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AN A PRIORI ERROR ANALYSIS OF A STRAIN GRADIENT MODEL USING C^0 INTERIOR PENALTY METHODS

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Abstract In this work we study, from the numerical point of view, a strain gradient model. It can be written as a linear fourth-order in space and second-order in time partial differential equation which leads to a parabolic variational equation in terms of the velocity field. Then, a fully discrete approximation is provided by using the implicit Euler scheme to discretize the time derivatives and the so-called C^0 interior penalty method for the spatial approximation. A priori error estimates are obtained, and from them it follows the convergence of the approximations (under suitable regularity conditions). Finally, some two-dimensional numerical simulations are shown to demonstrate the numerical behaviour.

Keywords Strain gradient theory, finite elements, C^0 interior penalty methods, a priori error estimates, numerical simulations.

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1. Introduction

Since the first results obtained by Toupin [21] and Mindlin [20] the strain gradient theory has been studied in depth. Its main idea is to include higher gradients of displacement in the basic postulates of the elasticity theory. In [21] the nonlinear form was considered, and its linear version was studied in [20]. It is well-known that this strain gradient theory of elasticity is adequate to deal with problems related to the size effects. Some classical examples are, for instance, auxetic materials, bones, honeycomb structures as well as some types of composites. The number of contributions published over the last fifty years is really huge (see, e.g., [1, 3, 7, 9–11, 14, 16–18] and the references cited therein).

In this work, we continue the research started in [17], where the thermal effects and the microtemperatures were considered. Here, our aim is to consider the finite element approximation using the so-called C^0 interior penalty formulation provided in [4] (and used later for many other problems as [2, 8, 12]). So, in order to simplify the calculations and the analysis, we restrict ourselves to the simpler case of the strain gradient elasticity.

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The paper is structured as follows. The mechanical problem and its weak form are presented in Section 2. Then, the fully discrete approximations are introduced in Section 3 by using the implicit Euler scheme and a C^0 interior penalty method to approximate the time derivatives and the spatial variable, respectively. A priori error estimates and a convergence order are shown. Finally, some numerical simulations are presented in Section 4 to demonstrate the accuracy of the finite element approximation.

2. The mechanical problem and its variational formulation

Let B be a two-dimensional polygonal domain with boundary ∂B and outward unit normal vector $\boldsymbol{n} = (n_i)_{i=1}^2$. We assume that the body occupying the set $\overline{B}$ is being acted upon by a volume force of density $\mathbf{f} = (f_i)_{i=1}^2$. The spatial variable is represented by $\mathbf{x}$ and the time variable by t; being the final time denoted by T, although the dependence of the functions on these variables is omitted for the sake of clarity. As usual, a subscript after a comma represents its spatial derivative with respect to that variable, i.e. $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$, and the time derivatives are represented as a dot over each variable for the first order or two dots for the second order. Moreover, the repeated index notation is used to indicate summation.

If we denote by $\mathbf{u} = (u_1, u_2)$ the displacement field, the corresponding mechanical problem that we will numerically study in this paper is the following.

Problem P. *Find the displacement field* $\mathbf{u} : \bar{B} \times [0, T] \to \mathbb{R}^2$ *such that, for* $i = 1, 2,$

$$(\mu - \nu_1 \Delta)\Delta u_i + (\lambda + \mu - \nu_2 \Delta) u_{j,ji} + \rho f_i = \rho \ddot{u}_i \quad \text{in} \quad B \times (0, T), \tag{2.1}$$

$$u_i = \frac{\partial u_i}{\partial \boldsymbol{n}} = 0 \quad \text{on} \quad \partial B \times (0, T), \tag{2.2}$$

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in B. \tag{2.3}$$

Here, ρ denotes the material density (from now on, in order to simplify the writing, we assume that it is equal to one), λ and μ are the Lame's parameters, ν_1 and ν_2 are constitutive coefficients related to the strain gradient model and Δ represents the Laplacian operator. Finally, $\mathbf{u}^0 = (u_i^0)_{i=1}^2$ and $\mathbf{v}^0 = (v_i^0)_{i=1}^2$ are the initial conditions for the displacement and velocity fields.

In order to provide the numerical analysis of this problem in the next section, we will assume the following conditions on the constitutive parameters:

$$\lambda > 0, \quad \mu > 0, \quad \nu_1 > 0, \quad \nu_2 > 0. \tag{2.4}$$

We note that these conditions are slightly more restrictive that those imposed in [17]; however, we used them for the sake of simplicity.

Now, we obtain the variational form of Problem P. Therefore, we denote by $Y = L^2(B)$, $H = [L^2(B)]^2$, $Q = [L^2(B)]^{2\times 2}$ and $V = [H_0^2(B)]^2$.

By using Green's formula and boundary conditions (2.2), we write the variational formulation of Problem P in terms of the velocity field $\mathbf{v} = \dot{\mathbf{u}}$.

Problem VP. *Find the velocity field* $\mathbf{v} : [0, T] \to V$ *such that, for a.e.* $t \in (0, T)$ *and for all* $\mathbf{w} \in V$,

$$\big(\dot{\mathbf{v}}(t), \mathbf{w}\big)_H + \nu_1 \big(\Delta \mathbf{u}(t), \Delta \mathbf{w}\big)_H + \nu_2 \big(\nabla \mathrm{div}\, \mathbf{u}(t), \nabla \mathrm{div}\, \mathbf{w}\big)_H$$

$$+\mu(\nabla \mathbf{u}(t), \nabla \mathbf{w})_Q + (\lambda+\mu)(\operatorname{div}\mathbf{u}(t), \operatorname{div}\mathbf{w})_Y = \big(\mathbf{f}(t),\, \mathbf{w}\big)_H. \tag{2.5}$$

The following theorem which states the existence of a unique solution to Problem VP was proved in [15].

Theorem 2.1. *Under the assumptions (2.4) there exists a unique solution to Problem VP with the following regularity:*

$$\mathbf{u} \in C^2([0,T];H) \cap C^1([0,T];V).$$

We omit the details for the sake of clarity because it is not the aim of this work.

3. Fully discrete approximation by using C^0 interior penalty methods

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we denote by $\mathcal{T}^h$ a regular triangulation of B in the sense of [6]. Thus, according to [4] we construct the finite dimensional space V^h given by

$$V^h = \{\mathbf{w}^h \in [C(\overline{B})]^2 \cap [H_0^1(B)]^2\ ;\ \mathbf{w}^h_{|Tr} \in [P_2(Tr)]^2 \quad \forall Tr \in \mathcal{T}^h\},$$

where $P_2(Tr)$ represents the space of polynomials of degree less or equal to two in the element Tr. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\mathbf{u}_0^h$ and $\mathbf{v}_0^h$, are given by

$$\mathbf{u}_0^h = \mathcal{P}^h \mathbf{u}^0, \quad \mathbf{v}_0^h = \mathcal{P}^h \mathbf{v}^0, \tag{3.1}$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over V^h (see, e.g., [6]).

We will also use the following notations:

- h_{T_r}: diameter of element T_r (so, $h = \max\limits_{h_{T_r}\in\mathcal{T}^h} h_{T_r}$).
- $\mathcal{E}_i^h$: set of all interior edges of $\mathcal{T}^h$.
- $\mathcal{E}_b^h$: set of all boundary edges of $\mathcal{T}^h$.
- $\mathcal{E}^h$: set of all edges of $\mathcal{T}^h$ (that is, $\mathcal{E}^h = \mathcal{E}_i^h \cup \mathcal{E}_b^h$).

Let $e \in \mathcal{E}_i^h$ be the edge shared by two neighboring triangles T_{r+} and T_{r-} (i.e. $e \in \partial T_{r+} \cap \partial T_{r-}$). Moreover, let us denote by $\boldsymbol{n}_+$ the unit normal vector of e pointing from T_{r+} to T_{r-}, and $\boldsymbol{n}_-$ the unit normal vector of e pointing from T_{r-} to T_{r+} (which implies that $\boldsymbol{n}_+ = -\boldsymbol{n}_-$). For any scalar valued function $f \in H^2(T_r)$, we define the jumps $[\![\cdot]\!]$ and the averages $\{\!\{\cdot\}\!\}$ across the edge e as follows:

$$\left[\!\!\left[\frac{\partial f}{\partial \boldsymbol{n}}\right]\!\!\right] = \frac{\partial f_+}{\partial \boldsymbol{n}_e} - \frac{\partial f_-}{\partial \boldsymbol{n}_e}, \quad \left\{\!\!\left\{\frac{\partial^2 f}{\partial \boldsymbol{n}^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 f_+}{\partial \boldsymbol{n}_e^2} + \frac{\partial^2 f_-}{\partial \boldsymbol{n}_e^2}\right).$$

If $e \in \mathcal{E}_b^h$ (that is, e is part of the boundary of B), then the jumps and the averages are given by ($\boldsymbol{n}_e$ represents now the unit normal vector pointing outside B):

$$\left[\!\!\left[\frac{\partial f}{\partial \boldsymbol{n}}\right]\!\!\right] = -\frac{\partial f}{\partial \boldsymbol{n}_e}, \quad \left\{\!\!\left\{\frac{\partial^2 f}{\partial \boldsymbol{n}^2}\right\}\!\!\right\} = \frac{\partial^2 f}{\partial \boldsymbol{n}_e^2}.$$

Secondly, we consider a partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$.

Problem VP^{hk}. *Find the discrete velocity field* $\mathbf{v}^{hk} = \{\mathbf{v}_n^{hk}\}_{n=0}^N \subset V^h$ *such that, for* $n = 1, 2, \ldots, N$ *and for all* $\mathbf{w}^h \in V^h$,

$$\big((\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk})/k, \mathbf{w}^h\big)_H + A^h(\mathbf{u}_n^{hk}, \mathbf{w}^h) = \big(\mathbf{f}_n, \mathbf{w}^h\big)_H, \tag{3.2}$$

where the discrete displacement field $\mathbf{u}_n^{hk}$ *is obtained as*

$$\mathbf{u}_n^{hk} = k \sum_{j=1}^{n} \mathbf{v}_j^{hk} + \mathbf{u}_0^h.$$

In equation (3.2) the bilinear form $A^h(\cdot,\cdot)$ is given by

$$\begin{aligned} A^h(\mathbf{u},\mathbf{w}) = \sum_{T_r\in\mathcal{T}^h} \int_{T_r} & \nu_1 \Delta\mathbf{u}\cdot\Delta\mathbf{w} + \nu_2 \nabla \mathrm{div}\,\mathbf{u}\cdot\nabla\mathrm{div}\,\mathbf{w} + \mu\nabla\mathbf{u} : \nabla\mathbf{w} \\ & + (\lambda+\mu)\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{w}\, d\mathbf{x} + \sum_{e\in\mathcal{E}^h} \frac{\alpha}{|e|} \int_e \left[\!\!\left[\frac{\partial \mathbf{u}}{\partial \boldsymbol{n}} \right]\!\!\right] \left[\!\!\left[\frac{\partial \mathbf{w}}{\partial \boldsymbol{n}} \right]\!\!\right] d\sigma \\ & + \sum_{e\in\mathcal{E}_b^h} \int_e \left\{\!\!\left\{ \frac{\partial^2 \mathbf{w}}{\partial \boldsymbol{n}^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial \mathbf{u}}{\partial \boldsymbol{n}} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 \mathbf{u}}{\partial \boldsymbol{n}^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial \mathbf{w}}{\partial \boldsymbol{n}} \right]\!\!\right] d\sigma, \end{aligned}$$

where $|e|$ denotes the measure of e and α represents a penalty parameter.

Remark 3.1. This problem is solved using the finite element framework FEniCS [13, 19]. The computations are performed in a 3.30Ghz PC with 16Gb of RAM memory, where a typical run with $h = k = 0.015$ takes around 29 seconds of CPU time.

In [4] the authors proved that the bilinear form $A^h(\cdot,\cdot)$ is continuous and, if the penalty parameter α is assumed sufficiently large, coercive on the finite element space V^h with respect to the discrete norm $\|\cdot\|_h$ defined as

$$\begin{aligned} \|\mathbf{w}\|_h^2 = \sum_{T_r\in\mathcal{T}^h} \Big(|\mathbf{w}|^2_{[H^2(T_r)]^2} + |\mathbf{w}|^2_{[H^1(T_r)]^2} \Big) + \sum_{e\in\mathcal{E}^h} |e| \left\| \left\{\!\!\left\{ \frac{\partial \mathbf{w}^2}{\partial \boldsymbol{n}^2} \right\}\!\!\right\} \right\|^2_{L^2(e)} \\ + \sum_{e\in\mathcal{E}^h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial \mathbf{w}}{\partial \boldsymbol{n}} \right]\!\!\right] \right\|^2_{L^2(e)}. \end{aligned}$$

That is, if α is assumed large enough then there exist two positive constants C_1 and C_2 such that

$$|A^h(\mathbf{u},\mathbf{w})| \le C_1 \|\mathbf{u}\|_h \|\mathbf{w}\|_h, \quad A^h(\mathbf{w},\mathbf{w}) \ge C_2 \|\mathbf{w}\|_h^2.$$

We note that this norm $\|\cdot\|_h$ is equivalent to the seminorm $|\cdot|_{H^2(B,\mathcal{T}^h)}$ given by

$$|\mathbf{w}|^2_{[H^2(B,\mathcal{T}^h)]^2} = \sum_{T_r\in\mathcal{T}^h} |\mathbf{w}|^2_{[H^2(T_r)]^2} + \sum_{e\in\mathcal{E}^h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial \mathbf{w}}{\partial \boldsymbol{n}} \right]\!\!\right] \right\|^2_{L^2(e)}.$$

Moreover, as it is pointed out in [4], the size of the penalty parameter α depends only on the shape regularity of $\mathcal{T}^h$.

In the rest of this section we assume that conditions (2.4) hold.

Now, we will obtain some a priori error estimates. First, proceeding as in [4] we find that the solution to Problem VP satisfies the following variational equation:

$$\big(\dot{\mathbf{v}}_n, \mathbf{w}^h\big)_H + A^h(\mathbf{u}_n, \mathbf{w}^h) = \big(\mathbf{f}_n, \mathbf{w}^h\big)_H. \tag{3.3}$$

Subtracting variational (3.3) and (3.2) it follows that

$$\big(\dot{\mathbf{v}}_n - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk})/k, \mathbf{w}^h\big)_H + A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{w}^h) = 0,$$

and so we have, for all $\mathbf{w}^h \in V^h$,

$$\begin{aligned}&\big(\dot{\mathbf{v}}_n - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk})/k, \mathbf{v}_n - \mathbf{v}_n^{hk}\big)_H + A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk})\\ &\quad = \big(\dot{\mathbf{v}}_n - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk})/k, \mathbf{v}_n - \mathbf{w}^h\big)_H + A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{w}^h).\end{aligned}$$

Taking into account that

$$\begin{aligned}\big(\dot{\mathbf{v}}_n - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk})/k, \mathbf{v}_n - \mathbf{v}_n^{hk}\big)_H &= \big(\dot{\mathbf{v}}_n - (\mathbf{v}_n - \mathbf{v}_{n-1})/k, \mathbf{v}_n - \mathbf{v}_n^{hk}\big)_H\\ &\quad + \big((\mathbf{v}_n - \mathbf{v}_{n-1} - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk}))/k, \mathbf{v}_n - \mathbf{v}_n^{hk}\big)_H,\\ \big((\mathbf{v}_n - \mathbf{v}_{n-1} - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk}))/k, \mathbf{v}_n - \mathbf{v}_n^{hk}\big)_H &\geq \frac{1}{2k}\Big[\|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk}\|_H^2\Big],\\ A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) &= A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, \dot{\mathbf{u}}_n - (\mathbf{u}_n - \mathbf{u}_{n-1})/k)\\ &\quad + A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, (\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}))/k),\\ A^h(\mathbf{u}_n - \mathbf{u}_n^{hk}, (\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}))/k) &\geq \frac{1}{2k}\Big[\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_h^2 - \|\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}\|_h^2\Big],\end{aligned}$$

using several times Cauchy's inequality $ab \leq \epsilon a^2 + \dfrac{1}{4\epsilon} b^2$, $a, b, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, it follows that

$$\begin{aligned}&\frac{1}{2k}\Big[\|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk}\|_H^2\Big] + \frac{1}{2k}\Big[\|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_h^2 - \|\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}\|_h^2\Big]\\ &\leq C\Big(\|\dot{\mathbf{v}}_n - (\mathbf{v}_n - \mathbf{v}_{n-1})/k\|_H^2 + \|\dot{\mathbf{u}}_n - (\mathbf{u}_n - \mathbf{u}_{n-1})/k\|_h^2 + \|\mathbf{v}_n - \mathbf{w}^h\|_H^2\\ &\quad + \|\mathbf{v}_n - \mathbf{w}^h\|_h^2 + \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_h^2\\ &\quad + \big((\mathbf{v}_n - \mathbf{v}_{n-1} - (\mathbf{v}_n^{hk} - \mathbf{v}_{n-1}^{hk}))/k, \mathbf{v}_n - \mathbf{w}^h\big)_H\Big).\end{aligned}$$

Summing up to n we have

$$\begin{aligned}&\|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_h^2\\ &\leq Ck\sum_{j=1}^{n}\Big(\|\dot{\mathbf{v}}_j - (\mathbf{v}_j - \mathbf{v}_{j-1})/k\|_H^2 + \|\mathbf{v}_j - \mathbf{w}_j^h\|_H^2\\ &\quad + \|\dot{\mathbf{u}}_j - (\mathbf{u}_j - \mathbf{u}_{j-1})/k\|_h^2 + \|\mathbf{v}_j - \mathbf{w}_j^h\|_h^2 + \|\mathbf{v}_j - \mathbf{v}_j^{hk}\|_H^2 + \|\mathbf{u}_j - \mathbf{u}_j^{hk}\|_h^2\\ &\quad + \big((\mathbf{v}_j - \mathbf{v}_{j-1} - (\mathbf{v}_j^{hk} - \mathbf{v}_{j-1}^{hk}))/k, \mathbf{v}_j - \mathbf{w}_j^h\big)_H\Big) + C(\|\mathbf{v}^0 - \mathbf{v}_0^h\|_H^2 + \|\mathbf{u}^0 - \mathbf{u}_0^h\|_h^2).\end{aligned}$$

Now, keeping in mind that

$$
\begin{aligned}
&k\sum_{j=1}^{n}\big((\mathbf{v}_j-\mathbf{v}_{j-1}-(\mathbf{v}_j^{hk}-\mathbf{v}_{j-1}^{hk}))/k,\mathbf{v}_j-\mathbf{w}_j^h\big)_H\\
=&(\mathbf{v}^0-\mathbf{v}_0^h,\mathbf{v}_n-\mathbf{w}_n^h)_H+(\mathbf{v}_0^h-\mathbf{v}^0,\mathbf{v}_1-\mathbf{w}_1^h)_H\\
&+\sum_{j=1}^{n-1}(\mathbf{v}_j-\mathbf{v}_j^{hk},\mathbf{v}_j-\mathbf{w}_j^h-(\mathbf{v}_{j+1}-\mathbf{w}_{j+1}^h))_H,
\end{aligned}
$$

applying a discrete version of Gronwall's inequality (see [5] for details) we have the following.

Theorem 3.1. *Let the assumptions (2.4) hold. If we denote by* $\mathbf{v}$ *and* $\mathbf{v}^{hk}$ *the respective solutions to problems* VP *and* VP^{hk}*, then we have the following a priori error estimates, for all* $\mathbf{w}^h=\{\mathbf{w}_j^h\}_{j=0}^N\subset V^h$*,*

$$
\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\mathbf{v}_n-\mathbf{v}_n^{hk}\|_H^2+\|\mathbf{u}_n-\mathbf{u}_n^{hk}\|_h^2\Big\}\\
\le& Ck\sum_{j=1}^{N}\Big(\|\dot{\mathbf{v}}_j-(\mathbf{v}_j-\mathbf{v}_{j-1})/k\|_H^2+\|\mathbf{v}_j-\mathbf{w}_j^h\|_H^2+\|\dot{\mathbf{u}}_j-(\mathbf{u}_j-\mathbf{u}_{j-1})/k\|_h^2+\|\mathbf{v}_j-\mathbf{w}_j^h\|_h^2\Big)\\
&+C\max_{0\le n\le N}\|\mathbf{v}_n-\mathbf{w}_n^h\|_H^2+\frac{C}{k}\sum_{j=1}^{N-1}\|\mathbf{v}_j-\mathbf{w}_j^h-(\mathbf{v}_{j+1}-\mathbf{w}_{j+1}^h)\|_H^2\\
&+C\Big(\|\mathbf{v}^0-\mathbf{v}_0^h\|_H^2+\|\mathbf{u}^0-\mathbf{u}_0^h\|_h^2\Big),
\end{aligned}
\tag{3.4}
$$

where $C>0$ *is a positive constant assumed to be independent of the discretization parameters* h *and* k*.*

Now, assume that the solution to Problem VP has the additional regularity:

$$
\mathbf{v}\in C^1([0,T];H)\cap C([0,T];[H^{2+r}(B)]^2)\cap H^1(0,T;[H^1(B)]^2),\tag{3.5}
$$

where we denote by $r\in(1/2,1]$ the index of elliptic regularity (see [4]), then from estimates (3.4) we have the following result which shows the convergence order.

Corollary 3.1. *Let the assumptions of Theorem 3.1 and the additional regularity (3.5) hold. Then, there exists a positive constant* C*, independent of the discretization parameters* h *and* k *such that*

$$
\max_{0\le n\le N}\Big\{\|\mathbf{v}_n-\mathbf{v}_n^{hk}\|_H+\|\mathbf{u}_n-\mathbf{u}_n^{hk}\|_h\Big\}\le Ch^r+k.
$$

4. Numerical results

We consider in all the numerical experiments the domain $B=[0,1]\times[0,1]$, and we denote the spatial coordinates as $\mathbf{x}=(x,\,y)$.

To check the numerical convergence we define an exact solution as follows:

$$
\mathbf{u}(\mathbf{x},t)=(10t^2\,(-x+1)^3\,(-y+1)^3\,e^tx^3y^3,10t^2\,(-x+1)^3\,(-y+1)^3\,e^tx^3y^3),
$$

and the source function $\mathbf{f}$ is obtained from equation (2.1).

4.1. Penalty parameter

Since there is no way of determining the optimal parameter α of the interior penalty method, we perform a parametric study. Thus, we solve Problem VP^{hk} in the domain B given above, with the source function defined from the previous exact solution, and for a final time $T = 1$. The discretization parameters used in this example are: $k = 0.00390625$ and $h = 0.00552427$. We compare the numerical solution with the exact one, computing the total error Err from several terms:

$$Err = \max_{0 \le n \le N} \Big\{ \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H + \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_h \Big\}.$$

We obtain the results presented in Figure 1, showing that the error stabilizes for values of α above 150, approximately. From now on, we consider the value of 250 for this parameter, ensuring that it has no influence on the error. We recall that the theoretical value should be large enough.

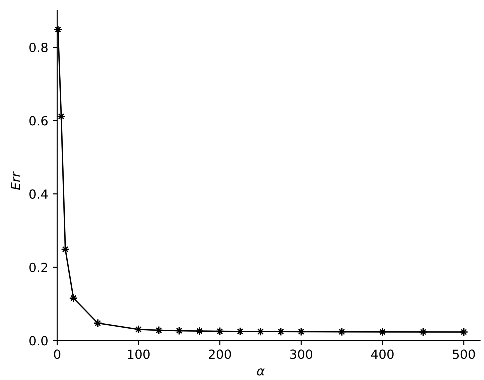

Figure 1. Evolution of the error with respect to parameter α.

4.2. Numerical convergence

As a second example, in order to show the convergence of the discrete solution with a numerical example, we solve the discrete problem in the unit square with the source function defined previously. The final time is considered $T = 1$ again.

In Table 1 we show the error for some values of the discretization parameters h and k. In Figure 2 we show the evolution of the error with respect to parameter $h+k$. We observe that the algorithm converges although the numerical convergence seems to be quadratic.

4.3. Numerical example

In this final example, we will show the solution of the aforementioned problem. We consider the same spatial domain as before and final time $T = 1$. The discretization parameters for this case are $h = 0.01104$ and $k = 0.00195$. In Figure 3 we show the solution for the first components of the velocity and displacement fields at final time. We can see both variables have the same shape, because the time derivative

Table 1. Numerical errors for some discretization parameters.

$h \downarrow k \rightarrow$	2^{-2}	2^{-4}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$7.071e-01$	6.36003	6.31253	6.30359	6.29637	6.29422	6.29232	6.29093
$3.536e-01$	1.57240	1.43045	1.41183	1.40150	1.40019	1.39958	1.39930
$1.768e-01$	1.30932	1.14905	1.12077	1.10044	1.09723	1.09568	1.09494
$8.839e-02$	0.67034	0.49520	0.46257	0.43866	0.43501	0.43329	0.43247
$4.419e-02$	0.42573	0.24647	0.21259	0.18673	0.18259	0.18069	0.17983
$2.210e-02$	0.33152	0.14850	0.11394	0.08733	0.08289	0.08077	0.07992
$1.105e-02$	0.29300	0.10887	0.07393	0.04695	0.04239	0.04019	0.03949
$5.524e-03$	0.27532	0.09101	0.05605	0.02907	0.02451	0.02231	0.02176
$3.536e-03$	0.27001	0.08502	0.05000	0.02299	0.01843	0.01623	0.01573

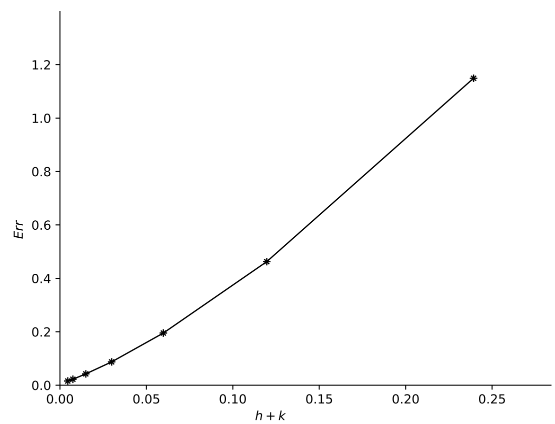

Figure 2. Numerical convergence of the approximation.

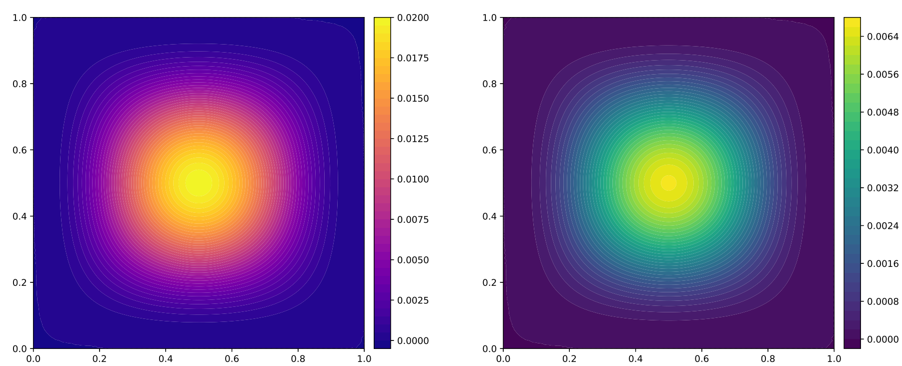

Figure 3. First components of the velocity (left) and displacement (right) fields.

of the theoretical solution has the same spatial dependence. Since the borders of the plate are fixed, the maximum deformation appears in the center as expected.

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