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LIMIT CYCLE BIFURCATIONS IN A CLASS OF PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS UNDER NON-SMOOTH PERTURBATIONS

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Abstract This paper deals with the problem of limit cycles of a class of piecewise smooth integrable differential systems with switching line $x = 0$. The generating functions of the associated first order Melnikov function satisfy two different Picard-Fuchs equations. By using the property of Chebyshev space, we obtain an upper bound for the number of limit cycles bifurcating from the period annulus under non-smooth perturbations of polynomials of degree n. Finally, we present a concrete example to illustrate the theoretical result.

Keywords Piecewise smooth differential system, Melnikov function, limit cycle, Picard-Fuchs equation, Chebyshev space.

MSC(2010) 34C07, 34C05.

1. Introduction

One of the main problems in the qualitative theory of piecewise smooth differential systems is the study of its limit cycles and their distributions, and many methodologies have been developed, such as Melnikov function method [4,6,8,10,13,14,17,18], averaging method [1,3,5,11,15]. Picard-Fuchs equation is an important tool to calculate Melnikov function, see [9,16,19–21]. In this paper, we will study limit cycle bifurcations of a class of perturbed piecewise smooth integrable differential systems. The generating functions of the associated first order Melnikov function satisfy two different Picard-Fuchs equations.

Consider a perturbed piecewise smooth integrable differential system

$$(\dot{x},\ \dot{y}) = \begin{cases} \big(P^+(x,y) + \varepsilon f^+(x,y),\ Q^+(x,y) + \varepsilon g^+(x,y)\big), & x \geq 0, \\ \big(P^-(x,y) + \varepsilon f^-(x,y),\ Q^-(x,y) + \varepsilon g^-(x,y)\big), & x < 0, \end{cases} \tag{1.1}$$

where $P^{\pm}(x,y), Q^{\pm}(x,y) \in C^{\infty}$, $0 < |\varepsilon| \ll 1$,

$$f^{\pm}(x,y) = \sum_{i+j=0}^{n} a^{\pm}_{i,j} x^i y^j, \quad g^{\pm}(x,y) = \sum_{i+j=0}^{n} b^{\pm}_{i,j} x^i y^j, \quad i,j \in \mathbb{N}.$$

Assume that system $(1.1)|_{\varepsilon=0}$ has a first integral $H^+(x,y)$ (resp. $H^-(x,y)$) for $x \geq 0$ (resp. $x < 0$) and has an integrating factor $\mu^+(x,y)$ (resp. $\mu^-(x,y)$) for

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$x \geq 0$ (resp. $x < 0$). This system has two subsystems

$$\begin{cases} \dot{x} = P^+(x,y) + \varepsilon f^+(x,y), \\ \dot{y} = Q^+(x,y) + \varepsilon g^+(x,y), \end{cases} \quad x \geq 0 \tag{1.2}$$

and

$$\begin{cases} \dot{x} = P^-(x,y) + \varepsilon f^-(x,y), \\ \dot{y} = Q^-(x,y) + \varepsilon g^-(x,y), \end{cases} \quad x < 0. \tag{1.3}$$

In order to establish the first order Melnikov function of system (1.1), one must first make the following assumptions as in [13]:

(H1) There exist an interval Σ, and two points $A(h) = (0, a(h))$ and $B(h) = (0, b(h))$ such that for all $h \in \Sigma$

$$H^+(A(h)) = H^+(B(h)) = h, \quad H^-(A(h)) = H^-(B(h)), \quad a(h) \neq b(h).$$

(H2) $(1.2)|_{\varepsilon=0}$ has an orbital arc Γ_h^+ starting from $A(h)$ and ending at $B(h)$ defined by $H^+(x,y) = h,\ x \geq 0$; the system $(1.3)|_{\varepsilon=0}$ has an orbital arc Γ_h^- starting from $B(h)$ and ending at $A(h)$ defined by $H^-(x,y) = H^-(B(h)),\ x < 0$.

Under the above two assumptions (H1) and (H2), system $(1.1)|_{\varepsilon=0}$ has a family of periodic orbits $\Gamma_h = \Gamma_h^+ \cup \Gamma_h^-,\ h \in \Sigma$. $\Gamma_h,\ h \in \Sigma$ is also called a period annulus of system $(1.1)|_{\varepsilon=0}$. If (1.1) has a limit cycle Γ_ε satisfying

$$\lim_{\varepsilon \to 0} \Gamma_\varepsilon = \Gamma_h$$

for some $h \in \Sigma$, we say that the limit cycle Γ_ε is bifurcated from the period annulus. For definition, we assume that the orbits Γ_h for $h \in \Sigma$ are oriented in clockwise sense; see Figure 1. From [12, 13], one knows that the first order Melnikov function of system (1.1) takes the form

$$\begin{aligned} M(h) = & \int_{\Gamma_h^+} \mu^+(x,y)\big[g^+(x,y)dx - f^+(x,y)dy\big] \\ & + \frac{H_y^+(A)}{H_y^-(A)} \int_{\Gamma_h^-} \mu^-(x,y)\big[g^-(x,y)dx - f^-(x,y)dy\big] \\ := & \Phi(h) + \Psi(h),\ h \in \Sigma. \end{aligned} \tag{1.4}$$

It was proved in [6, 13] that the number of limit cycles in (1.1) bifurcating for $|\varepsilon|$ small enough from the period annulus of the unperturbed system is bounded by the number of isolated zeros of $M(h)$ if $M(h) \not\equiv 0$ in Σ. In fact, recently Han and Yang [7] proved that if the function $M(h)$ has at most k zeros in Σ, multiplicities taken into account, then system (1.1) has at most k limit cycles bifurcated from the period annulus, multiplicities taken into account.

For system (1.1), we make the following assumptions:

(H3) $\Phi(h) = \alpha(h)I_1(h) + \beta(h)I_2(h)$ and $\Psi(h) = \gamma(h)J_1(h) + \delta(h)J_2(h)$, where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are polynomials of h with $\deg\alpha(h) \leq n_1$, $\deg\beta(h) \leq n_2$, $\deg\gamma(h) \leq m_1$, $\deg\delta(h) \leq m_2$, and $n_1 \geq n_2$, $m_1 \geq m_2$, $I_i(h)$ and $J_i(h)$ $(i = 1, 2)$ are integrals as the form $\int_\Gamma x^l y^k dx$ or $\int_\Gamma x^l y^k dy$, and Γ is the integral path Γ_h^+ or Γ_h^-, $l, k \in \mathbb{Z}$. Moreover, $I_1'(h) \neq 0$ and $J_1(h) \neq 0$ for $h \in \Sigma$.

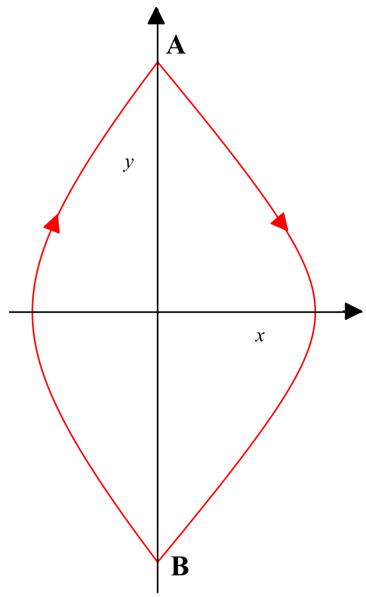

Figure 1. The closed orbits of system $(1.1)|_{\varepsilon=0}$.

(H4) The vector functions $V_1 = (I_1(h), I_2(h))^T$ and $V_2 = (J_1(h), J_2(h))^T$ satisfy the following Picard-Fuchs equations

$$V_1(h) = (B_1 h + C_1)V_1'(h), \quad V_2(h) = (B_2 h + C_2)V_2'(h), \tag{1.5}$$

respectively, where B_i and C_i $(i = 1, 2)$ are 2×2 constant matrices. Moreover, $\det|E - B_i| \neq 0$ for $i = 1, 2$, here E is the 2×2 identity matrix.

Under the assumptions (H1)-(H4), $M(h)$ in (1.4) can be rewritten as

$$M(h) = \big[\alpha(h)I_1(h) + \beta(h)I_2(h)\big] + \big[\gamma(h)J_1(h) + \delta(h)J_2(h)\big], \ h \in \Sigma.$$

Our main result is the following theorem.

Theorem 1.1. *Suppose that (H1)-(H4) hold, then the number of limit cycles of system (1.1) bifurcating from the period annulus is not more than* $3n_1 + 21m_1 + 33$ *for* $h \in \Sigma$*, taking into account their multiplicities.*

Remark 1.1. If $\mu^+(x, y) = \mu^-(x, y) = 1$, that is, system $(1.1)|_{\varepsilon=0}$ is a piecewise smooth Hamiltonian system, then Theorem 1.1 also holds.

2. Proof of Theorem 1.1

In order to prove the main result, we first introduce some definitions and helpful results in the literature, see [2].

Definition 2.1. The real vector space of functions V is said to be Chebyshev in interval $\mathbb{I}$ provided that every function $S \in V \setminus \{0\}$ has at most $\dim V - 1$ zeros, taking into account the multiplicity.

Proposition 2.1. *The solution space X of*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = 0 \tag{2.1}$$

is a Chebyshev space on $\mathbb{I}$ if and only if there exists a nowhere vanishing solution $x_0(t) \in X$ $(x_0(t) \neq 0,\ \forall t \in \mathbb{I})$.

Proposition 2.2. *Suppose that the solution space of (2.1) is a Chebyshev space and let $R(t)$ be an analytic function on $\mathbb{I}$ having l zeros, taking into account the multiplicity. Then every solution $x(t)$ of*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = R(t)$$

has at most $l + 2$ zeros on $\mathbb{I}$.

In the following, we denote by $\#\{\varphi(h) = 0, h \in (a, b)\}$ the number of isolated zeros of $\varphi(h)$ on (a, b) taking into account the multiplicity, and we also denote

$$G_1(h) = \det(B_1h + C_1), \quad G_2(h) = \det(B_2h + C_2).$$

Lemma 2.1. *(i) Let $\omega_1(h) = \frac{I_2'(h)}{I_1'(h)}$, $h \in \Sigma$, then $\omega_1(h)$ satisfies the Riccati equation*

$$G_1(h)\omega_1'(h) = -b_{12}^*(h)\omega_1^2(h) + \big(b_{22}^*(h) - b_{11}^*(h)\big)\omega_1(h) + b_{21}^*(h), \tag{2.2}$$

where $b_{12}^*(h)$, $b_{22}^*(h)$, $b_{11}^*(h)$ and $b_{21}^*(h)$ are polynomials of h of degree not more than 1.

(ii) Let $\omega_2(h) = \frac{J_2(h)}{J_1(h)}$, $h \in \Sigma$, then $\omega_2(h)$ satisfies the Riccati equation

$$G_2(h)\omega_2'(h) = -c_{12}^*(h)\omega_2^2(h) + \big(c_{22}^*(h) - c_{11}^*(h)\big)\omega_2(h) + c_{21}^*(h), \tag{2.3}$$

where $c_{12}^*(h)$, $c_{22}^*(h)$, $c_{11}^*(h)$ and $c_{21}^*(h)$ are polynomials of h of degree not more than 1.

Proof. If $G_1(h) \neq 0$, then, in view of (1.5), one has

$$G_1(h)V_1''(h) = (B_1h + C_1)^*(E - B_1)V_1'(h) := \begin{pmatrix} b_{11}^*(h) & b_{12}^*(h) \\ b_{21}^*(h) & b_{22}^*(h) \end{pmatrix} V_1'(h), \tag{2.4}$$

where E is a 2×2 identity matrix, $(B_1h + C_1)^*$ is the adjoint matrix of $B_1h + C_1$. It is easy to get that $\deg b_{ij}^*(h) \leq 1$ and $\deg G_1(h) \leq 2$. A direct calculation shows that

$$\begin{aligned} \omega_1'(h) =& \frac{I_2''(h)}{I_1'(h)} - \omega_1(h)\frac{I_1''(h)}{I_1'(h)} \\ =& \frac{1}{G_1(h)}\Big[- b_{12}^*(h)\omega_1^2(h) + \big(b_{22}^*(h) - b_{11}^*(h)\big)\omega_1(h) + b_{21}^*(h)\Big]. \end{aligned}$$

If $G_1(h) = 0$, (2.2) also holds. (2.3) can be proved similarly. This completes the proof. $\square$

Lemma 2.2. *Assume that $G_1(h) \neq 0$ for $h \in \Sigma$. Then $\Psi(h)$ has at most $3m_1 + 2$ zeros on Σ, taking into account the multiplicity.*

Proof. Since $J_1(h) \neq 0$ for $h \in \Sigma$, let $\chi_2(h) = \frac{\Psi(h)}{J_1(h)} = \gamma(h) + \delta(h)\omega_2(h)$. It follows from (2.3) that

$$G_2(h)\delta(h)\chi_2'(h) = -c_{12}^*(h)\chi_2(h)^2 + F_2(h)\chi_2(h) + F_1(h), \tag{2.5}$$

where

$$\begin{aligned} F_1(h) =& G_2(h)\big(\gamma'(h)\delta(h) - \gamma(h)\delta'(h)\big) + c_{21}^*(h)\delta(h)^2 - c_{12}^*(h)\gamma(h)^2 \\ &- (c_{22}^*(h) - c_{11}^*(h))\gamma(h)\delta(h), \\ F_2(h) =& G_2(h)\delta'(h) + 2c_{12}^*(h)\gamma(h) + (c_{22}^*(h) - c_{11}^*(h))\delta(h) \end{aligned}$$

with $\deg F_1(h) \leq 2m_1 + 1$. From Lemma 4.4 in [22], one gets

$$\#\{\chi_2(h) = 0, h \in \Sigma\} \leq \#\{\delta(h) = 0, h \in \Sigma\} + \#\{F_1(h) = 0, h \in \Sigma\} + 1.$$

Hence,

$$\#\{\Psi(h) = 0, h \in \Sigma\} = \#\{\chi_2(h) = 0, h \in \Sigma\} \leq 3m_1 + 2.$$

This completes the proof. □

Lemma 2.3. *If $K = m_1 + m_2 + 3$, then, for $h \in \Sigma$, there exist polynomials $P_2(h)$, $P_1(h)$ and $P_0(h)$ of h with degree respectively K, $K-1$ and $K-2$ such that $L(h)\Psi(h) = 0$, where*

$$L(h) = P_2(h)\frac{d^2}{dh^2} + P_1(h)\frac{d}{dh} + P_0(h). \tag{2.6}$$

Proof. By (1.5), we have

$$V_2'(h) = (E - B_2)^{-1}(B_2 h + C_2)V_2''(h),$$

where E is a 2×2 identity matrix. Hence,

$$\begin{aligned} \Psi(h) =& \tau(h)V_2(h) = \tau(h)(B_2 h + C_2)V_2'(h) \\ =& \tau(h)(B_2 h + C_2)(E - B_2)^{-1}(B_2 h + C_2)V_2''(h) \\ :=& \Theta_{m_1+2}(h)J_1''(h) + \Theta_{m_2+2}(h)J_2''(h), \end{aligned}$$

where $\tau(h) = (\gamma(h), \delta(h))$, $\Theta_{m_1+2}(h)$ denotes a polynomial in h of degree at most $m_1 + 2$ and etc.. For $\Psi'(h)$, we have

$$\begin{aligned} \Psi'(h) =& \tau'(h)V_2(h) + \tau(h)V_2'(h) \\ =& \Big(\tau'(h)(B_2 h + C_2) + \tau(h)\Big)(E - B_2)^{-1}(B_2 h + C_2)V_2''(h) \\ :=& \Theta_{m_1+1}(h)J_1''(h) + \Theta_{m_2+1}(h)J_2''(h). \end{aligned}$$

In a similar way, we have

$$\Psi''(h) := \Theta_{m_1}(h)J_1''(h) + \Theta_{m_2}(h)J_2''(h).$$

Therefore,

$$\begin{aligned}
L(h)\Psi(h) =&P_2(h)\Psi''(h)+P_1(h)\Psi'(h)+P_0(h)\Psi(h)\\
=&P_2(h)\Big[\Theta_{m_1}(h)J_1''(h)+\Theta_{m_2}(h)J_2''(h)\Big]\\
&+P_1(h)\Big[\Theta_{m_1+1}(h)J_1''(h)+\Theta_{m_2+1}(h)J_2''(h)\Big]\\
&+P_0(h)\Big[\Theta_{m_1+2}(h)J_1''(h)+\Theta_{m_2+2}(h)J_2''(h)\Big]\\
=&\Big[P_2(h)\Theta_{m_1}(h)+P_1(h)\Theta_{m_1+1}(h)+P_0(h)\Theta_{m_1+2}(h)\Big]J_1''(h)\\
&+\Big[P_2(h)\Theta_{m_2}(h)+P_1(h)\Theta_{m_2+1}(h)+P_0(h)\Theta_{m_2+2}(h)\Big]J_2''(h)\\
:=&X(h)J_1''(h)+Y(h)J_2''(h),
\end{aligned}$$

where $X(h)$ and $Y(h)$ are polynomials of h with $\deg X(h)\le K+m_1$ and $\deg Y(h)\le K+m_2$. Let

$$P_2(h)=\sum_{k=0}^{K}p_{2,k}h^k,\quad P_1(h)=\sum_{m=0}^{K-1}p_{1,m}h^m,\quad P_0(h)=\sum_{l=0}^{K-2}p_{0,l}h^l \tag{2.7}$$

are polynomials of h with coefficients $p_{2,k},\ p_{1,m}$ and $p_{0,l}$ to be determined such that $L(h)\Psi(h)=0$ for

$$0\le k\le K,\quad 0\le m\le K-1,\quad 0\le l\le K-2. \tag{2.8}$$

Assume that

$$X(h)=\sum_{i=0}^{K+m_1}x_ih^i,\quad Y(h)=\sum_{j=0}^{K+m_2}y_jh^j,$$

where x_i and y_j are expressed by $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ of (2.7) linearly. Let

$$\begin{cases}x_i=0,\\ y_j=0,\end{cases}\quad 0\le i\le K+m_1,\quad 0\le j\le K+m_2, \tag{2.9}$$

then (2.9) is a homogenous linear equations with at most $2K+m_1+m_2+2$ equations about $3K$ variables of $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ for k, m and l satisfy (2.8). Since $3K-(2K+m_1+m_2+2)=1$, there exist $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ such that (2.9) holds. This ends the proof. □

Proof of Theorem 1.1. For the sake of clearness, we split the proof into three steps.

(1) For $h\in\Sigma$, $L(h)I(h)=R(h)$, where $L(h)$ is defined by (2.6),

$$R(h)=\frac{1}{G_1(h)}\Big[\Theta_{K+n_1+1}(h)I_1'(h)+\Lambda_{K+n_1+1}(h)I_2'(h)\Big], \tag{2.10}$$

where $\Theta_l(h)$ and $\Lambda_l(h)$ denote polynomials in h of degree at most l and etc..

In fact, from (1.5) and (2.4), one has

$$\begin{aligned}
\Phi(h) =&\sigma(h)V_1(h)=\sigma(h)(B_1h+C_1)V_1'(h)\\
:=&\Theta_{n_1+1}(h)I_1'(h)+\Lambda_{n_2+1}(h)I_2'(h),
\end{aligned}$$

$$\begin{aligned}\Phi'(h) =&\sigma'(h)V_1(h)+\sigma(h)V_1'(h)=[\sigma'(h)(B_1h+C_1)+\sigma(h)]V_1'(h)\\ :=&\Theta_{n_1}(h)I_1'(h)+\Lambda_{n_2}(h)I_2'(h),\\ \Phi''(h) =&\Theta_{n_1}'(h)I_1'(h)+\Lambda_{n_2}'(h)I_2'(h)+\Theta_{n_1}(h)I_1''(h)+\Lambda_{n_2}(h)I_2''(h)\\ :=&\frac{1}{G_1(h)}\big[\Theta_{n_1+1}(h)I_1'(h)+\Lambda_{n_1+1}(h)I_2'(h)\big],\end{aligned} \tag{2.11}$$

where $\sigma(h)=(\alpha(h),\beta(h))$. From Lemma 2.3, we have

$$L(h)I(h)=L(h)\Phi(h)=P_2(h)\Phi''(h)+P_1(h)\Phi'(h)+P_0(h)\Phi(h). \tag{2.12}$$

Substituting (2.11) into (2.12) gives (2.10).

(2) Zeros of $R(h)$ for $h\in\Sigma$.

By (2.10), we obtain

$$R(h)=\frac{I_1'(h)}{G_1(h)}\Big[\Theta_{K+n_1+1}(h)+\Lambda_{K+n_1+1}(h)\omega_1(h)\Big].$$

Noting that $I_1'(h)\neq 0$ for $h\in\Sigma$, we have for $h\in\Sigma$

$$\#\{R(h)=0\}\le\#\{\Theta_{K+n_1+1}(h)+\Lambda_{K+n_1+1}(h)\omega_1(h)=0\}+2.$$

Let $\chi_1(h)=\Theta_{K+n_1+1}(h)+\Lambda_{K+n_1+1}(h)\omega_1(h)$, by (2.2), we obtain

$$G_1(h)\Lambda_{K+n_1+1}(h)\chi_1'(h)=-b_{12}^*(h)\chi_1(h)^2+\Theta_{K+n_1+2}(h)\chi_1(h)+\Theta_{2K+2n_1+3}(h).$$

From Lemma 4.4 in [22], we have for $h\in\Sigma$

$$\#\{\chi_1(h)=0\}\le\#\{\Lambda_{K+n_1+1}(h)=0\}+\#\{\Theta_{2K+2n_1+3}(h)=0\}+1.$$

Hence,

$$\#\{R(h)=0,h\in\Sigma\}\le 3K+3n_1+7.$$

(3) Zeros of $I(h)$ for $h\in\Sigma$.

By Lemma 2.2, we have $\Psi(h)$ has at most $3m_1+2$ zeros on Σ. We assume that

$$P_2(\tilde h_i)=0,\ \Psi(\bar h_j)=0,\ \tilde h_i,\bar h_j\in\Sigma,\ 1\le i\le K,\ 1\le j\le 3m_1+2.$$

Denote $\tilde h_i$ and $\bar h_j$ as h_m^*, and reorder them such that $h_m^*<h_{m+1}^*$ for $m=1,2,\cdots,K+3m_1+2$. Let

$$\Delta_s=(h_s^*,h_{s+1}^*),\ s=0,1,\cdots,K+3m_1+2,$$

where h_0^* is the left end point of Σ and $h_{K+3m_1+3}^*$ is the right end point of Σ. Then $P_2(h)\neq 0$ and $\Psi(h)\neq 0$ for $h\in\Delta_s$ and $L(h)\Psi(h)=0$. By Proposition 2.1, the solution space of

$$L(h)=P_2(h)\Big(\frac{d^2}{dh^2}+\frac{P_1(h)}{P_2(h)}\frac{d}{dh}+\frac{P_0(h)}{P_2(h)}\Big)$$

is a Chebyshev space on Δ_s. By Proposition 2.2, $I(h)$ has at most $2+l_s$ zeros for $h \in \Delta_s$, where l_s is the number of zeros of $R(h)$ on Δ_s. Therefore, we obtain for $h \in \Sigma$

$$\begin{aligned}\#\{I(h)=0\} \leq & \#\{R(h)=0\} + 2\cdot \text{the number of the intervals of } \Delta_s \\ & + \text{the number of the end points of } \Delta_s \\ \leq & 3K + 3n_1 + 7 + 2(K + 3m_1 + 3) + K + 3m_1 + 2 \\ \leq & 3n_1 + 21m_1 + 33.\end{aligned}$$

This completes the proof Theorem 1.1.

3. Application

In this section, we will present a piecewise smooth differential system with the form of (1.1). Consider the following perturbed piecewise smooth Hamiltonian systems

$$\begin{aligned} &\begin{cases}\dot{x} = y + \varepsilon f^{+}(x,y), \\ \dot{y} = x - 1 + \varepsilon g^{+}(x,y),\end{cases} \quad x \geq 0, \\ &\begin{cases}\dot{x} = y + \varepsilon f^{-}(x,y), \\ \dot{y} = x + 1 + \varepsilon g^{-}(x,y),\end{cases} \quad x < 0,\end{aligned} \tag{3.1}$$

where

$$f^{\pm}(x,y) = \sum_{i+j=0}^{n} x^i y^j, \quad g^{\pm}(x,y) = \sum_{i+j=0}^{n} x^i y^j, \quad i,j \in \mathbb{N}.$$

When $\varepsilon = 0$, the corresponding Hamiltonian functions for (3.1) are

$$H^{+}(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + x, \quad x \geq 0, \tag{3.2}$$

and

$$H^{-}(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 - x, \quad x < 0. \tag{3.3}$$

When $\varepsilon = 0$, (3.1) has a family of periodic orbits as follows

$$\begin{aligned}\Gamma_h = & \{(x,y)|H^{+}(x,y) = h, x \geq 0\} \cup \{(x,y)|H^{-}(x,y) = h, x < 0\} \\ := & \Gamma_h^{+} \cup \Gamma_h^{-},\end{aligned}$$

with $h \in (0, \frac{1}{2})$, see Figure 2.

Theorem 3.1. *An upper bound for the number of limit cycles of system (3.1) is* $3[\frac{n}{2}] + 21[\frac{n-1}{2}] + 33$.

Now we study the algebraic structure of the first order Melnikov function $M(h)$ of system (3.1). Obviously, $H_y^{+}(0,y) = H_y^{-}(0,y)$. Hence

$$\begin{aligned}M(h) = & \int_{\Gamma_h^{+}} g^{+}(x,y)dx - f^{+}(x,y)dy + \int_{\Gamma_h^{-}} g^{-}(x,y)dx - f^{-}(x,y)dy \\ = & \Phi(h) + \Psi(h).\end{aligned} \tag{3.4}$$

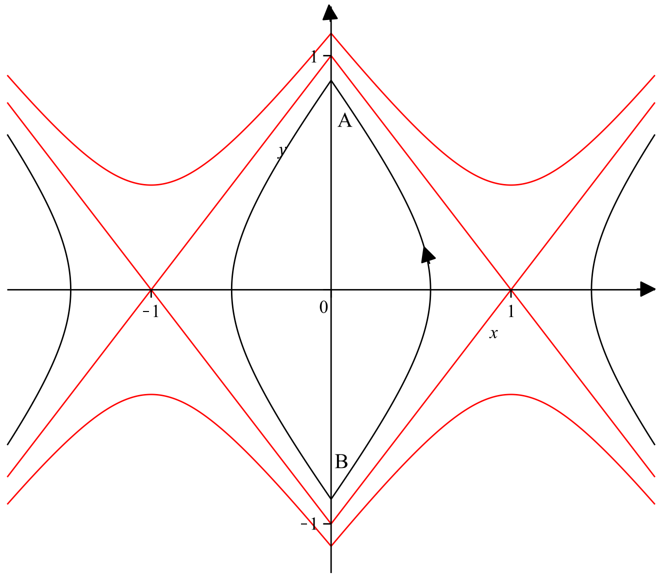

Figure 2. The closed orbits of system $(3.1)|_{\varepsilon=0}$.

For $h \in (0, \frac{1}{2})$, we denote

$$I_{i,j}(h) = \int_{\Gamma_h^+} x^i y^j dy, \quad J_{i,j}(h) = \int_{\Gamma_h^-} x^i y^j dy,$$

and $I_1(h) = I_{0,0}(h)$, $I_2(h) = I_{1,0}(h)$, $J_1(h) = J_{0,0}(h)$ and $J_2(h) = J_{1,0}(h)$. It is easy to get that $I_1'(h) \neq 0$ and $J_1(h) \neq 0$ for $h \in (0, \frac{1}{2})$. The orbits $\Gamma_h^{\pm}$ are symmetric with respect to the x-axis. Thus, $I_{i,2j+1}(h) = J_{i,2j+1}(h) \equiv 0$. So we only need to consider $I_{i,2j}(h)$ and $J_{i,2j}(h)$. We first prove the following results.

Lemma 3.1. *If $h \in (0, \frac{1}{2})$, then*

$$\Phi(h) = \alpha(h)I_1(h) + \beta(h)I_2(h), \quad \Psi(h) = \gamma(h)J_1(h) + \delta(h)J_2(h), \tag{3.5}$$

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are polynomials of h with

$$\deg \alpha(h), \deg \gamma(h) \leq [\frac{n}{2}], \ \deg \beta(h), \deg \delta(h) \leq [\frac{n-1}{2}].$$

Proof. Let D be the interior of $\Gamma_h^+ \cup \overrightarrow{AB}$, see Figure 2. Using the Green's Formula, one has

$$\begin{aligned}\int_{\Gamma_h^+} x^i y^j dx &= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j dx - \int_{\overrightarrow{AB}} x^i y^j dx \\ &= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j dx = j \iint\limits_D x^i y^{j-1} dxdy,\end{aligned}$$

$$\int_{\Gamma_h^+} x^{i+1} y^{j-1} dy = \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^{i+1} y^{j-1} dy = -(i+1) \iint\limits_D x^i y^{j-1} dxdy.$$

Hence,

$$\int_{\Gamma_h^+} x^i y^j dx = -\frac{j}{i+1}\int_{\Gamma_h^+} x^{i+1} y^{j-1} dy. \tag{3.6}$$

In a similar way, one gets

$$\int_{\Gamma_h^-} x^i y^j dx = -\frac{j}{i+1}\int_{\Gamma_h^-} x^{i+1} y^{j-1} dy. \tag{3.7}$$

By a straightforward calculation and noting that (3.6) and (3.7), we obtain

$$\begin{aligned}
M(h) &= \int_{\Gamma_h^+} \sum_{i+j=0}^{n} b_{i,j}^{+} x^i y^j dx - \int_{\Gamma_h^+} \sum_{i+j=0}^{n} a_{i,j}^{+} x^i y^j dy \\
&\quad + \int_{\Gamma_h^-} \sum_{i+j=0}^{n} b_{i,j}^{-} x^i y^j dx - \int_{\Gamma_h^-} \sum_{i+j=0}^{n} a_{i,j}^{-} x^i y^j dy \\
&= -\sum_{i+j=1,j\geq 1}^{n} \frac{j}{i+1} b_{i,j}^{+} \int_{\Gamma_h^+} x^{i+1} y^{j-1} dy - \int_{\Gamma_h^+} \sum_{i+j=0}^{n} a_{i,j}^{+} \int_{\Gamma_h^+} x^i y^j dy \\
&\quad - \sum_{i+j=1,j\geq 1}^{n} \frac{j}{i+1} b_{i,j}^{-} \int_{\Gamma_h^-} x^{i+1} y^{j-1} dy - \sum_{i+j=0}^{n} a_{i,j}^{-} \int_{\Gamma_h^-} x^i y^j dy \\
&= \sum_{i+j=0}^{n} \xi_{i,j} I_{i,j}(h) + \sum_{i+j=0}^{n} \eta_{i,j} J_{i,j}(h),
\end{aligned}$$

where $\xi_{i,j}$ and $\eta_{i,j}$ are constants which can be expressed by the coefficients of $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$.

Without loss of generality, we only prove the first equality in (3.5). The second one can be shown similarly. To establish the relations between the integrals $I_{i,j}(h)$, we take the equation $H^+(x,y) = h$, and differentiate both sides with respect to y. One has

$$y - x\frac{\partial x}{\partial y} + \frac{\partial x}{\partial y} = 0. \tag{3.8}$$

Multiplying (3.8) by the one-form $x^i y^{j-1} dy$ and integrating, one obtains the relation

$$I_{i,j} = \frac{j-1}{i+1} I_{i+1,j-2} - \frac{j-1}{i+2} I_{i+2,j-2},\ i \geq 0, j \geq 1. \tag{3.9}$$

Similarly, multiplying $H^+(x,y) = h$ both sides by $x^{i-3} y^j dy$ and integrating over Γ_h^+, we get another relation

$$I_{i,j} = -2hI_{i-2,j} + 2I_{i-1,j} + I_{i-2,j+2},\ i \geq 2, j \geq 0. \tag{3.10}$$

Elementary manipulations reduce equations (3.9) and (3.10) to

$$I_{i,j} = -\frac{i}{i+j+1}\left[2hI_{i-2,j} - \frac{2i+j-1}{i-1} I_{i-1,j}\right],\ i \geq 2, j \geq 0 \tag{3.11}$$

and

$$I_{i,j} = \frac{j-1}{i+j+1}\left[2hI_{i,j-2} - \frac{i}{i+1} I_{i+1,j-2}\right],\ i \geq 0, j \geq 1. \tag{3.12}$$

Easy computation using the above two equalities gives

$$\begin{cases} I_{0,2}(h) = \frac{2}{3}hI_{0,0}(h), \\ I_{2,0}(h) = -\frac{4}{3}hI_{0,0}(h) + 2I_{1,0}(h), \\ I_{1,2}(h) = \frac{1}{6}hI_{0,0}(h) + (\frac{1}{2}h - \frac{1}{4})I_{1,0}(h), \\ I_{3,0}(h) = -\frac{5}{2}hI_{0,0}(h) - (\frac{3}{2}h - \frac{15}{4})I_{1,0}(h). \end{cases} \tag{3.13}$$

Then, the result about $\Phi(h)$ in (3.5) follows directly by induction using (3.11), (3.12) and (3.13). The proof for $\Psi(h)$ follows by using the same arguments, so we omit for the sake of brevity and readability. This ends the proof. $\square$

Lemma 3.2. *The vector functions $(I_1(h), I_2(h))^T$ and $(J_1(h), J_2(h))^T$ satisfy the following Picard-Fuchs equations*

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 2h & 0 \\ h & h - \frac{1}{2} \end{pmatrix} \begin{pmatrix} I_1' \\ I_2' \end{pmatrix} \tag{3.14}$$

and

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 2h & 0 \\ -h & h - \frac{1}{2} \end{pmatrix} \begin{pmatrix} J_1' \\ J_2' \end{pmatrix}, \tag{3.15}$$

respectively.

Proof. We only prove (3.14). (3.15) can be proved similarly. According to (3.2) one has

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 + x = h. \tag{3.16}$$

Differentiating the above equation with respect to h gives $\frac{\partial x}{\partial h} = \frac{1}{1-x}$, which implies which implies

$$I_{i,j}' = i\int_{\Gamma_h^+} \frac{x^{i-1}y^j}{1-x}dx. \tag{3.17}$$

Hence,

$$I_{i,j} = \frac{1}{i+1}I_{i+1,j}' - \frac{1}{i+2}I_{i+2,j}'. \tag{3.18}$$

Multiplying both side of (3.16) by h, one gets

$$hI_{i,j}' = \frac{1}{2}I_{i,j+2}' + \frac{i}{i+1)}I_{i+1,j}' - \frac{i}{2(i+2)}I_{i+2,j}'. \tag{3.19}$$

On the other hand, in view of (3.6) and (3.17), one has for $i \geq 1, j \geq 0$

$$\begin{aligned} I_{i,j} =& -\frac{i}{j+1}\int_{\Gamma_h^+} x^{i-1}y^{j+1}dx \\ =&\frac{i}{j+1}\int_{\Gamma_h^+} \frac{x^{i-1}y^{j+2}}{1-x}dx \\ =&\frac{1}{j+1}I_{i,j+2}'. \end{aligned} \tag{3.20}$$

Thus, by (3.18), (3.19) and (3.20), we obtain

$$I_{i,j} = \frac{1}{i+j+1}\Big(2hI'_{i,j} - \frac{i}{i+1}I'_{i+1,j}\Big),\ i \geq 1, j \geq 0, \tag{3.21}$$

which yields

$$I_{1,0} = hI'_{1,0} - \frac{1}{4}I'_{2,0}.$$

It follows from (3.18) that

$$I_{0,0} = I'_{1,0} - \frac{1}{2}I'_{2,0}.$$

The result then follows from the above two equalities and the second equality of (3.13). This completes the proof. □

Proof of Theorem 3.1. By Lemmas 3.1 and 3.2, it is easy to check that the conditions (H1)-(H4) hold for system (3.1). Hence, by Theorem 1.1 we obtain that the number of limit cycles of system (3.1) bifurcating from the period annulus is not more than $3[\frac{n}{2}]+21[\frac{n-1}{2}]+33$ for $h \in (0, \frac{1}{2})$, taking into account their multiplicities.

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