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LIMITING DYNAMICS OF NON-AUTONOMOUS STOCHASTIC GINZBURG-LANDAU EQUATIONS ON THIN DOMAINS

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Abstract We examine the limiting dynamics of a class of non-autonomous stochastic Ginzburg-Landau equations driven by multiplicative noise and deterministic non-autonomous terms defined on thin domains. The existence and uniqueness of tempered pullback random attractors are established for the stochastic Ginzburg-Landau systems defined on $(n+1)$-dimensional narrow domain. In addition, the upper semicontinuity of these attractors is obtained when a family of $(n+1)$-dimensional thin domains collapses onto an n-dimensional domain.

Keywords Stochastic Ginzburg-Landau equation, thin domain, random attractor, upper semicontinuity.

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1. Introduction

The Ginzburg-Landau equation is an important nonlinear evolution equation, which is used to simplify mathematical models for pattern formation in mechanics, physics and chemistry (see [2,5,8] for more details, particularly, physical backgrounds). For the deterministic Ginzburg-Landau equation, the long time behavior of solutions was investigated in [15, 16, 27]. For the stochastic Ginzburg-Landau equation, the study of the random attractor can be found in [17, 26, 28]. Our main interest in this work is to study the dynamics of the stochastic system (1.1) defined on the thin domain $\mathcal{O}_\varepsilon$ for small ε and explore the limiting behavior of the system as $\varepsilon \to 0$.

In this paper, we investigate the asymptotic behavior of solutions of the following non-autonomous stochastic Ginzburg-Landau equations driven by multiplicative noise on $\mathcal{O}_\varepsilon$ for $t > \tau$ with $\tau \in \mathbb{R}$

$$\begin{cases} d\hat{u}^\varepsilon - (1+\mathrm{i}\mu)\Delta\hat{u}^\varepsilon dt + \rho\hat{u}^\varepsilon dt = \left(f(t,x,\hat{u}^\varepsilon) + G(t,x)\right)dt + \hat{u}^\varepsilon \circ dW, \quad x \in \mathcal{O}_\varepsilon, \\ \dfrac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \end{cases} \tag{1.1}$$

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with initial condition

$$\hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon_\tau(x), \quad x \in \mathcal{O}_\varepsilon, \tag{1.2}$$

where $\hat{u}^\varepsilon(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathcal{O}_\varepsilon$. In (1.1), i is the imaginary unit, μ, ρ are real constants and $\rho > 0$. ν_ε is the unit outward normal vector to $\partial\mathcal{O}_\varepsilon$. W is a two-sided real-valued Wiener process on a probability space, the symbol $\circ$ indicates that the equation is understood in the sense of Stratonovich integration. The so-called thin domain $\mathcal{O}_\varepsilon$ is given by

$$\mathcal{O}_\varepsilon = \{x = (x^*, x_{n+1}) | \quad x^* = (x_1, x_2, \cdots, x_n) \in \mathcal{Q},\ 0 < x_{n+1} < \varepsilon g(x^*)\}$$

with $0 < \varepsilon \leq 1$ and $g \in C^2(\overline{\mathcal{Q}}, (0, +\infty))$, where $\mathcal{Q}$ is a smooth bounded domain in $\mathbb{R}^n$. Since $g \in C^2(\overline{\mathcal{Q}}, (0, +\infty))$, there exist two positive constants β_1 and β_2 such that

$$\beta_1 \leq g(x^*) \leq \beta_2, \quad \forall\ x^* \in \overline{\mathcal{Q}}. \tag{1.3}$$

Denote $\mathcal{O} = \mathcal{Q} \times (0,1)$ and $\widetilde{\mathcal{O}} = \mathcal{Q} \times (0, \beta_2)$ which contains $\mathcal{O}_\varepsilon$ for $0 < \varepsilon \leq 1$. The nonlinearity f and the body force G satisfy some conditions which will be specified later.

As $\varepsilon \to 0$, the thin domain $\mathcal{O}_\varepsilon$ collapses to an n-dimensional domain. In this paper, we will see that the limiting behavior of the equation is determined by the following system on the lower dimensional spatial domain $\mathcal{Q}$, for $t > \tau$ with $\tau \in \mathbb{R}$ and $y^* = (y_1, \cdots, y_n) \in \mathcal{Q}$,

$$\begin{cases} du^0 - (1+\mathrm{i}\mu)\dfrac{1}{g}\displaystyle\sum_{i=1}^{n}(g u^0_{y_i})_{y_i} dt + \rho u^0 dt = \big(f(t, y^*, 0, u^0) + G(t, y^*, 0)\big)\, dt + u^0 \circ dW, \\ \dfrac{\partial u^0}{\partial \nu_0} = 0, \quad y^* \in \partial\mathcal{Q}, \end{cases} \tag{1.4}$$

with initial condition

$$u^0(\tau, y^*) = u^0_\tau(y^*), \quad y^* \in \mathcal{Q}, \tag{1.5}$$

where ν_0 is the unit outward normal vector to $\partial\mathcal{Q}$. Note that $u^0_{y_i}$ means $\frac{\partial u^0}{\partial y_i}$ in (1.4) and similar notation will be used throughout this paper.

The study of the asymptotic behavior of deterministic PDEs defined in thin domains was first initiated by Hale and Raugel [9, 10]. Then their results were extended to various problem, see for instance, [1, 4, 7, 12, 13, 18–22].

The Ginzburg-Landau equation is an important nonlinear evolution equation, which is used to simplify mathematical models for pattern formation in mechanics, physics and chemistry. For the deterministic Ginzburg-Landau equation, the long time behavior of solutions was investigated in [15, 16, 27]. For the stochastic Ginzburg-Landau equation, some recent studies can be found in [6, 11, 17, 23, 26, 28]. Particularly, in [14], the authors studied the stochastic Ginzburg-Landau equation on thin domain $\mathcal{O}_\varepsilon \subset \mathbb{R}^2$. Our main interest in this work is to study the dynamics of the stochastic system (1.1) defined on the thin domain $\mathcal{O}_\varepsilon \subset \mathbb{R}^{n+1}$ for small ε with an additional nonlinear term $f(t, x, \hat{u}^\varepsilon)$ under some further restrictions, and explore the limiting behavior of the system as $\varepsilon \to 0$. Our study is a natural extension of the work done in [14] and provides complementary understanding of the dynamics of the stochastic Ginzburg-Landau equation.

We organize the paper as follows. In Section 2, we establish the existence of a continuous cocycle in $L^2(\mathcal{O})$ for the stochastic equation defined on the fixed domain

$\mathcal{O}$, which is converted from (1.1) and (1.2). We also describe the existence of a continuous cocycle in $L^2(\mathcal{Q})$ for the stochastic equation (1.4) and (1.5). In Section 3, we deduce all necessary uniform estimates of the solutions. In Section 4, we prove the existence and uniqueness of tempered attractors for the stochastic equation. Section 5 deals with the upper semicontinuity of these attractors.

2. Cocycles for stochastic Ginzburg-Landau systems

In this section, we will defined a continuous cocycle for the following non-autonomous Ginzburg-Landau systems deriven by multiplicative white noise for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$ and $t > \tau$,

$$\begin{cases} d\hat{u}^\varepsilon - (1 + \mathrm{i}\mu)\Delta\hat{u}^\varepsilon dt + \rho\hat{u}^\varepsilon dt = (f(t, x, \hat{u}^\varepsilon) + G(t, x))\, dt + \hat{u}^\varepsilon \circ dW, \\ \dfrac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \\ \hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon_\tau(x), \end{cases} \tag{2.1}$$

where $\tau \in \mathbb{R}$, $\mu, \rho > 0$ are constants, $G \in L^2_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$. W is a two-sided real-valued Wiener process defined on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ equipped with the compact-open topology, $\mathcal{F} = \mathfrak{B}(\Omega)$ is the Borel sigma-algebra of Ω, $\mathbb{P}$ is the Wiener measure, and $\{\theta_t\}_{t\in\mathbb{R}}$ is the measure-preserving transformation group on Ω given by $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}$. In this paper, f is a nonlinear function, and in the various lemmas that follow we assume f satisfies the following conditions: for all $x \in \widetilde{\mathcal{O}}$, $u \in \mathbb{C}$ and $t, s \in \mathbb{R}$,

$$\mathrm{Re} f(t, x, u)\bar{u} \le -\gamma|u|^p + \psi_1(t, x), \tag{2.2}$$

$$\left|\frac{\partial f(t, x, u)}{\partial u}\right| \le \beta, \tag{2.3}$$

$$\left|\frac{\partial f}{\partial x}(t, x, u)\right| \le \psi_2(t, x), \tag{2.4}$$

where $p \ge 2$, $\psi_1 \in L^1_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, $\psi_2 \in L^2_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, γ and β are positive constants, . In particular, those conditions also hold for the well-known non-gauge interaction function $f = (\alpha + i\beta)|u|^2 u$ (or $f = (1 + i\mu)|u|^{2\sigma} u$ with $\sigma > 0$, particularly for $\sigma = 1$, for many related works), which is widely used for Ginzburg-Landau equations (see [2, 5, 8] for example). In the function f, α represents the nonlinear saturation, which is required to be positive, while β represents the strength of nonlinear dispersion effects.

Next, we transfer the problem (2.1) into a boundary value problem on the fixed domain $\mathcal{O}$. For $0 < \varepsilon \le 1$, we define a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x^*, x_{n+1}) = (x^*, \frac{x_{n+1}}{\varepsilon g(x^*)})$ for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$. Let $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1})$. Then we have $x^* = y^*$, $x_{n+1} = \varepsilon g(y^*) y_{n+1}$. By some calculations, we find that the Jacobian matrix of T_ε is given by

$$J=\frac{\partial(y_1,\cdots,y_{n+1})}{\partial(x_1,\cdots,x_{n+1})}=\begin{pmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & 0\\ -\frac{y_{n+1}}{g}g_{y_1} & -\frac{y_{n+1}}{g}g_{y_2} & \cdots & -\frac{y_{n+1}}{g}g_{y_n} & \frac{1}{\varepsilon g(y^*)}\end{pmatrix}.$$

The determinant of J is $|J|=\frac{1}{\varepsilon g(y^*)}$. Let J^* be the transport of J. Then we have

$$JJ^*=\begin{pmatrix}1 & 0 & \cdots & 0 & -\frac{y_{n+1}}{g}g_{y_1}\\ 0 & 1 & \cdots & 0 & -\frac{y_{n+1}}{g}g_{y_2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & -\frac{y_{n+1}}{g}g_{y_n}\\ -\frac{y_{n+1}}{g}g_{y_1} & -\frac{y_{n+1}}{g}g_{y_2} & \cdots & -\frac{y_{n+1}}{g}g_{y_n} & \sum_{i=1}^{n}\left(\frac{y_{n+1}}{g}g_{y_i}\right)^2+\left(\frac{1}{\varepsilon g(y^*)}\right)^2\end{pmatrix}.$$

It follows from [9] that the gradient operator and the Laplace operator in the original variable $x\in\mathcal{O}_\varepsilon$ and the new variable $y\in\mathcal{O}$ are related by

$$\nabla_x\hat{u}(x)=J^*\nabla_y u(y)\text{ and }\triangle_x\hat{u}(x)=|J|\mathrm{div}_y(|J|^{-1}JJ^*\nabla_y u(y))=\frac{1}{g}\mathrm{div}_y(P_\varepsilon u(y)),$$

where $\hat{u}(x)=u(y)$, ∇_x is the gradient operator in $x\in\mathcal{O}_\varepsilon$, $\triangle_x$ is the Laplace operator in $x\in\mathcal{O}_\varepsilon$, div_y is the divergence operator, ∇_y is the gradient operator in $y\in\mathcal{O}$, and P_ε is the operator given by

$$P_\varepsilon u(y)=\begin{pmatrix} gu_{y_1}-g_{y_1}y_{n+1}u_{y_{n+1}}\\ \vdots\\ gu_{y_n}-g_{y_n}y_{n+1}u_{y_{n+1}}\\ -\sum_{i=1}^{n}y_{n+1}g_{y_i}u_{y_i}+\frac{1}{\varepsilon^2 g}\left(1+\sum_{i=1}^{n}(\varepsilon y_{n+1}g_{y_i})^2\right)u_{y_{n+1}}\end{pmatrix}.$$

In the sequel, for $x=(x^*,x_{n+1})\in\mathcal{O}_\varepsilon$, $y=(y^*,y_{n+1})\in\mathcal{O}$ and $t,s\in\mathbb{R}$, we denote by

$$\begin{aligned}&u^\varepsilon(y)=\hat{u}^\varepsilon(x),\quad f(t,x,s)=f(t,x^*,x_{n+1},s),\\&f_\varepsilon(t,y^*,y_{n+1},s)=f(t,y^*,\varepsilon g(y^*)y_{n+1},s),\quad f_0(t,y^*,s)=f(t,y^*,0,s),\\&G_\varepsilon(t,y^*,y_{n+1})=G(t,y^*,\varepsilon g(y^*)y_{n+1}),\quad G_0(t,y^*)=f(t,y^*,0).\end{aligned}$$

Then, the problem (2.1) is equivalent to the following system for $y=(y^*,y_{n+1})\in\mathcal{O}$

and $t > \tau$,

$$\begin{cases} du^\varepsilon - (1+\mathrm{i}\mu)\dfrac{1}{g}\mathrm{div}_y(P_\varepsilon u^\varepsilon)dt + \rho u^\varepsilon dt = \left(f(t,y,u^\varepsilon) + G_\varepsilon(t,y)\right)dt + u^\varepsilon \circ dW, \\ P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial\mathcal{O}, \\ u^\varepsilon(\tau,y) = u^\varepsilon_\tau(y) = \hat{u}^\varepsilon_\tau(T_\varepsilon^{-1}(y)), \end{cases} \tag{2.5}$$

where ν is the unit outward normal vector to $\partial\mathcal{O}$.

To write the problem (2.5) as an abstract system, we introduce an inner product $(\cdot,\cdot)_{H_g(\mathcal{O})}$ on $L^2(\mathcal{O})$ by

$$(u,v)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} gu\bar{v}dy, \quad \text{for all} \ \ u,v \in L^2(\mathcal{O})$$

and denote by $H_g(\mathcal{O})$ the space equipped with this inner product. Since g is a continuous function on $\overline{\mathcal{Q}}$ and satisfies (1.3), one can easily show that $H_g(\mathcal{O})$ is a Hilbert space with norm equivalent to the natural norm of $L^2(\mathcal{O})$. For $0 < \varepsilon \le 1$, we introduce a bilinear form $a_\varepsilon(\cdot,\cdot) : H^1(\mathcal{O}) \times H^1(\mathcal{O}) \to \mathbb{C}$, given by

$$a_\varepsilon(u,v) = (J^*\nabla_y u, J^*\nabla_y v)_{H_g(\mathcal{O})} \quad \text{for } u,v \in H^1(\mathcal{O}), \tag{2.6}$$

where

$$J^*\nabla_y u = \left(u_{y_1} - \frac{g_{y_1}}{g}y_{n+1}u_{y_{n+1}}, \cdots, u_{y_n} - \frac{g_{y_n}}{g}y_{n+1}u_{y_{n+1}}, \frac{1}{\varepsilon g}u_{y_{n+1}}\right).$$

Let $H^1_\varepsilon(\mathcal{O})$ be the space $H^1(\mathcal{O})$ endowed with norm

$$\|u\|_{H^1_\varepsilon(\mathcal{O})} = \left(\|u\|^2_{H^1(\mathcal{O})} + \frac{1}{\varepsilon^2}\|u_{y_{n+1}}\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}. \tag{2.7}$$

It yields from [9] that there exist positive constants ε_0, η_1 and η_2 such that for all $0 < \varepsilon < \varepsilon_0$ and $u \in H^1(\mathcal{O})$,

$$\eta_1\|u\|^2_{H^1_\varepsilon(\mathcal{O})} \le a_\varepsilon(u,u) + \|u\|^2_{L^2(\mathcal{O})} \le \eta_2\|u\|^2_{H^1_\varepsilon(\mathcal{O})}. \tag{2.8}$$

Denote by A_ε the linear self-adjoint operator

$$A_\varepsilon u = -\frac{1}{g}\mathrm{div}_y(P_\varepsilon u), \quad u \in D(A_\varepsilon) = \left\{u \in H^2(\mathcal{O}) : P_\varepsilon u \cdot \nu = 0 \text{ on } \partial\mathcal{O}\right\}.$$

Then, we have

$$a_\varepsilon(u,v) = (A_\varepsilon u, v)_{H_g(\mathcal{O})}, \quad \forall u \in D(A_\varepsilon), \quad \forall v \in H^1(\mathcal{O}). \tag{2.9}$$

Note that system (2.5) can be rewritten as, for $y \in \mathcal{O},\ t > \tau$,

$$\begin{cases} \dfrac{du^\varepsilon}{dt} + (1+\mathrm{i}\mu)A_\varepsilon u^\varepsilon + \rho u^\varepsilon = f_\varepsilon(t,y,u^\varepsilon) + G_\varepsilon(t,y) + u^\varepsilon \circ \dfrac{dW}{dt}, \\ u^\varepsilon(\tau) = u^\varepsilon_\tau. \end{cases} \tag{2.10}$$

For system (1.4)-(1.5), we introduce an inner product $(\cdot,\cdot)_{H_g(\mathcal{Q})}$ on $L^2(\mathcal{Q})$ by

$$(u,v)_{H_g(\mathcal{Q})} = \int_{\mathcal{Q}} gu\bar{v}dy^*, \quad \text{for all} \ \ u,v \in L^2(\mathcal{Q})$$

and denote by $H_g(\mathcal{Q})$ the space $L^2(\mathcal{Q})$ equipped with this product. Let $a_0(\cdot,\cdot) : H^1(\mathcal{Q}) \times H^1(\mathcal{Q}) \to \mathbb{C}$ be a bilinear form given by

$$a_0(u,v) = \int_{\mathcal{Q}} g\nabla u \cdot \nabla \bar{v} dy^*.$$

Denote by A_0 the unbounded operator on $H_g(\mathcal{Q})$ with domain $D(A_0) = \{u \in H^2(\mathcal{Q}), \frac{\partial u}{\partial \nu_0} = 0 \text{ on } \partial\mathcal{Q}\}$ as defined by

$$A_0 u = -\frac{1}{g}\sum_{i=1}^{n}(g u_{y_i})_{y_i}, \quad u \in D(A_0).$$

Then, we have $a_0(u,v) = (A_0 u, v)_{H_g(\mathcal{Q})}, \quad \forall u \in D(A_0), \ \forall v \in H^1(\mathcal{Q})$. Therefore, system (1.4)-(1.5) can be rewritten as, for $y^* \in \mathcal{Q}, \ t > \tau$,

$$\begin{cases} \dfrac{du^0}{dt} + (1+\mathrm{i}\mu)A_0 u^0 + \rho u^0 = f_0(t,y^*,u^0) + G_0(t,y^*) + u^0 \circ \dfrac{dW}{dt}, \\ u^0(\tau) = u^0_\tau. \end{cases} \tag{2.11}$$

In the rest of this paper, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega,\mathcal{F})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}. \tag{2.12}$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system. It follows from [3] that there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset of full measure (still denoted by Ω) such that

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0 \ \text{ for every } \omega \in \Omega. \tag{2.13}$$

We now convert the stochastic system into a deterministic non-autonomous one. Let $v^\varepsilon(t,\tau,\omega,v^\varepsilon_\tau) = z(t,\omega)u^\varepsilon(t,\tau,\omega,u^\varepsilon_\tau)$ with $z(t,\omega) = e^{-\omega(t)}$. Then v^ε satisfies, for $y \in \mathcal{O}, \ t > \tau$,

$$\begin{cases} \dfrac{dv^\varepsilon}{dt} + (1+\mathrm{i}\mu)A_\varepsilon v^\varepsilon + \rho v^\varepsilon = z(t,\omega) f_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon) + z(t,\omega)G_\varepsilon(t,y), \\ v^\varepsilon(\tau) = v^\varepsilon_\tau. \end{cases} \tag{2.14}$$

Since (2.14) is a deterministic equation which is parametrized by $\omega \in \Omega$, by a Galerkin method, one can show that if f satisfies (2.2)–(2.4), then for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v^\varepsilon_\tau \in L^2(\mathcal{O})$, system (2.14) has a unique solution $v^\varepsilon(\cdot,\tau,\omega,v^\varepsilon_\tau) \in C([\tau,\infty), L^2(\mathcal{O})) \cap L^2((\tau,\tau+T), H^1(\mathcal{O}))$ for every $T > 0$. Furthermore, one may show that $v^\varepsilon(t,\tau,\omega,v^\varepsilon_\tau)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O})))$-measure in $\omega \in \Omega$ and continuous in v^ε_τ with respect to the norm of $L^2(\mathcal{O})$. We now define a mapping $\Psi_\varepsilon : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathcal{O}) \to L^2(\mathcal{O})$ for problem (2.10). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v^\varepsilon_\tau \in L^2(\mathcal{O})$. Let

$$\Psi_\varepsilon(t,\tau,\omega,u^\varepsilon_\tau) = u^\varepsilon(t+\tau,\tau,\theta_{-\tau}\omega,u^\varepsilon_\tau) = \frac{1}{z(t+\tau,\theta_{-\tau}\omega)} v^\varepsilon(t+\tau,\tau,\theta_{-\tau}\omega,v^\varepsilon_\tau), \tag{2.15}$$

where $v_\tau^\varepsilon = z(\tau, \theta_{-\tau}\omega)u_\tau^\varepsilon$. As stated in [24], the mapping Ψ_ε is a continuous cocycle on $L^2(\mathcal{O})$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Let $R_\varepsilon : L^2(\mathcal{O}_\varepsilon) \to L^2(\mathcal{O})$ be an affine mapping of the form $(R_\varepsilon \hat{u}(y)) = \hat{u}(T_\varepsilon^{-1}y), \quad \forall \hat{u} \in L^2(\mathcal{O}_\varepsilon)$. Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\hat{u}_\tau^\varepsilon \in L^2(\mathcal{O}_\varepsilon)$, we can define a continuous cocycle $\hat{\Psi}_\varepsilon$ for problem (2.1) by the formula $\hat{\Psi}_\varepsilon(t, \tau, \omega, u_\tau^\varepsilon) = R_\varepsilon^{-1}\Psi_\varepsilon(t, \tau, \omega, R_\varepsilon u_\tau^\varepsilon)$, where Ψ_ε is the continuous cocycle for problem (2.10) on $L^2(\mathcal{O})$.

Similarly, let $v^0(t, \tau, \omega, v_\tau^0) = z(t, \omega)u^0(t, \tau, \omega, u_\tau^0)$. Then system (2.11) can be transformed into the following equation on $\mathcal{Q}$ with $y^* \in \mathcal{Q},\ t > \tau$,

$$\begin{cases} \dfrac{dv^0}{dt} + (1 + \mathrm{i}\mu)A_0 v^0 + \rho v^0 = z(t, \omega) f_0(t, y^*, z^{-1}(t, \omega)v^0) + z(t, \omega)G_0(t, y^*), \\ v^0(\tau) = v_\tau^0. \end{cases} \tag{2.16}$$

It follows from above arguments that system (2.11) generates a continuous cocycle $\Psi_0(t, \tau, \omega, u_\tau^0)$ in the space $L^2(\mathcal{Q})$.

Denote by $X_\varepsilon = L^2(\mathcal{O}_\varepsilon)$, $X_0 = L^2(\mathcal{Q})$ and $X_1 = L^2(\mathcal{O})$. For each $i = \varepsilon, 0$ or 1, let $D_i = \{D_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of nonempty subsets of X_i. Then, D_i is called tempered (or subexponentially growing) if for every $c > 0$, $\lim\limits_{t\to-\infty} e^{ct}\|D_i(\tau + t, \theta_t\omega)\|_{X_i} = 0$ holds, where $\|D_i\|_{X_i} = \sup_{x\in D_i} \|x\|_{X_i}$. This definition is a straightforward extension of the concept of tempered random subsets for autonomous random dynamical systems. We also denote by $\mathcal{D}_i$ the collection of all families of tempered nonempty subsets of X_i, i.e.,

$$\mathcal{D}_i = \{D_i = \{D_i(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D_i \text{ is tempered in } X_i\}.$$

The following condition will be needed when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\frac{1}{4}\rho s} \left(\|G(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \right) ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{2.17}$$

When constructing tempered pullback attractors for the cocycle Ψ_ε, we will assume for any $\sigma > 0$,

$$\lim_{r\to-\infty} e^{\sigma r} \int_{-\infty}^{0} e^{\frac{1}{4}\rho s} \left(\|G(s + r, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s + r, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \right) ds = 0. \tag{2.18}$$

3. Uniform estimates of solutions

In this section, we derive uniform estimates of solutions for system (2.14). We first derive the estimates of solutions for problem (2.14) in $H_g(\mathcal{O})$.

Lemma 3.1. *Assume* (2.2) *and* (2.17) *hold. There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D_1) > 0$, *independent of* ε, *such that for all* $t \geq T$, *the solution* v^ε *of system* (2.14) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned} &\|v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \\ \leq &M z^{-2}(-\tau, \omega) \int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s, \omega) \Big(1 + \|G(s + \tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s + \tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big) ds \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)}\Big(\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}\\&+z^2(s,\theta_{-\tau}\omega)\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}\Big)ds\\ \le& Mz^{-2}(-\tau,\omega)\int_{-\infty}^0 e^{\frac{1}{2}\rho s}z^2(s,\omega)\Big(1+\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds,\end{aligned} \tag{3.2}$$

where $v^\varepsilon_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ *and* M *is a positive constant depending on* ρ*, but independent of* τ,ω,ε *and* D_1*.*

Proof. Taking the inner product of (2.14) with v^ε in $H_g(\mathcal{O})$ and taking the real part, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon\|^2_{H_g(\mathcal{O})}+\mathrm{Re}(1+\mathrm{i}\mu)(A_\varepsilon v^\varepsilon,v^\varepsilon)_{H_g(\mathcal{O})}+\rho\|v^\varepsilon\|^2_{H_g(\mathcal{O})}\\=&z(t,\omega)\mathrm{Re}(f_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon),v^\varepsilon)_{H_g(\mathcal{O})}+z(t,\omega)\mathrm{Re}(G_\varepsilon(t,y),v^\varepsilon)_{H_g(\mathcal{O})}.\end{aligned} \tag{3.3}$$

For the second term on the left-hand side of (3.3), applying (2.9), we have that

$$\mathrm{Re}(1+\mathrm{i}\mu)(A_\varepsilon v^\varepsilon,v^\varepsilon)_{H_g(\mathcal{O})}=a_\varepsilon(v^\varepsilon,v^\varepsilon). \tag{3.4}$$

For the first term on the right-hand side of (3.3), using (2.2) and (1.3), we obtain that

$$\begin{aligned}&z(t,\omega)\mathrm{Re}(f_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon),v^\varepsilon)_{H_g(\mathcal{O})}\\=&z(t,\omega)\mathrm{Re}\int_{\mathcal{O}} gf_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon)\overline{v^\varepsilon}dy\\=&z^2(t,\omega)\mathrm{Re}\int_{\mathcal{O}} gf_\varepsilon(t,y^*,\varepsilon g(y^*)y_{n+1},u^\varepsilon)\overline{u^\varepsilon}dy\\ \le&-\gamma z^2(t,\omega)\int_{\mathcal{O}} g|u^\varepsilon|^p dy+z^2(t,\omega)\int_{\mathcal{O}} g\psi_1(t,y^*,\varepsilon g(y^*)y_{n+1})dy\\ \le&-\gamma\beta_1 z^2(t,\omega)\int_{\mathcal{O}}|u^\varepsilon|^p dy+cz^2(t,\omega)\|\psi_1(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}.\end{aligned} \tag{3.5}$$

Applying Hölder inequality and Young inequality, the last term on the right-hand side of (3.3) is bounded by

$$\begin{aligned}z(t,\omega)\mathrm{Re}(G_\varepsilon(t,y),v^\varepsilon)_{H_g(\mathcal{O})}&\le z(t,\omega)\|G_\varepsilon(t,y)\|_{H_g(\mathcal{O})}\|v^\varepsilon\|_{H_g(\mathcal{O})}\\&\le\frac{1}{2}\rho\|v^\varepsilon\|^2_{H_g(\mathcal{O})}+\frac{1}{2\rho}z^2(t,\omega)\|G_\varepsilon(t,y)\|^2_{H_g(\mathcal{O})}\\&\le\frac{1}{2}\rho\|v^\varepsilon\|^2_{H_g(\mathcal{O})}+cz^2(t,\omega)\|G(t,y)\|^2_{L^\infty(\widetilde{\mathcal{O}})}.\end{aligned} \tag{3.6}$$

By (3.3)–(3.6), we obtain

$$\begin{aligned}&\frac{d}{dt}\|v^\varepsilon\|^2_{H_g(\mathcal{O})}+\rho\|v^\varepsilon\|^2_{H_g(\mathcal{O})}+2a_\varepsilon(v^\varepsilon,v^\varepsilon)+2\gamma\beta_1 z^2(t,\omega)\int_{\mathcal{O}}|u^\varepsilon|^p dy\\ \le& cz^2(t,\omega)\left(\|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\right).\end{aligned} \tag{3.7}$$

Multiplying (3.7) by $e^{\frac{1}{2}\rho t}$ and integrating the resulting inequality on $(\tau - t, \tau)$ with $\tau \geq 0$, we obtain, for every $\omega \in \Omega$,

$$\begin{aligned}
&\|v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \\
&+ 2\int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} a_\varepsilon(v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t}))ds \\
&+ \frac{1}{2}\rho \int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} \|v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} ds \\
&+ 2\gamma\beta_1 \int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} z^2(s, \omega)\|u^\varepsilon(s, \tau - t, \omega, u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})} ds \\
\leq & e^{-\frac{1}{2}\rho t}\|v^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + ce^{-\frac{1}{2}\rho\tau}\int_{-\infty}^{\tau} e^{\frac{1}{2}\rho s} z^2(s, \omega) \left(\|G(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\right) ds.
\end{aligned} \tag{3.8}$$

Now, replacing ω by $\theta_{-\tau}\omega$ in (3.8), we get

$$\begin{aligned}
&\|v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \\
&+ 2\int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} a_\varepsilon(v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}))ds \\
&+ \frac{1}{2}\rho \int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} ds \\
&+ 2\gamma\beta_1 \int_{\tau-t}^{\tau} e^{\frac{1}{2}\rho(s-\tau)} z^2(s, \theta_{-\tau}\omega)\|u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})} ds \\
\leq & e^{-\frac{1}{2}\rho t}\|v^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} \\
&+ ce^{-\frac{1}{2}\rho\tau} \int_{-\infty}^{\tau} e^{\frac{1}{2}\rho s} z^2(s, \theta_{-\tau}\omega) \left(\|G(s, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\right) ds \\
\leq & e^{-\frac{1}{2}\rho t}\|v^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + cz^{-2}(-\tau, \omega) \int_{-\infty}^{0} e^{\frac{\rho s}{2}} z^2(s, \omega) \big(\|G(s + \tau, y)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \\
&+ \|\psi_1(s + \tau, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\big) ds.
\end{aligned} \tag{3.9}$$

Note that $v^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta_{-t}\omega)$ and D_1 is tempered. It follows that there exists $T = T(\tau, \omega, D_1) > 0$ such that for all $t \geq T$,

$$e^{-\frac{1}{2}\rho t}\|v^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} \leq e^{-\frac{1}{2}\rho t}\|D_1(\tau - t, \theta_{-t}\omega)\|^2_{H_g(\mathcal{O})} \leq z^{-2}(-\tau, \omega) \int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s, \omega) ds.$$

The lemma then follows immediately from (2.8) and (3.9). □

As a consequence of Lemma 3.1, we obtain the following inequality which is useful for deriving the uniform estimates of solutions in $H^1_\varepsilon(\mathcal{O})$.

Lemma 3.2. *Assume that* (2.2) *and* (2.17) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D_1) \geq 1$, *independent of* ε, *such that for all* $t \geq T$, *the solution* v^ε *of system* (2.14) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\int_{\tau-1}^{\tau} \|v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} + z^2(s, \theta_{-\tau}\omega)\|u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})} ds$$

$$\leq M z^{-2}(-\tau,\omega)\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s,\omega)\Big(1+\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds, \tag{3.10}$$

where $v^\varepsilon_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ and M is a positive constant depending on ρ, but independent of τ,ω,ε and D_1.

Proof. Since $e^{\frac{1}{2}\rho(\tau-1)}\leq e^{\frac{1}{2}\rho s}\leq e^{\frac{1}{2}\rho\tau}$ for all $\tau-1\leq s\leq\tau$, we have, for $t\geq 1$,

$$\begin{aligned}
&e^{-\frac{\rho}{2}}\int_{\tau-1}^{\tau}\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}+z^2(s,\theta_{-\tau}\omega)\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}ds\\
&\leq\int_{\tau-1}^{\tau}e^{\frac{1}{2}\rho(s-\tau)}\Big(\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}\\
&\quad+z^2(s,\theta_{-\tau}\omega)\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}\Big)ds\\
&\leq\int_{\tau-t}^{\tau}e^{\frac{1}{2}\rho(s-\tau)}\Big(\|v^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}\\
&\quad+z^2(s,\theta_{-\tau}\omega)\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}\Big)ds.
\end{aligned}$$

Together with Lemma 3.1, the desired result follows. □

We need the following inequality to deduce uniform estimates of solutions v^ε in $H^1_\varepsilon(\mathcal{O})$.

Lemma 3.3. *Assume that (2.2)–(2.4) hold. Then we have for $u\in D(A_\varepsilon)$*

$$Re(f_\varepsilon(t,y,u),A_\varepsilon u)_{H_g(\mathcal{O})}\leq M\left(a_\varepsilon(u,u)+\|\psi_2\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right),$$

where M is a positive constant independent of ε.

Proof. By (2.6) and (2.9), we infer that

$$\begin{aligned}
&\mathrm{Re}\,(f_\varepsilon(t,y,u),A_\varepsilon u)_{H_g(\mathcal{O})}=\mathrm{Re}\ a_\varepsilon\,(f_\varepsilon(t,y,u),u)\\
&=\mathrm{Re}\sum_{i=1}^{n}\int_{\mathcal{O}}\Big(f_{\varepsilon y_i}+f_{\varepsilon u}u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}(f_{\varepsilon y_{n+1}}+f_{\varepsilon u}u_{y_{n+1}})\Big)\Big(\bar{u}_{y_i}-\frac{g_{y_i}}{g}y_{n+1}\bar{u}_{y_{n+1}}\Big)gdy\\
&\quad+\mathrm{Re}\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g}(f_{\varepsilon y_{n+1}}(t,y,u)+f_{\varepsilon u}(t,y,u)u_{y_{n+1}})\bar{u}_{y_{n+1}}dy\\
&=\sum_{i=1}^{n}\int_{\mathcal{O}}f_{\varepsilon u}(t,y,u)\Big|u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}u_{y_{n+1}}\Big|^2gdy\\
&\quad+\mathrm{Re}\sum_{i=1}^{n}\int_{\mathcal{O}}\Big(f_{\varepsilon y_i}(t,y,u)-\frac{g_{y_i}}{g}y_{n+1}f_{\varepsilon y_{n+1}}(t,y,u)\Big)\Big(\bar{u}_{y_i}-\frac{g_{y_i}}{g}y_{n+1}\bar{u}_{y_{n+1}}\Big)gdy\\
&\quad+\mathrm{Re}\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g}f_{\varepsilon y_{n+1}}(t,y,u)\bar{u}_{y_{n+1}}dy+\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g^2}f_{\varepsilon u}(t,y,u)|u_{y_{n+1}}|^2gdy.
\end{aligned}$$

Together with (2.3) and (2.4), one has

$$\mathrm{Re}\,(f_\varepsilon(t,y,u),A_\varepsilon u)_{H_g(\mathcal{O})}=\mathrm{Re}a_\varepsilon\,(f_\varepsilon(t,y,u),u)$$

$$\begin{aligned}
&\leq \beta\ a_\varepsilon(u,u) + \int_{\mathcal{O}} \frac{1}{\varepsilon^2 g} \left| f_{\varepsilon y_{n+1}}(t,y,u) \right| \left| u_{y_{n+1}} \right| dy \\
&\quad + \sum_{i=1}^{n} \int_{\mathcal{O}} \left| f_{\varepsilon y_i}(t,y,u) - \frac{g_{y_i}}{g} y_{n+1} f_{\varepsilon y_{n+1}}(t,y,u) \right| \left| u_{y_i} - \frac{g_{y_i}}{g} y_{n+1} u_{y_{n+1}} \right| g dy \\
&\leq \beta\ a_\varepsilon(u,u) + \frac{1}{2}\ a_\varepsilon(u,u) + \frac{1}{2} \int_{\mathcal{O}} \frac{1}{\varepsilon^2 g^2} f^2_{\varepsilon y_{n+1}}(t,y,u) g dy \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \int_{\mathcal{O}} \Big(f_{\varepsilon y_i}(t,y,u) - \frac{g_{y_i}}{g} y_{n+1} f_{\varepsilon y_{n+1}}(t,y,u) \Big)^2 g dy \\
&\leq \Big(\beta + \frac{1}{2} \Big)\ a_\varepsilon(u,u) + c \|\psi_2\|^2_{L^\infty(\widetilde{\mathcal{O}})}.
\end{aligned}$$

This completes the proof. □

Lemma 3.4. *Assume that* (2.2)–(2.4) *and* (2.17) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau,\omega,D_1) \geq 1$, *independent of* ε, *such that for all* $t \geq T$, *the solution* v^ε *of system* (2.14) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned}
&\|v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} \\
&\leq M z^{-2}(-\tau,\omega) \int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s,\omega) \Big(1 + \|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \\
&\quad + \|\psi_1(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \Big) ds, \qquad (3.11)
\end{aligned}$$

where $v^\varepsilon_{\tau-t} \in D_1(\tau-t,\theta_{-t}\omega)$ *and* M *is a positive constant depending on* ρ, *but independent of* τ,ω,ε *and* D_1.

Proof. Taking the inner product of (2.14) with $A_\varepsilon v^\varepsilon$ in $H_g(\mathcal{O})$ and taking the real part, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\ a_\varepsilon(v^\varepsilon,v^\varepsilon) + \|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})} + \rho\ a_\varepsilon(v^\varepsilon,v^\varepsilon) \\
&= z(t,\omega)\mathrm{Re}(f_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon), A_\varepsilon v^\varepsilon)_{H_g(\mathcal{O})} \\
&\quad + z(t,\omega)\mathrm{Re}(G_\varepsilon(t,y), A_\varepsilon v^\varepsilon)_{H_g(\mathcal{O})}. \qquad (3.12)
\end{aligned}$$

For the first term of the right-hand side of (3.12), by Lemma 3.3, we have

$$\begin{aligned}
&z(t,\omega)\mathrm{Re}(f_\varepsilon(t,y,z^{-1}(t,\omega)v^\varepsilon), A_\varepsilon v^\varepsilon)_{H_g(\mathcal{O})} \\
&= z^2(t,\omega)\mathrm{Re}(f_\varepsilon(t,y,u^\varepsilon), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})} \\
&\leq c z^2(t,\omega) \Big(\ a_\varepsilon(u^\varepsilon,u^\varepsilon) + \|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \Big) \\
&= c a_\varepsilon(v^\varepsilon,v^\varepsilon) + c z^2(t,\omega)\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}. \qquad (3.13)
\end{aligned}$$

For the second term of the right-hand side of (3.12), applying Young's inequality, we get

$$\begin{aligned}
z(t,\omega)\mathrm{Re}(G_\varepsilon(t,y), A_\varepsilon v^\varepsilon)_{H_g(\mathcal{O})} &\leq \frac{1}{2}\|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})} + \frac{1}{2} z^2(t,\omega)\|G_\varepsilon(t,y)\|^2_{H_g(\mathcal{O})} \\
&\leq \frac{1}{2}\|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})} + c z^2(t,\omega)\|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}. \qquad (3.14)
\end{aligned}$$

By (3.12)–(3.14), we deduce

$$\begin{aligned}&\frac{d}{dt}\ a_\varepsilon(v^\varepsilon, v^\varepsilon) + \|A_\varepsilon v^\varepsilon\|^2_{H_g(\mathcal{O})} + 2\rho a_\varepsilon(v^\varepsilon, v^\varepsilon)\\ \le& c a_\varepsilon(v^\varepsilon, v^\varepsilon) + c z^2(t,\omega)\left(\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right),\end{aligned} \tag{3.15}$$

which implies that

$$\frac{d}{dt}\ a_\varepsilon(v^\varepsilon, v^\varepsilon) \le c a_\varepsilon(v^\varepsilon, v^\varepsilon) + c z^2(t,\omega)\left(\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right). \tag{3.16}$$

Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $s \in (\tau - 1, \tau)$, by integrating (3.16) on (s, τ) we know that

$$\begin{aligned}&a_\varepsilon\left(v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau-t})\right)\\ \le& a_\varepsilon\left(v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t})\right)\\ &+ c\int_s^\tau a_\varepsilon\left(v^\varepsilon(\xi, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(\xi, \tau - t, \omega, v^\varepsilon_{\tau-t})\right) d\xi\\ &+ c\int_s^\tau z^2(\xi,\omega)\left(\|\psi_2(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) d\xi.\end{aligned}$$

We now integrate the above with respect to s on $(\tau - 1, \tau)$ to obtain

$$\begin{aligned}&a_\varepsilon\left(v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(\tau, \tau - t, \omega, v^\varepsilon_{\tau-t})\right)\\ \le& \int_{\tau-1}^\tau a_\varepsilon\left(v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t})\right) ds\\ &+ c\int_{\tau-1}^\tau a_\varepsilon\left(v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \omega, v^\varepsilon_{\tau-t})\right) ds\\ &+ c\int_{\tau-1}^\tau z^2(s,\omega)\left(\|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds.\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$, we obtain that

$$\begin{aligned}&a_\varepsilon\left(v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\right)\\ \le&(c+1)\int_{\tau-1}^\tau a_\varepsilon\left(v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t}), v^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\right) ds\\ &+ c\int_{\tau-1}^\tau z^2(s,\theta_{-\tau}\omega)\left(\|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds.\end{aligned} \tag{3.17}$$

Note that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned}&\int_{\tau-1}^\tau z^2(s,\theta_{-\tau}\omega)\left(\|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds\\ =&z^{-2}(-\tau,\omega)\int_{\tau-1}^\tau z^2(s-\tau,\omega)\left(\|\psi_2(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds\\ =&z^{-2}(-\tau,\omega)\int_{-1}^0 z^2(s,\omega)\left(\|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds\\ \le&e^{\frac{1}{2}\rho}z^{-2}(-\tau,\omega)\int_{-1}^0 e^{\frac{1}{2}\rho s}z^2(s,\omega)\left(\|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right) ds\end{aligned}$$

$$\leq e^{\frac{1}{2}\rho} z^{-2}(-\tau,\omega)\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s,\omega)\left(\|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds.$$

Let $T=T(\tau,\omega,D_1)\geq 1$ be the positive number established in Lemma 3.2. Then it follows from (3.17), (3.17) and Lemma 3.2 that, for all $t\geq T$ and for all $\omega\in\Omega$,

$$\begin{aligned}
&a_\varepsilon\left(v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t}),v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\right)\\
\leq & c z^{-2}(-\tau,\omega)\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s,\omega)\Big(1+\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\\
&+\|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds,
\end{aligned}$$

which, together with Lemma 3.1, completes the proof. □

4. Existence of pullback random attractors

In this section, we establish the existence of $\mathcal{D}_1$-pullback attractor for the cocycle Ψ_ε associated with the stochastic problem (2.10) and $\mathcal{D}_0$-pullback attractor for the cocycle Ψ_0 associated with the stochastic problem (2.11), respectively. We first show that problem (2.10) has tempered pullback absorbing set as stated below.

Lemma 4.1. *Suppose that* (2.2)–(2.4), (2.17) *and* (2.18) *hold. Then, there exists* $\varepsilon_0>0$ *such that for every* $0<\varepsilon<\varepsilon_0$, *the continuous cocycle* Ψ_ε *associated with problem* (2.10) *has a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $K\in\mathcal{D}_1$ *which is given by, for every* $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$ $K(\tau,\omega)=\left\{u\in L^2(\mathcal{O}):\|u\|^2_{L^2(\mathcal{O})}\leq L(\tau,\omega)\right\}$, *where*

$$\begin{aligned}
L(\tau,\omega)=&M\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^2(s,\omega)\Big(1+\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\\
&+\|\psi_2(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds
\end{aligned}$$

and M *is a positive constant depending on* ρ, *but independent of* τ, ω, ε *and* D_1.

Proof. Given $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$, define a new family $\widetilde{D}_1$ for D_1 as

$$\begin{aligned}
\widetilde{D}_1=&\Big\{\widetilde{D}_1(\tau,\omega):\widetilde{D}_1(\tau,\omega)=\{v\in L^2(\mathcal{O}):\|v\|_{L^2(\mathcal{O})}\leq z^{-1}(-\tau,\omega)\|D_1(\tau,\omega)\|_{L^2(\mathcal{O})}\},\\
&\tau\in\mathbb{R},\omega\in\Omega\Big\}.
\end{aligned}$$

For any $D_1\in\mathcal{D}_1$, by (2.13) one can check that $\widetilde{D}_1$ also belongs to $\mathcal{D}_1$, i.e., $\widetilde{D}_1$ is tempered. For any $u^\varepsilon_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$, we find that $v^\varepsilon_{\tau-t}=z(\tau-t,\theta_{-\tau}\omega)u^\varepsilon_{\tau-t}$ satisfies

$$\begin{aligned}
\|v^\varepsilon_{\tau-t}\|_{L^2(\mathcal{O})}&=\|z(\tau-t,\theta_{-\tau}\omega)u^\varepsilon_{\tau-t}\|_{L^2(\mathcal{O})}\\
&\leq z^{-1}(t-\tau,\theta_{-t}\omega)\|D_1(\tau-t,\theta_{-t}\omega)\|_{L^2(\mathcal{O})}.
\end{aligned}\tag{4.1}$$

By (4.1), we obtain that $v^\varepsilon_{\tau-t}\in\widetilde{D}_1(\tau-t,\theta_{-t}\omega)$. Since $\widetilde{D}_1\in\mathcal{D}_1$, by Lemma 3.4, there exists $T=T(\tau,\omega,D_1)\geq 1$ such that for all $t\geq T$,

$$\|v^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}$$

$$\leq M z^{-2}(-\tau,\omega)\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^{2}(s,\omega)\Big(1+\|G(s+\tau,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} + \|\psi_2(s+\tau,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds. \tag{4.2}$$

Notice that $v^{\varepsilon}(t,\tau,\omega,v^{\varepsilon}_{\tau}) = z(t,\omega)u^{\varepsilon}(t,\tau,\omega,u^{\varepsilon}_{\tau})$. This implies

$$\begin{aligned} v^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,v^{\varepsilon}_{\tau-t}) &= z(\tau,\theta_{-\tau}\omega)u^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,u^{\varepsilon}_{\tau-t}) \\ &= z^{-1}(-\tau,\omega)u^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,u^{\varepsilon}_{\tau-t}), \end{aligned}$$

which along with (4.2) implies that for $u^{\varepsilon}_{\tau-t} \in D_1(\tau-t,\theta_{-t}\omega)$

$$\|u^{\varepsilon}(\tau,\tau-t,\theta_{-\tau}\omega,u^{\varepsilon}_{\tau-t})\|^{2}_{H^{1}_{\varepsilon}(\mathcal{O})} \leq L(\tau,\omega). \tag{4.3}$$

Therefore, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D_1\in\mathcal{D}_1$, there exists $T=T(\tau,\omega,D_1)\geq 1$, independent of ε, such that for all $t\geq T$, $\Psi_{\varepsilon}(t,\tau-t,\theta_{-t}\omega,D_1(\tau-t,\theta_{-t}\omega))\subseteq K(\tau,\omega)$.

Next, we prove $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is tempered. Given $\sigma>0$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, we deduce

$$\begin{aligned} &e^{\sigma r}\|K(\tau+r,\theta_r\omega)\|_{L^2(\mathcal{O})} \leq e^{\sigma r}L(\tau+r,\theta_r\omega) \\ =&M e^{\sigma r}\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} z^{2}(s,\theta_r\omega)\Big(1+\|G(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} \\ &+\|\psi_1(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} + \|\psi_2(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds \\ =&M e^{\sigma r}\int_{-\infty}^{0} e^{\frac{1}{2}\rho s} e^{2(\omega(r)-\omega(r+s))}\Big(1+\|G(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} \\ &+\|\psi_1(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} + \|\psi_2(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds. \end{aligned}$$

Let $0<c<\min\{\rho/8,\sigma/4\}$. By (2.13), for every $\omega\in\Omega$, there exists $T_1=T_1(\omega)<0$ such that for all $r\leq T_1$ and $s<0$, $|\omega(r)|\leq -cr$, $|\omega(r+s)|\leq -c(r+s)$. Consequently, we infer that for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$

$$\begin{aligned} &\limsup_{r\to-\infty} e^{\sigma r}\|K(\tau+r,\theta_r\omega)\|_{L^2(\mathcal{O})} \\ \leq& M\limsup_{r\to-\infty} e^{(\sigma-4c)r}\int_{-\infty}^{0} e^{(\frac{1}{2}\rho-2c)s}\Big(1+\|G(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} \\ &+\|\psi_1(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} + \|\psi_2(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds \\ \leq& M\limsup_{r\to-\infty} e^{(\sigma-4c)r}\int_{-\infty}^{0} e^{\frac{1}{4}\rho s}\Big(1+\|G(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} \\ &+\|\psi_2(s+\tau+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds \\ \leq& M e^{-\frac{1}{4}\rho\tau}\limsup_{r\to-\infty} e^{(\sigma-4c)r}\int_{-\infty}^{\tau} e^{\frac{1}{4}\rho s}\Big(1+\|G(s+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}+\|\psi_1(s+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})} \\ &+\|\psi_2(s+r,\cdot)\|^{2}_{L^{\infty}(\widetilde{\mathcal{O}})}\Big) ds. \end{aligned}$$

from which, together with (2.17) and (2.18), we deduce that

$$\lim_{r\to-\infty} e^{\sigma r}\|K(\tau+r,\theta_r\omega)\|_{L^2(\mathcal{O})}=0$$

and hence $K(\tau,\omega)$ is tempered in $L^2(\mathcal{O})$. On the other hand, it is evident that, for every $\tau\in\mathbb{R}$, $L(\tau,\cdot):\Omega\to\mathbb{R}$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable. Consequently, K is a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ_ε in $\mathcal{D}_1$. □

Theorem 4.1. *Suppose that (2.2)–(2.4), (2.17) and (2.18) hold. Then there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$, the continuous cocycle Ψ_ε has a unique $\mathcal{D}_1$-pullback attractor $\mathcal{A}_\varepsilon=\{\mathcal{A}_\varepsilon(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_1$ in $L^2(\mathcal{O})$. In addition, if G,f,ψ_1,ψ_2 are T-periodic with respect to t with $T>0$, then the attractor $\mathcal{A}_\varepsilon$ is also T-periodic.*

Proof. From Lemma 4.1, we know that Ψ_ε has a closed measurable $\mathcal{D}_1$-pullback absorbing set K. Applying (4.3) and the compact embedding $H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$, we get that Ψ_ε is $\mathcal{D}_1$-pullback asymptotically compact in $L^2(\mathcal{O})$. Hence, we obtain the existence of a unique $\mathcal{D}_1$-pullback attractor for the cocycle Ψ_ε following from [25] immediately. If G,f,ψ_1,ψ_2 are T-periodic with respect to t, then the continuous cocycle Ψ_ε and the absorbing set K are also T-periodic, which implies the T-periodicity of the attractor. □

Similar results also hold for the solutions of the problem (2.11), more precisely, we have

Theorem 4.2. *Suppose that (2.2)–(2.4), (2.17) and (2.18) hold. Then the continuous cocycle Ψ_0 has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0=\{\mathcal{A}_0(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}_0$ in $L^2(\mathcal{O})$. In addition, if G,f,ψ_1,ψ_2 are T-periodic with respect to t with $T>0$, then the attractor $\mathcal{A}_0$ is also T-periodic.*

5. Upper-semicontinuity of random attractors

In this section, we establish the upper semicontinuity of the random attractor $\mathcal{A}_\varepsilon$. To get started, we derive the uniform estimates of solutions.

Lemma 5.1. *Suppose that (2.2)–(2.4) hold. Then there exists $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$, $\tau\in\mathbb{R}$, $\omega\in\Omega$, $T>0$ and $v^\varepsilon_\tau\in H_g(\mathcal{O})$, the solution v^ε of (2.14) satisfies, for all $t\in[\tau,\tau+T]$,*

$$\begin{aligned}&\int_\tau^t \|v^\varepsilon(s,\tau,\omega,v^\varepsilon_\tau)\|^2_{H^1_\varepsilon(\mathcal{O})}ds\\ \le& M\|v^\varepsilon_\tau\|^2_{H_g(\mathcal{O})}+M\int_\tau^{\tau+T}\left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds,\end{aligned}$$

where M is a positive constant depending on τ,ω,ρ and T, but independent of ε.

Proof. Multiplying (3.7) by $e^{\frac12\rho t}$ and then integrating the resulting inequality on (τ,t), we deduce that for every $\omega\in\Omega$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\|v^\varepsilon(t,\tau,\omega,v^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})}+2\int_\tau^t e^{\frac12\rho(s-t)}a_\varepsilon(v^\varepsilon(s,\tau,\omega,v^\varepsilon_\tau),v^\varepsilon(s,\tau,\omega,v^\varepsilon_\tau))ds\\ &+\frac12\rho\int_\tau^t e^{\frac12\rho(s-t)}\|v^\varepsilon(s,\tau,\omega,v^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})}ds\end{aligned}$$

$$
\begin{aligned}
&+ 2\gamma\beta_1 \int_\tau^t e^{\frac{1}{2}\rho(s-t)} z^2(s,\omega) \|u^\varepsilon(s,\tau,\omega,u_\tau^\varepsilon)\|^p_{L^p(\mathcal{O})} ds \\
\le& e^{-\frac{1}{2}\rho(t-\tau)} \|v_\tau^\varepsilon\|^2_{H_g(\mathcal{O})} + c\int_\tau^t e^{\frac{1}{2}\rho(s-t)} z^2(s,\omega) \left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \right) ds \\
\le& \|v_\tau^\varepsilon\|^2_{H_g(\mathcal{O})} + c\int_\tau^{\tau+T} z^2(s,\omega) \left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \right) ds \\
\le& \|v_\tau^\varepsilon\|^2_{H_g(\mathcal{O})} + c \max_{\tau\le s\le \tau+T} z^2(s,\omega) \int_\tau^{\tau+T} \left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \right) ds,
\end{aligned}
\tag{5.1}
$$

which along with the same argument as Lemma 3.2 completes the proof. □

Similarly, we can obtain the following estimates.

Lemma 5.2. *Suppose that* (2.2)–(2.4) *hold. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ *and* $v_\tau^0 \in H_g(\mathcal{O})$, *the solution* v^0 *of* (2.16) *satisfies, for all* $t \in [\tau, \tau+T]$,

$$
\begin{aligned}
&\int_\tau^t \|v^0(s,\tau,\omega,v_\tau^0)\|^2_{H^1(\mathcal{O})} ds \\
\le& M\|v_\tau^0\|^2_{H_g(\mathcal{O})} + M\int_\tau^{\tau+T} \left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \right) ds,
\end{aligned}
$$

where M *is a positive constant depending on* τ, ω, ρ *and* T, *but independent of* ε.

Given $u \in L^2(\mathcal{O})$, let $\mathcal{M}u$ be the average function of u in y_{n+1} as defined by

$$
\mathcal{M}u = \int_0^1 u(y^*, y_{n+1}) dy_{n+1}.
$$

Then following result on the average function can be found in [9].

Lemma 5.3. *If* $u \in H^1(\mathcal{O})$, *then* $\mathcal{M}u \in H^1(\mathcal{Q})$ *and* $\|u - \mathcal{M}u\|_{H_g(\mathcal{O})} \le c\varepsilon\|u\|_{H^1_\varepsilon(\mathcal{O})}$, *which* c *is a constant, independent of* ε.

In the sequel, we further assume the functions f and G satisfy

$$
\|f_\varepsilon(t,\cdot,s) - f_0(t,\cdot,s)\|_{L^2(\mathcal{O})} \le \varphi_1(t)\varepsilon, \quad \text{for all } t, s \in \mathbb{R}, \tag{5.2}
$$

$$
\|G_\varepsilon(t,\cdot) - G_0(t,\cdot)\|_{L^2(\mathcal{O})} \le \varphi_2(t)\varepsilon, \quad \text{for all } t \in \mathbb{R}, \tag{5.3}
$$

where $\varphi_1(t), \varphi_2(t) \in L^2_{loc}(\mathbb{R})$. Since $L^2(\mathcal{Q})$ can be embedded naturally into $L^2(\mathcal{O})$ as the subspace of functions independent of y_{n+1}, we can consider the cocycle Ψ_0 as a mapping from $L^2(\mathcal{Q})$ into $L^2(\mathcal{O})$. In this sense, we can compare Ψ_0 and Ψ_ε.

Theorem 5.1. *Suppose that* (2.2)–(2.4) *and* (5.2)–(5.3) *hold. Given* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and a positive number* $\eta(\tau,\omega)$, *if* $u_\tau^\varepsilon \in H^1_\varepsilon(\mathcal{O})$*such that* $\|u_\tau^\varepsilon\|_{H^1_\varepsilon(\mathcal{O})} \le \eta(\tau,\omega)$, *then, for any* $t \ge \tau$,

$$
\lim_{\varepsilon\to 0} \|\Psi_\varepsilon(t,\tau,\omega,u_\tau^\varepsilon) - \Psi_0(t,\tau,\omega,\mathcal{M}u_\tau^\varepsilon)\|_{L^2(\mathcal{O})} = 0.
$$

Proof. Taking the inner product of (2.16) with $g\phi$, where $\phi \in H^1(\mathcal{Q})$, we infer that

$$
\int_{\mathcal{Q}} g\frac{dv^0}{dt}\bar{\phi} dy^* + (1+\mathrm{i}\mu)\sum_{i=1}^n \int_{\mathcal{Q}} g v^0_{y_i}\bar{\phi}_{y_i} dy^* + \rho\int_{\mathcal{Q}} g v^0 \bar{\phi} dy^*
$$

$$=z(t,\omega)\int_{\mathcal{Q}} gf(t,y^*,0,z^{-1}(t,\omega)v^0)\bar{\phi}dy^* + z(t,\omega)\int_{\mathcal{Q}} gG(t,y^*,0)\bar{\phi}dy^*.$$

If $\xi \in H^1(\mathcal{O})$, then $\int_0^1 \xi(y^*,y_{n+1})dy_{n+1} \in H^1(\mathcal{Q})$. So for any $\xi \in H^1(\mathcal{O})$, we have

$$\begin{aligned}&\left(\frac{dv^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)\sum_{i=1}^n \left(v^0_{y_i},\xi_{y_i}\right)_{H_g(\mathcal{O})} + \rho\left(v^0,\xi\right)_{H_g(\mathcal{O})}\\ =&z(t,\omega)\left(f(t,y^*,0,z^{-1}(t,\omega)v^0),\xi\right)_{H_g(\mathcal{O})} + z(t,\omega)\left(G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})}.\end{aligned}$$

Since v^0 is independent of y_{n+1}, the above equality gives, for any $\xi \in H^1(\mathcal{O})$ and $0<\varepsilon\le 1$,

$$\begin{aligned}&\left(\frac{dv^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)a_\varepsilon\left(v^0,\xi\right) + \rho\left(v^0,\xi\right)_{H_g(\mathcal{O})}\\ =&z(t,\omega)\left(f(t,y^*,0,z^{-1}(t,\omega)v^0),\xi\right)_{H_g(\mathcal{O})}\\ &+ z(t,\omega)\left(G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})} - (1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}v^0_{y_i}, y_{n+1}\xi_{y_{n+1}}\right)_{H_g(\mathcal{O})}.\end{aligned}\tag{5.4}$$

Due to (5.4) and (2.14), we obtain for any $\xi \in H^1(\mathcal{O})$

$$\begin{aligned}&\left(\frac{dv^\varepsilon}{dt}-\frac{dv^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)a_\varepsilon\left(v^\varepsilon-v^0,\xi\right) + \rho\left(v^\varepsilon-v^0,\xi\right)_{H_g(\mathcal{O})}\\ =&z(t,\omega)\left(f_\varepsilon(t,y^*,y_{n+1},z^{-1}(t,\omega)v^\varepsilon) - f(t,y^*,0,z^{-1}(t,\omega)v^0),\xi\right)_{H_g(\mathcal{O})}\\ &+ z(t,\omega)\left(G_\varepsilon(t,y^*,y_{n+1}) - G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})}\\ &+ (1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}v^0_{y_i}, y_{n+1}\xi_{y_{n+1}}\right)_{H_g(\mathcal{O})}.\end{aligned}\tag{5.5}$$

Setting $\xi = v^\varepsilon - v^0$, then taking the real part, (5.5) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v^\varepsilon-v^0\|^2_{H_g(\mathcal{O})} + a_\varepsilon\left(v^\varepsilon-v^0,v^\varepsilon-v^0\right) + \rho\|v^\varepsilon-v^0\|^2_{H_g(\mathcal{O})}\\ =&z(t,\omega)\mathrm{Re}\left(f_\varepsilon(t,y^*,y_{n+1},z^{-1}(t,\omega)v^\varepsilon) - f(t,y^*,0,z^{-1}(t,\omega)v^0),v^\varepsilon-v^0\right)_{H_g(\mathcal{O})}\\ &+ z(t,\omega)\mathrm{Re}\left(G_\varepsilon(t,y^*,y_{n+1}) - G(t,y^*,0),v^\varepsilon-v^0\right)_{H_g(\mathcal{O})}\\ &+ \mathrm{Re}(1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}v^0_{y_i}, y_{n+1}(v^\varepsilon_{y_{n+1}}-v^0_{y_{n+1}})\right)_{H_g(\mathcal{O})}.\end{aligned}\tag{5.6}$$

By (2.3) and (5.2), we have

$$\begin{aligned}&z(t,\omega)\mathrm{Re}\Big(f_\varepsilon(t,y^*,y_{n+1},z^{-1}(t,\omega)v^\varepsilon) - f(t,y^*,0,z^{-1}(t,\omega)v^0),v^\varepsilon-v^0\Big)_{H_g(\mathcal{O})}\\ =&z(t,\omega)\mathrm{Re}\Big(\mathcal{F}_1,v^\varepsilon-v^0\Big)_{H_g(\mathcal{O})} + z(t,\omega)\mathrm{Re}\Big(\mathcal{F}_2,v^\varepsilon-v^0\Big)_{H_g(\mathcal{O})}\\ \le&\beta\|v^\varepsilon-v^0\|^2_{H_g(\mathcal{O})} + c\varepsilon\varphi_1^2(t) + c\varepsilon z^2(t,\omega)\left(\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \|v^0\|^2_{H_g(\mathcal{O})}\right),\end{aligned}\tag{5.7}$$

where

$$\mathcal{F}_1 = f(t,y^*,\varepsilon g(y^*)y_{n+1},z^{-1}(t,\omega)v^\varepsilon) - f(t,y^*,\varepsilon g(y^*)y_{n+1},z^{-1}(t,\omega)v^0),$$

$$\mathcal{F}_2 = f(t, y^*, \varepsilon g(y^*) y_{n+1}, z^{-1}(t,\omega) v^0) - f(t, y^*, 0, z^{-1}(t,\omega) v^0).$$

By (5.3), we obtain

$$\begin{aligned}
& z(t,\omega) \mathrm{Re}\Big(G_\varepsilon(t, y^*, y_{n+1}) - G(t, y^*, 0), v^\varepsilon - v^0\Big)_{H_g(\mathcal{O})} \\
\le & z(t,\omega) \|G_\varepsilon(t, y^*, y_{n+1}) - G(t, y^*, 0)\|_{H_g(\mathcal{O})} \|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})} \\
\le & c\varphi_2(t) \varepsilon z(t,\omega) \|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})} \\
\le & c\varepsilon \varphi_2^2(t) + c\varepsilon z^2(t,\omega) \left(\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \|v^0\|^2_{H_g(\mathcal{O})} \right). \qquad (5.8)
\end{aligned}$$

Finally, by (2.7), we get

$$\begin{aligned}
& \mathrm{Re}(1 + \mathrm{i}\mu) \sum_{i=1}^{n} \left(\frac{g_{y_i}}{g} v^0_{y_i}, y_{n+1}(v^\varepsilon_{y_{n+1}} - v^0_{y_{n+1}}) \right)_{H_g(\mathcal{O})} \\
= & \mathrm{Re}(1 + \mathrm{i}\mu) \sum_{i=1}^{n} \left(g_{y_i} v^0_{y_i}, y_{n+1}(v^\varepsilon_{y_{n+1}} - v^0_{y_{n+1}}) \right)_{L^2(\mathcal{O})} \\
\le & c\varepsilon \|v^0\|_{H^1(\mathcal{Q})} \|v^\varepsilon - v^0\|^2_{H^1_\varepsilon(\mathcal{O})} \\
\le & c\varepsilon \left(\|v^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^0\|^2_{H^1(\mathcal{Q})} \right). \qquad (5.9)
\end{aligned}$$

From (5.6)–(5.9), we obtain that, for $t \ge \tau$,

$$\begin{aligned}
\frac{d}{dt} \|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})} \le & \, 2\beta \|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})} + c\varepsilon \left(\|v^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^0\|^2_{H^1(\mathcal{Q})} \right) \\
& + c\varepsilon \left(\varphi_1^2(t) + \varphi_2^2(t) \right) + c\varepsilon z^2(t,\omega) \left(\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \|v^0\|^2_{H_g(H_g(\mathcal{O}))} \right). \qquad (5.10)
\end{aligned}$$

Multiplying (5.10) by $e^{-2\beta t}$ and then integrating the resulting inequality on (τ, t), we deduce

$$\begin{aligned}
& \|v^\varepsilon(t) - v^0(t)\|^2_{H_g(\mathcal{O})} \\
\le & \, e^{2\beta(t-\tau)} \|v^\varepsilon(\tau) - v^0(\tau)\|^2_{H_g(\mathcal{O})} + c\varepsilon e^{2\beta(t-\tau)} \int_\tau^t \left(\|v^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|v^0\|^2_{H^1(\mathcal{Q})} \right) ds \\
& + c\varepsilon e^{2\beta(t-\tau)} \int_\tau^t \left(\varphi_1^2(s) + \varphi_2^2(s) \right) ds \\
& + c\varepsilon e^{2\beta(t-\tau)} \max_{\tau \le s \le t} z^2(s,\omega) \int_\tau^t \left(\|v^\varepsilon\|^2_{H_g(\mathcal{O})} + \|v^0\|^2_{H_g(H_g(\mathcal{O}))} \right) ds. \qquad (5.11)
\end{aligned}$$

By Lemma 5.1 and Lemma 5.2, we have that there exists a positive constant $\varrho = \varrho(\tau, \omega, \gamma, T)$ such that for all $t \in [\tau, \tau + T]$ with $T > 0$,

$$\begin{aligned}
& \|v^\varepsilon(t) - v^0(t)\|^2_{H_g(\mathcal{O})} \\
\le & \, e^{2\beta T} \|v^\varepsilon(\tau) - v^0(\tau)\|^2_{H_g(\mathcal{O})} + \varrho \varepsilon e^{2\beta T} \Big[\|v^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \|v^0_\tau\|^2_{H_g(\mathcal{Q})} \\
& + \int_\tau^{\tau+T} \left(\varphi_1^2(s) + \varphi_2^2(s) \right) ds + \int_\tau^{\tau+T} \left(1 + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \right) ds \Big].
\end{aligned}$$

Together with Lemma 5.3, for all $t \in [\tau, \tau + T]$, we have

$$
\begin{aligned}
&\|u^\varepsilon(t,\tau,\omega,u^\varepsilon_\tau) - u^0(t,\tau,\omega,\mathcal{M}u^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})} \\
&= z^{-2}(t,\omega)\|v^\varepsilon(t,\tau,\omega,z(\tau,\omega)u^\varepsilon_\tau) - v^0(t,\tau,\omega,z(\tau,\omega)\mathcal{M}u^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})} \\
&\le e^{2\beta T} z^{-2}(t,\omega)\|z(\tau,\omega)u^\varepsilon_\tau - z(\tau,\omega)\mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} \\
&\quad + \varrho\varepsilon e^{2\beta T} z^{-2}(t,\omega)\Big[\|z(\tau,\omega)u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \|z(\tau,\omega)\mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{Q})} \\
&\quad + \int_\tau^{\tau+T}\left(\varphi_1^2(s)+\varphi_2^2(s)\right)ds + \int_\tau^{\tau+T}\Big(1+\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds\Big] \\
&\le c\varepsilon^2 z^2(\tau,\omega)z^{-2}(t,\omega)\|u^\varepsilon_\tau\|^2_{H^1_\varepsilon(\mathcal{O})} + \varrho\varepsilon e^{2\beta T} z^{-2}(t,\omega)\Big[\|z(\tau,\omega)u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} \\
&\quad + \|z(\tau,\omega)\mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{Q})} + \int_\tau^{\tau+T}\left(\varphi_1^2(s)+\varphi_2^2(s)\right)ds \\
&\quad + \int_\tau^{\tau+T}\Big(1+\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds\Big].
\end{aligned}
$$

Together with the assumption $\|u^\varepsilon_\tau\|_{H^1_\varepsilon(\mathcal{O})} \le \eta(\tau,\omega)$, we obtain the desired result. □

We finally establish the upper semicontinuity of random attractors as $\varepsilon \to 0$.

Theorem 5.2. *Suppose that* (2.2)–(2.4), (2.17), (2.18) *and* (5.2)–(5.3) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, $\lim_{\varepsilon\to 0} dist_{L^2(\mathcal{O})}\big(\mathcal{A}_\varepsilon(\tau,\omega),\mathcal{A}_0(\tau,\omega)\big) = 0$.

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by the invariance of $\mathcal{A}_\varepsilon$ and (4.3), we find that there exists $\varepsilon_0 > 0$ such that

$$\|u\|^2_{H^1_\varepsilon(\mathcal{O})} \le L(\tau,\omega) \quad \text{for all } 0<\varepsilon<\varepsilon_0 \text{ and } u \in \mathcal{A}_\varepsilon(\tau,\omega), \tag{5.12}$$

where $L(\tau,\omega)$ is the positive constant in (4.3), which is independent of ε. Let $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be the $\mathcal{D}_1$-pullback absorbing set of Ψ_ε obtained in Lemma 4.1 and denote by $K_0 = \{K_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ with $K_0(\tau,\omega) = \{\mathcal{M}u : u \in K(\tau,\omega)\}$. Then K_0 is tempered in $L^2(\mathcal{Q})$, and hence $K_0 \in \mathcal{D}_0$. Since $\mathcal{A}_0$ is the $\mathcal{D}_0$-pullback attractor of Ψ_0 in $L^2(\mathcal{Q})$, given $\eta > 0$, we infer that there exists $T = T(\eta,\tau,\omega) \ge 1$ such that

$$\text{dist}_{L^2(\mathcal{Q})}(\Psi_0(T,\tau-T,\theta_{-T}\omega,K_0(\tau-T,\theta_{-T}\omega)),\mathcal{A}_0(\tau,\omega)) < \frac{1}{2}\eta. \tag{5.13}$$

By the invariance of $\mathcal{A}_\varepsilon(\tau,\omega)$, we see that for any $x_\varepsilon \in \mathcal{A}_\varepsilon(\tau,\omega)$, there exists $y_\varepsilon \in \mathcal{A}_\varepsilon(\tau-T,\theta_{-T}\omega)$ such that

$$x_\varepsilon = \Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon). \tag{5.14}$$

By (5.12) and Theorem 5.1, we obtain

$$\lim_{\varepsilon\to 0}\|\Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon) - \Psi_0(T,\tau-T,\theta_{-T}\omega,\mathcal{M}y_\varepsilon)\|_{L^2(\mathcal{O})} = 0,$$

and hence there exists $\varepsilon_1 \in (0,\varepsilon_0)$ such that for all $\varepsilon < \varepsilon_1$,

$$\|\Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon) - \Psi_0(T,\tau-T,\theta_{-T}\omega,\mathcal{M}y_\varepsilon)\|_{L^2(\mathcal{O})} < \frac{1}{2}\eta. \tag{5.15}$$

Since $y_\varepsilon \in \mathcal{A}_\varepsilon(\tau - T, \theta_{-T}\omega)$ and $\mathcal{A}_\varepsilon(\tau - T, \theta_{-T}\omega) \subseteq K(\tau - T, \theta_{-T}\omega)$, we know $\mathcal{M}y_\varepsilon \in K_0(\tau - T, \theta_{-T}\omega)$, which along with (5.13) implies

$$\operatorname{dist}_{L^2(\mathcal{Q})}(\Psi_0(T, \tau - T, \theta_{-T}\omega, \mathcal{M}y_\varepsilon), \mathcal{A}_0(\tau, \omega)) < \frac{1}{2}\eta. \tag{5.16}$$

By (5.15) and (5.16), we have, for all $\varepsilon < \varepsilon_1$,

$$\operatorname{dist}_{L^2(\mathcal{O})}(\Psi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon), \mathcal{A}_0(\tau, \omega)) < \eta. \tag{5.17}$$

By (5.14) and (5.17), we deduce, for all $\varepsilon < \varepsilon_1$, $\operatorname{dist}_{L^2(\mathcal{O})}(x_\varepsilon, \mathcal{A}_0(\tau, \omega)) < \eta$, for all $x_\varepsilon \in \mathcal{A}_\varepsilon(\tau, \omega)$. This indicates that for all $\varepsilon < \varepsilon_1$, $\operatorname{dist}_{L^2(\mathcal{O})}(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) \leq \eta$, as desired. □

References

[1] F. Antoci and M. Prizzi, *Reaction-diffusion equations on unbounded thin domains*, Topol. Methods Nonlinear Anal., 2001, 18, 283–302.

[2] I. S. Aranson and L. Kramer, *The world of the complex Ginzburg-Landau equation,* Rev. Mod. Phys., 2002, 74, 99–143.

[3] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, 1998.

[4] J. Arrieta, A. Carvalho, M. Pereira and R. P. Da Silva, *Semilinear parabolic problems in thin domains with a highly oscillatory boundary,* Nonlinear Anal., 2011, 74, 5111–5132.

[5] M. Bartuccelli, P. Constantin, C. R. Doering, J. D. Gibbon and M. Gisselfält, *On the possibility of soft and hard turbulence in the complex Ginzburg-Landau equation,* Physica D, 1990, 44, 421–444.

[6] S. Becker and A. Jentzen, *Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg-Landau equations*, Stochastic Process and their Applications, 2019, 129, 28–69.

[7] I. Ciuperca, *Reaction-diffusion equations on thin domains with varying order of thinness*, J. Differ. Equ., 1996, 126, 244–291.

[8] C. R. Doering, J. D. Gibbon and C. D. Levermore, *Weak and strong solutions of the complex Ginzburg-Landau equation*, Physica D, 1994, 71, 285–318.

[9] J. K. Hale and G. Raugel, *Reaction-diffusion equations on thin domains*, J. Math. Pures Appl., 1992, 71, 33–95.

[10] J. K. Hale and G. Raugel, *A reaction-diffusion equation on a thin L-shaped domain*, Proc. Roy. Soc. Edinburgh Sect. A., 1995, 125, 283–327.

[11] M. Hoshino, Y. Inahama and N. Naganuma, *Stochastic complex Ginzburg-Landau equation with space-time white noise*, Electronic J. Probability, 2017, 22, 1–68.

[12] D. Li, K. Lu, B. Wang and X. Wang, *Limiting dynamics for non-autonomous stochastic retarded reaction-diffusion equations on thin domain*, Discrete Contin. Dyn. Syst. A., 2019, 39, 3717–3747.

[13] D. Li, B. Wang and X. Wang, *Limiting behavior of non-autonomous stochastic reaction-diffusion equations on thin domains*, J. Differ. Equ., 2017, 262, 1575–1602.

[14] D. Li and X. Wang, *Asymptotic behavior of stochastic complex Ginzburg-Landau equations with deterministic non-autonomous forcing on thin domains*, Discrete Contin. Dyn. Syst. Ser. B., 2019, 24, 449–465.

[15] F. Li and B. You, *Global attractors for the complex Ginzburg-Landau equation*, J. Mathe. Anal. Appl., 2011, 415, 14–24.

[16] S. Lv and Q. Lu, *Exponential attractor for the 3D Ginzburg-Landau type equation*, Nonlinear Anal: Theory, Methods and Applications, 2007, 67, 3116–3135.

[17] Y. Lv and J. Sun, *Asymptotic behavior of stochastic discrete complex Ginzburg-Landau equations*, Physica D: Nonlinear Phenomena, 2006, 221, 157–169.

[18] Y. Morita, *Stable solutions to the Ginzburg-Landau equation with magnetic effect in a thin domain*, Jpn. J. Ind. Appl. Math., 2004, 21, 129–147.

[19] M. Prizzi and K. P. Rybakowski, *Recent results on thin domain problems, II*, Topol. Methods Nonlinear Anal., 2002, 19, 199–219.

[20] M. Prizzi and K. P. *Rybakowski,The effect of domain squeezing upon the dynamics of reaction-diffusion equations*, J. Differ. Equ., 2001, 237, 271–320.

[21] G. Raugel and G. Sell, *Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc., 1993, 6, 503–568.

[22] L. Shi, R. Wang, K. Lu and B. Wang, *Asymptotic behavior of stochastic Fitzhugh-Nagumo Systems on unbounded thin domain*, J. Differ. Equ., 2019, 7, 4373–4409.

[23] D. Shimotsuma, T. Yokota and K. Yoshii, *Existence and decay estimates of solutions to complex Ginzburg-Landau type equations*, J. Differ. Equ., 2016, 260, 3119–3149.

[24] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Differ. Equ., 2012, 253, 1544–1583.

[25] B. Wang, *Random attractors for non-autonomous stochastic wave equations with multiplicative noise*, Discrete Contin. Dyn. Syst., 2014, 34, 269–300.

[26] G. Wang, B. Guo and Y. Li, *The asymptotic behavior of the stochastic Ginzburg-Landau equation with additive noise*, Appl. Math. Comput., 2008, 198, 849–857.

[27] C. Zhao and S. Zhou, *Limit behavior of global attractors for the complex Ginzburg-Landau equation on infinite lattices*, Appl. Math. Lett., 2008, 21, 628–635.

[28] Q. Zhang, *Random attractors for a Ginzburg-Landau equation with additive noise*, Chaos, Solitons and Fractals, 2009, 39, 463–472.