \frac{1}{2}, 1)$ to get the following 3-stage SPRK method denoted by the Butcher tableau

$$\begin{array}{c|ccc|ccc|ccc|ccc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{6} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ & 0 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & -\frac{1}{6} \\ \hline & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}. \tag{4.10}$$

Due to Theorem 3.2 and Theorem 4.1, the SPRK methods (4.9) and (4.10) are symplectic and have convergence order 1 for solving (1.1), which will be verified in the next section.

5. Numerical examples

In this section, the symplectic SPRK methods (4.9) and (4.10) retrieved from the symplectic CSSPRK method (4.7) will be applied to solving three stochastic Hamiltonian systems to verify the convergence order results and show the superiority of symplectic SPRK methods. For convenience, we denote the 2-stage symplectic SPRK method (4.9) by SPRK-2 and the 3-stage symplectic SPRK method (4.10) by SPRK-3.

Example 5.1. Consider the Kubo stochastic oscillator

$$\begin{cases} dp(t) = -aq(t)dt - bq(t) \circ dW(t), \ t \in [0, T], \\ dq(t) = ap(t)dt + bp(t) \circ dW(t), \ t \in [0, T]. \end{cases} \tag{5.1}$$

(5.1) is a stochastic Hamiltonian system (1.1) with $H(p, q) = \dfrac{a(p^2 + q^2)}{2}$, $\tilde{H}(p, q) = \dfrac{b(p^2 + q^2)}{2}$. This example is often used to demonstrate the convergence order of a numerical method. Compared to those equations whose exact solutions cannot be expressed explicitly, (5.1) has the following explicit exact solution

$$\begin{aligned} p(t) &= p_0 \cos(at + bW(t)) - q_0 \sin(at + bW(t)), \\ q(t) &= p_0 \sin(at + bW(t)) + q_0 \cos(at + bW(t)), \end{aligned}$$

where $p_0 = p(0)$, $q_0 = q(0)$ are initial values, so that the derived convergence order results are more convincing.

We employ SPRK-2 and SPRK-3 to solve (5.1). Choose the initial values $p_0 = 0.5$, $q_0 = 0$ and the coefficients $a = 1$, $b = 0.5$. Figure 1 demonstrates the convergence rates of SPRK-2 and SPRK-3 for solving (5.1), where we use 1000 independent sample paths, and for each path, SPRK-2 and SPRK-3 are implemented with five different step sizes: $h = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$, respectively. We calculate the sample errors at the terminal $T = 1$ by

$$\Big(\sum_{i=1}^{1000} \sqrt{|p(1, \omega_i) - p_N(\omega_i)|^2 + |q(1, \omega_i) - q_N(\omega_i)|^2}\Big)/1000,$$

and show the results in a log-log plot in Figure 1. By comparing with the reference line with slope 1, we see SPRK-2 and SPRK-3 are of convergence order 1, which coincides with Theorem 4.1.

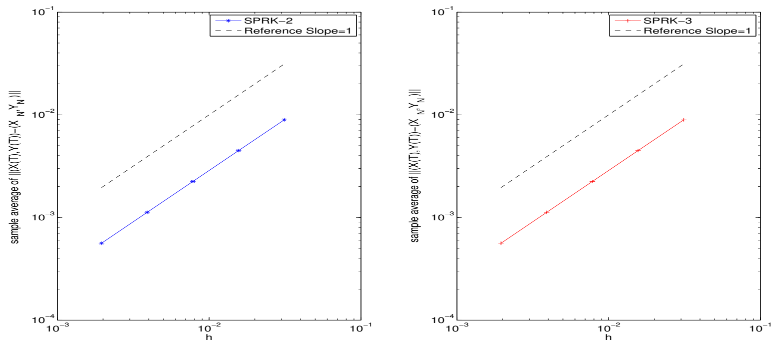

Figure 1. The convergence rates of SPRK-2 and SPRK-3 for solving (5.1). Left: SPRK-2; Right: SPRK-3.

For illustrating the superiority of symplectic SPRK methods, in the next two examples, we introduce a common non-symplectic SPRK method (denoted by NS-SPRK for short) as

$$\begin{array}{c|cccc} & 0 & 1 & 0.5 & 0.5 \\ \hline & 1 & 1 & 1 & 1 \end{array}, \tag{5.2}$$

and make a comparison.

Example 5.2. The stochastic Kepler problem

Consider the stochastic Kepler problem

$$\begin{cases} d\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{q_1}{(q_1^2+q_2^2)^{\frac{3}{2}}} \\ -\dfrac{q_2}{(q_1^2+q_2^2)^{\frac{3}{2}}} \\ p_1 \\ p_2 \end{pmatrix} (dt + \beta \circ dW_t), \\ p_1(0) = p_{10},\ p_2(0) = p_{20},\ q_1(0) = q_{10},\ q_2(0) = q_{20}, \end{cases} \tag{5.3}$$

where β denotes the noise intensity. The system (5.3) possesses a quadratic invariant $I(p_1, p_2, q_1, q_2) = q_1 p_2 - q_2 p_1$ representing the angular momentum.

We choose the initial values $p_{10} = 0,\ p_{20} = \sqrt{\frac{1+e}{1-e}},\ q_{10} = 1 - e,\ q_{20} = 0$ with $e = 0.3$, and the noise intensity $\beta = 0.1$. Figure 2 reports the numerical solutions with respect to (q_1, q_2) of a sample phase trajectory of (5.3) simulated by SPRK-2, SPRK-3 and NS-SPRK on the interval $[0, 200]$ with step size $h = 0.1$, respectively, from which we find the numerical solution of NS-SPRK spirals outward while the numerical solutions of SPRK-2 and SPRK-3 could better simulate the behaviour of the exact solution. Figure 3 exhibits the errors $|I(p_n, q_n) - I(p_0, q_0)|$ of SPRK-2, SPRK-3 and NS-SPRK on the interval $[0, 200]$ with step size $h = 0.1$. It is shown the symplectic SPRK methods SPRK-2 and SPRK-3 can preserve this kind of quadratic invariants very well while NS-SPRK cannot.

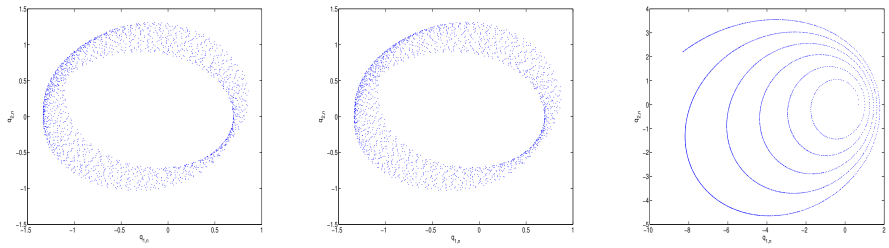

Figure 2. Numerical solutions with respect to (q_1, q_2) of (5.3) computed by the three numerical methods with $h = 0.1$. Left: SPRK-2; Middle: SPRK-3; Right: NS-SPRK.

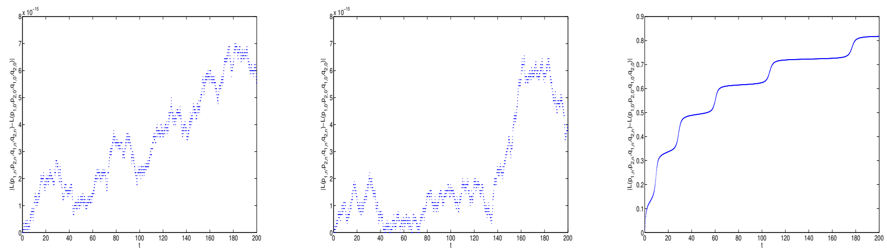

Figure 3. Errors in the angular momentum computed by the three numerical methods for the system (5.3) with $h = 0.1$. Left: SPRK-2; Middle: SPRK-3; Right: NS-SPRK.

Example 5.3. We consider the mathematical pendulum problem [11] with the additive noise, which is given by

$$\begin{aligned} dp(t) &= -\sin q(t)dt + \sigma \circ dW(t), & p(0) &= p_0, \\ dq(t) &= p(t)dt, & q(0) &= q_0. \end{aligned} \tag{5.4}$$

It is easy to check that (5.4) is a stochastic Hamiltonian system (1.1) with $H(p, q) = \frac{1}{2}p^2 - \cos q$, $\tilde{H}(p, q) = -\sigma q$. The Hamiltonian $H(p, q) = \frac{1}{2}p^2 - \cos q$ has a linear growth moment, that is,

$$\mathbb{E}(H(p(t), q(t))) = \mathbb{E}(H(p_0, q_0)) + \frac{1}{2}\sigma^2 t, \tag{5.5}$$

where $(p(t), q(t))$ is the exact solution of (5.4).

We apply SPRK-2, SPRK-3 and NS-SPRK to (5.4) to test the abilities of the three numerical methods in keeping the property (5.5). Choose the step size $h = 0.1$, the initial values $p_0 = 1$, $q_0 = 0$ and the coefficient $\sigma = 0.1$. Figure 4 reports the moments computed by SPRK-2, SPRK-3 and NS-SPRK on the interval $[0, 100]$, where we choose 1000 independent sample paths to simulate $\mathbb{E}(H(p_n, q_n))$. It turns out that SPRK-2 and SPRK-3 show good behaviour in keeping the linear growth of the moment while NS-SPRK doesn't keep the property.

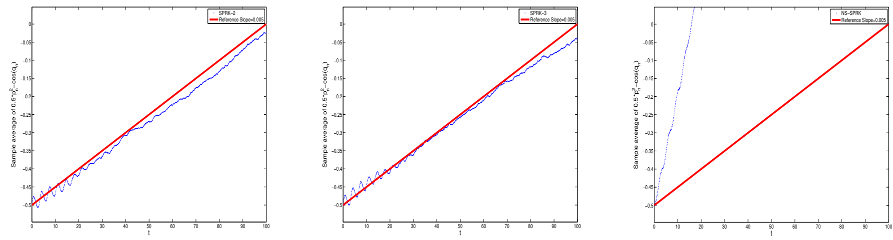

Figure 4. Moments computed by the three numerical methods for the system (5.4) with $h = 0.1$. Left: SPRK-2; Middle: SPRK-3; Right: NS-SPRK.

The three examples above report that symplectic SPRK methods can preserve some qualities of the original stochastic Hamiltonian systems while common non-symplectic methods cannot, so symplectic SPRK methods could better simulate the original stochastic Hamiltonian systems, which also shows the importance of constructing symplectic SPRK methods.

6. Conclusions

In this paper, we present a novel way of constructing symplectic SPRK methods for stochastic Hamiltonian systems. First, we propose CSSPRK methods for general partitioned SDEs. Based on the stochastic B-series theory, we derive the order conditions. Then we apply the CSSPRK methods to stochastic Hamiltonian systems to obtain the symplectic conditions. Moreover, we prove the classical SPRK methods retrieved from the symplectic CSSPRK methods with any quadrature formula are still symplectic, which means various symplectic SPRK methods could be easily constructed by use of different quadrature formulas. As an example, a concrete symplectic CSSPRK method with order 1 and two retrieved classical SPRK methods are obtained. Finally, numerical experiments are given to verify the theoretical results.

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