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TRAVELING WAVE SOLUTIONS OF TWO TYPES OF GENERALIZED BREAKING SOLITON EQUATIONS*

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Abstract In this paper, the bifurcation theory of dynamical system is applied to study traveling waves of two types of generalized breaking soliton (GBS) equations which include many famous partial differential equation models. Without any parameter constraints, we investigate their traveling wave systems in detail from the geometric point of view. Due to the existence of a two dimensional invariant manifold, all bounded and unbounded orbits are identified and studied in different parameter bifurcation sets. Furthermore, by calculating complicated elliptic integrals along these orbits, we obtain exact expressions of all possible single wave solutions of two types of GBS equations.

Keywords GBS equations, traveling waves, bifurcation, dynamical system.

MSC(2010) 58F15, 58F17, 53C35.

1. Introduction

In this paper, we consider a (2+1)-dimensional generalized breaking soliton equation which was proposed in the form [35]

$$\begin{cases} u_t + au_{xxx} + bu_{xxy} + cuu_x + duv_x + ru_xv = 0, \\ u_y = v_x, \end{cases} \tag{1.1}$$

where the subscripts denote partial derivatives, $u(x, y, t)$ and $v(x, y, t)$ are two physical potentials, x, y and t are three independent variables, a, b, c, d, r are real parameters. Equation (1.1) can be turned into some famous PDE models, for example the KdV equation ($c = 6a$, $d = 4b$, $r = 2b$) and the Bogoyavlensky-Konoplechenko (BK) equation ($c = 6a$, $d = r = 4b$) [14, 18, 19, 29, 41].

In 2015, Xu applied the singularity analysis to equation (1.1) to derive its integrable conditions. She pointed out that equation (1.1) passed the Painlevé test

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under the condition that $c = \dfrac{3ar}{b}$, $d = 2r$, i.e. the equation

$$\begin{cases} u_t + au_{xxx} + bu_{xxy} + \dfrac{3ar}{b}uu_x + 2ruv_x + ru_x v = 0, \\ u_y = v_x, \end{cases} \tag{1.2}$$

is Painlevé integrable. With the binary Bell polynomials, she gave some integrable properties of equation (1.2), such as bilinear form, N-soliton solution [17], bilinear Bäcklund transformation, Lax pair and infinite conservation laws [35]. The completely integrable equation (1.2) attracted people's great attention. In 2016, Wazwaz obtained its many types of solutions including multiple soliton solutions, bell solitary wave solution and some periodic solutions by using the simplified Hirota method and some solitary wave ansatze methods, such as the tanh method and the tan method [34]. In the same year, based on the bilinear Bäcklund transformation and the multidimensional Riemann theta function, Zhao and Han derived the one and two periodic wave solutions of equation (1.2) and discussed the dynamical behaviors of the quasiperiodic wave solutions [42]. In 2017, according to the truncated Painlevé expansion and the consistent Riccati expansion method, the nonlocal symmetry of equation (1.2) was derived and the interaction solutions between solitons and cnoidal waves were discussed [5]. Recently, people begin to focus on another special form of equation (1.1) ($c = 6a, d = r = 3b$). In 2018, by deriving a bilinear form of it and using the extended homoclinic test theory, Yan *et al.* constructed its soliton solutions, homoclinic breather waves and rogue waves solutions [38]. Later, through appropriate choice of the functions in the bilinear forms, two types of mixed soliton and rogue wave solutions of it were constructed. Meanwhile, the rogue wave structures and the interaction characteristics between the soliton and rogue wave were studied [15, 20].

With the transformatiom $u = F_x$ and $v = F_y$, equation (1.1) can be turned into

$$F_{xt} + aF_{xxxx} + bF_{xxxy} + cF_xF_{xx} + dF_xF_{xy} + rF_{xx}F_y = 0, \tag{1.3}$$

which can be regarded as a more generalized model of breaking soliton equation and includes many well-known partial differential equations. When $a = c = 0$, $b = 1$, $d = -4$ and $r = -2$, it degenerates to the (2+1)-dimensional breaking soliton equation

$$F_{xt} + F_{xxxy} - 2F_{xx}F_y - 4F_xF_{xy} = 0, \tag{1.4}$$

which was first established by Calogero and Degasperis [9, 10] in 1977. If $a = c = 0$, $b = 1$, $d = 4$ and $r = 4$, then equation (1.3) becomes another breaking soliton equation

$$F_{xt} + F_{xxxy} + 4F_{xx}F_y + 4F_xF_{xy} = 0, \tag{1.5}$$

which has been studied by Bogoyavlenskii [10]. Equations (1.4) and (1.5) can be used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis. Meanwhile, equations (1.4) and (1.5) are well known as classic breaking soliton equations since the phenomenon of overturning (whiplash) of the wave front occurs which leads to the solution becoming multivalued [4, 23]. The Painlevé property, dromion-like structures and various exact solutions of them have been studied by people widely [1, 7, 8, 12, 13, 21, 25, 26, 28, 30, 32, 36, 37, 39]. When $a = c = 0$, $b = 1$, $d = 4$ and $r = 2$, equation (1.3) degenerates to the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$F_{xt} + F_{xxxy} + 2F_{xx}F_y + 4F_xF_{xy} = 0. \tag{1.6}$$

Its bilinear form, lump solutions and some exact solutions have been given [3, 6, 11, 24, 27, 31, 33]. Noting that equations (1.4), (1.5) and (1.6) can be rewritten in a unified form

$$F_{xt} + F_{xxxy} + k_1F_{xx}F_y + k_2F_xF_{xy} = 0,$$

where k_1, k_2 are real constants, by using the extended homoclinic test approach, sine-cosine method and modified simple equation method [16], people constructed some exact solutions of it, such as soliton solutions and breather type solutions [2, 11, 22].

Though there have been so many profound results about traveling waves of equations (1.1) and (1.3) which contribute to understanding of nonlinear physical phenomena and wave propagation, there exist some problems unsolved. Firstly, we note that above conclusions about equations (1.1) and (1.3) are all given under some special parameter conditions. It means that wave phenomena described by them without parameter constraints need study further. In addition, although some direct methods mentioned above can be applied to obtain traveling wave solutions of equations (1.1) and (1.3) concisely and efficiently, some traveling wave solutions could be still lost, especially for the unbounded traveling wave solutions. These problems arouse our great interest in surveying the traveling waves of equations (1.1) and (1.3) again. In this paper, without any parameter constraints, we try to give explicit expressions of all possible single wave solutions of them. From the geometric point of view, we investigate the phase space structure of traveling wave systems of equations (1.1) and (1.3) by using method of dynamical system. Due to the existence of a two dimensional integrable invariant manifold, we achieve the goal by calculating some complicated elliptic integrals. Our conclusions will further promote the study of analytic solutions and numerical solutions of equations (1.1) and (1.3) as well as understanding the complicated nonlinear wave phenomena.

2. Traveling wave system and bifurcation analysis

Firstly, we begin with equation (1.3). With the traveling wave transformation

$$F = F(x, y, t) = f(\xi) = f(x + my - nt),$$

we obtain the traveling wave system of equation (1.3)

$$(a + mb)f^{''''} + (c + md + mr)f^{''}f^{'} - nf^{''} = 0, \tag{2.1}$$

where $'$ stands for $d/d\xi$, $m \neq 0$ represents the wave numbers in the y direction and $n \neq 0$ is the wave speed. Integrating (2.1) once, we get

$$(a + mb)f''' + \frac{1}{2}(c + md + mr)f'^2 - nf' = e,$$

which has the equivalent system

$$\begin{cases} f^{'} = p, \\ p^{'} = q, \\ q^{'} = -\dfrac{c+md+mr}{2(a+mb)}p^2 + \dfrac{n}{a+mb}p + \dfrac{e}{a+mb}, \end{cases} \tag{2.2}$$

where e is an integral constant. Interestingly, system (2.2) has a 2-dimensional invariant manifold in $\mathbb{R}^3$. Flows on it are governed by the last two equations in system (2.2), i.e.

$$\begin{cases} p^{'} = q, \\ q^{'} = \dfrac{-(c+md+mr)p^2 + 2np + 2e}{2(a+mb)}, \end{cases} \tag{2.3}$$

which is exactly a Hamiltonian system with the energy function

$$H(p,q) = \frac{1}{2}q^2 + \frac{A}{6(a+mb)}p^3 - \frac{n}{2(a+mb)}p^2 - \frac{e}{a+mb}p, \tag{2.4}$$

where $A = c + m(d+r)$.

Theorem 2.1. *The equilibria of system (2.3) have the following properties:*

- *If $n^2 + 2Ae > 0$, system (2.3) has two equilibria $B_1(\frac{n+\sqrt{n^2+2eA}}{A}, 0)$ and $B_2(\frac{n-\sqrt{n^2+2eA}}{A}, 0)$. Further, B_1 is a center and B_2 is a saddle when $a+bm > 0$, whereas B_1 is a saddle and B_2 is a center when $a+bm < 0$.*
- *If $n^2 + 2Ae = 0$, system (2.3) has a unique cusp $B_3(\frac{n}{A}, 0)$.*
- *If $n^2 + 2Ae < 0$, system (2.3) has no equilibrium.*

Proof. For the case $n^2+2Ae \neq 0$, it is not difficult for one to check corresponding conclusions above by a direct computation and the qualitative theory of differential equations [40]. Here we omit it for simplicity.

For the case $n^2+2Ae = 0$, system (2.3) has a unique equilibrium $B_3(\frac{n}{A}, 0)$ with a nilpotent matrix

$$M(B_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It means that B_3 is a degenerated equilibrium. In order to judge its type further, we make a homeomorphic transformation

$$\varphi = p - \frac{n}{A},\ \psi = q,$$

which converts system (2.3) into the normal form below

$$\begin{cases} \varphi^{'} = \psi, \\ \psi^{'} = a_k\varphi^k[1+p(\varphi)] + b_n\varphi^k\psi[1+q(\varphi)] + \psi^2 g(\varphi,\psi) = \dfrac{-A\varphi^2}{2(a+mb)}, \end{cases}$$

where $k = 2$, $a_k = -\dfrac{A}{2(a+bm)}$, $b_n = 0$, $p(\varphi) = 0$, $q(\varphi) = 0$ and $g(\varphi,\psi) = 0$. According to the qualitative theory of differential equation [40, Theorem 7.3, Chapter 2], B_3 is a cusp as shown in figure 1. □

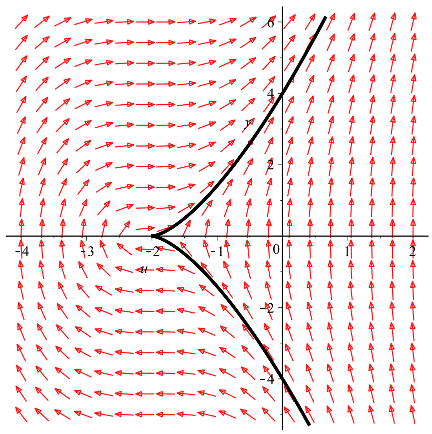


Figure 1. Simulation of cusp B_3 for $a = c = 0, b = 1, d = -4, r = -2, m = \frac{1}{6}, n = e = 2$.

Next, it needs to discuss global phase portraits of system (2.3) in different parameter bifurcation sets $\{(a,b,c,d,r,m,n,e)|n^2+2Ae>0\}$, $\{(a,b,c,d,r,m,n,e)|n^2+2Ae=0\}$ and $\{(a,b,c,d,r,m,n,e)|n^2+2Ae<0\}$. According to the properties of Hamiltonian system [40] and energy function (2.4), we have the following results.

Case 1. For $n^2+2Ae>0$ and $a+bm>0$, there is a homoclinic orbit γ connecting the saddle B_2. Inside the homoclinic loop $\gamma \cup B_2$ there exists a family of periodic orbits

$$\Gamma(h) = \{H(p,q) = h, h \in (h(B_1), h(B_2))\},$$

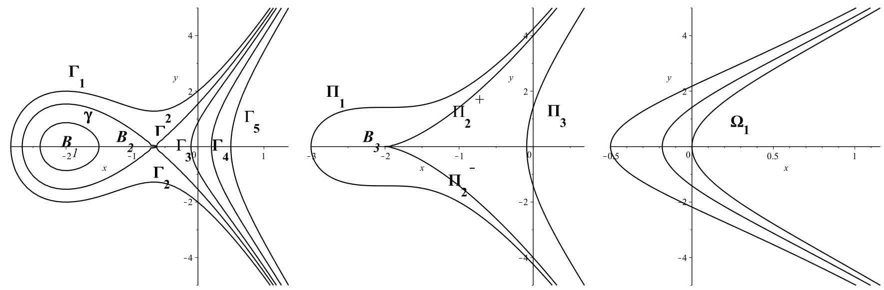
which surround center B_1, where

$$h(B_1) = \frac{-2n^3 - (2n^2+4eA)\sqrt{n^2+2eA} - 6neA}{6A^2(a+mb)},$$
$$h(B_2) = \frac{-2n^3 + (2n^2+4eA)\sqrt{n^2+2eA} - 6neA}{6A^2(a+mb)}.$$

Moreover, $\Gamma(h)$ tends to B_1 as h$\to h(B_1)$ and tends to γ as h$\to h(B_2)$. Outside of the homoclinic loop $\gamma \cup B_2$, all orbits are unbounded, as shown in figure 2(a).

Case 2. For $n^2+2Ae=0$, all orbits of system (2.3) are unbounded. There are two special orbits Π_2^+ and Π_2^- which are different from others. The α-limit set of Π_2^+ and the ω-limit set of Π_2^- correspond to the unique cusp B_3, as shown in figure 2(b).

Case 3. For $n^2+2Ae<0$, system (2.3) has only one type of orbits. All of them are unbounded, as shown in figure 2(c).



(a) $n^2 + 2Ae > 0$ (b) $n^2 + 2Ae = 0$ (c) $n^2 + 2Ae < 0$

Figure 2. The phase portraits of system (2.3) in different parameter bifurcation sets.

Remark 2.1. Noting that the phase portrait of system (2.3) for the case $n^2+2Ae > 0$ and $a + bm > 0$ is topologically equivalent to that for the case $n^2 + 2Ae > 0$ and $a + bm < 0$, here we only give the conclusions about the former for simplicity.

3. Exact solutions of system (2.3)

In this section, we try to seek explicit expressions of all solutions of system (2.3), including bounded and unbounded ones.

3.1. Homoclinic solution and periodic solutions of system (2.3)

From the results in section 2, when $n^2 + 2Ae > 0$, system (2.3) has two types of bounded orbits, namely the homoclinic orbit and the periodic orbits. Firstly, we begin with the periodic orbits.

(B1) For $n^2 + 2Ae > 0$, $a + bm > 0$ and $A < 0$, according to the energy function (2.4), any one of the periodic orbits $\Gamma(h)$ can be expressed by

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_1)(p_2-p)(p_3-p)},$$

where p_1, p_2 and p_3 are reals and satisfy the constraint condition $\frac{n+2\sqrt{n^2+2eA}}{A} < p_1 < p < p_2 < \frac{n-\sqrt{n^2+2eA}}{A} < p_3$. Assuming that its period is 2T and choosing initial value $p(0) = p_1$, we have

$$\int_{p_1}^{p} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_1)(p_2-p)(p_3-p)}} = \int_0^{\xi} d\xi, \qquad 0 < \xi < T,$$

$$-\int_{p}^{p_1} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_1)(p_2-p)(p_3-p)}} = \int_{\xi}^{0} d\xi, \quad -T < \xi < 0,$$

which can be rewritten as

$$\int_{p_1}^{p} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_1)(p_2-p)(p_3-p)}} =| \xi |, \quad -T<\xi<T.$$

By calculating the elliptic integral

$$\int_{p_1}^{p} \frac{dp}{\sqrt{(p-p_1)(p_2-p)(p_3-p)}} = g\cdot \mathrm{sn}^{-1}\left(\sqrt{\frac{p-p_1}{p_2-p_1}}, k\right),$$

where $g=\dfrac{2}{\sqrt{p_3-p_1}}$, $k^2=\dfrac{p_2-p_1}{p_3-p_1}$, we get the expression of periodic solution of system (2.3)

$$p_{b11}(\xi)=p_1+(p_2-p_1)\mathrm{sn}^2\left(\sqrt{\frac{-A(p_3-p_1)}{12(a+mb)}}\xi, \sqrt{\frac{p_2-p_1}{p_3-p_1}}\right), \quad -T<\xi<T.$$

If $n^2+2Ae>0$, $a+bm<0$ and $A>0$, one can check that periodic solution of system (2.3) is the same form as $p_{b11}(\xi)$.

For $n^2+2Ae>0$, $a+bm>0$ and $A>0$, by choosing the initial value $p(0)=p_2^{'}$ and adopting similar calculation, we get the second type of expression of periodic solution of system (2.3)

$$p_{b12}(\xi)=p_1^{'}+\frac{(p_2^{'}-p_1^{'})(p_3^{'}-p_1^{'})}{(p_3^{'}-p_1^{'})-(p_3^{'}-p_2^{'})\mathrm{sn}^2\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi, \sqrt{\frac{p_3^{'}-p_2^{'}}{p_3^{'}-p_1^{'}}}\right)}, \quad -T<\xi<T,$$

where $p_1^{'}$, $p_2^{'}$ and $p_3^{'}$ are reals and satisfy the constraint condition $p_1^{'}<\frac{n-\sqrt{n^2+2eA}}{A}<p_2^{'}<p<p_3^{'}<\frac{n+2\sqrt{n^2+2eA}}{A}$. If $n^2+2Ae>0$, $a+bm<0$ and $A<0$, one can check that the periodic solution of system (2.3) is the same form as $p_{b12}(\xi)$.

(B2) For $n^2+2Ae>0$, $a+bm>0$ and $A<0$, the homoclinic orbit γ can be expressed by

$$q=\pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p_5-p)^2(p-p_4)},$$

where $p_4=\dfrac{n+2\sqrt{n^2+2eA}}{A}$ and $p_5=\dfrac{n-\sqrt{n^2+2eA}}{A}$ satisfy the condition that $p_4<p<p_5$. Choosing $p(0)=p_4$, we have

$$\int_{p_4}^{p} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p_5-p)\sqrt{p-p_4}} = \int_0^{\xi} d\xi, \qquad \xi>0,$$

$$-\int_{p}^{p_4} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p_5-p)\sqrt{p-p_4}} = \int_{\xi}^{0} d\xi, \quad \xi<0,$$

which can be rewritten as

$$\int_{p_4}^{p} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p_5-p)\sqrt{p-p_4}} =\mid \xi \mid, \quad -\infty < \xi < +\infty.$$

Noting that

$$\int_{p_4}^{p} \frac{dp}{(p_5-p)\sqrt{p-p_4}} = -\frac{1}{\sqrt{p_5-p_4}} \ln \frac{\sqrt{p_5-p_4}-\sqrt{p-p_4}}{\sqrt{p_5-p_4}+\sqrt{p-p_4}},$$

we get the bell-shaped bounded solution of the system (2.3)

$$p_{b21}(\xi) = p_4 + \frac{(p_5-p_4)\left(1-\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}|\xi|\right)\right)^2}{\left(1+\exp\left(\sqrt{\frac{-A(p_5-p_4)}{3(a+mb)}}|\xi|\right)\right)^2}, \quad -\infty < \xi < +\infty,$$

which can be further simplified to

$$p_{b21}(\xi) = p_4 + \frac{(p_5-p_4)\left(1-\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2}{\left(1+\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2}, \quad -\infty < \xi < +\infty. \tag{3.1}$$

If $n^2+2Ae>0$, $a+bm<0$ and $A>0$, one can check that the bell-shaped bounded solution of the system (2.3) is the same form as $p_{b21}(\xi)$.

For $n^2+2Ae>0$, $a+bm>0$ and $A>0$, similarly, by choosing the initial value $p(0)=p_5^{'}$, we obtain another type of bell-shaped bounded solution of the system (2.3)

$$p_{b22}(\xi) = p_5^{'} - \frac{(p_5^{'}-p_4^{'})\left(1-\exp\left(\sqrt{\frac{A(p_5^{'}-p_4^{'})}{3(a+mb)}}\xi\right)\right)^2}{\left(1+\exp\left(\sqrt{\frac{A(p_5^{'}-p_4^{'})}{3(a+mb)}}\xi\right)\right)^2}, \quad -\infty < \xi < +\infty, \tag{3.2}$$

where $p_4^{'} = \dfrac{n-\sqrt{n^2+2eA}}{A}$ and $p_5^{'} = \dfrac{n+2\sqrt{n^2+2eA}}{A}$ satisfy the condition that $p_4^{'} < p < p_5^{'}$. If $n^2+2Ae>0$, $a+bm<0$ and $A<0$, one can check that the bell-shaped bounded solution of the system (2.3) is the same form as $p_{b22}(\xi)$.

3.2. Unbounded solutions of system (2.3)

Next, we try to derive exact expressions of all unbounded solutions of system (2.3). According to the parameter bifurcation sets in section 2, we need to discuss them in three cases.

Remark 3.1. In fact, similar to bounded solutions of system (2.3), the expressions of unbounded solutions of system (2.3) for the case $a+bm<0$ and $A>0$ ($a+bm<0$ and $A<0$) are same as that for the case $a+bm>0$ and $A<0$ ($a+bm>0$ and $A>0$). So, we only give the expressions of unbounded solutions of system (2.3) below for cases $a+bm>0$, $A<0$ and $a+bm>0$, $A>0$ for simplicity.

(I) First of all, we start with the case that $n^2+2Ae>0$. This case incudes five subcases (U1-U5) according to different level curves of energy function (2.4).

(U1) For $n^2+2Ae>0$, $a+bm>0$ and $A<0$, we consider the first type of unbounded orbits Γ^2 and Γ_2 as shown in figure 2(a), where their energy is equal to energy of saddle B_2, as well as energy of homoclinic orbit γ. They can be respectively expressed by

$$q=\pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_5)^2(p-p_4)},$$

where $p_4<p_5<p<+\infty$. Given initial value $p(0)=+\infty$, we have

$$\int_p^{+\infty}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p-p_5)\sqrt{p-p_4}}=\int_\xi^0 d\xi,\quad \xi<0,$$

$$-\int_{+\infty}^p\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p-p_5)\sqrt{p-p_4}}=\int_0^\xi d\xi,\quad \xi>0,$$

which can be rewritten as

$$\int_p^{+\infty}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p_5-p)\sqrt{p-p_4}}=\mid\xi\mid,\quad \xi\neq 0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{(p-p_5)\sqrt{p-p_4}}=-\frac{1}{\sqrt{p_5-p_4}}\ln\frac{\sqrt{p-p_4}-\sqrt{p_5-p_4}}{\sqrt{p-p_4}+\sqrt{p_5-p_4}},$$

we get the expression of the first type of unbounded solution of system (2.3)

$$p_{u11}(\xi)=p_4+\frac{(p_5-p_4)\left(1+\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\mid\xi\mid\right)\right)^2}{\left(1-\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\mid\xi\mid\right)\right)^2},\quad \xi\neq 0,$$

which can be further simplified to

$$p_{u11}(\xi)=p_4+\frac{(p_5-p_4)\left(1+\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2}{\left(1-\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2},\quad \xi\neq 0. \tag{3.3}$$

For $n^2+2Ae>0$, $a+bm>0$ and $A>0$, by choosing the initial value $p(0)=-\infty$, we can get the first type of unbounded solution of system (2.3) as follows

$$p_{u12}(\xi)=p_5^{'}-\frac{(p_5^{'}-p_4^{'})\left(1+\exp\left(\sqrt{\frac{A(p_5^{'}-p_4^{'})}{3(a+mb)}}\xi\right)\right)^2}{\left(1-\exp\left(\sqrt{\frac{A(p_5^{'}-p_4^{'})}{3(a+mb)}}\xi\right)\right)^2},\quad \xi\neq 0, \tag{3.4}$$

where $-\infty < p < p_4^{'} < p_5^{'}$.

(U2) For $n^2 + 2Ae > 0$, $a + bm > 0$ and $A < 0$, we consider the second type of unbounded orbits where their energy is lower than energy of saddle B_2, but higher than energy of center B_1. Any one of them, for example the orbit Γ_3 as shown in figure 2(a), can be expressed by

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_6)(p-p_7)(p-p_8)},$$

where p_6, p_7, p_8 are reals and relationship $p_6 < \frac{n+\sqrt{n^2+2eA}}{A} < p_7 < \frac{n-\sqrt{n^2+2eA}}{A} < p_8 < p < +\infty$ holds. Choosing $p(0) = +\infty$, we have

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_6)(p-p_7)(p-p_8)}} = \int_\xi^0 d\xi, \quad \xi < 0,$$

$$-\int_{+\infty}^p \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_6)(p-p_7)(p-p_8)}} = \int_0^\xi d\xi, \quad \xi > 0,$$

which can be rewritten as

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_6)(p-p_7)(p-p_8)}} =\mid \xi \mid, \quad \xi \neq 0.$$

By calculating the elliptic integral

$$\int_p^{+\infty} \frac{dp}{\sqrt{(p-p_6)(p-p_7)(p-p_8)}} = g \cdot \mathrm{sn}^{-1}\left(\sqrt{\frac{(p_8-p_6)}{(p-p_6)}}, k\right),$$

where $g = \frac{2}{\sqrt{p_8-p_6}}$, $k^2 = \frac{p_7-p_6}{p_8-p_6}$, we get the second type of unbounded solution of system (2.3) as follows

$$p_{u21}(\xi) = p_6 + \frac{p_8-p_6}{\mathrm{sn}^2\left(\sqrt{-\frac{A(p_8-p_6)}{12(a+mb)}}\xi, \sqrt{\frac{p_7-p_6}{p_8-p_6}}\right)}, \quad -\xi_1 < \xi < \xi_1, \quad \xi \neq 0, \tag{3.5}$$

where $\xi_1 = 4\sqrt{-\frac{3(a+mb)}{A(p_8-p_6)}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{p_7-p_6}{p_8-p_6}\cdot\sin^2\theta}}$.

For $n^2+2Ae > 0$, $a+bm > 0$ and $A > 0$, by choosing the initial value $p(0) = -\infty$, we can get the second type of unbounded solution of it

$$p_{u22}(\xi) = p_8^{'} - \frac{p_8^{'}-p_6^{'}}{\mathrm{sn}^2\left(\sqrt{\frac{A(p_8^{'}-p_6^{'})}{12(a+mb)}}\xi, \sqrt{\frac{p_8^{'}-p_7^{'}}{p_8^{'}-p_6^{'}}}\right)}, \quad -\xi_1^{'} < \xi < \xi_1^{'}, \quad \xi \neq 0, \tag{3.6}$$

where $\xi_1^{'}=4\sqrt{\frac{3(a+mb)}{A(p_8^{'}-p_6^{'})}}\cdot\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{p_8^{'}-p_7^{'}}{p_8^{'}-p_6^{'}}\cdot\sin^2\theta}}$ and $-\infty<p<p_6^{'}<\frac{n-\sqrt{n^2+2eA}}{A}<p_7^{'}<\frac{n+\sqrt{n^2+2eA}}{A}<p_8^{'}$.

(U3) For $n^2+2Ae>0$, $a+bm>0$ and $A<0$, we consider the third type of unbounded orbit Γ_4 shown in figure 2(a), where its energy is equal to energy of center B_1. It can be expressed by

$$q=\pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_9)^2(p-p_{10})},$$

where $p_9=\frac{n+\sqrt{n^2+2eA}}{A}$ and $p_{10}=\frac{n-2\sqrt{n^2+2eA}}{A}$ are reals and relationship $p_9<p_{10}<p<+\infty$ holds. Choosing the initial value $p(0)=+\infty$, we have the following integral expressions

$$\int_p^{+\infty}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p-p_9)\sqrt{(p-p_{10})}}=\int_\xi^0 d\xi,\quad \xi<0,$$

$$-\int_{+\infty}^p\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p-p_9)\sqrt{(p-p_{10})}}=\int_0^\xi d\xi,\quad \xi>0,$$

which can be rewritten as

$$\int_p^{+\infty}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p-p_9)\sqrt{(p-p_{10})}}=\mid\xi\mid,\quad \xi\neq 0.$$

Noting that

$$\int_p^{+\infty}\frac{dp}{(p-p_9)\sqrt{(p-p_{10})}}=\frac{1}{\sqrt{p_{10}-p_9}}\left(\pi-2\arctan\sqrt{\frac{p-p_{10}}{p_{10}-p_9}}\right),$$

we get the third type of unbounded solution of system (2.3)

$$p_{u31}(\xi)=p_{10}+(p_{10}-p_9)\cdot\cot^2\left(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi\right),\quad -\xi_2<\xi<\xi_2,\ \ \xi\neq 0,\quad (3.7)$$

where $\xi_2=\sqrt{-\frac{12(a+mb)}{A(p_{10}-p_9)}}\cdot\pi$.

For $n^2+2Ae>0$, $a+bm>0$ and $A>0$, by choosing the initial value $p(0)=-\infty$, we can get the third type of unbounded solution of system (2.3)

$$p_{u32}(\xi)=p_9^{'}-(p_{10}^{'}-p_9^{'})\cdot\cot^2\left(\sqrt{\frac{A(p_{10}^{'}-p_9^{'})}{12(a+mb)}}\xi\right),\quad -\xi_2^{'}<\xi<\xi_2^{'},\ \ \xi\neq 0,\quad (3.8)$$

where $\xi_2^{'} = \sqrt{\dfrac{12(a+mb)}{A(p_{10}^{'} - p_9^{'})}} \cdot \pi$, $p_9^{'} = \dfrac{n - 2\sqrt{n^2+2eA}}{A}$, $p_{10}^{'} = \dfrac{n + \sqrt{n^2+2eA}}{A}$ and $-\infty < p < p_9^{'} < p_{10}^{'}$.

(U4) For $n^2 + 2Ae > 0$, $a + bm > 0$ and $A < 0$, we consider the fourth type of unbounded orbits, where their energy is lower than energy of center B_1. Any one of them, for example the orbit Γ_5 shown in figure 2(a), has the form

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{11})[p^2 + (p_{11} - \frac{3n}{A})p + (p_{11}^2 - \frac{3n}{A}p_{11} - \frac{6e}{A})]},$$

where $p_{10} < p_{11} < p < +\infty$. Taking $p(0) = +\infty$, we have

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{11})[p^2+(p_{11}-\frac{3n}{A})p+(p_{11}^2-\frac{3n}{A}p_{11}-\frac{6e}{A})]}} = \int_\xi^0 d\xi, \quad \xi < 0.$$

$$-\int_{+\infty}^{p} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{11})[p^2+(p_{11}-\frac{3n}{A})p+(p_{11}^2-\frac{3n}{A}p_{11}-\frac{6e}{A})]}} = \int_0^\xi d\xi, \ \xi < 0.$$

By calculating the elliptic integral

$$\int_p^{+\infty} \frac{dp}{\sqrt{(p-p_{11})[p^2+(p_{11}-\frac{3n}{A})p+(p_{11}^2-\frac{3n}{A}p_{11}-\frac{6e}{A})]}} = g \cdot \mathrm{cn}^{-1}\left(\frac{p-p_{11}-B}{p-p_{11}+B}, k\right),$$

where $B^2 = 3p_{11}^2 - \dfrac{6np_{11}}{A} - \dfrac{6e}{A}, g = \dfrac{1}{\sqrt{B}}$ and $k^2 = \dfrac{2AB - 3Ap_{11} - 3n}{4AB}$, we get the fourth unbounded solution of system (2.3)

$$p_{u41}(\xi) = p_{11} - \sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}} + \frac{2\sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2 - 6np_{11} - 6e}{A}}}\xi\right)}, \quad (3.9)$$

$$-\xi_3 < \xi < \xi_3, \quad \xi \neq 0,$$

where $\xi_3 = \dfrac{4 \cdot \sqrt[4]{\frac{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}}{A}}}{\sqrt{-\frac{A}{3(a+mb)}}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{\sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}} - \frac{3}{2}p_{11} - \frac{3n}{2A}}{2\sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}}} \cdot \sin^2\theta}}$.

For $n^2+2Ae > 0$, $a+bm > 0$ and $A > 0$, by choosing the initial value $p(0) = -\infty$, we can get the fourth unbounded solution of system (2.3)

$$p_{u42}(\xi) = p_{11}^{'} + \sqrt{3(p_{11}^{'})^2 - \frac{6n}{A}p_{11}^{'} - \frac{6e}{A}} - \frac{2\sqrt{3(p_{11}^{'})^2 - \frac{6n}{A}p_{11}^{'} - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p_{11}^{'})^2 - 6np_{11}^{'} - 6e}{A}}}\xi\right)}, \quad (3.10)$$

$$-\xi_3^{'} < \xi < \xi_3^{'}, \quad \xi \neq 0,$$

where $\xi_3^{'} = \dfrac{4\cdot\sqrt[4]{\frac{3(p_{11}^{'})^2-\frac{6n}{A}p_{11}^{'}-\frac{6e}{A}}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{3(p_{11}^{'})^2-\frac{6n}{A}p_{11}^{'}-\frac{6e}{A}}-\frac{3}{2}p_{11}^{'}-\frac{3n}{2A}}{2\sqrt{3(p_{11}^{'})^2-\frac{6n}{A}p_{11}^{'}-\frac{6e}{A}}}\cdot\sin^2\theta}}$

and $-\infty < p < p_{11}^{'} < p_9^{'}$.

(U5) For $n^2+2Ae>0$, $a+bm>0$ and $A<0$, we consider the fifth type of unbounded orbits, where their energy is higher than energy of saddle B_2. Any one of them, for example the orbit Γ_1 shown in figure 2(a), can be expressed by

$$q=\pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{12})[p^2+(p_{12}-\frac{3n}{A})p+(p_{12}^2-\frac{3n}{A}p_{12}-\frac{6e}{A})]},$$

where $p_{12} < \dfrac{n+2\sqrt{n^2+2eA}}{A}$ and relationship $p_{12}<p<+\infty$ holds. Choosing $p(0)=+\infty$, we have

$$\int_p^{+\infty}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{12})[p^2+(p_{12}-\frac{3n}{A})p+(p_{12}^2-\frac{3n}{A}p_{12}-\frac{6e}{A})]}}=\int_\xi^0 d\xi,\quad \xi<0.$$

$$-\int_{+\infty}^{p}\frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{12})[p^2+(p_{12}-\frac{3n}{A})p+(p_{12}^2-\frac{3n}{A}p_{12}-\frac{6e}{A})]}}=\int_0^{\xi} d\xi,\ \ \xi>0.$$

Similar calculation leads to the fifth unbounded solution of system (2.3)

$$p_{u51}(\xi)=p_{12}-\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}+\frac{2\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}}{1-\mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi\right)},\tag{3.11}$$

$$-\xi_4<\xi<\xi_4,\quad \xi\neq 0,$$

where $\xi_4 = \dfrac{4\cdot\sqrt[4]{\frac{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}{A}}}{\sqrt{-\frac{A}{3(a+mb)}}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}-\frac{3}{2}p_{12}-\frac{3n}{2A}}{2\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}}\cdot\sin^2\theta}}.$

For $n^2+2Ae>0$, $a+bm>0$ and $A>0$, by taking the initial value $p(0)=-\infty$, we can get the fifth unbounded solution of system (2.3) as follows

$$p_{u52}(\xi)=p_{12}^{'}+\sqrt{3(p_{12}^{'})^2-\frac{6n}{A}p_{12}^{'}-\frac{6e}{A}}-\frac{2\sqrt{3(p_{12}^{'})^2-\frac{6n}{A}p_{12}^{'}-\frac{6e}{A}}}{1-\mathrm{cn}\left(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p_{12}^{'})^2-6np_{12}^{'}-6e}{A}}}\xi\right)},\tag{3.12}$$

$$-\xi_4^{'}<\xi<\xi_4^{'},\quad \xi\neq 0,$$

where $\xi_4^{'} = \dfrac{4\cdot\sqrt[4]{\frac{3(p_{12}^{'})^2-\frac{6n}{A}p_{12}^{'}-\frac{6e}{A}}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{3(p_{12}^{'})^2-\frac{6n}{A}p_{12}^{'}-\frac{6e}{A}}-\frac{3}{2}p_{12}^{'}-\frac{3n}{2A}}{2\sqrt{3(p_{12}^{'})^2-\frac{6n}{A}p_{12}^{'}-\frac{6e}{A}}}\cdot\sin^2\theta}},$

$-\infty < p < p_{12}^{'}$ and $p_{12}^{'} > \dfrac{n + 2\sqrt{n^2 + 2eA}}{A}$.

(II) Next, we discuss the case that $n^2 + 2Ae = 0$. This case includes two subcases (U6-U7) according to different level curves of energy function.

(U6) For $n^2 + 2Ae = 0$, $a + bm > 0$ and $A < 0$, we consider the unbounded orbits Π_2^+ and Π_2^- shown in figure 2(b), where their energy is equal to energy of the cusp B_3, which can be expressed by

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}(p - \frac{A}{n})\sqrt{p - \frac{A}{n}},$$

where $\dfrac{A}{n} < p < +\infty$. Similarly, choosing $p(0) = +\infty$, we have

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p - \frac{A}{n})\sqrt{p - \frac{A}{n}}} = \int_\xi^0 d\xi, \quad \xi < 0,$$

$$-\int_{+\infty}^p \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}(p - \frac{A}{n})\sqrt{p - \frac{A}{n}}} = \int_0^\xi d\xi, \quad \xi > 0.$$

By a direct calculation, we get the sixth unbounded solution of system (2.3)

$$p_{u61}(\xi) = \frac{A}{n} - \frac{12(a+mb)}{A\xi^2}, \quad \xi \neq 0. \tag{3.13}$$

For $n^2+2Ae = 0$, $a+bm > 0$ and $A > 0$, by choosing the initial value $p(0) = -\infty$, we can get the sixth unbounded solution

$$p_{u62}(\xi) = \frac{A}{n} + \frac{12(a+mb)}{A\xi^2}, \quad \xi \neq 0, \tag{3.14}$$

where $-\infty < p < \dfrac{A}{n}$.

(U7) For $n^2+2Ae = 0$, $a+bm > 0$ and $A < 0$, we consider other unbounded orbits, for example Π_1 and Π_3 shown in figure 2(b), which can be uniformly expressed by

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p - p_{13})[p^2 + (p_{13} - \frac{3n}{A})p + (p_{13}^2 - \frac{3n}{A}p_{13} - \frac{6e}{A})]},$$

where $p_{13} < p < +\infty$ and $p_{13} \neq \dfrac{A}{n}$. Choosing $p(0) = +\infty$, we have

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{13})[p^2+(p_{13}-\frac{3n}{A})p+(p_{13}^2-\frac{3n}{A}p_{13}-\frac{6e}{A})]}} = \int_\xi^0 d\xi, \quad \xi<0.$$

$$-\int_{+\infty}^p \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{13})[p^2+(p_{13}-\frac{3n}{A})p+(p_{13}^2-\frac{3n}{A}p_{13}-\frac{6e}{A})]}} = \int_0^\xi d\xi, \ \xi>0.$$

A direct calculation leads to the seventh unbounded solution of system (2.3)

$$p_{u71}(\xi) = p_{13} - \sqrt{3p_{13}^2 - \frac{6n}{A}p_{13} - \frac{6e}{A}} + \frac{2\sqrt{3p_{13}^2 - \frac{6n}{A}p_{13} - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2 - 6np_{13} - 6e}{A}}}\xi\right)}, \tag{3.15}$$

$\xi_5 < \xi < \xi_5, \quad \xi \neq 0,$

where $\xi_5 = \dfrac{4 \cdot \sqrt[4]{\frac{3p_{13}^2 - \frac{6n}{A}p_{13} - \frac{6e}{A}}{A}}}{\sqrt{-\frac{A}{3(a+mb)}}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{\sqrt{3p_{13}^2 - \frac{6n}{A}p_{13} - \frac{6e}{A}} - \frac{3}{2}p_{13} - \frac{3n}{2A}}{2\sqrt{3p_{13}^2 - \frac{6n}{A}p_{13} - \frac{6e}{A}}} \cdot \sin^2\theta}}.$

For $n^2 + 2Ae = 0$, $a + bm > 0$ and $A > 0$, by giving the initial value $p(0) = -\infty$, we can get the seventh unbounded solution

$$p_{u72}(\xi) = p_{13}' + \sqrt{3(p_{13}')^2 - \frac{6n}{A}p_{13}' - \frac{6e}{A}} - \frac{2\sqrt{3(p_{13}')^2 - \frac{6n}{A}p_{13}' - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p_{13}')^2 - 6np_{13}' - 6e}{A}}}\xi\right)}, \tag{3.16}$$

$\xi_5' < \xi < \xi_5', \quad \xi \neq 0,$

where $\xi_5' = \dfrac{4 \cdot \sqrt[4]{\frac{3(p_{13}')^2 - \frac{6n}{A}p_{13}' - \frac{6e}{A}}{A}}}{\sqrt{\frac{A}{3(a+mb)}}} \cdot \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{\sqrt{3(p_{13}')^2 - \frac{6n}{A}p_{13}' - \frac{6e}{A}} - \frac{3}{2}p_{13}' - \frac{3n}{2A}}{2\sqrt{3(p_{13}')^2 - \frac{6n}{A}p_{13}' - \frac{6e}{A}}} \cdot \sin^2\theta}}$

and $p_{13}' \neq \dfrac{A}{n}$.

(III) Finally, we discuss the case that $n^2 + 2Ae < 0$.

(U8) For $n^2 + 2Ae < 0$, $a + bm > 0$ and $A < 0$, any one of orbits can be expressed by

$$q = \pm\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p - p_{14})[p^2 + (p_{14} - \frac{3n}{A})p + (p_{14}^2 - \frac{3n}{A}p_{14} - \frac{6e}{A})]},$$

where $p_{14} < p < +\infty$. Choosing $p(0) = +\infty$, we have

$$\int_p^{+\infty} \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{14})[p^2+(p_{14}-\frac{3n}{A})p+(p_{14}^2-\frac{3n}{A}p_{14}-\frac{6e}{A})]}} = \int_\xi^0 d\xi, \quad \xi<0,$$

$$-\int_{+\infty}^p \frac{dp}{\sqrt{-\frac{A}{3(a+mb)}}\sqrt{(p-p_{14})[p^2+(p_{14}-\frac{3n}{A})p+(p_{14}^2-\frac{3n}{A}p_{14}-\frac{6e}{A})]}} = \int_0^\xi d\xi, \ \xi>0.$$

Thus, we obtain the last type of unbounded solutions of system (2.3)

$$p_{u81}(\xi) = p_{14} - \sqrt{3p_{14}^2 - \frac{6n}{A}p_{14} - \frac{6e}{A}} + \frac{2\sqrt{3p_{14}^2 - \frac{6n}{A}p_{14} - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2 - 6np_{14} - 6e}{A}}}\xi\right)}, \tag{3.17}$$

$\xi_6 < \xi < \xi_6, \quad \xi \neq 0,$

where $\xi_6 = \dfrac{4\cdot\sqrt[4]{\frac{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}{A}}}{\sqrt{-\frac{A}{3(a+mb)}}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}-\frac{3}{2}p_{14}-\frac{3n}{2A}}{2\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}}\cdot\sin^2\theta}}$.

For $n^2+2Ae<0$, $a+bm>0$ and $A>0$, by giving the initial value $p(0)=-\infty$, we can get the eighth unbounded solution

$$p_{u82}(\xi)=p_{14}^{'}+\sqrt{3(p_{14}^{'})^2-\frac{6n}{A}p_{14}^{'}-\frac{6e}{A}}-\frac{2\sqrt{3(p_{14}^{'})^2-\frac{6n}{A}p_{14}^{'}-\frac{6e}{A}}}{1-\mathrm{cn}\left(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p_{14}^{'})^2-6np_{14}^{'}-6e}{A}}}\xi\right)}, \tag{3.18}$$

$\xi_6^{'}<\xi<\xi_6^{'},\quad \xi\neq 0,$

where $\xi_6^{'} = \dfrac{4\cdot\sqrt[4]{\frac{3(p_{14}^{'})^2-\frac{6n}{A}p_{14}^{'}-\frac{6e}{A}}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{\sqrt{3(p_{14}^{'})^2-\frac{6n}{A}p_{14}^{'}-\frac{6e}{A}}-\frac{3}{2}p_{14}^{'}-\frac{3n}{2A}}{2\sqrt{3(p_{13}^{'})^2-\frac{6n}{A}p_{14}^{'}-\frac{6e}{A}}}\cdot\sin^2\theta}}$.

4. Traveling wave solutions of equation (1.3)

According to equation (2.2), to get the final traveling wave solutions of equation (1.3), we need to integrate the obtained solutions of system (2.3) once again with respect to ξ.

Remark 4.1. From section 3, we note that some expressions of solutions of system (2.3) are similar, for example p_{u41}, p_{u42}, p_{u5}, p_{u52}, p_{u71}, p_{u72}, p_{u81} and p_{u82}. To avoid repetitive calculations, we only give detailed calculation for seven types of traveling wave solutions of equation (1.3) which correspond to solutions p_{b11}, p_{b12}, p_{b21}, p_{u21}, p_{u31}, p_{u41} and p_{u61} respectively. The rest of traveling wave solution of equation (1.3) will be listed in the appendix 1.

(S1) Noting that

$$\int \mathrm{sn}^2(u)du=\frac{1}{k^2}(u-E(u)),$$

we integrate the periodic solution p_{b11} of system (2.3) with respect to ξ to calculate the first type of traveling wave solution of equation (1.3) as follows

$$\begin{aligned}
f_1(\xi) &= \int p_{b11}(\xi)d\xi\\
&= \int [p_1+(p_2-p_1)\mathrm{sn}^2\left(\sqrt{\frac{-A(p_3-p_1)}{12(a+mb)}}\xi\right)]d\xi\\
&= p_1\cdot\xi+(p_2-p_1)\cdot\sqrt{-\frac{12(a+mb)}{A(p_3-p_1)}}\\
&\quad\times\int[\mathrm{sn}^2\left(\sqrt{-\frac{A(p_3-p_1)}{12(a+mb)}}\xi\right)d\left(\sqrt{-\frac{A(p_3-p_1)}{12(a+mb)}}\xi\right)]\\
&= p_3\cdot\xi-(\sqrt{\frac{12(a+mb)(p_3-p_1)}{-A}})E\left(\sqrt{-\frac{A(p_3-p_1)}{12(a+mb)}}\xi\right)+C_1
\end{aligned}$$

where $-T < \xi < T$ and C_1 is a constant.
(S2) For another type of periodic solution $p_{b12}(\xi)$ of system (2.3), noting that

$$\int \frac{du}{1 \pm k \cdot \mathrm{sn}(u)} = \frac{1}{k'^2}[E(u) + k(1 \mp k \cdot \mathrm{sn}(u))cd(u)],$$

where $k' = \sqrt{1-k^2}$, we have

$$\begin{aligned}
f_2(\xi) &= \int p_{b12}(\xi)d\xi \\
&= \int [p_1^{'} + \frac{(p_2^{'} - p_1^{'})(p_3^{'} - p_1^{'})}{(p_3^{'} - p_1^{'}) - (p_3^{'} - p_2^{'})\mathrm{sn}^2\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi\right)}]d\xi \\
&= \int [p_1^{'} + \frac{p_2^{'} - p_1^{'}}{1 - \frac{p_3^{'}-p_2^{'}}{p_3^{'}-p_1^{'}}\mathrm{sn}^2\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi\right)}]d\xi \\
&= \int [p_1^{'} + \frac{p_2^{'} - p_1^{'}}{1 - k^2 \cdot \mathrm{sn}^2\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi\right)}]d\xi \\
&= \int \{p_1^{'} + \frac{p_2^{'}-p_1^{'}}{2}[\frac{1}{1-k \cdot \mathrm{sn}\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi\right)} + \frac{1}{1+k \cdot \mathrm{sn}\left(\sqrt{\frac{A(p_3^{'}-p_1^{'})}{12(a+mb)}}\xi\right)}]\}d\xi \\
&= p_1^{'} \cdot \xi + \sqrt{\frac{12(a+mb)(p_3^{'} - p_1^{'})}{A}} \\
&\quad \cdot [E\left(\sqrt{\frac{A(p_3^{'} - p_1^{'})}{12(a+mb)}}\xi\right) + \sqrt{\frac{p_3^{'} - p_2^{'}}{p_3^{'} - p_1^{'}}} \cdot cd\left(\sqrt{\frac{A(p_3^{'} - p_1^{'})}{12(a+mb)}}\xi\right)] + C_2,
\end{aligned}$$

where $-T < \xi < T$ and C_2 is a constant.
(S3) Integrating p_{b21} directly, we get the third type of traveling wave solution of equation (1.3)

$$\begin{aligned}
f_3(\xi) &= \int p_{b21}(\xi)d\xi \\
&= \int [p_4 + \frac{(p_5 - p_4)\left(1 - \exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2}{\left(1 + \exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right)\right)^2}]d\xi \\
&= p_5 \cdot \xi + 4\sqrt{-\frac{3(p_5 - p_4)(a+mb)}{A}} \cdot \frac{1}{\exp\left(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi\right) + 1} + C_3
\end{aligned}$$

where $-\infty < \xi < \infty$ and C_3 is a constant.

(S4) Integrating p_{u21} leads to

$$f_4(\xi) = \int p_{u21}(\xi)d\xi = \int [p_6 + \frac{p_8 - p_6}{\mathrm{sn}^2\left(\sqrt{-\frac{A(p_8-p_6)}{12(a+mb)}}\xi\right)}]d\xi.$$

From the fact that

$$\int \frac{du}{\mathrm{sn}^2(u)} = \int ns^2(u)du = u - E(u) - \mathrm{dn}(u)\cdot cs(u).$$

The fourth type of traveling wave solution of equation (1.3) can be given by

$$\begin{aligned} f_4(\xi) = p_8\cdot\xi - \sqrt{\frac{12(p_8-p_6)(a+mb)}{-A}}\cdot [E\left(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi\right) \\ - \mathrm{dn}\left(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi\right) cs\left(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi\right)] + C_4, \end{aligned}$$

where $-\xi_1 < \xi < \xi_1$, $\xi \neq 0$ and C_4 is a constant.

(S5) Integrating p_{u31} directly, we get the fifth type of traveling wave solution of equation (1.3)

$$\begin{aligned} f_5(\xi) =& \int p_{u31}(\xi)d\xi \\ =& \int [p_{10} + (p_{10}-p_9)\cdot\cot^2\left(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi\right)]d\xi \\ =& p_9\cdot\xi - \sqrt{-\frac{12(a+mb)(p_{10}-p_9)}{A}}\cdot\cot\left(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi\right) + C_5 \end{aligned}$$

where $-\xi_2 < \xi < \xi_2$, $\xi \neq 0$ and C_5 is a constant.

(S6) Integrating p_{u41} leads to

$$\begin{aligned} f_6(\xi) =& \int p_{u41}(\xi)d\xi \\ =& \int [p_{11} - \sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}} + \frac{2\sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}}}{1 - \mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2 - 6np_{11} - 6e}{A}}}\xi\right)}]d\xi. \end{aligned}$$

Noting that

$$\int \frac{du}{1-\mathrm{cn}(u)} = u - E(u) - \frac{\mathrm{dn}(u)\cdot\mathrm{sn}(u)}{1-\mathrm{cn}(u)},$$

we have

$$\begin{aligned} f_6(\xi) = (p_{11} + \sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}})\cdot\xi \\ - \frac{2\cdot\sqrt{3p_{11}^2 - \frac{6n}{A}p_{11} - \frac{6e}{A}}\sqrt[4]{\frac{3Ap_{11}^2 - 6np_{11} - 6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot [E\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2 - 6np_{11} - 6e}{A}}}\xi\right) \end{aligned}$$

$$+\frac{\mathrm{dn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi\right)\mathrm{sn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi\right)}{1-\mathrm{cn}\left(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi\right)}]+C_6$$

where $-\xi_3 < \xi < \xi_3$, $\xi \neq 0$ and C_6 is a constant.

(S7) At last, integrating p_{u61} directly leads to the seventh type of traveling wave solution of equation (1.3)

$$f_7(\xi) = \int p_{u61}(\xi)d\xi = \int (\frac{A}{n} - \frac{12(a+mb)}{A\xi^2})d\xi = \frac{A}{n}\cdot\xi + \frac{12(a+mb)}{A\xi} + C_7,$$

where $\xi \neq 0$ and C_7 is a constant.

5. Discussion and conclusion

In this paper, the dynamical system method is applied to study the traveling waves of equation (1.3). It allows detailed analysis of the phase space geometry of the traveling wave system of equation (1.3) so that all bounded and unbounded orbits are identified clearly and investigated carefully. Due to it, we obtain all single wave solutions of equation (1.3) without any loss. It is shown that our method is a powerful approach to deal with traveling waves of a PDE and can be applied to other PDE models.

Finally, as application of our results, we give all single wave solutions of equation (1.1) in Appendix 2 by the transformation $u = F_x$ and $v = F_y$.

Remark 5.1. From the fact that $u = F_x$, $v = F_y$ and traveling wave transformation $F(x, y, t) = f(\xi) = f(x + my - nt)$, we have $v(\xi) = m \cdot u(\xi)$. So, for the sake of simplicity, we only list solutions $u(\xi)$ of equation (1.1) in Appendix 2.

Appendix 1

Table 1. All single wave solutions of equation (1.3)

Condition1	*Condition2*	TWS	*Range of p*	*Range of ξ*
$n^2+2Ae>0$	$a+mb>0$, $A<0$	$f_1(\xi)=p_3\xi-(\sqrt{\frac{12(a+mb)(p_3-p_1)}{-A}})E(\sqrt{-\frac{A(p_3-p_1)}{12(a+mb)}}\xi)+C_1$	$\frac{n+2\sqrt{n^2+2eA}}{A}<p_1<p<p_2<\frac{n-\sqrt{n^2+2eA}}{A}<p_3$	$-T<\xi<T$
		$f_2(\xi)=p_5\xi+4\sqrt{-\frac{3(p_5-p_4)(a+mb)}{A}}\cdot\frac{1}{\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi)+1}+C_2$	$p_4=\frac{n+2\sqrt{n^2+2eA}}{A}<p<p_5=\frac{n-\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$f_3(\xi)=p_5\xi+4\sqrt{-\frac{3(a+mb)(p_5-p_4)}{A}}\cdot\frac{1}{\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi)-1}+C_3$	$p_4<p_5<p<+\infty$	$\xi\neq0$
		$f_4(\xi)=p_8\cdot\xi-\sqrt{\frac{12(p_8-p_6)(a+mb)}{-A}}\cdot[E(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)-dn(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)cs(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)]+C_4$	$p_6<\frac{n+\sqrt{n^2+2eA}}{A}<p_7<\frac{n-\sqrt{n^2+2eA}}{A}<p_8<p<+\infty$	$-\xi_1<\xi<\xi_1,\ \xi\neq0$
		$f_5(\xi)=p_9\cdot\xi-\sqrt{-\frac{12(a+mb)(p_{10}-p_9)}{A}}\cdot cot(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi)+C_5$	$p_9=\frac{n+\sqrt{n^2+2eA}}{A}<p_{10}=\frac{n-2\sqrt{n^2+2eA}}{A}<p<+\infty$	$-\xi_2<\xi<\xi_2,\ \xi\neq0$
		$f_6(\xi)=(p_{11}+\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}]+C_6$	$p_{10}<p_{11}<p<+\infty$	$-\xi_3<\xi<\xi_3,\ \xi\neq0$
		$f_7(\xi)=(p_{12}+\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}]+C_7$	$p_{12}<\frac{n+2\sqrt{n^2+2eA}}{A},p_{12}<p<+\infty$	$-\xi_4<\xi<\xi_4,\ \xi\neq0$
	$a+mb<0$, $A>0$	$f_1(\xi)=p_3\xi-(\sqrt{\frac{12(a+mb)(p_3-p_1)}{-A}})E(\sqrt{-\frac{A(p_3-p_1)}{12(a+mb)}}\xi)+C_1$	$\frac{n-2\sqrt{n^2+2eA}}{A}<p_1<p<p_2<\frac{n+\sqrt{n^2+2eA}}{A}<p_3$	$-T<\xi<T$
		$f_2(\xi)=p_5\xi+4\sqrt{-\frac{3(p_5-p_4)(a+mb)}{A}}\cdot\frac{1}{\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi)+1}+C_2$	$p_4=\frac{n-2\sqrt{n^2+2eA}}{A}<p<p_5=\frac{n+\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$f_3(\xi)=p_5\xi+4\sqrt{-\frac{3(a+mb)(p_5-p_4)}{A}}\cdot\frac{1}{\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi)-1}+C_3$	$p_4<p_5<p<+\infty$	$\xi\neq0$
		$f_4(\xi)=p_8\cdot\xi-\sqrt{\frac{12(p_8-p_6)(a+mb)}{-A}}\cdot[E(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)-dn(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)cs(\sqrt{\frac{-A(p_8-p_6)}{12(a+mb)}}\xi)]+C_4$	$p_6<\frac{n-\sqrt{n^2+2eA}}{A}<p_7<\frac{n+\sqrt{n^2+2eA}}{A}<p_8<p<+\infty$	$-\xi_1<\xi<\xi_1,\ \xi\neq0$
		$f_5(\xi)=p_9\cdot\xi-\sqrt{-\frac{12(a+mb)(p_{10}-p_9)}{A}}\cdot cot(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi)+C_5$	$p_9=\frac{n-\sqrt{n^2+2eA}}{A}<p_{10}=\frac{n+2\sqrt{n^2+2eA}}{A}<p<+\infty$	$-\xi_2<\xi<\xi_2,\ \xi\neq0$
		$f_6(\xi)=(p_{11}+\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}]+C_6$	$p_{10}<p_{11}<p<+\infty$	$-\xi_3<\xi<\xi_3,\ \xi\neq0$
		$f_7(\xi)=(p_{12}+\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}]+C_7$	$p_{12}<\frac{n-2\sqrt{n^2+2eA}}{A},p_{12}<p<+\infty$	$-\xi_4<\xi<\xi_4,\ \xi\neq0$
$n^2+2Ae=0$	$a+mb>0$, $A<0$	$f_8(\xi)=\frac{A}{n}\cdot\xi+\frac{12(a+mb)}{A\xi}+C_8$	$\frac{A}{n}<p<+\infty$	$\xi\neq0$
		$f_9(\xi)=f_9(\xi)=(p_{13}+\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}]+C_9$	$p_{13}\neq\frac{A}{n},p_{13}<p<+\infty$	$-\xi_5<\xi<\xi_5,\ \xi\neq0$
	$a+mb>0$, $A<0$	$f_8(\xi)=\frac{A}{n}\cdot\xi+\frac{12(a+mb)}{A\xi}+C_8$	$\frac{A}{n}<p<+\infty$	$\xi\neq0$
		$f_9(\xi)=(p_{13}+\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}]+C_9$	$p_{13}\neq\frac{A}{n},p_{13}<p<+\infty$	$-\xi_5<\xi<\xi_5,\ \xi\neq0$
$n^2+2Ae<0$	$a+mb>0$, $A<0$	$f_{10}(\xi)=(p_{14}+\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{14}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)}]+C_{10}$	$p_{14}<p<+\infty$	$-\xi_6<\xi<\xi_6,\ \xi\neq0$
	$a+mb<0$, $A>0$	$f_{10}(\xi)=(p_{14}+\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}})\cdot\xi-\frac{2\cdot\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}{\sqrt{\frac{-A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)sn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{14}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)}]+C_{10}$	$p_{14}<p<+\infty$	$-\xi_6<\xi<\xi_6,\ \xi\neq0$

Condition1	*Condition2*	TWS	*Range of p*	*Range of* ξ
$n^2+2Ae>0$	$a+mb>0$, $A>0$	$f'_1(\xi)=p'_1\cdot\xi+\sqrt{\frac{12(a+mb)(p'_3-p'_1)}{A}}[E(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)+\sqrt{\frac{p'_3-p'_2}{p'_3-p'_3}}\cdot cd(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)]+C'_1$	$p'_1<\frac{n-\sqrt{n^2+2eA}}{A}<p'_2<p<p'_3<\frac{n+2\sqrt{n^2+2eA}}{A}$	$-T<\xi<T$
		$f'_2(\xi)=p'_4\xi-4\sqrt{\frac{3(p_5-p_4)(a+mb)}{A}}\cdot\frac{1}{\exp(\sqrt{\frac{A(p_5-p_4)}{3(a+mb)}}\xi)+1}+C'_2$	$p'_4=\frac{n-\sqrt{n^2+2eA}}{A}<p<p'_5=\frac{n+2\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$f'_3(\xi)=p'_4\cdot\xi-4\sqrt{\frac{3(a+mb)(p_5-p_4)}{A}}\cdot\frac{1}{\exp(\sqrt{\frac{A(p_5-p_4)}{3(a+mb)}}\xi)-1}+C'_3$	$-\infty<p<p'_4<p'_5$	$\xi\neq0$
		$f'_4(\xi)=p'_6\cdot\xi+\sqrt{\frac{12(p'_8-p'_6)(a+mb)}{A}}\cdot[E(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)+dn(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)cs(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)]+C'_4$	$-\infty<p<p'_6<\frac{n-\sqrt{n^2+2eA}}{A}<p'_7<\frac{n+\sqrt{n^2+2eA}}{A}<p'_8$	$-\xi'_1<\xi<\xi'_1,\ \ \xi\neq0$
		$f'_5(\xi)=p'_{10}\cdot\xi+\sqrt{\frac{12(a+mb)(p'_{10}-p'_9)}{A}}\cdot cot(\sqrt{\frac{A(p'_{10}-p'_9)}{12(a+mb)}}\xi)+C'_5$	$-\infty<p<p'_9=\frac{n-2\sqrt{n^2+2eA}}{A}<p'_{10}=\frac{n+\sqrt{n^2+2eA}}{A}$	$-\xi'_2<\xi<\xi'_2,\ \ \xi\neq0$
		$f'_6(\xi)=(p'_{11}-\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}]+C'_6$	$-\infty<p<p'_{11}<p'_9$	$-\xi'_3<\xi<\xi'_3,\ \ \xi\neq0$
		$f'_7(\xi)=(p'_{12}-\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}]+C'_7$	$p'_{12}>\frac{n+2\sqrt{n^2+2eA}}{A},-\infty<p<p'_{12}$	$-\xi'_4<\xi<\xi'_4,\ \ \xi\neq0$
	$a+mb<0$, $A<0$	$f'_1(\xi)=p'_1\cdot\xi+\sqrt{\frac{12(a+mb)(p'_3-p'_1)}{A}}[E(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)+\sqrt{\frac{p'_3-p'_2}{p'_3-p'_3}}\cdot cd(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)]+C'_1$	$p'_1<\frac{n+\sqrt{n^2+2eA}}{A}<p'_2<p<p'_3<\frac{n-2\sqrt{n^2+2eA}}{A}$	$-T<\xi<T$
		$f'_2(\xi)=p'_4\xi-4\sqrt{\frac{3(p_5-p_4)(a+mb)}{A}}\cdot\frac{1}{\exp(\sqrt{\frac{A(p_5-p_4)}{3(a+mb)}}\xi)+1}+C'_2$	$p'_4=\frac{n+\sqrt{n^2+2eA}}{A}<p<p'_5=\frac{n-2\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$f'_3(\xi)=p'_4\cdot\xi-4\sqrt{\frac{3(a+mb)(p_5-p_4)}{A}}\cdot\frac{1}{\exp(\sqrt{\frac{A(p_5-p_4)}{3(a+mb)}}\xi)-1}+C'_3$	$-\infty<p<p'_4<p'_5$	$\xi\neq0$
		$f'_4(\xi)=p'_6\cdot\xi+\sqrt{\frac{12(p'_8-p'_6)(a+mb)}{A}}\cdot[E(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)+dn(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)cs(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)]+C'_4$	$-\infty<p<p'_6<\frac{n+\sqrt{n^2+2eA}}{A}<p'_7<\frac{n-\sqrt{n^2+2eA}}{A}<p'_8$	$-\xi'_1<\xi<\xi'_1,\ \ \xi\neq0$
		$f'_5(\xi)=p'_{10}\cdot\xi+\sqrt{\frac{12(a+mb)(p'_{10}-p'_9)}{A}}\cdot cot(\sqrt{\frac{A(p'_{10}-p'_9)}{12(a+mb)}}\xi)+C'_5$	$-\infty<p<p'_9=\frac{n+2\sqrt{n^2+2eA}}{A}<p'_{10}=\frac{n-\sqrt{n^2+2eA}}{A}$	$-\xi'_2<\xi<\xi'_2,\ \ \xi\neq0$
		$f'_6(\xi)=(p'_{11}-\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}]+C'_6$	$-\infty<p<p'_{11}<p'_9$	$-\xi'_3<\xi<\xi'_3,\ \ \xi\neq0$
		$f'_7(\xi)=(p'_{12}-\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}]+C'_7$	$p'_{12}>\frac{n-2\sqrt{n^2+2eA}}{A},-\infty<p<p'_{12}$	$-\xi'_4<\xi<\xi'_4,\ \ \xi\neq0$
$n^2+2Ae=0$	$a+mb>0$, $A>0$	$f'_8(\xi)=\frac{A}{n}\cdot\xi-\frac{12(a+mb)}{A\xi}+C'_8$	$-\infty<p<\frac{A}{n}$	$\xi\neq0$
		$f'_9(\xi)=(p'_{13}-\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}]+C'_9$	$p_{13}\neq\frac{A}{n},-\infty<p<p'_{13}$	$-\xi'_5<\xi<\xi'_5,\ \ \xi\neq0$
	$a+mb<0$, $A<0$	$f'_8(\xi)=\frac{A}{n}\cdot\xi-\frac{12(a+mb)}{A\xi}+C'_8$	$-\infty<p<\frac{A}{n}$	$\xi\neq0$
		$f'_9(\xi)=(p'_{13}-\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}]+C'_9$	$p'_{13}\neq\frac{A}{n},-\infty<p<p'_{13}$	$-\xi'_5<\xi<\xi'_5,\ \ \xi\neq0$
$n^2+2Ae<0$	$a+mb>0$, $A>0$	$f'_{10}(\xi)=(p'_{14}-\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{14}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}]+C'_{10}$	$-\infty<p<p'_{14}$	$-\xi'_6<\xi<\xi'_6,\ \ \xi\neq0$
	$a+mb<0$, $A<0$	$f'_{10}(\xi)=(p'_{14}-\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}})\cdot\xi+\frac{2\cdot\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{14}-6e}{A}}}{\sqrt{\frac{A}{3(a+mb)}}}\cdot[E(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)+\frac{dn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)sn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}]+C'_{10}$	$-\infty<p<p'_{14}$	$-\xi'_6<\xi<\xi'_6,\ \ \xi\neq0$

Appendix 2

Table 2. All single wave solutions of equation (1.1)

Condition1	*Condition2*	TWS	*Range of p*	*Range of ξ*
$n^2+2Ae>0$	$a+mb>0$, $A<0$	$u_1(\xi)=p_1+(p_2-p_1)sn^2(\sqrt{\frac{-A(p_3-p_1)}{12(a+mb)}}\xi)$	$\frac{n+2\sqrt{n^2+2eA}}{A}<p_1<p<p_2<\frac{n-\sqrt{n^2+2eA}}{A}<p_3$	$-T<\xi<T$
		$u_2(\xi)=p_4+\frac{(p_5-p_4)(1-\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}{(1+\exp(\sqrt{\frac{-A(p_5-p_4)}{3(a+mb)}}\xi))^2}$	$p_4=\frac{n+2\sqrt{n^2+2eA}}{A}<p<p_5=\frac{n-\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$u_3(\xi)=p_4+\frac{(p_5-p_4)(1+\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}{(1-\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}$	$p_4<p_5<p<+\infty$	$\xi\neq 0$
		$u_4(\xi)=p_6+\frac{p_8-p_6}{sn^2(\sqrt{-\frac{A(p_8-p_6)}{12(a+mb)}}\xi)}$	$p_6<\frac{n+\sqrt{n^2+2eA}}{A}<p_7<\frac{n-\sqrt{n^2+2eA}}{A}<p_8<p<+\infty$	$-\xi_1<\xi<\xi_1,\ \ \xi\neq 0$
		$u_5(\xi)=p_{10}+(p_{10}-p_9)\cdot cot^2(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi)$	$p_9=\frac{n+\sqrt{n^2+2eA}}{A}<p_{10}=\frac{n-2\sqrt{n^2+2eA}}{A}<p<+\infty$	$-\xi_2<\xi<\xi_2,\ \ \xi\neq 0$
		$u_6(\xi)=p_{11}-\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}+\frac{2\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}$	$p_{10}<p_{11}<p<+\infty$	$-\xi_3<\xi<\xi_3,\ \ \xi\neq 0$
		$u_7(\xi)=p_{12}-\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}+\frac{2\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}$	$p_{12}<\frac{n+2\sqrt{n^2+2eA}}{A},p_{12}<p<+\infty$	$-\xi_4<\xi<\xi_4,\ \ \xi\neq 0$
	$a+mb<0$, $A>0$	$u_1(\xi)=p_1+(p_2-p_1)sn^2(\sqrt{\frac{-A(p_3-p_1)}{12(a+mb)}}\xi)$	$\frac{n-2\sqrt{n^2+2eA}}{A}<p_1<p<p_2<\frac{n+\sqrt{n^2+2eA}}{A}<p_3$	$-T<\xi<T$
		$u_2(\xi)=p_4+\frac{(p_5-p_4)(1-\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}{(1+\exp(\sqrt{\frac{-A(p_5-p_4)}{3(a+mb)}}\xi))^2}$	$p_4=\frac{n-2\sqrt{n^2+2eA}}{A}<p<p_5=\frac{n+\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$u_3(\xi)=p_4+\frac{(p_5-p_4)(1+\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}{(1-\exp(\sqrt{-\frac{A(p_5-p_4)}{3(a+mb)}}\xi))^2}$	$p_4<p_5<p<+\infty$	$\xi\neq 0$
		$u_4(\xi)=p_6+\frac{p_8-p_6}{sn^2(\sqrt{-\frac{A(p_8-p_6)}{12(a+mb)}}\xi)}$	$p_6<\frac{n-\sqrt{n^2+2eA}}{A}<p_7<\frac{n+\sqrt{n^2+2eA}}{A}<p_8<p<+\infty$	$-\xi_1<\xi<\xi_1,\ \ \xi\neq 0$
		$u_5(\xi)=p_{10}+(p_{10}-p_9)\cdot cot^2(\sqrt{-\frac{A(p_{10}-p_9)}{12(a+mb)}}\xi)$	$p_9=\frac{n-\sqrt{n^2+2eA}}{A}<p_{10}=\frac{n+2\sqrt{n^2+2eA}}{A}<p<+\infty$	$-\xi_2<\xi<\xi_2,\ \ \xi\neq 0$
		$u_6(\xi)=p_{11}-\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}+\frac{2\sqrt{3p_{11}^2-\frac{6n}{A}p_{11}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{11}^2-6np_{11}-6e}{A}}}\xi)}$	$p_{10}<p_{11}<p<+\infty$	$-\xi_3<\xi<\xi_3,\ \ \xi\neq 0$
		$u_7(\xi)=p_{12}-\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}+\frac{2\sqrt{3p_{12}^2-\frac{6n}{A}p_{12}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{12}^2-6np_{12}-6e}{A}}}\xi)}$	$p_{12}<\frac{n-2\sqrt{n^2+2eA}}{A},p_{12}<p<+\infty$	$-\xi_4<\xi<\xi_4,\ \ \xi\neq 0$
$n^2+2Ae=0$	$a+mb>0$, $A<0$	$u_8(\xi)=\frac{A}{n}-\frac{12(a+mb)}{A\xi^2}$	$\frac{A}{n}<p<+\infty$	$\xi\neq 0$
		$u_9(\xi)=p_{13}-\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}+\frac{2\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}$	$p_{13}\neq\frac{A}{n},p_{13}<p<+\infty$	$-\xi_5<\xi<\xi_5,\ \ \xi\neq 0$
	$a+mb>0$, $A<0$	$u_8(\xi)=\frac{A}{n}-\frac{12(a+mb)}{A\xi^2}$	$\frac{A}{n}<p<+\infty$	$\xi\neq 0$
		$u_9(\xi)=p_{13}-\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}+\frac{2\sqrt{3p_{13}^2-\frac{6n}{A}p_{13}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{13}^2-6np_{13}-6e}{A}}}\xi)}$	$p_{13}\neq\frac{A}{n},p_{13}<p<+\infty$	$-\xi_5<\xi<\xi_5,\ \ \xi\neq 0$
$n^2+2Ae<0$	$a+mb>0$, $A<0$	$u_{10}(\xi)=p_{14}-\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}+\frac{2\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)}$	$p_{14}<p<+\infty$	$-\xi_6<\xi<\xi_6,\ \ \xi\neq 0$
	$a+mb<0$, $A>0$	$u_{10}(\xi)=p_{14}-\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}+\frac{2\sqrt{3p_{14}^2-\frac{6n}{A}p_{14}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{-\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3Ap_{14}^2-6np_{14}-6e}{A}}}\xi)}$	$p_{14}<p<+\infty$	$-\xi_6<\xi<\xi_6,\ \ \xi\neq 0$

*Condition*1	*Condition*2	TWS	*Range of p*	*Range of ξ*
$n^2+2Ae>0$	$a+mb>0$, $A>0$	$u'_1(\xi)=p'_1+\frac{(p'_2-p'_1)(p'_3-p'_1)}{(p'_3-p'_1)-(p'_3-p'_2)sn^2(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)}$	$p'_1<\frac{n-\sqrt{n^2+2eA}}{A}<p'_2<p<p'_3<\frac{n+2\sqrt{n^2+2eA}}{A}$	$-T<\xi<T$
		$u'_2(\xi)=p'_5-\frac{(p'_5-p'_4)(1-\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}{(1+\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}$	$p'_4=\frac{n-\sqrt{n^2+2eA}}{A}<p<p'_5=\frac{n+2\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$u'_3(\xi)=p'_5-\frac{(p'_5-p'_4)(1+\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}{(1-\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}$	$-\infty<p<p'_4<p'_5$	$\xi\neq 0$
		$u'_4(\xi)=p'_8-\frac{p'_8-p'_6}{sn^2(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)}$	$-\infty<p<p'_6<\frac{n-\sqrt{n^2+2eA}}{A}<p'_7<\frac{n+\sqrt{n^2+2eA}}{A}<p'_8$	$-\xi'_1<\xi<\xi'_1,\ \ \xi\neq 0$
		$u'_5(\xi)=p'_9-(p'_{10}-p'_9)cot^2(\sqrt{\frac{A(p'_{10}-p'_9)}{12(a+mb)}}\xi)$	$-\infty<p<p'_9=\frac{n-2\sqrt{n^2+2eA}}{A}<p'_{10}=\frac{n+\sqrt{n^2+2eA}}{A}$	$-\xi'_2<\xi<\xi'_2,\ \ \xi\neq 0$
		$u'_6(\xi)=p'_{11}+\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}$	$-\infty<p<p'_{11}<p'_9$	$-\xi'_3<\xi<\xi'_3,\ \ \xi\neq 0$
		$u'_7(\xi)=(p'_{12}+\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)}$	$p'_{12}>\frac{n+2\sqrt{n^2+2eA}}{A},-\infty<p<p'_{12}$	$-\xi'_4<\xi<\xi'_4,\ \ \xi\neq 0$
	$a+mb<0$, $A<0$	$u'_1(\xi)=p'_1+\frac{(p'_2-p'_1)(p'_3-p'_1)}{(p'_3-p'_1)-(p'_3-p'_2)sn^2(\sqrt{\frac{A(p'_3-p'_1)}{12(a+mb)}}\xi)}$	$p'_1<\frac{n+\sqrt{n^2+2eA}}{A}<p'_2<p<p'_3<\frac{n-2\sqrt{n^2+2eA}}{A}$	$-T<\xi<T$
		$u'_2(\xi)=p'_5-\frac{(p'_5-p'_4)(1-\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}{(1+\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}$	$p'_4=\frac{n+\sqrt{n^2+2eA}}{A}<p<p'_5=\frac{n-2\sqrt{n^2+2eA}}{A}$	$-\infty<\xi<+\infty$
		$u'_3(\xi)=p'_5-\frac{(p'_5-p'_4)(1+\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}{(1-\exp(\sqrt{\frac{A(p'_5-p'_4)}{3(a+mb)}}\xi))^2}$	$-\infty<p<p'_4<p'_5$	$\xi\neq 0$
		$u'_4(\xi)=p'_8-\frac{p'_8-p'_6}{sn^2(\sqrt{\frac{A(p'_8-p'_6)}{12(a+mb)}}\xi)}$	$-\infty<p<p'_6<\frac{n+\sqrt{n^2+2eA}}{A}<p'_7<\frac{n-\sqrt{n^2+2eA}}{A}<p'_8$	$-\xi'_1<\xi<\xi'_1,\ \ \xi\neq 0$
		$u'_5(\xi)=p'_9-(p'_{10}-p'_9)cot^2(\sqrt{\frac{A(p'_{10}-p'_9)}{12(a+mb)}}\xi)$	$-\infty<p<p'_9=\frac{n+2\sqrt{n^2+2eA}}{A}<p'_{10}=\frac{n-\sqrt{n^2+2eA}}{A}$	$-\xi'_2<\xi<\xi'_2,\ \ \xi\neq 0$
		$u'_6(\xi)=p'_{11}+\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{11})^2-\frac{6n}{A}p'_{11}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{11})^2-6np'_{11}-6e}{A}}}\xi)}$	$-\infty<p<p'_{11}<p'_9$	$-\xi'_3<\xi<\xi'_3,\ \ \xi\neq 0$
		$u'_7(\xi)=(p'_{12}+\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{12})^2-\frac{6n}{A}p'_{12}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{12})^2-6np'_{12}-6e}{A}}}\xi)}$	$p'_{12}>\frac{n-2\sqrt{n^2+2eA}}{A},-\infty<p<p'_{12}$	$-\xi'_4<\xi<\xi'_4,\ \ \xi\neq 0$
$n^2+2Ae=0$	$a+mb>0$, $A>0$	$u'_8(\xi)=\frac{A}{n}+\frac{12(a+mb)}{A\xi^2}$	$-\infty<p<\frac{A}{n}$	$\xi\neq 0$
		$u'_9(\xi)=p'_{13}+\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}$	$p_{13}\neq\frac{A}{n},-\infty<p<p'_{13}$	$-\xi'_5<\xi<\xi'_5,\ \ \xi\neq 0$
	$a+mb<0$, $A<0$	$u'_8(\xi)=\frac{A}{n}+\frac{12(a+mb)}{A\xi^2}$	$-\infty<p<\frac{A}{n}$	$\xi\neq 0$
		$u'_9(\xi)=p'_{13}+\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{13})^2-\frac{6n}{A}p'_{13}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{13})^2-6np'_{13}-6e}{A}}}\xi)}$	$p'_{13}\neq\frac{A}{n},-\infty<p<p'_{13}$	$-\xi'_5<\xi<\xi'_5,\ \ \xi\neq 0$
$n^2+2Ae<0$	$a+mb>0$, $A>0$	$u'_{10}(\xi)=p'_{14}+\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}$	$-\infty<p<p'_{14}$	$-\xi'_6<\xi<\xi'_6,\ \ \xi\neq 0$
	$a+mb<0$, $A<0$	$u'_{10}(\xi)={}'_{14}+\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}-\frac{2\sqrt{3(p'_{14})^2-\frac{6n}{A}p'_{14}-\frac{6e}{A}}}{1-cn(\frac{\sqrt{\frac{A}{3(a+mb)}}}{\sqrt[4]{\frac{3A(p'_{14})^2-6np'_{14}-6e}{A}}}\xi)}$	$-\infty<p<p'_{14}$	$-\xi'_6<\xi<\xi'_6,\ \ \xi\neq 0$

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