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MULTIPARAMETRIC SOLUTIONS TO THE GARDNER EQUATION AND THE DEGENERATE RATIONAL CASE

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Abstract We construct solutions to the Gardner equation in terms of trigonometric and hyperbolic functions, depending on several real parameters. Using a passage to the limit when one of these parameters goes to 0, we get, for each positive integer N, rational solutions as a quotient of polynomials in x and t depending on $2N$ parameters. We construct explicit expressions of these rational solutions for orders $N=1$ until $N=3$.

We easily deduce solutions to the mKdV equation in terms of wronskians as well as rational solutions depending on $2N$ real parameters.

Keywords Gardner equation, Wronskians, rational solutions.

MSC(2010) 35-XX, 35Cxx, 35Qxx, 74Jxx, 76-XX.

1. Introduction

We consider the Gardner equation in the following normalization

$$u_t - 6u(1+u)u_x + u_{xxx} = 0, \tag{1.1}$$

where the subscripts x and t denote partial derivatives.

The introduction of this equation is attributed to Gardner [12] in 1968. It was first considered [12] as an auxiliary mathematical tool in the derivation of the infinite set of local conservation laws of the Korteweg de Vries equation. This equation is a fundamental mathematical model for the description of weakly nonlinear dispersive waves. It can describe nonlinear wave effects in several physical contexts: for example, in plasma physics [13,15], fluid flows [5], quantum fluid dynamics [4], in dusty plasmas [8], in ocean and atmosphere [6]. It is fundamental tool to describe large-amplitude internal waves [1, 5, 7]. These waves are such that their vertical amplitudes underwater can exceed 170 meters as described in [2].

Many methods have been used to solve this equation, as for example the Hirota method [16], the series expansion method [3], the mapping method [10] or the method of leading-order analysis [17].

Here, we used the Darboux transformation to construct different type of solutions. We give a representation of solutions in terms of a quotient of a wronskian of order $N+1$ by a wronskian of order N. We get what we will call N-order solutions which depend on $2N$ real parameters in terms of trigonometric or hyperbolic

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functions. Then we construct rational solutions in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as a quotient of polynomials in x and t, depending on $2N$ parameters. We give explicit solutions in the simplest cases $N = 1, 2, 3$.

We easily deduce solutions to the mKdV equation and their corresponding rational solutions depending on $2N$ real parameters.

2. Nth-order solutions to the Gardner equation in terms of wronskians

2.1. First type of solutions

2.1.1. Nth-order solutions in terms of wronskians of sine functions

We consider the Gardner equation

$$u_t - 6u(1+u)u_x + u_{xxx} = 0.$$

In the following, we will use the wronskian of order N of the functions $f_1, \ldots, f_N$ which is the determinant denoted $W(f_1, \ldots, f_N)$, defined by $\det(\partial_x^{i-1} f_j)_{1\le i\le N,\, 1\le j\le N}$, ∂_x^i being the partial derivative of order i with respect to x and $\partial_x^0 f_j$ being the function f_j.

We consider a_j, b_j, arbitrary real numbers $1 \le j \le N$. Then, we have the following statement :

Theorem 2.1. *Let f_j, f be the functions defined by*

$$\begin{aligned} f_j(x,t) &= \sin\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), \qquad \text{for } 1 \le i \le N, \\ f(x,t) &= \exp\left(\frac{1}{2}(x-t)\right) \end{aligned} \tag{2.1}$$

then the function u defined by

$$u(x,t) = \partial_x \ln\left(\frac{W(f_1, \ldots, f_N, f)}{W(f_1, \ldots, f_N)}\right) - \frac{1}{2} \tag{2.2}$$

is a solution to the Gardner equation (1.1) depending on $2N$ real parameters a_j, b_j, $1 \le j \le N$.

Proof. The corresponding Lax pair to the Korteweg de Vries (2.3) (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.3}$$

is given by

$$\begin{cases} -\phi_{xx} + u\phi = \lambda\phi, \\ \phi_t = -4\phi_{xxx} + 6v\phi_x + 3v_x u\phi. \end{cases} \tag{2.4}$$

This system is known to be covariant by the Darboux transformation. If $\phi_1, \ldots, \phi_N, \phi$ are solutions of the system (2.4) respectively associated to $\lambda_1, \ldots, \lambda_N, \lambda$, then $\phi[N]$

defined by $\phi[N] = \dfrac{W(\phi_1, \ldots, \phi_N, \phi)}{W(\phi_1, \ldots, \phi_N)}$ is another solution of this system (2.4) associated to λ where u is replaced by $u[N] = u - 2(\ln W(\phi_1, \ldots, \phi_N)_{xx}$.
If we choose $u = 0$, then the functions $\phi_j = f_j$ defined in (2.1) verify the following system

$$\begin{cases} -\phi_{xx} = \lambda\phi, \\ \phi_t = -4\phi_{xx}. \end{cases} \tag{2.5}$$

Then the solution of the system (2.4) associated to λ can be written as $\varphi(x,t) = \dfrac{W(f_1, \ldots, f_N, f)}{W(f_1, \ldots, _N)}$.

It is well known from [9] that if φ is a solution of the system (2.4), then the function w defined by $w(x,t) = \partial_x \ln(\varphi(x,t))$ is a solution to the equation

$$w_t - 6(\lambda + w^2)w_x + w_{xxx} = 0. \tag{2.6}$$

It is also well known that if w is a solution of the mKdV equation (2.7)

$$w_t - 6w^2 w_x + w_{xxx} = 0. \tag{2.7}$$

then, the function u defined by $u = w - \dfrac{1}{2}$ is a solution to the Gardner equation (1.1). Replacing w by $u + \dfrac{1}{2}$ in (2.6), we get the relation

$$u_t - 6(\lambda + \frac{1}{4} + u + u^2)u_x + u_{xxx} = 0.$$

By the choice of the function f defined in (2.1), the term $\lambda + \dfrac{1}{4}$ in factor of u_x is canceled, and u is a solution of the Gardner equation.

Thus the function u defined by $u(x,t) = \partial_x \ln\left(\dfrac{W(f_1, \ldots, f_N, f)}{W(f_1, \ldots, f_N)}\right) - \dfrac{1}{2}$ is a solution of the Gardner equation, which proves the result. $\square$

Remark 2.1. This result looks like to this given in [14], but with another similar normalization of the Gardner equation

$$u_t + 6u(1-u)u_x + u_{xxx} = 0. \tag{2.8}$$

In this paper [14], the generating functions are hyperbolic and the choice of the general solution $f(x,t)$ gives another types of solutions to the Gardner equation. Only solutions of orders 1 and 2 have been explicitly presented.

2.1.2. Some examples of solutions to the Gardner equation

To shortened the paper, we only give the solutions of order 1, 2 and 3 in the case of generating trigonometric sinus functions.

Solution of order 1

Example 2.1. The function u defined by

$$u(x,t) = 1/2\,\frac{{a_1}^2}{\sin(1/2\,a_1x+1/2\,{a_1}^3t+b_1)(\sin(1/2\,a_1x+1/2\,{a_1}^3t+b_1)-\cos(1/2\,a_1x+1/2\,{a_1}^3t+b_1)a_1)}.$$

is a solution to the Gardner equation (1.1) with a_1, b_1 arbitrarily real parameters.

Solution of order 2

Example 2.2. The function u defined by

$$u(x,t) = \frac{n(x,t)}{d(x,t)}, \tag{2.9}$$

with

$$\begin{aligned}
n(x,t) = & - {a_2}^4 - {a_1}^4 + {a_2}^4(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 \\
& + {a_1}^4(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 - {a_2}^2{a_1}^2(\cos(1/2a_2x + 1/2{a_2}^3t + b_2))^2 \\
& - {a_2}^2{a_1}^2(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 + 2\,{a_2}^2{a_1}^2 \\
& + {a_1}^4\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)a_2\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& - \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1){a_1}^3{a_2}^2\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1) \\
& - {a_1}^2\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2){a_2}^3\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& + \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)a_1{a_2}^4\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)
\end{aligned}$$

and

$$\begin{aligned}
d(x,t) = & - 2\,(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2{a_2}^2 \\
& + 2\,(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2{a_2}^2(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 \\
& - 2\,\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2){a_2}^3 \\
& + 2\,\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& \times {a_2}^3(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 + 4\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1) \\
& \times \cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)a_2\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& \times \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)a_1 + 2\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1) \\
& \times \cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2){a_2}^3\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& \times \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)a_1 + 2\,\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2){a_1}^2 \\
& \times \cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)a_2 - 2\,\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2){a_1}^2 \\
& \times \cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)a_2(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 \\
& - 2\,{a_2}^2{a_1}^2(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 \\
& + 4\,{a_2}^2{a_1}^2(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 \\
& + 2\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)a_1{a_2}^2 \\
& - 2\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)a_1{a_2}^2 \\
& \times (\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 - 2\,(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2{a_1}^2 \\
& + 2\,(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2{a_1}^2(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 \\
& - 2\,{a_2}^2{a_1}^2(\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1))^2 - 2\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1) \\
& \times \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1){a_1}^3 + 2\,\sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1) \\
& \times \cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1){a_1}^3(\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2))^2 \\
& + 2\,\cos(1/2\,a_1x + 1/2\,{a_1}^3t + b_1){a_1}^3\sin(1/2\,a_2x + 1/2\,{a_2}^3t + b_2) \\
& \times \sin(1/2\,a_1x + 1/2\,{a_1}^3t + b_1)\cos(1/2\,a_2x + 1/2\,{a_2}^3t + b_2)a_2
\end{aligned}$$

is a solution to the Gardner equation (1.1) with a_1, a_2, b_1, b_2 arbitrarily real parameters.

Solution of order 3
For this third order, we only present solution in the particular case where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$ to shorten the paper.

Example 2.3. The function u defined by

$$u(x,t) = \frac{n(x,t)}{d(x,t)}, \tag{2.10}$$

with

$$\begin{aligned}
n(x,t) = & -30\,\sin(1/2\,x+1/2\,t)\cos(1/2\,x+1/2\,t)-54\,\sin(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(3/2\,x+\frac{27}{2}\,t)+96\,\sin(x+4\,t)\cos(x+4\,t)+192\,(\cos(x+4\,t))^2\\
&+60\,(\cos(1/2\,x+1/2\,t))^2+108\,\sin(1/2\,x+1/2\,t)\cos(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(1/2\,x+1/2\,t)\sin(3/2\,x+\frac{27}{2}\,t)-252\,\sin(1/2\,x+1/2\,t)\cos(x+4\,t)\\
&\times\cos(1/2\,x+1/2\,t)\sin(x+4\,t)-108\,\cos(x+4\,t)\sin(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(3/2x+\frac{27}{2}t)\sin(x+4t)+84(\cos(1/2x+1/2t))^2(\cos(3/2x+\frac{27}{2}t))^2\\
&-102(\cos(x+4t))^2(\cos(3/2x+\frac{27}{2}t))^2-234(\cos(1/2x+1/2t))^2(\cos(x+4t))^2\\
&-108\sin(x+4t)\cos(x+4\,t)(\cos(1/2\,x+1/2\,t))^2-24\,\sin(1/2\,x+1/2\,t)\\
&\times\cos(1/2\,x+1/2\,t)(\cos(3/2\,x+\frac{27}{2}\,t))^2+54\,\sin(1/2\,x+1/2\,t)\\
&\times\cos(1/2\,x+1/2\,t)(\cos(x+4\,t))^2-18\,(\cos(x+4\,t))^2\sin(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(3/2\,x+\frac{27}{2}\,t)+12\,(\cos(3/2\,x+\frac{27}{2}\,t))^2\sin(x+4\,t)\cos(x+4\,t)\\
&+72\,(\cos(1/2\,x+1/2\,t))^2\sin(3/2\,x+\frac{27}{2}\,t)\cos(3/2\,x+\frac{27}{2}\,t)
\end{aligned}$$

and,

$$\begin{aligned}
d(x,t) = & 128\,(\cos(x+4\,t))^2+81\,(\cos(3/2\,x+\frac{27}{2}\,t))^2+5\,(\cos(1/2\,x+1/2\,t))^2\\
&+276\,\sin(1/2\,x+1/2\,t)\cos(x+4\,t)\sin(3/2\,x+\frac{27}{2}\,t)\cos(1/2\,x+1/2\,t)\\
&\times\cos(3/2\,x+\frac{27}{2}\,t)\sin(x+4\,t)+54\,\sin(1/2\,x+1/2\,t)\cos(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(1/2\,x+1/2\,t)\sin(3/2\,x+\frac{27}{2}\,t)-56\,\sin(1/2\,x+1/2\,t)\cos(x+4\,t)\\
&\times\cos(1/2\,x+1/2\,t)\sin(x+4\,t)-216\,\cos(x+4\,t)\sin(3/2\,x+\frac{27}{2}\,t)\\
&\times\cos(3/2\,x+\frac{27}{2}\,t)\sin(x+4\,t)-54\,\sin(1/2\,x+1/2\,t)(\cos(x+4\,t))^2
\end{aligned}$$

$$
\begin{aligned}
&\times \cos(3/2\,x+\frac{27}{2}\,t)\cos(1/2\,x+1/2\,t)\sin(3/2\,x+\frac{27}{2}\,t)\\
&+56\,\sin(1/2\,x+1/2\,t)\cos(x+4\,t)(\cos(3/2\,x+\frac{27}{2}\,t))^2\cos(1/2\,x+1/2\,t)\\
&\times \sin(x+4\,t)+216\,(\cos(1/2\,x+1/2\,t))^2\cos(3/2\,x+\frac{27}{2}\,t)\sin(x+4\,t)\\
&\times \cos(x+4\,t)\sin(3/2\,x+\frac{27}{2}\,t)-86\,(\cos(1/2\,x+1/2\,t))^2(\cos(3/2\,x+\frac{27}{2}\,t))^2\\
&-209\,(\cos(x+4\,t))^2(\cos(3/2\,x+\frac{27}{2}\,t))^2-84\,\sin(x+4\,t)\cos(x+4\,t)\\
&\times (\cos(1/2\,x+1/2\,t))^2(\cos(3/2\,x+\frac{27}{2}\,t))^2+156\,\sin(3/2\,x+\frac{27}{2}\,t)\\
&\times \cos(3/2\,x+\frac{27}{2}\,t)(\cos(1/2\,x+1/2\,t))^2(\cos(x+4\,t))^2\\
&+204\,\sin(1/2\,x+1/2\,t)\cos(1/2\,x+1/2\,t)(\cos(x+4\,t))^2(\cos(3/2\,x+\frac{27}{2}\,t))^2\\
&+214\,(\cos(x+4\,t))^2(\cos(3/2\,x+\frac{27}{2}\,t))^2(\cos(1/2\,x+1/2\,t))^2\\
&-133\,(\cos(1/2\,x+1/2\,t))^2(\cos(x+4\,t))^2+30\,\sin(x+4\,t)\cos(x+4\,t)\\
&\times (\cos(1/2\,x+1/2\,t))^2-108\,\sin(1/2\,x+1/2\,t)\cos(1/2\,x+1/2\,t)\\
&\times (\cos(3/2\,x+\frac{27}{2}\,t))^2-96\,\sin(1/2\,x+1/2\,t)\cos(1/2\,x+1/2\,t)(\cos(x+4\,t))^2\\
&-96\,(\cos(x+4\,t))^2\sin(3/2\,x+\frac{27}{2}\,t)\cos(3/2\,x+\frac{27}{2}\,t)\\
&+54\,(\cos(3/2\,x+\frac{27}{2}\,t))^2\sin(x+4\,t)\cos(x+4\,t)\\
&-60\,(\cos(1/2\,x+1/2\,t))^2\sin(3/2\,x+\frac{27}{2}\,t)\cos(3/2\,x+\frac{27}{2}\,t)
\end{aligned}
$$

is a solution to the Gardner equation (1.1).

2.2. Other types of solutions

We obtain similar results with other types of generating functions whose proofs are identical.

2.2.1. Solutions with cosine generating functions

Theorem 2.2. *Let g_j, g be the following functions*

$$
\begin{aligned}
&g_j(x,t)=\cos\left(\frac{1}{2}a_jx+\frac{1}{2}a_j^3t+b_j\right), \qquad \text{for } 1\le i\le N,\\
&g(x,t)=\exp\left(\frac{1}{2}(x-t)\right)
\end{aligned}
\tag{2.11}
$$

then the function u defined by

$$
u(x,t)=\partial_x\ln\left(\frac{W(g_1,\ldots,g_N,g)}{W(g_1,\ldots,g_N)}\right)-\frac{1}{2} \tag{2.12}
$$

is a solution to the Gardner equation (1.1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

2.2.2. Solutions with hyperbolic generating functions

Theorem 2.3. *Let h_j, h be the following functions*

$$\begin{aligned} h_j(x,t) &= \sinh\left(\frac{1}{2}a_j x - \frac{1}{2}a_j^3 t + b_j\right), \qquad \text{for } 1 \leq i \leq N, \\ h(x,t) &= \exp\left(\frac{1}{2}(x-t)\right) \end{aligned} \tag{2.13}$$

then the function u defined by

$$u(x,t) = \partial_x \ln\left(\frac{W(h_1,\dots,h_N,h)}{W(h_1,\dots,h_N)}\right) - \frac{1}{2} \tag{2.14}$$

is a solution to the Gardner equation (1.1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

Theorem 2.4. *Let k_j, k be the following functions*

$$\begin{aligned} k_j(x,t) &= \cosh\left(\frac{1}{2}a_j x - \frac{1}{2}a_j^3 t + b_j\right), \qquad \text{for } 1 \leq i \leq N, \\ k(x,t) &= \exp\left(\frac{1}{2}(x-t)\right) \end{aligned} \tag{2.15}$$

then the function u defined by

$$u(x,t) = \partial_x \ln\left(\frac{W(k_1,\dots,k_N,k)}{W(k_1,\dots,k_N)}\right) - \frac{1}{2} \tag{2.16}$$

is a solution to the Gardner equation (1.1) with a_j, b_j $1 \leq j \leq N$ arbitrarily real parameters.

3. Rational solutions to the Gardner equation

3.1. Rational solutions to the Gardner equation as a limit

To obtain rational solutions to the Gardner equation (1.1), we are going to perform a limit when a parameter e tends to 0. For this we replace all parameters a_j and b_j, $1 \leq j \leq N$ by $\hat{a}_j = \sum_{k=1}^{N} a_k (je)^{2k-1}$ and $\hat{b}_j = \sum_{k=1}^{N} b_k (je)^{2k-1}$ with e an arbitrary real parameter.
We get the following result :

Theorem 3.1. *Let ψ_j, ψ be the functions*

$$\psi_j(x,t,e) = \sin\left(\frac{1}{2}\sum_{k=1}^{N} a_k (je)^{2k-1} x + \frac{1}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^3 t + \sum_{k=1}^{N} b_k (je)^{2k-1}\right),$$

for $1 \leq j \leq N$,

$$\psi(x,t) = \exp\left(\frac{1}{2}(x-t)\right)$$

then the function u *defined by*

$$u(x,t) = \lim_{e\to 0} \partial_x \ln\left(\frac{W(\psi_1,\ldots,\psi_N,\psi)}{W(\psi_1,\ldots,\psi_N)}\right) - \frac{1}{2} \tag{3.1}$$

is a rational solution to the Gardner equation (1.1).

Proof. It is a direct consequence of the result of the previous section. □

We have similar results with generating cosine or hyperbolic functions.

3.2. Degenerate rational solutions to the Gardner equation

We can give the expression of the rational solutions of the Gardner equation avoiding the presence of a limit. For this we consider another type of functions.
We get the following result :

Theorem 3.2. *Let* ψ, φ_j, φ *be the functions*

$$\psi(x,t,e) = \sin\left(\frac{1}{2}\left(\sum_{k=1}^{N} a_k e^{2k-1}\right)x + \frac{1}{2}\left(\sum_{k=1}^{N} a_k e^{2k-1}\right)^3 t + \sum_{k=1}^{N} b_k e^{2k-1}\right),$$

$$\varphi_j(x,t) = \frac{\partial^{2j-1}\psi(x,t,0)}{\partial_{2j-1}e},\ \textit{for}\ 1 \leq j \leq N,$$

$$\varphi(x,t) = \exp\left(\frac{1}{2}(x-t)\right)$$

then the function v *defined by*

$$v(x,t) = \partial_x \ln\left(\frac{W(\varphi_1,\ldots,\varphi_N,\varphi)}{W(\varphi_1,\ldots,\varphi_N)}\right) - \frac{1}{2} \tag{3.2}$$

is a rational solution to the Gardner equation (1.1) depending on $2N$ *parameters* a_j, b_j, $1 \leq j \leq N$.

Proof. It is sufficient to combine the columns of the determinant of the previous theorem and to take a passage to the limit when e tends to 0 for each column. □

So we obtain an infinite hierarchy of rational solutions to the Gardner equation depending on the integer N.

In the following we give some examples of rational solutions.

These results are consequences of the previous result.

But, it is also possible to prove it directly in replacing the expressions of each of the solutions given in the corresponding equation and check that the relation is verified.

3.3. First order rational solutions

We have the following result at order $N = 1$:

Example 3.1. The function v defined by

$$v(x,t)=\frac{2{a_1}^2}{(a_1x+2\,b_1)\,(-2\,a_1+2\,b_1+a_1x)}, \tag{3.3}$$

is a rational solution to the Gardner equation (1.1) with a_1, b_1, arbitrarily real parameters.

3.4. Second order rational solutions

Example 3.2. The function v defined by

$$v(x,t)=\frac{n(x,t)}{d(x,t)}, \tag{3.4}$$

with

$$\begin{aligned}n(x,t)=&-6\,a_1a_2(-{a_1}^5a_2-{a_2}^5a_1+2{a_2}^3{a_1}^3)x^4-6a_1a_2(4{a_2}^5a_1+4{a_1}^5a_2-8\,{a_2}^3{a_1}^3)x^3\\&-6\,a_1a_2(-48\,{a_2}^3t{a_1}^3-48\,b_2{a_1}^3+48\,b_2a_1{a_2}^2+24\,{a_1}^5ta_2+48\,b_1a_2{a_1}^2\\&-48\,b_1{a_2}^3+24\,{a_2}^5ta_1)x-6\,a_1a_2(-48\,b_2a_1{a_2}^2-48\,b_1a_2{a_1}^2\\&+48\,{a_2}^3t{a_1}^3+48\,b_2{a_1}^3-24\,{a_1}^5ta_2+48\,b_1{a_2}^3-24\,{a_2}^5ta_1),\end{aligned}$$

and,

$$\begin{aligned}d(x,t)=&({a_1}^3a_2-a_1{a_2}^3)^2x^6+({a_1}^3a_2-a_1{a_2}^3)(-6\,{a_1}^3a_2+6\,a_1{a_2}^3)x^5\\&+({a_1}^3a_2-a_1{a_2}^3)(12\,{a_1}^3a_2-12\,a_1{a_2}^3)x^4+2(-12\,{a_2}^3ta_1-24\,b_2a_1\\&+12\,{a_1}^3ta_2+24b_1a_2)({a_1}^3a_2-a_1{a_2}^3)x^3+(-12\,{a_2}^3ta_1-24\,b_2a_1\\&+12\,{a_1}^3ta_2+24\,b_1a_2)(-6\,{a_1}^3a_2+6\,a_1{a_2}^3)x^2+(-12\,{a_2}^3ta_1-24\,b_2a_1\\&+12\,{a_1}^3ta_2+24\,b_1a_2)(12\,{a_1}^3a_2-12\,a_1{a_2}^3)x\\&+(-12\,{a_2}^3ta_1-24\,b_2a_1+12\,{a_1}^3ta_2+24\,b_1a_2)^2\end{aligned}$$

is a rational solution to the Gardner equation (1.1) dependant on 4 real parameters a_1, a_2, b_1, b_2.

3.5. Rational solutions of order three

The explicit solution depending on 6 real parameters being too long, we give only the rational solution without parameters. We get the following rational solutions given by :

Example 3.3. The function v defined by

$$v(x,t)=\frac{n(x,t)}{d(x,t)}, \tag{3.5}$$

with

$$\begin{aligned}n(x,t)=&12\,x^{10}-120\,x^9+360\,x^8+8640\,tx^5+64800\,t^2x^4-259200\,t^2x^3\\&+518400\,t^2x^2+518400\,t^3x-518400\,t^3\end{aligned}$$

and,

$$\begin{aligned}d(x,t) =&x^{12}-12\,x^{11}+60\,x^{10}+(120\,t-120)x^9-1080\,tx^8+4320\,tx^7\\&+(-1440t^2-60t(120-60t)-1440t)x^6-12960t^2x^5+(720t^2(120-60t)\\&-60t(1440t+720t^2))x^3+259200t^3x^2-518400t^3x+720t^2(1440t+720t^2)\end{aligned}$$

is a rational solution to the Gardner equation (1.1).

4. Case of the mKdV equation

4.1. Solutions in terms of wronskians

As a consequence of the previous study, we easily deduce solutions to the modified Korteweg-de Vries (mKdV) equation in the following normalization (2.7)

$$w_t - 6w^2 w_x + w_{xxx} = 0.$$

We have the following result :

Theorem 4.1. *Let f_j,f be the following functions*

$$\begin{aligned}f_j(x,t) &= \sin\left(\frac{1}{2}a_j x + \frac{1}{2}a_j^3 t + b_j\right), \qquad \text{for } 1 \leq i \leq N,\\ f(x,t) &= \exp\left(\frac{1}{2}(x-t)\right)\end{aligned} \tag{4.1}$$

then the function w defined by

$$w(x,t) = \partial_x \ln\left(\frac{W(f_1,\dots,f_N,f)}{W(f_1,\dots,f_N)}\right) \tag{4.2}$$

is a solution to the mKdV equation (2.7) depending on $2N$ real parameters a_j, b_j, $1 \leq j \leq N$.

4.2. Rational solutions

We get the following statement :

Theorem 4.2. *Let ψ_j, ψ be the functions*

$$\psi_j(x,t,e) = \sin\left(\frac{1}{2}\sum_{k=1}^{N} a_k (je)^{2k-1} x + \frac{1}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^3 t + \sum_{k=1}^{N} b_k (je)^{2k-1}\right),$$

for $1 \leq j \leq N$,

$$\psi(x,t) = \exp\left(\frac{1}{2}(x-t)\right)$$

then the function u defined by

$$u(x,t) = \lim_{e\to 0} \partial_x \ln\left(\frac{W(\psi_1,\dots,\psi_N,\psi)}{W(\psi_1,\dots,\psi_N)}\right) \tag{4.3}$$

is a rational solution to the mKdV equation (2.7).

We can also give the expression of the rational solutions of the mKdV equation without the presence of a limit. We get the following result : We get the following result :

Theorem 4.3. *Let ψ, φ_j, φ be the functions*

$$\psi(x,t,e)=\sin\left(\frac{1}{2}\left(\sum_{k=1}^{N}a_k e^{2k-1}\right)x+\frac{1}{2}\left(\sum_{k=1}^{N}a_k e^{2k-1}\right)^3 t+\sum_{k=1}^{N}b_k e^{2k-1}\right),$$

$$\varphi_j(x,t)=\frac{\partial^{2j-1}\psi(x,t,0)}{\partial_{2j-1}e},\ \textit{for } 1\leq j\leq N,$$

$$\varphi(x,t)=\exp\left(\frac{1}{2}(x-t)\right)$$

then the function v defined by

$$v(x,t)=\partial_x \ln\left(\frac{W(\varphi_1,\ldots,\varphi_N,\varphi)}{W(\varphi_1,\ldots,\varphi_N)}\right) \tag{4.4}$$

is a rational solution to the mKdV equation (2.7) depending on $2N$ parameters a_j, b_j, $1\leq j\leq N$.

5. Conclusion

We have given two types of representations of solutions to the Gardner equation. First, solutions as a quotient of a wronskian of order $N+1$ by a wronskian of order N depending on $2N$ real parameters have been constructed. Then performing a passage to the limit when one parameter goes to 0 we get rational solutions to the Gardner equation depending on $2N$ real parameters.

So we obtain an infinite hierarchy of multiparametric families of rational solutions to the Gardner equation as a quotient of a polynomials in x and t depending on $2N$ real parameters.

As a byproduct, we easily deduce solutions to the mKdV equation in terms of wronkians and rational solutions, depending on $2N$ real parameters.

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