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INTEGRABILITY AND LIMIT CYCLES OF A SYMMETRIC 3-DIM QUADRATIC SYSTEM*

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Abstract We study periodic solutions and first integrals in a three-dimensional quadratic system of ODEs. Coefficient conditions for existence of centers on center manifolds are obtained. Some bounds of the number of limit cycles bifurcating from the centers under small perturbations are given.

Keywords Integrability, limit cycle, center, center manifold, polynomial system.

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1. Introduction

Quadratic autonomous systems of ordinary differential equation represent an important object to study in the qualitative theory of ODEs. The theory of planar quadratic systems, that is, systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are quadratic polynomials, is well developed and the classification of phase portraits of this system is almost completed [1], however the question of the second part of Hilbert's 16th problem about the upper bound for the number of limit cycles remains unanswered even for system (1.1).

The quadratic three dimensional system, that is, the system

$$\begin{aligned}\dot{x} =& a_0 + a_1x + a_4x^2 + a_2y + a_5xy + a_6y^2 + a_3z + a_9xz + a_7yz + a_8z^2,\\ \dot{y} =& b_0 + b_1x + b_4x^2 + b_2y + b_5xy + b_6y^2 + b_3z + b_9xz + b_7yz + b_8z^2,\\ \dot{z} =& c_0 + c_1x + c_4x^2 + c_2y + c_5xy + c_6y^2 + c_3z + c_9xz + c_7yz + c_8z^2,\end{aligned} \tag{1.2}$$

is much less investigated. It is practically impossible to study system (1.2) in full generality, so only some subfamilies of the system have been studied. In particular, integrability of some subfamilies of (1.2) was studied in [2, 3, 6, 12, 18].

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Limit cycles of (1.2) were studied in many works. An example of a quadratic system with infinitely many non-isolated limit cycles was presented in [5]. Twelve limit cycles in a subfamily of system (1.2) with $\mathbb{Z}_3$ symmetry were constructed in [16]. An example of a subfamily of (1.2) having 10 small amplitude limit cycles was given in [9] and recently it has been shown in [17] that 12 limit cycles can bifurcate from two foci at the center manifold in the system

$$\begin{aligned}\dot{x} =& xy,\\ \dot{y} =& -(B_1+d)y + z - (x^2+z^2)/2 + B_1yz - B_2y^2,\\ \dot{z} =& -C_3z - C_1y - C_2y^2 + C_1yz + C_3z^2.\end{aligned} \tag{1.3}$$

Polynomial systems having some kind of symmetry have attracted much attention (see e.g. [9, 16, 19] and the references given there). In this paper we study integrability and limit cycle bifurcations in a family of system (1.2) with "YOU" symmetry, which is different from the one considered in [9] and [17]. Simple computations show that system (1.2) is symmetric with respect to $x \to -x$ and has singular points $E_1 = (1,0,1)$ and $E_2 = (-1,0,1)$ if and only if

$$\begin{aligned}&a_0 = a_4 = a_2 = a_6 = a_3 = a_7 = a_8 = b_1 = b_5 = b_9 = c_1 = c_5 = c_9 = 0,\\ &b_0 = -b_4 - b_3 - b_8,\\ &c_0 = -c_3 - c_4 - c_8.\end{aligned} \tag{1.4}$$

To simplify the further computations we set

$$a_1 = a_9 = 0,\ a_5 = -2b_4,\ b_3 = -2b_8.$$

Then, under conditions (1.4) system (1.2) is written as

$$\begin{aligned}\dot{x} =& -2b_4xy,\\ \dot{y} =& -b_4 + b_8 + b_4x^2 + b_2y + b_6y^2 - 2b_8z + b_7yz + b_8z^2,\\ \dot{z} =& -c_3 - c_4 - c_8 + c_4x^2 + c_2y + c_6y^2 + c_3z + c_7yz + c_8z^2.\end{aligned} \tag{1.5}$$

If $b_4 = b_8 = -\frac{1}{2},\ c_4 = 0, c_3 = -c_8,\ c_6 = -c_2$, then system (1.5) is the same as system (1.3). We obtain another subfamily of (1.5) setting in (1.5) $b_2 = c_7 = 0$, that is,

$$\begin{aligned}\dot{x} =& -2b_4xy,\\ \dot{y} =& -b_4 + b_8 + b_4x^2 + b_6y^2 - 2b_8z + b_7yz + b_8z^2,\\ \dot{z} =& -c_3 - c_4 - c_8 + c_4x^2 + c_2y + c_6y^2 + c_3z + c_8z^2.\end{aligned}$$

We can rescale all equations of this system by $-2b_4$, or, equivalently, we set in it $b_4 = -1/2$. Additionally, we set $c_3 = -1 - 2c_8$ obtaining after moving the origin to the point E_1 the system

$$\begin{aligned}\dot{x} =& y + xy,\\ \dot{y} =& -x - \frac{x^2}{2} + b_7y + b_6y^2 + b_7yz + b_8z^2,\\ \dot{z} =& -z + 2c_4x + c_4x^2 + c_2y + c_6y^2 + c_8z^2.\end{aligned} \tag{1.6}$$

In the present paper we study first integrals and periodic solutions of system (1.6). We first obtain some conditions on the parameters of the system under which it has families of periodic solutions on the center manifolds passing through the points E_1 and E_2. In fact, in such situation the system has analytic first integrals in the neighborhoods of E_1 and E_2. Then, we study bifurcations of the families of periodic solutions and estimate the number of limit cycles which appear near E_1 and E_2 after small perturbations.

2. Preliminaries

Consider a three-dimensional system of the form

$$\begin{aligned}\dot{u} &= \alpha u - v + P(u,v,w;\mu) = \widetilde{P}(u,v,w;\mu)\\ \dot{v} &= u + \alpha v + Q(u,v,w;\mu) = \widetilde{Q}(u,v,w;\mu)\\ \dot{w} &= -\lambda w + R(u,v,w;\mu) = \widetilde{R}(u,v,w;\mu),\end{aligned} \tag{2.1}$$

where P, Q, R are power series without constant and linear terms, which are convergent in a neighborhood of the origin, $\mu = (\mu_1, \ldots, \mu_p)$, and the parameter space is some algebraic set $E = (\lambda; \mu, \alpha)$ which is a subset of $(\lambda; \mu, \alpha) = \mathbb{R}^* \times \mathbb{R}^p \times \mathbb{R}$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

In *this section we assume that* $\alpha = 0$. Then the Jacobian of (2.1) at the origin has the eigenvalues $\pm i$ and λ. According to the Center Manifold Theorem (see e.c. [7]), system (2.1) has a local center manifold W^c passing through the origin defined by a function $w = f(u,v)$ and tangent to the (u,v)-plane at the origin. The phase portrait in a neighborhood of the origin on W^c can be, depending on the nonlinear terms P, Q and R, either a center, in which case every trajectory (other than the origin itself) is an oval surrounding the origin, or a focus, in which case, every trajectory spirals towards the origin or every trajectory spirals away from the origin as the time increases.

Let

$$\mathfrak{X} = \widetilde{P}\tfrac{\partial}{\partial u} + \widetilde{Q}\tfrac{\partial}{\partial v} + \widetilde{R}\tfrac{\partial}{\partial w} \tag{2.2}$$

be the vector field corresponding to system (2.1).

The following theorem is proved in [4, §13].

Theorem 2.1 (Lyapunov Center Theorem)**.** *The origin is a center for* $\mathfrak{X}|W^c$ *if and only if* $\mathfrak{X}$ *admits a real analytic local first integral in a neighborhood of the origin in* $\mathbb{R}^3$*. Moreover when there exists a center the local center manifold* W^c *is analytic.*

For system (2.1) one can always find a function

$$\Phi(u,v,w) = u^2 + v^2 + \sum_{j+k+\ell=3}^{\infty} \phi_{jk\ell} u^j v^k w^\ell \tag{2.3}$$

such that

$$\mathfrak{X}\Phi = \sum_{i=1}^{\infty} v_{2i+1}(u^2+v^2)^{i+1}. \tag{2.4}$$

For any fixed value $\lambda \in \mathbb{R}^*$ the coefficients v_{2i+1} are polynomials in parameters of system (2.1) (see [14] for more details about the properties of v_{2i+1}). Obviously, for fixed values of the parameters function (2.3) is a first integral of (2.1) if and only if $v_{2i+1} = 0$ for all $i \in \mathbb{N}$.

However if system (2.1) depends on many parameters the calculation of v_{2i+1} becomes an extremely difficult computational problem. It has been observed already by Lyapunov that computations are easier, if we introduce complex phase variables setting

$$X = u + iv.$$

Then the first two equations in (2.1) are equivalent to a single equation

$$\dot{X} = iX + \mathcal{P}(X, \bar{X}, w).$$

Adjoining to this equation its complex conjugate, replacing $\bar{X}$ everywhere by Y, regarding Y as an independent complex variable, replacing w by Z and rescaling time $dt \to -idt$ we obtain the complexification of family (2.1) given by the system

$$\begin{aligned} \dot{X} &= \quad iX + \sum_{p+q+r=2}^{\infty} a_{pqr}(\mu) X^p Y^q Z^r \\ \dot{Y} &= -iY + \sum_{p+q+r=2}^{\infty} b_{pqr}(\mu) X^p Y^q Z^r \\ \dot{Z} &= -\lambda Z + \sum_{p+q+r=2}^{\infty} c_{pqr}(\mu) X^p Y^q Z^r, \end{aligned} \tag{2.5}$$

where $b_{qpr} = \bar{a}_{pqr}$ and the c_{pqr} are such that $\sum_{p+q+r=2}^{N} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$, for all $w \in \mathbb{R}$. Existence of a first integral $\Phi(u, v, w) = u^2 + v^2 + \cdots$ for a system in family (2.1) is equivalent to existence of a first integral

$$\Psi(x, y, z) = XY + \sum_{j+k+\ell=3} \psi_{jk\ell} X^j Y^k Z^\ell \tag{2.6}$$

for the corresponding system in family (2.5).

Let $\mathfrak{Z}$ denote the vector field on $\mathbb{C}^3$ corresponding to system (2.5). It is always possible to find a function Ψ of the form (2.6) such that

$$\mathfrak{Z}\Psi(X, Y, Z) = g_1(XY)^2 + g_2(XY)^3 + g_3(XY)^4 + \cdots, \tag{2.7}$$

see, for instance, [13] for more details. The polynomial g_k on the right hand side of (2.7) is called the kth focus quantity of system (2.5). Vanishing of all focus quantities is necessary and sufficient for existence of a first integral of the form (2.6).

Recall that the set of common zeros of a collection of polynomials that generate an ideal I is the variety $\mathbf{V}(I)$ of I. The following theorem gives a characterization of polynomial systems with a center on the center manifold. It is proved in [13] for the case of a fixed λ and in [14] for general case.

Theorem 2.2. *Let* (2.1) *(with* $\alpha = 0$*) be a family of polynomial differential equations on* $\mathbb{R}^3$*. For any system in the family let* $\mathfrak{X}$ *be the corresponding vector field* (2.2) *and let* W^c *be a local center manifold through the origin. Then there exists a variety* $V_{\mathcal{C}}$ *in the space* E *of admissible coefficients such that the origin is a center for* $\mathfrak{X}|W^c$ *if and only if the coefficients of the components of* $\mathfrak{X}$ *lie in* $V_{\mathcal{C}} \cap E$.

We call the variety $V_{\mathcal{C}}$ the center variety of system (2.1).

3. The center variety of system (1.6)

The eigenvalues of the Jacobian of system (1.6) at the origin are

$$\kappa_1 = \frac{1}{2}\left(b_7 - \sqrt{{b_7}^2 - 4}\right), \ \kappa_2 = \frac{1}{2}\left(\sqrt{{b_7}^2 - 4} + b_7\right), \ \kappa_3 = -1. \tag{3.1}$$

Thus, the system can have two-dimensional center manifolds passing through E_1 and E_2 only if $b_7 = 0$. Substituting this value in (1.6) we obtain the system

$$\begin{aligned} \dot{x} =& y + xy, \\ \dot{y} =& -x - x^2/2 + b_6 y^2 + b_8 z^2, \\ \dot{z} =& -z + 2c_4 x + c_4 x^2 + c_2 y + c_6 y^2 + c_8 z^2. \end{aligned} \tag{3.2}$$

Clearly, the eigenvalues of the linearized system are $\pm i$ and -1.

In this section we look for the center variety of system (3.2). To make the computations feasible we introduce new variables setting

$$X = x + iy, \quad \bar{X} = x - iy, \quad Z = z.$$

Then system (3.2) takes the form

$$\begin{aligned} \dot{X} =& -iX - \frac{1}{4} i b_6 X^2 + \frac{1}{2} i b_6 X\bar{X} - \frac{1}{4} i b_6 \bar{X}^2 + i b_8 Z^2 - \frac{3iX^2}{8} - \frac{iX\bar{X}}{4} + \frac{i\bar{X}^2}{8}, \\ \dot{\bar{X}} =& i\bar{X} + \frac{1}{4} i b_6 X^2 - \frac{1}{2} i b_6 X\bar{X} + \frac{1}{4} i b_6 \bar{X}^2 - i b_8 Z^2 - \frac{iX^2}{8} + \frac{iX\bar{X}}{4} + \frac{3i\bar{X}^2}{8}, \\ \dot{Z} =& -Z - \frac{ic_2 X}{2} + \frac{ic_2 \bar{X}}{2} + \frac{c_4 X^2}{4} + \frac{c_4 X\bar{X}}{2} + c_4 X + \frac{c_4 \bar{X}^2}{4} + c_4 \bar{X} \\ & - \frac{c_6 X^2}{4} + \frac{c_6 X\bar{X}}{2} - \frac{c_6 \bar{X}^2}{4} + c_8 Z^2. \end{aligned} \tag{3.3}$$

From the computational point of view it is more convenient to consider instead of (3.3) the more general system

$$\begin{aligned} \dot{X} =& X + (1/2 + A_4) X^2 - 2A_4 XY + A_4 Y^2 - B_6 Z^2, \\ \dot{Y} =& -Y - A_4 X^2 + 2A_4 XY - (1/2 + A_4) Y^2 + B_6 Z^2, \\ \dot{Z} =& -iZ + C_1 X + C_6 X^2 + C_2 Y + C_4 XY + C_6 Y^2 + C_8 Z^2. \end{aligned} \tag{3.4}$$

Simple calculations show that system (3.2) (equivalently (3.3)) is a subfamily of (3.4), namely, if we set in (3.4)

$$\begin{aligned} & x = X, \ y = \bar{X}, \ z = Z, \ A_4 = \frac{1}{8}(2b_6 - 1), B_6 = b_8, \ C_1 = \frac{1}{2}(c_2 + 2ic_4), \\ & C_2 = \frac{1}{2}(-c_2 + 2ic_4), \ C_4 = \frac{1}{2} i(c_4 + c_6). \ C_6 = \frac{1}{4} i(c_4 - c_6), \ C_8 = ic_8 \end{aligned} \tag{3.5}$$

and go back to the real phase space we obtain from (3.4) system (3.2) (taking into account time rescalling $dt \to idt$).

Theorem 3.1. *System* (3.2) *has an analytic local first integral in a neighborhood of the origin if one of the following conditions is fulfilled:*

1) $c_2+2c_4=c_6+2(b_6c_4+4b_8c_4^3+2c_4^2c_8)=0$
2) $c_6=b_6+2=c_2=-b_8+c_8^2=-\frac{1}{2}+c_4c_8=b_8c_4-\frac{c_8}{2}=0,$
3) $c_2=-\frac{b_6}{2}+c_4c_8+\frac{3}{4}=b_6c_4-\frac{c_4}{2}+c_6=-b_6c_8+b_8c_4+b_8c_6+\frac{c_8}{2}={b_6}^2-2b_6+2c_6c_8+\frac{3}{4}=\frac{b_6b_8}{2}-b_6{c_8}^2+b_8c_6c_8-\frac{3b_8}{4}+\frac{{c_8}^2}{2}=-b_6c_6c_8+\frac{b_6}{2}+b_8{c_6}^2-\frac{c_6c_8}{2}-\frac{1}{4}=b_6b_8c_6-b_6c_8-\frac{3b_8c_6}{2}+2c_6{c_8}^2+\frac{c_8}{2}=0,$
4) $c_2-2c_4=\frac{b_6}{2}+c_4c_8=b_8c_6-b_6c_8=b_6c_4-c_4+\frac{c_6}{2}={b_6}^2-b_6-c_6c_8=b_6b_8-b_8-{c_8}^2=0,$
5) $b_8=0.$

Proof. For system (3.4) using computer algebra system Mathematica we computed 6 first focus quantities $g_1,\dots,g_6$ (polynomials on the right hand side of (2.7) where $\mathfrak{Z}$ is the vector field defined by (3.4)).

The first two polynomials g_1 and g_2 are given in Appendix. However the size of polynomials g_i increases exponentially, so we do not present the other polynomials in the paper, however the interested reader can compute them using an available computer algebra system. Performing in the focus quantities g_i substitution (3.5) we obtain polynomials

$$v_{2i+1}=-g_i,\ \ (i=1,\dots,6), \tag{3.6}$$

where v_{2i+1} are real polynomial in variables b_8,c_2,b_6,c_4,c_6,c_8.

Let I be the ideal generated by the polynomials v_{2i+1}, $I=\langle v_3,v_5,\dots v_{13}\rangle$. Computing with the routine `minAssGTZ` [11, 15] of the computer algebra system Singular [10] the minimal associate primes of I over the field $\mathbb{Z}_{32003}$* we find that they are:

$$\begin{aligned}
I_1&=\langle c_2+2c_4,b_8c_4^3-16001c_4^2c_8+8001b_6c_4-12001c_6\rangle,\\
I_2&=\langle c_6,b_6+2,c_2,c_8^2-b_8,c_4c_8+16001,b_8c_4+16001c_8\rangle,\\
I_3&=\langle c_2,c_4c_8+16001b_6-8000,b_6c_4+16001c_4+c_6,b_8c_4+b_8c_6-b_6c_8-16001c_8,\\
&\quad b_6^2+2c_6c_8-2b_6-8000,b_8c_6c_8-b_6c_8^2-16001b_8b_6-16001c_8^2+8000b_8,\\
&\quad b_8c_6^2-b_6c_6c_8+16001c_6c_8-16001b_6-8001,\\
&\quad b_8b_6c_6+2c_6c_8^2+16000b_8c_6-b_6c_8-16001c_8\rangle,\\
I_4&=\langle c_2,c_4c_8+16001b_6-8000,b_6c_4-c_4-16001c_6,b_8c_4+b_8c_6-b_6c_8+16001c_8,\\
&\quad b_6^2+c_6c_8+15999b_6-16000,b_8c_6c_8-b_6c_8^2-16001b_8b_6+16001c_8^2+8000b_8,\\
&\quad b_8c_6^2-b_6c_6c_8-c_6c_8+b_6-1,b_8b_6c_6+c_6c_8^2+16000b_8c_6-2b_6c_8+2c_8\rangle,\\
I_5&=\langle c_4-c_6,b_6+16001,b_8c_6+16001c_8,c_2^2c_8+4c_6^2c_8+2c_6,b_8c_2^2+2c_6c_8+1\rangle,\\
I_6&=\langle c_8,c_6,c_4,b_6-1,b_8c_2^2+3\rangle,\\
I_7&=\langle c_2-2c_4,c_4c_8-16001b_6,b_8c_6-b_6c_8,b_6c_4-c_4-16001c_6,\\
&\quad b_6^2-c_6c_8-b_6,b_8b_6-c_8^2-b_8\rangle,\\
I_8&=\langle b_8\rangle,\\
I_9&=\langle c_2+15058c_4+3766c_6,c_6c_8-15999b_6-4003,c_4c_8-16000b_6-2,\\
&\quad b_6c_6+14589c_4-10355c_6,b_8c_6-17b_6c_8+24c_8,c_4^2+13269c_4c_6-3122c_6^2,\\
&\quad b_6c_4-5649c_4+6589c_6,b_8c_4-15999c_8,b_6^2+15999b_6+4002,
\end{aligned}$$

*We were not able to complete calculations over the field of rational numbers

$$b_8b_6 + 10666c_8^2 - 10669b_8, b_6c_8^2 - 5335c_8^2 - 14668b_8, c_8^4 + 9601b_8c_8^2 + 10401b_8^2\rangle.$$

Performing the rational reconstruction (lifting to the field of characteristic zero) with the algorithm of [23] we obtain conditions 1)-5) of the theorem and conditions 1)-3) of the proposition below (the reconstruction of the ideal I_9 gives a polynomial ideal with the empty real variety).

We now prove that under each of conditions of the theorem system (3.2) has an analytic first integral.

1) If the first condition is fulfilled, the system has the form

$$\begin{aligned} \dot{x} =& y + xy, \\ \dot{y} =& -x - x^2/2 + b_6y^2 + b_8z^2, \\ \dot{z} =& -z + 2c_4x + c_4x^2 - 2c_4y - 2(b_6c_4 + 4b_8c_4^3 + 2c_4^2c_8)y^2 + c_8z^2. \end{aligned} \tag{3.7}$$

The search for invariant surfaces gives the invariant planes $L_1 = 2c_4y + z$ and $L_2 = 1 + x$.

Clearly, L_1, that is, $z = -2c_4y$ is the center manifold of (3.7). System (3.7) reduced on the center manifold has the form

$$\dot{x} = y + xy, \quad \dot{y} = -x - x^2/2 + b_6y^2 + 4b_8c_4^2y^2. \tag{3.8}$$

It has the invariant line $l_1 = x+1$ and the invariant conic $l_2 = -1-2b_6x-8b_8c_4^2x - b_6x^2 - 4b_8c_4^2x^2 - 2b_6y^2 + 2b_6^2y^2 - 8b_8c_4^2y^2 + 16b_6b_8c_4^2y^2 + 32b_8^2c_4^4y^2$ which allow to construct the first integral

$$\Psi = l_1^{-1}l_2^{\frac{1}{2(b_6+4b_8c_4^2)}}$$

in the case when $b_6 + 4b_8c_4^2 \neq 0$. Then by Theorem 2.1 system (3.7) has an analytic first integral in a neighborhood of the origin.

If $b_6 + 4b_8c_4^2 = 0$ the system has the integrating factor $\mu = \frac{1}{1+x}$ which yields the first integral

$$\Psi = \frac{1}{2}\left(-\frac{1}{2}(x+1)^2 + \log(x+1) - y^2\right).$$

2) Note that from the fifth equation of the second case it follows that $c_8 \neq 0$. Thus, under conditions of this case the system can be written as

$$\begin{aligned} \dot{x} =& y + xy, \\ \dot{y} =& -x - \frac{1}{2}x^2 - 2y^2 + c_8^2z^2, \\ \dot{z} =& (2x + x^2 - 2c_8z + 2c_8^2z^2)/(2c_8). \end{aligned} \tag{3.9}$$

Looking for invariant surfaces we find that it has the center manifold defined by

$$2x + x^2 - 2y - 4xy - 2x^2y - 4c_8z - 4c_8xz - 2c_8x^2z + 2c_8^2z^2 = 0. \tag{3.10}$$

Performing the substitution

$$\begin{aligned} X =& x, \\ Y =& 2x + x^2 - 2y - 4xy - 2x^2y - 4c_8z - 4c_8xz - 2c_8x^2z + 2c_8^2z^2, \\ Z =& z \end{aligned}$$

we obtain the system

$$\begin{aligned}\dot X =&\frac{2c_8^2Z^2-2c_8(X(X+2)+2)Z+X^2+2X-Y}{2(X+1)},\\ \dot Y =&Y(-1+2c_8Z),\\ \dot Z =&-Z+\frac{X^2}{2c_8}+\frac{X}{c_8}+c_8Z^2\end{aligned}$$

with the center manifold $Y=0$. Denoting now X by x and Z by y we see that the system reduced on the center manifold is written as

$$\begin{aligned}\dot x =&-\frac{-2c_8^2y^2+2c_8x^2y+4c_8xy+4c_8y-x^2-2x}{2(x+1)}=p(x,y),\\ \dot y =&-y+\frac{x^2}{2c_8}+\frac{x}{c_8}+c_8y^2=q(x,y).\end{aligned} \tag{3.11}$$

Simple computation shows that the system

$$\dot x = p(x,y)(1+x), \qquad \dot y = q(x,y)(1+x)$$

is Hamiltonian with the first integral

$$\Psi = xy+\frac{{c_8}^2y^3}{3}-\frac{x^4}{8c_8}-\frac{x^3}{2c_8}-\frac{1}{2}c_8x^2y^2-\frac{x^2}{2c_8}-c_8xy^2-c_8y^2+\frac{x^2y}{2},$$

which is also a first integral of system (3.11). Therefore system (3.9) has a center on the center manifold (3.10).

3) In this case if $c_8\neq 0$ system (3.2) is written as

$$\begin{aligned}\dot x =&y+xy,\\ \dot y =&-x+\frac{4(1-2b_6){c_8}^2z^2}{(3-2b_6)^2}+b_6y^2-\frac{x^2}{2},\\ \dot z =&-z-\frac{\left(4{b_6}^2-8b_6+3\right)y^2}{8c_8}+\frac{(2b_6-3)x^2}{4c_8}+\frac{(2b_6-3)x}{2c_8}+c_8z^2.\end{aligned} \tag{3.12}$$

It has the center manifold

$$z=\frac{-3x+2b_6x+3y-2b_6y}{4c_8}.$$

Computations show that system (3.12) reduced on the center manifold has 3 invariant straight lines. Thus, as it is well know, it is analytically locally integrable. Therefore (3.12) has a local analytic integral as well.

If $c_8=0$ then the corresponding system is

$$\begin{aligned}\dot x =&y+xy,\\ \dot y =&-x-\frac{z^2}{2c_6^2}-\frac{x^2}{2}+\frac{3y^2}{2},\\ \dot z =&-z-2c_6x-c_6x^2+c_6y^2.\end{aligned} \tag{3.13}$$

On the center manifold it has the integrating factor $\mu = \frac{1}{(x+1)^2(x-y+1)}$ yielding the first integral

$$\Psi = \frac{y-1}{x+1} + \log(x-y+1) - 2\log(x+1).$$

4) If $c_8 \neq 0$ the system of this case is

$$\begin{aligned} \dot{x} =& y + xy, \\ \dot{y} =& -x + \frac{c_8^2 z^2}{b_6 - 1} + b_6 y^2 - \frac{x^2}{2}, \\ \dot{z} =& -z - \frac{b_6 x^2}{2c_8} - \frac{b_6 x}{c_8} + \frac{(b_6-1)b_6 y^2}{c_8} - \frac{b_6 y}{c_8} + c_8 z^2. \end{aligned} \tag{3.14}$$

The center manifold is defined by the equation

$$-2b_6 x - b_6 x^2 - 2b_6 y^2 + 2b_6^2 y^2 - 2c_8 z + 2c_8^2 z^2 = 0$$

and on the manifold system (3.2) is reduced either to

$$\dot{x} = y + xy, \quad \dot{y} = \frac{c_8(x+1)^2 - \sqrt{{c_8}^2\left(2b_6 x(x+2) - 4(b_6-1)b_6 y^2 + 1\right)}}{2(b_6-1)c_8},$$

if $c_8 > 0$, or to

$$\dot{x} = y + xy, \quad \dot{y} = \frac{\sqrt{{c_8}^2\left(2b_6 x(x+2) - 4(b_6-1)b_6 y^2 + 1\right)} + c_8(x+1)^2}{2(b_6-1)c_8},$$

if $c_8 < 0$. Clearly, both systems are unchanged under the transformation $y \to -y$, $t \to -t$, that means, they are time-reversible with respect to the line $x = 0$. Thus, the singularity at the origin is a center for both systems, yielding the local analytic integrability of (3.14). (See e.g. [22] for more details about time-reversibility.)

If $c_8 = 0$ the system of this case is the same as system (3.13).

5) If $b_8 = 0$ the system is decoupled. The system of the first two equations is time-reversible and, therefore, has a local analytic first integral. □

Remark 3.1. The complexification of system (3.2) was crucial for the proof of the theorem. Trying to compute function (2.3) for system (3.2) we were ably to find only three first focus quantities v_3, v_5, v_7 satisfying (2.4), whereas for the complex system (3.4) using the same computational facilities we have computed the first 6 focus quantities $g_1, \ldots, g_6$.

Remark 3.2. System (3.8), as well as the system on the center manifold of case 3) are quadratic systems. So, the obtained integrals are particular cases of known integrals of the quadratic system, see e.g. [20, 22].

Proposition 3.1. *System* (1.6) *has an algebraic center manifold passing through the origin if one of the following conditions is fulfilled:*

1) $c_2 = -\frac{b_6}{2} + c_4 c_8 + \frac{3}{4} = b_6 c_4 - c_4 + \frac{c_6}{2} = -b_6 c_8 + b_8 c_4 + b_8 c_6 - \frac{c_8}{2} = {b_6}^2 - \frac{5b_6}{2} + c_6 c_8 + \frac{3}{2} = \frac{b_6 b_8}{2} - b_6 c_8^2 + b_8 c_6 c_8 - \frac{3b_8}{4} - \frac{c_8^2}{2} = -b_6 c_6 c_8 + b_6 + b_8 {c_6}^2 - c_6 c_8 - 1 = b_6 b_8 c_6 - 2b_6 c_8 - \frac{3b_8 c_6}{2} + c_6 c_8^2 + 2c_8 = 0,$

2) $c_4 - c_6 = b_6 - \frac{1}{2} = b_8 c_6 - \frac{c_8}{2} = {c_2}^2 c_8 + 4c_6^2 c_8 + 2c_6 = b_8 c_2^2 + 2c_6 c_8 + 1 = 0$,

3) $c_8 = c_6 = c_4 = b_6 - 1 = b_8 c_2^2 + 3 = 0$.

Proof. 1) In this case if $c_8 \neq 0$ the system has the form

$$\begin{aligned}\dot{x} =& y + xy,\\ \dot{y} =& -x - \frac{2(2b_6+1)c_8^2 z^2}{(3-2b_6)^2} + b_6 y^2 - \frac{x^2}{2},\\ \dot{z} =& -z - \frac{\left(2b_6^2 - 5b_6 + 3\right) y^2}{2c_8} + \frac{(2b_6-3)x^2}{4c_8} + \frac{(2b_6-3)x}{2c_8} + c_8 z^2,\end{aligned}$$

and admits the center manifold

$$-18x+24b_6x-8b_6^2x-9x^2+12b_6x^2-4b_6^2x^2+18y-24b_6y+8b_6^2y-18y^2+42b_6y^2$$
$$-32b_6^2y^2+8b_6^3y^2-24c_8z+16b_6c_8z+12c_8yz-32b_6c_8yz+16b_6^2c_8yz+4c_8^2z^2+8b_6c_8^2z^2=0.$$

If $c_8 = 0$ the corresponding system is the same as system (3.13).

Simple computations yield that in case 2) the center manifold is $2c_2^3x + 4c_2^2c_6x + 8c_2c_6^2x + 16c_6^3x + c_2^3x^2 + 2c_2^2c_6x^2 + 4c_2c_6^2x^2 + 8c_6^3x^2 + 2c_2^3y - 4c_2^2c_6y + 8c_2c_6^2y - 16c_6^3y + c_2^3y^2 + 2c_2^2c_6y^2 + 4c_2c_6^2y^2 + 8c_6^3y^2 - 4c_2^2z - 16c_6^2z - 2c_2z^2 - 4c_6z^2 = 0$ and in case 3) it is $2c_2^2x + c_2^2x^2 + 2c_2^2y + 2c_2^2y^2 - 4c_2z + 4c_2xz + 2c_2x^2z - 6z^2 = 0$. □

Remark 3.3. We believe that under conditions of Proposition 1 system (1.6) has centers at the center manifold, however there remains an open problem to verify this claim. To prove it one have to show existence of analytic first integrals for the corresponding systems. It appears some new methods for proving existence of such integrals should be developed.

4. Limit cycles in system (1.6)

In this section we study the cyclicity of some centers on the center manifolds found above. We recall, that the cyclicity of a center or a weak focus is the maximum number of limit cycles that can bifurcate from it under small perturbations within a given family of polynomial systems.

Let F_k be a real analytic mapping $F : \mathbb{R}^* \times \mathbb{R}^p \to \mathbb{R}^k$ defined by

$$F_k = (v_3, v_5, \ldots, v_{2k+1}),$$

where v_{2s+1} are functions satisfying (2.4) (for each fixed $\lambda \in \mathbb{R}^*$ they are polynomials). We denote by $d_P F_k$ the $k \times (p+1)$ Jacobian matrix of F_k evaluated at $P \in E$.

In order to study the cyclicity of centers of system (1.6) we use the following theorem proven in [14].

Theorem 4.1. *Let C be an irreducible component of the center variety $V_{\mathcal{C}} \subset \mathbb{R}^{p+1}$ of a polynomial system* (2.1) *with $\alpha = 0$. Suppose $k \leq p+1$ and let $P = (\lambda^*; \mu^*) \in C \cap E$ be a point such that rank$(d_P F_k) = k$, i.e., is maximal. Then letting C also denote the set $(\lambda; \mu; 0) : (\lambda; \mu) \in C$ in $\mathbb{R}^{p+2}$, the following holds:*

(i) There exists a neighborhood U' of $P'(\lambda^*; \mu^*; 0) \in C \cap E' \in \mathbb{R}^{p+2}$ such that $C \cap U'$ is a submanifold of $\mathbb{R}^{p+2}$ of codimension at least $k+1$ and there exist

bifurcations of (2.1) *producing* k *small amplitude limit cycles from the origin for parameter values with* $(\lambda;\mu;\alpha)$ *sufficiently close to* P'.

(ii) If moreover the codimension of C *in* $\mathbb{R}^{p+2}$ *is* $k+1$ *then* P' *is a smooth point of* C *and the cyclicity of* P' *and also of any point in a relatively dense open subset of* C *is exactly* k.

From (3.1) we see that in system (1.6) the parameter b_7 plays the role of parameter α in system (2.1). We do not have the parameter λ in system (1.6) since the coefficient of z in the third equation of (1.6) is fixed (equal to -1). Thus, in the case of system (1.6), $p = 6$, but the space of parameters is the $\mathbb{R}^7$ (not $\mathbb{R}^8$, since we do not have the parameter λ).

Theorem 4.2. *For generic points of components 1), 2), 3), 4), 5) given in Theorem 3.1 the cyclicity of the origin of the corresponding system* (1.6) *is 2, 5, 4, 4, 1, respectively.*

Proof. 1) Computing the Jacobian matrix of the map $F_2 = (v_3, v_5)$, where $v_3 = -ig_1$, $v_5 = -ig_2$ are polynomials in variables $c_2, b_6, b_8, c_4, c_6, c_8$, obtained substituting expressions (3.5) into polynomials g_1, g_2 given in Appendix we see that its rank at the generic points of component 1) is 2. In the space of parameters $\mathbb{R}^7$ the component is parametrized by the equations

$$\begin{aligned} b_7 =& 0,\\ c_2 =& -2c_4,\\ c_6 =& -2(b_6c_4 + 4b_8c_4^3 + 2c_4^2c_8). \end{aligned}$$

Thus, the codimension of the component in $\mathbb{R}^7$ is 3. Therefore, by (ii) of Theorem 4.1 the cyclicity of a generic point of the component is 2.

2) Computing the Jacobian matrix of the map

$$F_5 = (v_3, v_5, v_7, v_9, v_{11}), \tag{4.1}$$

where $v_3, \ldots, v_{11}$ are polynomials defined by (3.6), we obtain that the rank of the Jacobian at generic points is 5. Since the codimension of the component in the space of parameters is 6, by (ii) of Theorem 4.1 the cyclicity of a generic point of the component is 5.

The proofs of other cases is similar. □

Remark 4.1. In the statement of Theorem 4.2 only limit cycles appearing from the origin are mentioned. Due to the symmetry of system (1.6) the same number of limit cycles bifurcate from the point $(-2, 0, 0)$.

Remark 4.2. Since in case 2) the Jacobian of map (4.1) depends only on one parameter, the minors of order 5 have simple expressions. Namely, they are

$$\begin{aligned} m_1 &= m_2 = m_3 = 0,\\ m_4 &= (3875229421122514896966985030203 75c_8)/32,\\ m_5 &= -(387522942112251489696698503020375/(64c_8)),\\ m_6 &= -((387522942112251489696698503020375c_8^2)/16). \end{aligned}$$

Thus, from the proof of the previous theorem we see that for any point of component 2) with $c_8 \neq 0$, there are perturbations of the system yielding 5 small limit cycles in a neighborhood of the origin of system (1.6), and, due to the symmetry, five other limit cycles around the point $(-2, 0, 0)$.

To summarize, we have found 8 necessary conditions for existence of local analytic first integral near the origin and the point $(-2, 0, 0)$ in system (1.6). The sufficiency of 5 of them is proven in Theorem 3.1, however there remains an open problem to prove that under 3 other conditions (the ones given in the statement of Proposition 1) the system has an analytic first integral as well.

The maximal cyclicity of the centers mentioned in Theorem (4.2) is five. In this connection there remains an open problem to study the maximal cyclicity of weak foci on the center manifolds. However we can not perform this study using our computational facilities. Using computations in the polynomial ring $\mathbb{Z}_{32003}[c_2, b_6, b_8, c_4, c_6, c_8]$ we checked that v_{13} does not vanishes identically on the variety of the ideal $\langle v_3, \dots, v_{11}$ in $\mathbb{Z}^6_{32003}$, this indicates that most probably v_{13} does not vanishes identically on the variety of the ideal $I = \langle v_3, \dots, v_{11}\rangle$ in $\mathbb{C}^6$, so the order of "complex" focus of system (1.6) can be 6. However we do not see a way to check if this property takes place for the case of the real variety of the ideal I, so it is an open problem to determine the maximal cyclicity of foci of system (1.6).

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Appendix

Two first focus quantities of system (3.4) are:

$$
\begin{aligned}
g_1 = {} & (1/10 + I/10)B_6((14 - 8I)A_4C_1^2 + (1 + I)C_1C_2 + (22 + 22I)A_4C_1C_2 \\
& - (9 - 2I)B_6C_1^3C_2 - (8 - 14I)A_4C_2^2 - (11 + 11I)B_6C_1^2C_2^2 + (2 - 9I)B_6C_1C_2^3 \\
& + 10C_1C_4 + (10I)C_2C_4 - (2 - 4I)C_1C_6 + (4 - 2I)C_2C_6 \\
& - (11 + 2I)C_1^2C_2C_8 - (2 + 11I)C_1C_2^2C_8); \\
g2 = {} & (1/1500)IB_6((80 + 80I)A_4^3((940 + 1667I)C_1 - (940 - 1667I)C_2)(C_1 + IC_2) \\
& - (9475 - 165I)B_6^3C_1^6C_2^2 + 10(150C_4^2 - 15C_2C_4((14 + 15I) + B_6C_2((23 + 2I)C_2 \\
& + (3 + I)C_4)) + C_2((-48 + 147I) + B_6C_2((65 + 124I)C_2 - (18 - 84I)B_6C_2^3 \\
& - (394 + 68I)C_4))C_6 + 30C_4C_6 - 4(-3 + (24 + 38I)B_6C_2^2)C_6^2) \\
& + (15 + 15I)C_2(C_2((-55 + 10I)C_4 - (24 - 10I)C_6) + (94 + 38I)B_6C_2^3C_6 \\
& - (8 - 4I)(25C_4^2 + (4 - 12I)C_4C_6 + (6 + 4I)C_6^2))C_8 + (1140 + 780I)C_2^3C_6C_8^2 \\
& - (2 + 4I)B_6^2C_1^5((186 + 48I)C_6 + C_2((-252 + 639I) + (5472 + 2805I)B_6C_2^2 \\
& + (935 - 4085I)C_2C_8)) + (1 + I)B_6C_1^4((18023 - 18023I)B_6^2C_2^4 \\
& + 5B_6C_2((1474 + 2033I)C_2 + (1026 - 1068I)C_4 + (6 - 704I)C_6) \\
& - (12504 + 2598I)B_6C_2^3C_8 + 5IC_8((3 + 9I)((-56 + 17I)C_2 + (2 + 32I)C_6) \\
& + (2880 + 769I)C_2^2C_8)) + (1 + 2I)C_1^2((-1961 + 3757I)B_6^3C_2^6 \\
& + (4 + 2I)B_6(-800IC_2^2 - (1143 + 487I)C_2C_4 + (15 + 45I)C_4^2 + (718 + 149I)C_2C_6
\end{aligned}
$$

$$+ (68 + 394I)C_4C_6 + (152 + 96I)C_6^2) - (2 - 4I)B_6C_2^2((909 + 1262I)C_2$$
$$- 3716C_4 - 2022C_6)C_8 - (7871 - 5187I)B_6C_2^4C_8^2$$
$$+ (1+I)B_6^2C_2^3((-4981 + 2592I)C_2 - (4946 - 3602I)C_4 + (2632 - 8742I)C_6$$
$$+ (492 + 5906I)C_2^2C_8) - (3 - 6I)C_8((-212 + 145I)C_2 + (65 - 45I)C_4$$
$$+ (34 - 14I)C_6 - 4C_2(45C_2 + (226 - 68I)C_4 + (45 + 39I)C_6)C_8$$
$$+ (578 + 150I)C_2^3C_8^2)) + (20 + 20I)A_4^2((-421 + 1249I)B_6C_1^4$$
$$+ C_1^3((-5706 - 8580I)B_6C_2 + (246 + 716I)C_8) + C_1^2((1367 + 1958I)$$
$$+ (4 - 4I)C_2(3154B_6C_2 + (374 - 359I)C_8)) + C_2((-3964 - 210I)C_4$$
$$+ (1812 + 1972I)C_6 + IC_2((-1367 + 1958I) + (421 + 1249I)B_6C_2^2$$
$$- (246 - 716I)C_2C_8)) + (1 + I)C_1((2087 + 1877I)C_4 - (1892 - 80I)C_6$$
$$+ C_2(3713I + (7143 - 1437I)B_6C_2^2 + (1436 - 1496I)C_2C_8)))$$
$$+ 5C_1((612 + 54I)B_6^2C_2^5 + 2B_6C_2^4((1047 + 21I)B_6C_4 + (355 + 349I)B_6C_6$$
$$+ (387 - 66I)C_8) - 6C_2(17 + 4B_6(25C_4^2 + 67C_4C_6 + 44C_6^2) + 178C_4C_8 + 28C_6C_8)$$
$$+ 6I((75 + 70I)C_4 - (49 - 16I)C_6 + (2 + 6I)(25C_4^2 + (4 + 12I)C_4C_6$$
$$+ (6 - 4I)C_6^2)C_8) + C_2^2(2B_6((487 + 1143I)C_4 - (149 + 718I)C_6)$$
$$+ 3C_8((212 + 145I) + (904 + 272I)C_4C_8 + (180 - 156I)C_6C_8))$$
$$+ 2C_2^3((57 + 39I)C_8^2 + B_6((149 + 334I) + (1587 + 1161I)C_4C_8$$
$$+ (1341 + 97I)C_6C_8))) + 2A_4((30 + 140I)B_6^2C_1^6$$
$$+ (5 + 15I)B_6C_1^5((163 - 1174I)B_6C_2 + (17 + 15I)C_8)$$
$$+ (1+3I)C_1^4(B_6((234 + 883I) + (21438 + 8684I)B_6C_2^2) - (850 + 7445I)B_6C_2C_8$$
$$+ (20 + 70I)C_8^2) + C_1^3(-2B_6((-3395 + 16800I)C_2 + 54433B_6C_2^3$$
$$+ (6175 - 545I)C_4 + (620 + 380I)C_6) + 5((-125 + 114I) + (1271 + 4951I)B_6C_2^2)C_8$$
$$+ (11680 - 8460I)C_2C_8^2) + 5((6 - 28I)B_6^2C_2^6 + 900C_4^2 + 536C_4C_6 + 224C_6^2$$
$$- 2C_2((1113 + 1384I)C_4 + (138 - 1078I)C_6) + IC_2^3((-218 + 2470I)B_6C_4$$
$$+ (152 + 248I)B_6C_6 - (114 - 125I)C_8) - (28 + 66I)B_6C_2^5C_8$$
$$- (1 - I)C_2^2((-216 + 341I) + (107 + 1747I)C_4C_8 + (470 + 542I)C_6C_8)$$
$$- (1 - I)C_2^4((83 + 400I)B_6 + (6 + 32I)C_8^2)) + C_1^2((-4614 - 72998I)B_6^2C_2^4$$
$$+ 10B_6C_2(6265C_2 + (1667 - 3451I)C_4 - (2044 - 2988I)C_6)$$
$$+ (6355 - 24755I)B_6C_2^3C_8 + 5((-125 + 557I) + 2I((820 + 927I)C_4$$
$$+ (36 + 506I)C_6)C_8 + C_2C_8((3275 - 2688I) + 5964C_2C_8)))$$
$$+ 5C_1((-2226 + 2768I)C_4 - (276 + 2156I)C_6 + C_2(-1774 + (3685 + 685I)B_6^2C_2^4$$
$$+ 2B_6C_2((679 + 3360I)C_2 + (1667 + 3451I)C_4 - (2044 + 2988I)C_6)$$
$$+ (4297 + 1999I)B_6C_2^3C_8 + C_8((3275 + 2688I)C_2 - 6228C_4 - 2008C_6$$
$$+ (2336 + 1692I)C_2^2C_8)))) + 2C_1^3((138 + 13749I)B_6^3C_2^5$$
$$+ (135 + 75I)C_8^2((1 - 2I)(C_2 + 2C_6) - (21 - 20I)C_2^2C_8) + B_6^2C_2^2((-2930 + 9220I)C_4$$
$$+ (11797 - 8319I)C_6 + C_2(-13883 - (4953 - 7551I)C_2C_8)) - 5B_6((345 - 30I)C_4$$
$$- (65 - 124I)C_6 + C_2((-149 + 334I) + C_8((909 - 1262I)C_2 - (1587 - 1161I)C_4$$

$- (1341 - 97I)C_6 + 3675C_2^2C_8))))).$

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