Journal of Applied Analysis and Computation
Volume 11, Number 5, October 2021, 2334–2354
Website:http://www.jaac-online.com
DOI:10.11948/20200383

ANALYTICAL SOLUTION FOR THE TWO-DIMENSIONAL LINEAR ADVECTION-DISPERSION EQUATION IN POROUS MEDIA VIA THE FOKAS METHOD*

Guenbo Hwang†

Abstract We present the analytical solution of the two-dimensional linear advection-dispersion equation (2-D LAD) in the quarter plane and the semi-infinite domain for two-dimensional solute transport in a porous medium. The governing equation includes terms describing advection, longitudinal and transverse dispersions and linear equilibrium adsorption. The analytical solution in terms of integrals in the complex plane is established by utilizing the unified transform method, also known as the Fokas method. The method hinges upon analysis of the divergence form of the governing equation and the so-called global relation, which is an algebraic relation coupling all known and unknown initial and boundary values. Particularly, the integral representation of the solution yields an accurate and fast numerical evaluation of the solution for the 2-D LAD equation. We demonstrate examples as an application of the developed solution and compare the analytical solution with numerical results.

Keywords Initial-boundary value problem, advection-dispersion equation, Fokas method, solute transport, environmental flow.

MSC(2010) 35A22, 35K10, 35K57, 35Q35.

1. Introduction

Analysis of contaminant transport problems in various porous media has been extensively studied during past decades. The advection-dispersion equation is commonly used as governing equation for solute transport of contaminants in porous media, which is also a long-standing analytical model for advective-diffusive transport problems in many areas of sciences and engineering such as heat and mass transfer and chemical or biological pollutant transport in environment [11, 21, 24, 33, 36, 42]. Thus, a plethora of analytical solutions of the advection-dispersion equation has been developed for analyzing dispersive transport problems in literatures. There are a number of analytical solutions for one-dimensional cases [3, 21, 24, 33, 36], whereas analytical solutions are relatively limited to the two- and three-dimensional advection-dispersion equation, which are more applicable in many physical circumstances [1, 4, 22, 30, 31].

The literature presents several analytical methods to solve the advection-dispersion equation on semi-infinite or finite domains, including separation of variables,

†The corresponding author. Email address: ghwang@daegu.ac.kr (G. Hwang)
Department of Mathematics, Daegu University, Gyeongsan Gyeongbuk 38453, Korea
*The author was supported by the Daegu University Research Grant 2020.

Green's function and certain integral transforms such as the Laplace and Fourier transforms or a combination of these integral transforms [1]. Particularly, the classic or generalized integral transform technique has been developed in [22, 23], providing an efficient and systematic approach to derive generalized analytical solutions of solute transport problems. Nevertheless, traditional approaches as well as the generalized integral transform technique may not be generally applicable for solving boundary value problems of the advection-dispersion equation. For example, the Fourier transform can provide a solution which is not uniformly convergent at boundaries [10, 13, 19] and the process of the Fourier and Laplace transforms technique may require complicated or difficult mathematical manipulations [3, 4, 22, 23]. The generalized integral transform may not be used in solving solute transport problems in finite or infinite domain subject to time-dependent inlet boundary conditions [3]. Thus, the aim of this study here is to develop a unified approach for solving the advection-dispersion equation of solute transport problems in various porous media.

The unified transform method, also known as the Fokas method, has been successfully used to solve boundary value problems for linear evolution equations [13, 19]. The Fokas method was developed to analyze boundary value problems for nonlinear integrable systems as a significant extension of the inverse scattering transform [14, 17]. Importantly, the method also yields a new approach to analyze a large class of partial differential equations (PDEs) such as nonlinear integrable systems, linear evolution equations [15, 18, 20], linear and nonlinear elliptic PDEs [8, 9, 16, 38] and linear and nonlinear integrable discrete equations [2, 35]. Recently, the Fokas method has been extended to analyze nonlinear droplet oscillations [40] and non self-adjoint diffusion problems with applications in statistical estimation [7]. Regarding the advection-dispersion equation, the Fokas method has been used for analyzing the one- and two-dimensional LAD equations [10, 18, 26] and the two-dimensional linear advection-dispersion equations (2-D LAD) in cylindrical coordinates [27]. In this paper, we focus on studying the two-dimensional solute transport in a porous medium. More precisely, we will derive an analytical solution of the 2-D LAD equation posed in the quarter plane and a semi-infinite domain by utilizing the Fokas method. The governing equation includes terms describing advection, longitudinal and transverse dispersions and linear equilibrium adsorption. It should be noted that the LAD equation can be considered as a generalization of the heat equation, and hence the present study is a further application as discussed in [18].

The present method has several advantages in solving boundary value problems for the LAD equation. The Fokas method is relatively simple, but effective to implement in solving boundary value problems with more general and complicated boundary conditions [5, 32] (see also [7] and references therein). Moreover the method provides an integral representation of the solution involving explicit exponential dependence on spatial and time variables. Thus, it enables one to understand and study long-time asymptotic behaviors of the solution [25–28, 34]. In addition, the integral representation of the solution provides a novel efficient method for computing numerically the analytical solution, called a hybrid analytical-numerical method [10].

The outline of this work is as following. In Sec. 2, we discuss the 2-D LAD equation as the relevant mathematical formulation of the 2-D solute transport in a porous medium and we present the analytical solution for the 2-D LAD equation posed on the quarter plane and a semi-infinite domain by means of the Fokas method. In Sec. 3, we demonstrate examples as computational applications for the developed solution. We end with concluding remarks in Sec. 4.

2. The two-dimensional linear advection-dispersion equation

Two-dimensional solute transport subjected to linear equilibrium adsorption, through a saturated porous medium with uniform steady flow is generally described by the two-dimensional linear advection-dispersion equation (2-D LAD) [31]

$$\frac{\partial C}{\partial t}+V\frac{\partial C}{\partial x}=D_L\frac{\partial^2 C}{\partial x^2}+D_T\frac{\partial^2 C}{\partial y^2},\quad 0<x<\infty,\quad 0<y<L,\tag{2.1}$$

where $C(x,y,t)$ represents the solute concentration, V is the average pore velocity and D_L and D_T represent the longitudinal and transverse dispersion coefficients, respectively. We here consider two cases that the length L is finite for the semi-infinite domain and is infinity for the quarter plane. We assume that the concentration $C(x,y,t)$ is sufficient smooth and rapidly decays for all t as $x\to\infty$ (and as $y\to\infty$ for $L=\infty$). Moreover, in order to solve eq. (2.1), it requires to prescribe initial and boundary conditions. We assume that the initial solute concentration is present at $t=0$

$$C(x,y,0)=C_0(x,y)$$

and in addition, we assume mass balance at $x=0$ and impermeable boundary conditions at $y=0$ (and $y=L$ for finite L), respectively,

$$\left(C-\frac{D_L}{V}\frac{\partial C}{\partial x}\right)(0,y,t)=0,\quad C_y(x,0,t)=C_y(x,L,t)=0.$$

2.1. Solution for the LAD equation on the quarter plane

We first study the 2-D LAD equation on the quarter plane ($L=\infty$). More specifically, letting $t'=Vt$ in eq. (2.1), we consider the following rescaled 2-D LAD equation posed on the quarter plane

$$\frac{\partial C}{\partial t}=b\frac{\partial^2 C}{\partial x^2}-\frac{\partial C}{\partial x}+a\frac{\partial^2 C}{\partial y^2},\quad x,y>0,\tag{2.2}$$

where we have omitted the prime for simplicity and $b=D_L/V$ and $a=D_T/V$. The initial and boundary conditions are given by

$$C(x,y,0)=C_0(x,y),\quad (C-bC_x)(0,y,t)=0,\quad C_y(x,0,t)=0.\tag{2.3}$$

Introduce the dispersion relations $\omega_1(k_1)$ and $\omega_2(k_2)$ defined by

$$\omega_1(k_1)=bk_1^2+ik_1,\quad \omega_2(k_2)=ak_2^2,\quad k_1,k_2\in\mathbb{C}\tag{2.4}$$

and it is convenient to write $k=(k_1,k_2)$, $z=(x,y)$ and

$$kz=k_1x+k_2y,\quad \omega(k)=\omega_1(k_1)+\omega_2(k_2).\tag{2.5}$$

We then write eq. (2.2) as

$$\left(\mathrm{e}^{-ikz+\omega(k)t}C\right)_t = \left[\mathrm{e}^{-ikz+\omega(k)t}\left((ibk_1-1)C+bC_x\right)\right]_x + \left[\mathrm{e}^{-ikz+\omega(k)t}\left(iak_2C+aC_y\right)\right]_y . \tag{2.6}$$

We denote the Fourier transform of $C(x,y,t)$ by $\hat{C}(k_1,k_2,t)$

$$\hat{C}(k_1,k_2,t) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \mathrm{e}^{-ikz}C(x,y,t). \tag{2.7}$$

Note that $\hat{C}(k_1,k_2,t)$ is well defined for $\operatorname{Im} k_{1,2} \leq 0$. Taking the Fourier transform in eq. (2.6), we find

$$\left(\mathrm{e}^{\omega(k)t}\hat{C}(k_1,k_2,t)\right)_t = -\int_0^\infty \mathrm{d}y\, \mathrm{e}^{-ik_2y+\omega(k)t} ibk_1C(0,y,t) - \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-ik_1x+\omega(k)t} iak_2C(x,0,t). \tag{2.8}$$

Integrating eq. (2.8) with respect to t, we find the global relation as

$$\mathrm{e}^{\omega(k)t}\hat{C}(k_1,k_2,t) = \hat{C}_0(k_1,k_2) - ibk_1\hat{g}_1(k_1,k_2,t) - iak_2\hat{f}_1(k_1,k_2,t), \quad \operatorname{Im} k_{1,2} \leq 0, \tag{2.9}$$

where $\hat{C}_0(k_1,k_2) = \hat{C}(k_1,k_2,0)$ and

$$\hat{g}_1(k_1,k_2,t) = \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-k_2\eta+\omega(k)s}C(0,\eta,s), \tag{2.10}$$

$$\hat{f}_1(k_1,k_2,t) = \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-k_1\xi+\omega(k)s}C(\xi,0,s). \tag{2.11}$$

Then, the solution $C(x,y,t)$ can be reconstructed by taking the inverse Fourier transform in the global relation (2.9),

$$C(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^\infty \mathrm{d}k_1 \int_{-\infty}^\infty \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,k_2) - \frac{1}{4\pi^2}\int_{-\infty}^\infty \mathrm{d}k_1 \int_{-\infty}^\infty \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t}\left(ibk_1\hat{g}_1(k_1,k_2,t)+iak_2\hat{f}_1(k_1,k_2,t)\right). \tag{2.12}$$

However, the representation of the solution given in (2.12) involves unknown functions $\hat{g}_1(k_1,k_2,t)$ and $\hat{f}_1(k_1,k_2,t)$ due to the unknown boundary values $C(0,y,t)$ and $C(x,0,t)$. Thus, for the explicit representation of the solution, we should determine the unknown functions $\hat{g}_1(k_1,k_2,t)$ and $\hat{f}_1(k_1,k_2,t)$. This can be done by using the global relation (2.9).

We first introduce the regions

$$D_j = \{k_j \in \mathbb{C} \mid \operatorname{Re}\omega_j(k_j) < 0\}, \quad j = 1,2, \tag{2.13}$$

and it is convenient to decompose the regions D_j $(j=1,2)$ into $D_j = D_j^+ \cup D_j^-$, where $D_j^+ = \{k_j \in \mathbb{C} \mid \operatorname{Re}\omega_j(k_j) < 0 \text{ and } \operatorname{Im} k_j > 0\}$ and $D_j^- = \{k_j \in \mathbb{C} \mid$

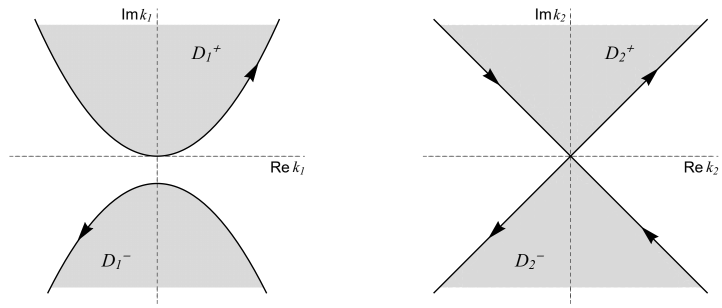

Figure 1. (Left) The region $D_1 = D_1^+ \cup D_1^-$ (shaded) in the complex k_1-plane, where $\mathrm{Re}\,\omega_1(k_1) < 0$ with $\omega_1(k_1) = bk_1^2 + ik_1$. (Right) The region $D_2 = D_2^+ \cup D_2^-$ (shaded) in the complex k_2-plane, where $\mathrm{Re}\,\omega_2(k_2) < 0$ with $\omega_2(k_2) = ak_2^2$.

$\mathrm{Re}\,\omega_j(k_j) < 0$ and $\mathrm{Im}\,k_j < 0\}$ (see Fig. 1). Thus, by the Cauchy theorem, we can deform the $(-\infty, \infty)$ to ∂D_j^+ and then eq. (2.12) can be written as

$$\begin{aligned} C(x,y,t) =& \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} \hat{C}_0(k_1,k_2) \\ & - \frac{1}{4\pi^2} \int_{\partial D_1^+} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} ibk_1 \hat{g}_1(k_1,k_2,t) \\ & - \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{\partial D_2^+} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} iak_2 \hat{f}_1(k_1,k_2,t). \end{aligned} \tag{2.14}$$

Note that under the transformation $k_1 \to -k_1 - i/b := \hat{k}_1$, the dispersion relations $\omega_1(k_1)$ is invariant. Thus, letting $k_1 \to \hat{k}_1$ in eq. (2.9), we find

$$\mathrm{e}^{\omega(k)t} \hat{C}(\hat{k}_1,k_2,t) = \hat{C}_0(\hat{k}_1,k_2) - (1 - ibk_1)\hat{g}_1(k_1,k_2,t) - iak_2 \hat{f}_1(\hat{k}_1,k_2,t), \tag{2.15}$$

which implies that

$$\begin{aligned} \hat{g}_1(k_1,k_2,t) =& \frac{1}{1 - ibk_1} \left(\hat{C}_0(\hat{k}_1,k_2) - iak_2 \hat{f}_1(\hat{k}_1,k_2,t) \right) \\ & - \frac{1}{1 - ibk_1} \mathrm{e}^{\omega(k)t} \hat{C}(\hat{k}_1,k_2,t). \end{aligned} \tag{2.16}$$

The term involving $\mathrm{e}^{\omega(k)t} \hat{C}(\hat{k}_1,k_2,t)$ in the above equation vanishes when substituting this term into eq. (2.14) thanks to the Cauchy theorem. On the other hand, $\omega_2(k_2)$ is invariant under the transformation $k_2 \to -k_2$. Hence, replacing $k_2 \to -k_2$, the global relation yields

$$\mathrm{e}^{\omega(k)t} \hat{C}(k_1,-k_2,t) = \hat{C}_0(k_1,-k_2) - ibk_1 \hat{g}_1(k_1,-k_2,t) + iak_2 \hat{f}_1(k_1,k_2,t), \tag{2.17}$$

and then we find

$$iak_2 \hat{f}_1(k_1,k_2,t) = -\hat{C}_0(k_1,-k_2) + ibk_1 \hat{g}_1(k_1,-k_2,t) + \mathrm{e}^{\omega(k)t} \hat{C}(k_1,-k_2,t). \tag{2.18}$$

As before, the term involving $\mathrm{e}^{\omega(k)t} \hat{C}(k_1,-k_2,t)$ in eq. (2.18) does not contribute in eq. (2.14) by the Cauchy theorem. Hence, substituting eqs. (2.16) and (2.18) into eq. (2.14), we find

$$\begin{aligned}C(x,y,t) =&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,k_2)\\&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,k_2)\\&+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,-k_2)\\&+\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\bigg[\frac{ibk_1}{1-ibk_1}iak_2\hat{f}_1(\hat{k}_1,k_2,t)\\&-ibk_1\hat{g}_1(k_1,-k_2,t)\bigg].\end{aligned}\tag{2.19}$$

On the other hand, replacing $k_1 \to \hat{k}_1$ and $k_2 \to -k_2$ in the global relation (2.9), we find

$$\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,-k_2,t)=\hat{C}_0(\hat{k}_1,-k_2)-(1-ibk_1)\hat{g}_1(k_1,-k_2,t)+iak_2\hat{f}_1(\hat{k}_1,k_2,t),\tag{2.20}$$

and then the squared bracket in the forth integral of eq. (2.19) becomes

$$\begin{aligned}&\frac{ibk_1}{1-ibk_1}iak_2\hat{f}_1(\hat{k}_1,k_2,t)-ibk_1\hat{g}_1(k_1,-k_2,t)\\=&-\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,-k_1)+\frac{ibk_1}{1-ibk_1}\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,-k_2,t).\end{aligned}\tag{2.21}$$

Using the above equation, the solution $C(x,y,t)$ can be written as

$$\begin{aligned}C(x,y,t) =&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,k_2)\\&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,k_2)\\&+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,-k_2)\\&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,-k_2),\end{aligned}\tag{2.22}$$

where we have used the fact that the term involving $\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,-k_2,t)$ in eq. (2.21) vanishes by the Cauchy theorem. Deforming the contours ∂D_1^+ and ∂D_2^+ into $(-\infty,\infty)$, finally we obatin the solution of eq. (2.2) as

$$\begin{aligned}C(x,y,t) =&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{N}_0(k_1,k_2)\\&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\hat{N}_0(\hat{k}_1,k_2),\end{aligned}\tag{2.23}$$

where $\hat{N}_0(k_1,k_2)$ is defined by

$$\hat{N}_0(k_1,k_2)=\hat{C}_0(k_1,k_2)+\hat{C}_0(k_1,-k_2).\tag{2.24}$$

2.2. Solution for the LAD equation in a semi-infinite domain

We now consider the 2-D LAD equation on the semi-infinite domain $\{0 < x,\ 0 < y < L\}$. Letting $y' = y/L$ and $t' = Vt$ in eq. (2.1), we study the following 2-D LAD equation posed on the semi-infinite domain

$$\frac{\partial C}{\partial t} = b\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} + a\frac{\partial^2 C}{\partial y^2}, \quad 0 < x, \quad 0 < y < 1, \tag{2.25}$$

where we have omitted the prime for simplicity and

$$b = \frac{D_L}{V}, \quad a = \frac{D_T}{VL^2}. \tag{2.26}$$

We consider the following initial and boundary conditions

$$C(x, y, 0) = C_0(x, y),\ \ (C - bC_x)(0, y, t) = 0,\ C_y(x, 0, t) = C_y(x, 1, t) = 0. \tag{2.27}$$

The implementation of the Fokas method in analyzing eq. (2.25) is similar as presented in Sec. 2.1 and hence we will abuse the notations used in this section. Note that eq. (2.6) is still valid for eq. (2.25). However, in this case, we let the Fourier transform be given by

$$\hat{C}(k_1, k_2, t) = \int_0^\infty \mathrm{d}x \int_0^1 \mathrm{d}y\, \mathrm{e}^{-ikz} C(x, y, t), \quad \mathrm{Im}\, k_1 \le 0. \tag{2.28}$$

Taking the Fourier transform in eq. (2.6), we find

$$\begin{aligned}\left(\mathrm{e}^{\omega(k)t}\hat{C}(k_1, k_2, t)\right)_t = &- \int_0^1 \mathrm{d}y\, \mathrm{e}^{-ik_2y+\omega(k)t} ibk_1 C(0, y, t)\\ &- \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-ik_1x+\omega(k)t} iak_2 C(x, 0, t)\\ &+ \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-ik_1x-ik_2+\omega(k)t} iak_2 C(x, 1, t).\end{aligned} \tag{2.29}$$

Integrating eq. (2.29) with respect to t, we obtain the global relation as

$$\begin{aligned}\mathrm{e}^{\omega(k)t}\hat{C}(k_1, k_2, t) =& \hat{C}_0(k_1, k_2) - ibk_1\hat{g}_1(k_1, k_2, t) - iak_2\hat{f}_1(k_1, k_2, t)\\ &+ iak_2\mathrm{e}^{-ik_2}\hat{f}_2(k_1, k_2, t), \quad \mathrm{Im}\, k_1 \le 0,\end{aligned} \tag{2.30}$$

where

$$\hat{g}_1(k_1, k_2, t) = \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}\eta\, \mathrm{e}^{-k_2\eta+\omega(k)s} C(0, \eta, s), \tag{2.31}$$

$$\hat{f}_1(k_1, k_2, t) = \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-k_1\xi+\omega(k)s} C(\xi, 0, s), \tag{2.32}$$

$$\hat{f}_2(k_1, k_2, t) = \int_0^t \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, \mathrm{e}^{-k_1\xi+\omega(k)s} C(\xi, 1, s). \tag{2.33}$$

Employing the inverse Fourier transform, the reconstruction formula of the solution can be found as

$$C(x, y, t) = \frac{1}{4\pi^2}\int_{-\infty}^\infty \mathrm{d}k_1 \int_{-\infty}^\infty \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1, k_2)$$

$$-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\Big[ibk_1\hat{g}_1(k_1,k_2,t)+iak_2\hat{f}_1(k_1,k_2,t)$$
$$-iak_2\mathrm{e}^{-ik_2}\hat{f}_2(k_1,k_2,t)\Big]. \tag{2.34}$$

We should determine the unknown functions $\hat{g}_1(k_1,k_2,t)$, $\hat{f}_1(k_1,k_2,t)$ and $\hat{f}_2(k_1,k_2,t)$ due to the unknown boundary values $C(0,y,t)$, $C(x,0,t)$ and $C(x,1,t)$. In this respect, we appropriately deform the contour $(-\infty,\infty)$ into $\partial D_j^{\pm}$ $(j=1,2)$ by the Cauchy theorem and then eq. (2.34) can be written as

$$\begin{aligned}C(x,y,t)=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{C}_0(k_1,k_2)\\&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}ibk_1\hat{g}_1(k_1,k_2,t)\\&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}iak_2\hat{f}_1(k_1,k_2,t)\\&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^-}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}iak_2\mathrm{e}^{-ik_2}\hat{f}_2(k_1,k_2,t).\end{aligned} \tag{2.35}$$

Note that letting $k_1\to\hat{k}_1=-k_1-i/b$, eq. (2.30) yields

$$\begin{aligned}\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,k_2,t)=&\hat{C}_0(\hat{k}_1,k_2)-(1-ibk_1)\hat{g}_1(k_1,k_2,t)\\&-iak_2\hat{f}_1(\hat{k}_1,k_2,t)+iak_2\mathrm{e}^{-ik_2}\hat{f}_2(\hat{k}_1,k_2,t),\end{aligned} \tag{2.36}$$

which implies that

$$\begin{aligned}\hat{g}_1(k_1,k_2,t)=&\frac{1}{1-ibk_1}\left(\hat{C}_0(\hat{k}_1,k_2)-iak_2\hat{f}_1(\hat{k}_1,k_2,t)+iak_2\mathrm{e}^{-ik_2}\hat{f}_2(\hat{k}_1,k_2)\right)\\&-\frac{1}{1-ibk_1}\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,k_2,t).\end{aligned} \tag{2.37}$$

Using eq. (2.37) in eq. (2.35), the second integral involving $\hat{g}_1(k_1,k_2,t)$ in eq. (2.35) can be written as

$$-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\Big[\hat{C}_0(\hat{k}_1,k_2,t)$$
$$-iak_2\hat{f}_1(\hat{k}_1,k_2,t)-iak_2\mathrm{e}^{-ik_2}\hat{f}_2(\hat{k}_1,k_2,t)\Big], \tag{2.38}$$

where we have used the fact that the term involving $\mathrm{e}^{\omega(k)t}\hat{C}(\hat{k}_1,k_2,t)$ in (2.37) vanishes by the Cauchy theorem. According to the regions, where each integrand in (2.38) is bounded and rapidly decaying as $k_{1,2}\to\infty$, we deform the contours and then we can write eq. (2.38) as

$$\begin{aligned}&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,k_2,t)\\&+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}iak_2\hat{f}_1(\hat{k}_1,k_2,t)\\&+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^-}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\frac{ibk_1}{1-ibk_1}iak_2\mathrm{e}^{-ik_2}\hat{f}_2(\hat{k}_1,k_2,t).\end{aligned} \tag{2.39}$$

On the other hand, letting $k_2 \to -k_2$, the global relation (2.30) yields

$$\begin{aligned}\mathrm{e}^{\omega(k)t}\hat{C}(k_1,-k_2,t) =& \hat{C}_0(k_1,-k_2) - ibk_1\hat{g}_1(k_1,-k_2,t)\\ &+ iak_2\hat{f}_1(k_1,k_2,t) - iak_2\mathrm{e}^{ik_2}\hat{f}_2(k_1,k_2).\end{aligned} \tag{2.40}$$

Solving eqs. (2.30) and (2.40) for $\hat{f}_j(k_1,k_2)$ ($j=1,2$), we find

$$\begin{aligned}iak_2\hat{f}_1(k_1,k_2,t) =& \frac{1}{\Delta(k_2)}\Big[\mathrm{e}^{ik_2}\hat{C}_0(k_1,k_2) - ibk_1\hat{g}_1(k_1,k_2,t) + \mathrm{e}^{-ik_2}\hat{C}_0(k_1,-k_2)\\ &- ibk_1\hat{g}_1(k_1,-k_2,t)\Big] - \frac{\mathrm{e}^{\omega(k)t}}{\Delta(k_2)}\Big[\mathrm{e}^{ik_2}\hat{C}(k_1,k_2,t)\\ &+ \mathrm{e}^{-ik_2}\hat{C}(k_1,-k_2,t)\Big],\end{aligned} \tag{2.41a}$$

$$\begin{aligned}iak_2\hat{f}_2(k_1,k_2,t) =& \frac{1}{\Delta(k_2)}\Big[\hat{C}_0(k_1,k_2) - ibk_1\hat{g}_1(k_1,k_2,t) + \hat{C}_0(k_1,-k_2)\\ &- ibk_1\hat{g}_1(k_1,-k_2,t)\Big] - \frac{\mathrm{e}^{\omega(k)t}}{\Delta(k_2)}\Big[\hat{C}(k_1,k_2,t) + \hat{C}(k_1,-k_2,t)\Big],\end{aligned} \tag{2.41b}$$

where $\Delta(k_2) = \mathrm{e}^{ik_2} - \mathrm{e}^{-ik_2}$ is the determinant of the linear system of eqs. (2.30) and (2.40). Replacing $k_1 \to \hat{k}_1$ in eq. (2.41a) and solving resulting equation with eq. (2.41a), we find

$$\begin{aligned}&\frac{ibk_1}{1-ibk_1}iak_2\hat{f}_1(\hat{k}_1,k_2,t)\\ =& iak_2\hat{f}_1(k_1,k_2,t) - \frac{1}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2) + \mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right]\\ &+ \frac{\mathrm{e}^{\omega(k)t}}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}(k_1,k_2,t) + \mathrm{e}^{-ik_2}\hat{N}(k_1,-k_2,t)\right],\end{aligned} \tag{2.42}$$

where $\hat{N}_0(k_1,k_2) = \hat{N}(k_1,k_2,0)$ with

$$\hat{N}(k_1,k_2,t) = \hat{C}(k_1,k_2,t) - \frac{ibk_1}{1-ibk_1}\hat{C}(\hat{k}_1,k_2,t). \tag{2.43}$$

Similarly, we can obtain

$$\begin{aligned}\frac{ibk_1}{1-ibk_1}iak_2\hat{f}_2(\hat{k}_1,k_2,t) =& iak_2\hat{f}_2(k_1,k_2,t) - \frac{1}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2) + \hat{N}_0(k_1,-k_2)\right]\\ &+ \frac{\mathrm{e}^{\omega(k)t}}{\Delta(k_2)}\left[\hat{N}(k_1,k_2,t) + \hat{N}(k_1,-k_2,t)\right].\end{aligned} \tag{2.44}$$

We now substitute eqs. (2.42) and (2.44) into eq. (2.39). First note that the terms involving $\mathrm{e}^{\omega(k)t}\hat{N}(k_1,\pm k_2,t)$ vanish by the Cauchy theorem when inserting eqs. (2.42) and (2.44) into eq. (2.39). Furthermore, the integrals regarding the terms $\hat{f}_{1,2}(k_1,k_2,t)$ in eqs. (2.42) and (2.44) cancel with the third and fourth integrals in (2.35). However, the function $\Delta(k_2)$ has simple zeros at $k_2 = m\pi$ ($m \in \mathbb{Z}$) and particularly, $k_2 = 0$ is in ∂D_2^+ and ∂D_2^-. Thus, we should deform the contours

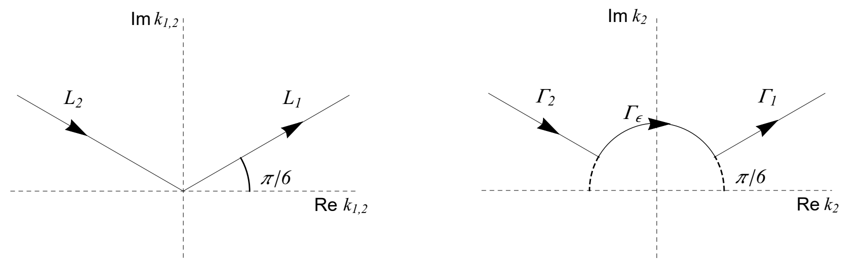

Figure 2. (Left) Efficient contour $L = L_1 \cup L_2$ for numerical integration. (Right) Efficient contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon$ for numerical integration (see the text for some details).

∂D_2^+ and ∂D_2^- to pass above and below $k_2 = 0$, which are denoted by $\partial \tilde{D}_2^+$ and $\partial \tilde{D}_2^-$, respectively. Finally, we find the solution of eq. (2.25) as

$$\begin{aligned} &C(x,y,t) \\ =&\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} \hat{N}_0(k_1,k_2) \\ &-\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{\partial\tilde{D}_2^+} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} \frac{1}{\Delta(k_2)} \left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2) + \mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right] \\ &-\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{\partial\tilde{D}_2^-} \mathrm{d}k_2\, \mathrm{e}^{ikz-\omega(k)t} \frac{\mathrm{e}^{-ik_2}}{\Delta(k_2)} \left[\hat{N}_0(k_1,k_2) + \hat{N}_0(k_1,-k_2)\right]. \end{aligned} \tag{2.45}$$

3. Application results

In this section, we present examples as applications of the developed solutions. Here, we discuss the case of the semi-infinite domain since analysis and results of the case for the quarter plane are similar. Throughout this section, we take the parameters as

$$V = 1, \quad L = 1, \quad D_T = 1 \tag{3.1}$$

and $D_L = 0.1$, 1 and 10 will be considered. More precisely, we study

$$\frac{\partial C}{\partial t} = b\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial y^2}, \quad 0 < x, \quad 0 < y < 1, \tag{3.2}$$

where $b = D_L$ and the initial and boundary conditions are given by eq. (2.27).

3.1. Point pulse release

We first consider the case that the initial value is given by

$$C(x,y,0) = C_0(x,y) = \delta(y)\delta(x-p), \tag{3.3}$$

where $\delta(\cdot)$ is the Dirac delta function and $p = 1$ is taken. This initial condition describes that a point pulse injection of the solute is initially provided at $y = 0$ and $x = p$. In this case, $\hat{C}_0(k_1,k_2) = \mathrm{e}^{-ik_1p}$ and $\hat{C}_0(-k_1 - i/b, k_2) = \mathrm{e}^{ik_1p-1/b}$ and hence we find

$$\hat{N}_0(k_1,k_2) = \hat{N}_0(k_1,-k_2) = \mathrm{e}^{-ik_1p} - \frac{ibk_1}{1-ibk_1}\mathrm{e}^{ik_1p-1/b}. \tag{3.4}$$

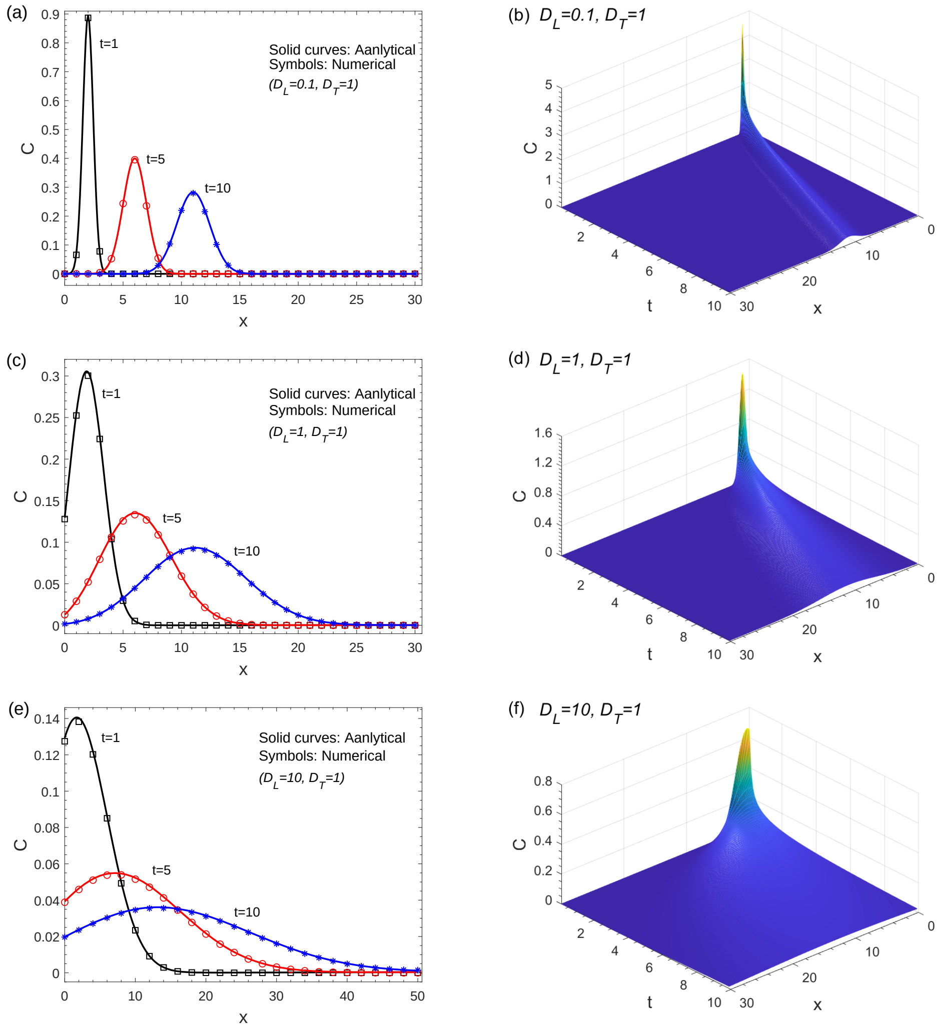

Figure 3. (a,c,e) Comparison of the analytical solutions (solid curves) and the numerical solutions (symbols) with the point pulse initial value in a longitudinal cross-section at $y = 0$ under the different dispersion coefficients. (b,d,f) Analytical solution profiles over the xt-plane, where $y = 0$ under the different dispersion coefficients.

Thus, the solution $C(x, y, t)$ can be found as

$$\begin{aligned} C(x,y,t) =& \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ikz-\omega(k)t}\hat{N}_0(k_1,k_2) \\ &-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{1}{\Delta(k_2)} \\ &\times\left(\mathrm{e}^{ik_2(y+1)}+\mathrm{e}^{ik_2(y-1)}+2\mathrm{e}^{-ik_2(y-1)}\right)\hat{N}_0(k_1,k_2), \end{aligned} \tag{3.5}$$

where $\hat{N}_0(k_{1,2})$ is given in eq. (3.4). Note that the analytical solution given in eq. (3.5) involves rather complicated integrals with respect to $k_{1,2}$ and it is not simple to evaluate explicitly. However numerical integrations can be done well effectively and efficiently. Here, we will use a hybrid analytical-numerical scheme, presented in [10] (also see [26, 27]). We first note that since the relevant integrands of k_j in eq. (3.5) are analytic, bounded and decaying rapidly as $k_j \to \infty$ in the region, where $\mathrm{Re}\,\omega_j(k_j) \geq 0$ and $\mathrm{Im}\,k_j \geq 0$ $(j = 1, 2)$, by the Cauchy theorem we can deform the contour $(-\infty, \infty)$ for the integrals in eq. (3.5) to the contour $L = L_1 \cup L_2$ (see Fig. 2), where

$$L_1 = \{k_j = r\mathrm{e}^{i\pi/6} \mid 0 \leq r\}, \quad L_2 = \{k_j = r\mathrm{e}^{5i\pi/6} \mid 0 \leq r\}. \tag{3.6}$$

Similarly, we can deform the contour $\partial\tilde{D}_2^+$ of the integral with respect to k_2 in eq. (3.5) to the contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon$ (see Fig. 2), where $\epsilon > 0$ and

$$\begin{aligned} &\Gamma_1 = \{k_2 = r\mathrm{e}^{i\pi/6} \mid \epsilon \leq r\}, \qquad \Gamma_2 = \{k_2 = r\mathrm{e}^{5i\pi/6} \mid \epsilon \leq r\}, \\ &\Gamma_\epsilon = \left\{k_2 = \epsilon\mathrm{e}^{i\theta} \mid \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}\right\}. \end{aligned} \tag{3.7}$$

Along these deformed contours L and Γ, the corresponding integrands decay rapidly as $k_j \to \infty$ $(j = 1, 2)$ due to the exponential terms and hence numerical integration converges very fast for large k_j. For the numerical integration, we use the adaptive Gauss-Kronrod quadrature method with $|r| \leq 100$ and $x \leq 50$. It can be shown that this quadrature rule converges exponentially with respect to the number of quadrature points N (see [10] and references therein). The exponential convergence of the relative error in L^2 norm is shown in Fig. 4, where N is increased by 50 quadrature points. In this figure, the rate of the convergence of the method shows no significant difference for the values of D_L.

Also, note that the integral over Γ_ϵ can be approximated by the residue contribution at $k_2 = 0$, that is,

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \mathrm{d}k_2\, \mathrm{e}^{-\omega_2(k_2)t} \frac{1}{\Delta(k_2)} \left(\mathrm{e}^{ik_2(y+1)} + \mathrm{e}^{ik_2(y-1)} + 2\mathrm{e}^{-ik_2(y-1)}\right) = -\frac{4\pi}{3}. \tag{3.8}$$

In Fig. 3(a,c,e), by using the hybrid analytical-numerical method, the analytical solutions given in eq. (3.5) are displayed as solid curves in a longitudinal cross-section at $y = 0$ under the different dispersion coefficients $D_L = 0.1$, 1 and 10. Note that since a small D_L value makes the parabolic boundary of D_1^+ wide, in this case we take the efficient contour $\tilde{L} = \tilde{L}_1 \cup \tilde{L}_2$, where

$$\tilde{L}_1 = \{k_1 = r\mathrm{e}^{iD_L\pi/6} \mid 0 \leq r\}, \quad \tilde{L}_2 = \{k_1 = r\mathrm{e}^{i(1-D_L/6)\pi} \mid 0 \leq r\}. \tag{3.9}$$

In addition, we also illustrated in Fig. 3(b,d,f) the analytical solution profiles in the longitudinal cross-section at $y = 0$ over the xt-plane.

For comparison, we solve eq. (3.2) numerically by using a classical finite difference method (FDM) namely, the centered and explicit finite difference method [37] (see also [26, 27]). Regarding the numerical solution of eq. (3.2), we used truncation of the infinite domain $(0 < x)$ to a sufficiently large finite one with $x \leq 50$. Fig. 3(a,c,e) show that the analytical solutions (solid curves) given by eq. (3.5) agree well with the numerical solutions (symbols) obtained by the FDM. In contrast to the exponentially convergence of the present method, the FDM converges algebraically with respect to step sizes of x, y and t [10]. Thus, the integral representation given in eq. (3.5) provides more efficient numerical scheme than the FDM.

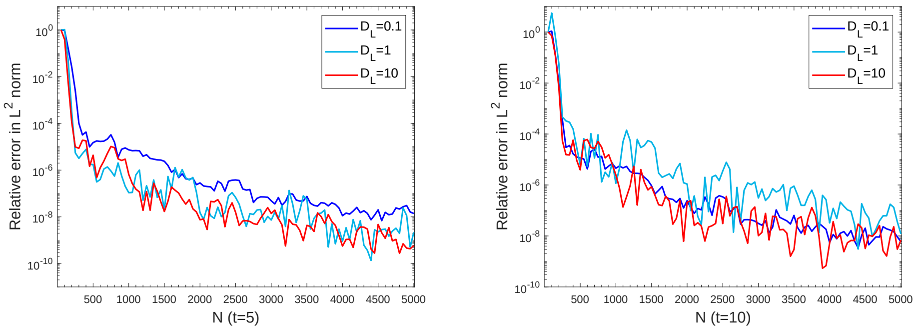

Figure 4. Exponential convergence of the Fokas method for the case of the point pulse initial value. Relative error in the L^2 norm versus the number of quadrature points N when (Left) $t = 5$ and (Right) $t = 10$ under different dispersion coefficients.

3.2. Gaussian pulse release

We consider the case that the initial value is given by two-dimensional Gaussian form

$$C_0(x, y) = \mathrm{e}^{-\frac{(x-p)^2}{4}-\frac{y^2}{4}}, \tag{3.10}$$

which is often used for solute transport in a porous medium [30]. Here, we take $p = 1$. In this case, we find

$$\hat{N}_0(k_1, k_2) = \hat{C}_1(k_1)\hat{C}_2(k_2) - \frac{ibk_1}{1 - ibk_1}\hat{C}_1(-k_1 - i/b)\hat{C}_2(k_2) \tag{3.11}$$

with

$$\hat{C}_1(k_1) = \mathrm{e}^{-k_1(k_1+ip)}\sqrt{\pi}\left[1 - i\mathrm{erfi}\left(k_1 + \frac{ip}{2}\right)\right], \tag{3.12}$$

$$\hat{C}_2(k_2) = \mathrm{e}^{-k_2^2}\sqrt{\pi}\left[\mathrm{erf}\left(\frac{1}{2} + ik_2\right) - i\mathrm{erfi}\left(k_2\right)\right], \tag{3.13}$$

where $\mathrm{erf}(\cdot)$ and $\mathrm{erfi}(\cdot)$ are respectively, the error and imaginary error functions. Substituting eq. (3.11) into eq. (2.45), we can find the explicit integral representation of the solution. We then employ the hybrid analytical-numerical scheme to integrate the resulting equation with respect to $k_{1,2}$ in a similar way as discussed in Sec. 3.1. The exponential convergence of the Gauss-Kronrod method was shown in Fig. 6, showing how effective and efficient the Fokas method is.

In Fig. 5(a,c,e), we compared the analytical solutions (displayed as solid curves) in a longitudinal cross-section at $y = 0$ with the numerical solutions (displayed as symbols) under the different dispersion coefficients $D_L = 0.1$, 1 and 10. These figures show that the analytical solutions obtained by the Fokas method with the hybrid analytical-numerical scheme are well consistent with the numerical results found by the FDM. We also demonstrated in Fig. 5(b,d,f) the analytical solution profiles in the longitudinal cross-section at $y = 0$ over the xt-plane.

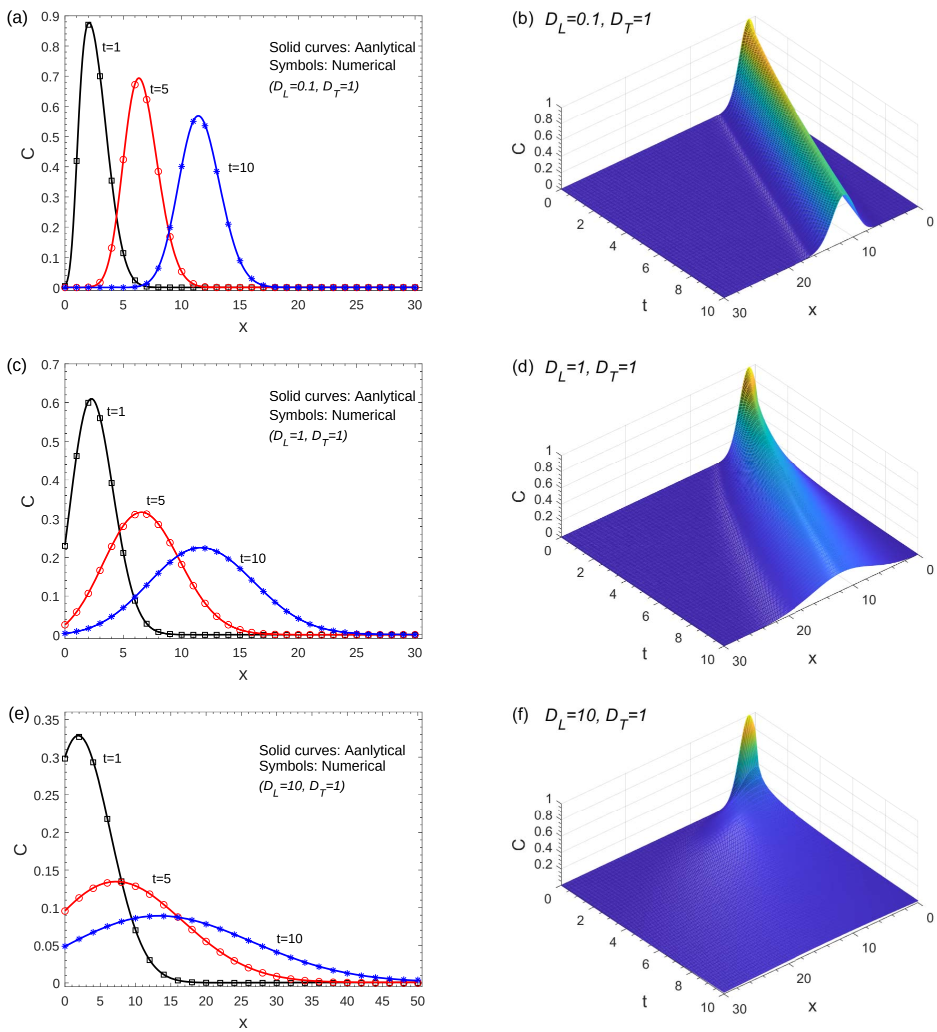

Figure 5. (a,c,e) Comparison of the analytical solutions (solid curves) and the numerical solutions (symbols) with the Gaussian initial value in a longitudinal cross-section at $y = 0$ under the different dispersion coefficients. (b,d,f) Analytical solution profiles over the xt-plane, where $y = 0$ under the different dispersion coefficients.

4. Concluding remarks

In conclusion, we have studied two-dimensional solute transport in a porous medium with advection, longitudinal and transverse dispersions and linear equilibrium adsorption, which can be modeled by the 2-D LAD equation posed on the quarter plane and in a semi-infinite domain. We have demonstrated the Fokas method to derive the analytical solution of the 2-D LAD equation as the integrals in the complex plane. The Fokas method is based on analysis of the divergence form and the global relation for the 2-D LAD equation. The developed solution has been compared with the numerical results, showing an excellent agreement.

The presented method has several advantages in studying the linear advection-

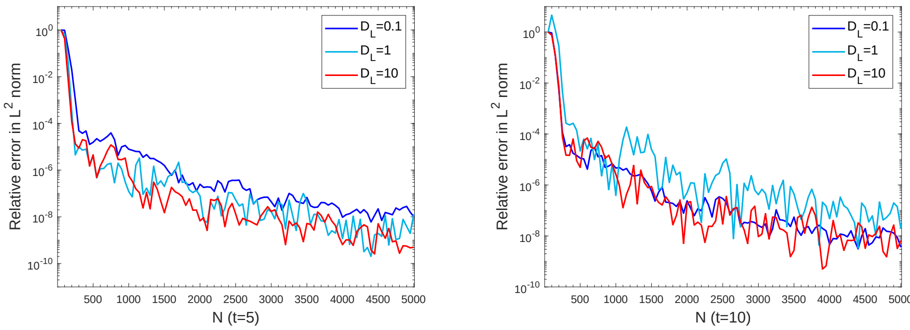

Figure 6. Exponential convergence of the Fokas method for the case of Gaussian initial value. Relative error in the L^2 norm versus the number of quadrature points N when (Left) $t = 5$ and (Right) $t = 10$ under different dispersion coefficients.

dispersion equation. The method is relatively simple, but effective to implement in solving the LAD equation. Importantly, the method provides the explicit integral representation of the solution, involving the time-dependence of exponential form, which leads to an effective numerical evaluation of the solution, called a hybrid analytical-numerical method. This hybrid method does not require to discretize the time domain and hence the presented method makes it efficient to understand and study the behavior of the long-time asymptotics [26, 27]. In addition, the present method converges exponentially with respect to the number of the quadrature points, which implies that the method is more efficient than finite difference methods. It is also remarked that in the case that the integral transforms of the initial and boundary values can not be computed explicitly, we can do numerically by using the Gauss-Kronrod quadrature method as discussed in [10].

The Fokas method has been extensively developed in solving boundary value problems and hence the presented method can be extended to analyze other physically important boundary value problems with more general and complicated boundary conditions as discussed in [6, 12, 29, 39, 41]. We will present these issues in the near future.

Acknowledgments

The author would like to thank anonymous reviewers, whose comments and suggestions helped improve the manuscript.

Appendix A.

In this section, we will verify that $C(x, y, t)$ given in eq. (2.45) solves eq. (2.25) satisfying the initial and boundary conditions. By substituting eq. (2.45) into eq. (2.25), it is straightforward to show that $C(x, y, t)$ solves the LAD equation (2.25). Thus, we will show that $C(x, y, t)$ given by eq. (2.45) satisfies the initial and boundary conditions given in eq. (2.27).

Initial condition. Note that eq. (2.45) at $t = 0$ is given by

$$
\begin{aligned}
C(x,y,0) = & \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz}\hat{N}_0(k_1,k_2) \\
& - \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{\partial\tilde{D}_2^+} \mathrm{d}k_2\, \frac{\mathrm{e}^{ikz}}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2) + \mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right] \\
& - \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{\partial\tilde{D}_2^-} \mathrm{d}k_2\, \frac{\mathrm{e}^{ikz-ik_2}}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2) + \hat{N}_0(k_1,-k_2)\right]. \quad \text{(A.1)}
\end{aligned}
$$

Regarding the first integral of the right-hand-side of eq. (A.1), we note that

$$
\begin{aligned}
& \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz}\hat{C}_0(k_1,k_2) \\
= & \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^1 \mathrm{d}\eta\, C_0(\xi,\eta) \int_{-\infty}^{\infty} \mathrm{d}k_1\, \mathrm{e}^{ik_1(x-\xi)} \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ik_2(y-\eta)}. \qquad \text{(A.2)}
\end{aligned}
$$

Using the following identity

$$
\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda(x-\xi)}\lambda = 2\pi\delta(x-\xi), \qquad \text{(A.3)}
$$

we find

$$
\begin{aligned}
\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz}\hat{C}_0(k_1,k_2) &= \int_{-\infty}^{\infty} \mathrm{d}\xi \int_0^1 \mathrm{d}\eta\, C_0(\xi,\eta)\delta(x-\xi)\delta(y-\eta) \\
&= C_0(x,y). \qquad \text{(A.4)}
\end{aligned}
$$

On the other hand, note that

$$
\begin{aligned}
& \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz}\frac{ibk_1}{1-ibk_1}\hat{C}_0(\hat{k}_1,k_2) \\
= & \frac{1}{4\pi^2}\int_0^{\infty} \mathrm{d}\xi \int_0^1 \mathrm{d}\eta\, \mathrm{e}^{-1/b}C_0(\xi,\eta) \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ik_2(y-\eta)} \\
& \times \int_{-\infty}^{\infty} \mathrm{d}k_1\, \mathrm{e}^{ik_1(x+\xi)}\frac{ibk_1}{1-ibk_1} = 0, \qquad \text{(A.5)}
\end{aligned}
$$

where we have used the fact that, by the Jordan lemma and eq. (A.3)

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}k_1\, \mathrm{e}^{ik_1(x+\xi)}\frac{ibk_1}{1-ibk_1} &= \int_{-\infty}^{\infty} \mathrm{d}k_1\, \mathrm{e}^{ik_1(x+\xi)}\left(\frac{1}{1-ibk_1} - 1\right) \\
&= -2\pi\delta(x+\xi) = 0 \quad (x>0). \qquad \text{(A.6)}
\end{aligned}
$$

Thus, we find

$$
\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}k_1 \int_{-\infty}^{\infty} \mathrm{d}k_2\, \mathrm{e}^{ikz}\hat{N}_0(k_1,k_2) = C_0(x,y). \qquad \text{(A.7)}
$$

On the other hand, the second and third integrals of the right-hand-side of eq. (A.1) vanish by using boundedness and analyticity of the integrands in the regions D_2^+ or D_2^-. Indeed, by the Jordan lemma, we find

$$
\int_{\partial\tilde{D}_2^+} \mathrm{d}k_2\, \mathrm{e}^{ik_2 y}\frac{\mathrm{e}^{\pm ik_2}}{\Delta(k_2)}\mathrm{e}^{\mp ik_2\eta} = 0, \qquad \text{(A.8)}
$$

which implies that

$$-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ikz}\,\frac{\mathrm{e}^{\pm ik_2}}{\Delta(k_2)}\hat{N}_0(k_1,\pm k_2)=0. \tag{A.9}$$

Hence the second integral of the right-hand-side of eq. (A.1) vanishes. In a similar way, using

$$\int_{\partial\tilde{D}_2^-}\mathrm{d}k_2\,\mathrm{e}^{ik_2(y-1)}\,\frac{\mathrm{e}^{\pm ik_2\eta}}{\Delta(k_2)}=0, \tag{A.10}$$

the third integral of the right-hand-side of eq. (A.1) equals to zero. Therefore, $C(x,y,0)=C_0(x,y)$, as desired.

Boundary condition at $x=0$. Differentiating eq. (2.45) with respect to x and evaluating the resulting equation at $x=0$, we find

$$\begin{aligned}
&(C-bC_x)(0,y,t)\\
=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}\,(1-ibk_1)\,\hat{N}_0(k_1,k_2)\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}\,\frac{1-ibk_1}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2)+\mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right]\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^-}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}\,\frac{(1-ibk_1)\mathrm{e}^{-ik_2}}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2)+\hat{N}_0(k_1,-k_2)\right].
\end{aligned} \tag{A.11}$$

Note that the first integral of the right-hand-side of eq. (A.11) can be written as

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}\,(1-ibk_1)\,\hat{N}_0(k_1,k_2)\\
=&-\frac{1}{4\pi^2}\int_{\partial D_1^-}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}(1-ibk_1)\hat{C}_0(k_1,k_2)\\
&-\frac{1}{4\pi^2}\int_{\partial D_1^+}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}ibk_1\hat{C}_0(\hat{k}_1,k_2).
\end{aligned} \tag{A.12}$$

Letting $k_1\to\hat{k}_1=-k_1-i/b$ in the second integral of the right-hand-side of eq. (A.12), the resulting integral cancels with the first integral of the right-hand-side of eq. (A.12). Thus, we find

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_2y-\omega(k)t}\,(1-ibk_1)\,\hat{N}_0(k_1,k_2)=0. \tag{A.13}$$

In a similar way, we can prove that the second and third integrals of the right-hand-side of eq. (A.11) vanish. Therefore, we can show that $(C-bC_x)\,(0,y,t)=0$.

Boundary conditions at $y=0$ and $y=1$. Differentiating eq. (2.45) with respect to y and evaluating the resulting equation at $y=0$, we find

$$C_y(x,0,t)$$

$$
\begin{aligned}
=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\hat{N}_0(k_1,k_2)\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{ik_2}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2)+\mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right]\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^-}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{ik_2\mathrm{e}^{-ik_2}}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2)+\hat{N}_0(k_1,-k_2)\right].
\end{aligned}
\tag{A.14}
$$

Replacing $k_2\to -k_2$ in the third integral of the right-hand-side of eq. (A.14) yields

$$
\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{ik_2\mathrm{e}^{ik_2}}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2)+\hat{N}_0(k_1,-k_2)\right].
\tag{A.15}
$$

Thus, eq. (A.14) can be written as

$$
\begin{aligned}
C_y(x,0,t)=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\hat{N}_0(k_1,k_2)\\
&+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\hat{N}_0(k_1,-k_2).
\end{aligned}
\tag{A.16}
$$

Replacing $k_2\to -k_2$ in the second integral of the right-hand-side of eq. (A.16), we find

$$
\begin{aligned}
&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\hat{N}_0(k_1,-k_2)\\
=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial D_2^-}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\hat{N}_0(k_1,k_2).
\end{aligned}
\tag{A.17}
$$

By the Cauchy theorem, we deform the contour ∂D_2^- in the right-hand-side of the above equation to the contour $(-\infty,\infty)$ in the negative orientation and then we know that the resulting integral cancels with the first integral of the right-hand-side of eq. (A.16). Thus, we can show that $C_y(x,0,t)=0$.

Differentiating eq. (2.45) with respect to y and evaluating the resulting equation at $y=1$, we find

$$
\begin{aligned}
&C_y(x,0,t)\\
=&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{-\infty}^{\infty}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}ik_2\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2)\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^+}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{ik_2\mathrm{e}^{ik_2}}{\Delta(k_2)}\left[\mathrm{e}^{ik_2}\hat{N}_0(k_1,k_2)+\mathrm{e}^{-ik_2}\hat{N}_0(k_1,-k_2)\right]\\
&-\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}k_1\int_{\partial\tilde{D}_2^-}\mathrm{d}k_2\,\mathrm{e}^{ik_1x-\omega(k)t}\frac{ik_2}{\Delta(k_2)}\left[\hat{N}_0(k_1,k_2)+\hat{N}_0(k_1,-k_2)\right].xx
\end{aligned}
\tag{A.18}
$$

In a similar way as discussed above, we can show that $C_y(x,1,t)=0$.

References

[1] V. Batu, *A generalized two-dimensional analytical solution for hydrodynamic dispersion in bounded media With the first-type boundary condition at the source*, Water Resour. Res., 1989, 25, 1125–1132.

[2] G. Biondini and G. Hwang, *Initial-boundary value problems for discrete evolution equations: discrete linear Schrödinger and integrable discrete nonlinear Schrödinger equations*, Inv. Probl., 2008, 24, 065011(1–44).

[3] J. Chen and C. Liu, *Generalized analytical solution for advection-dispersion equation in finite spatial domain with arbitrary time-dependent inlet boundary condition*, Hydro. Earth Syst. Sci., 2011, 15, 2471–2479.

[4] J. Chen, J. Chen, C. Liu, C. Liang and C. Lin, *Analytical solutions to two-dimensional advection-dispersion equation in cylindrical coordinates in finite domain subject to first- and third-type inlet boundary conditions*, J. Hydrology., 2011, 405, 522–531.

[5] M. J. Colbrook, *Extending the unified transform: curvilinear polygons and variable coefficients PDEs*, IMA J. Numer. Anal., 2020, 40, 976–1004.

[6] M. J. Colbrook, L. J. Ayton and A. S. Fokas, *The unified transform for mixed boundary condition problems in unbounded domains*, Proc. Roy, Soc. A., 2019, 475, 20180605.

[7] M. J. Colbrook, Z. I. Botev, K. Kuritz and S. MacNamara, *Kernel density estimation with linked boundary conditions*, Stud. Appl. Math., 2020, 145, 357–396.

[8] M. J. Colbrook, N. Flyer and B. Fornberg, *On the Fokas method for the solution of elliptic problems in both convex and non-convex polygonal domains*, J. Comput. Phys., 2018, 374, 996–1016.

[9] M. J. Colbrook, A. S. Fokas and P. Hashemzadeh, *A hybrid analytical-numerical technique for elliptic PDEs*, SIAM J. Sci. Comput., 2019, 41, A1066–A1090.

[10] F. R. J. de Barros, M. J. Colbrook and A. S. Fokas, *A hybrid analytical-numerical method for solving advection-dispersion problems on a half -line*, Int. J. Heat Mass Transf., 2019, 139, 482–491.

[11] G. Dagan, *Theory of solute transport by groundwater*, Ann. Rev. Fluid Mech., 1987, 19, 183–215.

[12] B. Deconinck, B. Pelloni and N. E. Sheils, *Non-steady-state heat conduction in composite walls*, Proc. Roy, Soc. A., 2014, 470, 20130605.

[13] B. Deconinck, T. Trogdon and V. Vasan, *The method of Fokas for solving linear partial differential equations*, SIAM Review., 2014, 56, 159–186.

[14] A. S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs*, Proc. Roy. Soc. London A., 1997, 453, 1411–1443.

[15] A. S. Fokas, *On the integrability of certain linear and nonlinear partial differential equations*, J. Math. Phys., 2000, 41, 4188–4237.

[16] A. S. Fokas, *Two dimensional linear PDEs in a convex polygon*, Proc. Roy. Soc. London A., 2001, 457, 371–393.

[17] A. S. Fokas, *Integrable nonlinear evolution equations on the half-line*, Comm. Math. Phys., 2002, 230, 1–39.

[18] A. S. Fokas, *A new transform method for evolution partial differential equations*, IMA J. Appl. Math., 2002, 67, 559–590.

[19] A. S. Fokas, *A Unified Approach to Boundary Value Problems*, (CBMS-NSF Regional Conference Series in Applied Mathematics) Philadelphia, SIAM., 2008.

[20] A. S. Fokas and B. Pelloni, *A transform method for linear evolution PDEs on a finite interval*, IMA J. Appl. Math., 2005, 70, 564–587.

[21] J. S. Pérez Guerrero, E. M. Pontedeiro, M. Th. van Genuchten and T. H. Skaggs, *Analytical solution of the one-dimensional advection-dispersion solute transport equation subject to time-dependent boundary conditions*, Chem. Eng. J., 2013, 221, 487–491.

[22] J. S. Pérez Guerrero, L. C. G. Pimentel, T. H. Skaggs and M. Th. van Genuchten, *Analytical solution of the advection-diffusion transport equation using a change-of-variable and integral transform technique*, Int. J. Heat Mass Transf., 2009, 52, 3297–3304.

[23] J. S. Pérez Guerrero, L. C. G. Pimentel, T. H. Skaggs and M. Th. van Genuchten, *Analytical solution for multi-species contaminant transport subject to sequential first-order decay reaction in finite media*, Transport Porous Med., 2009, 80, 373–357.

[24] M. T. van Genuchten and W. J. Alves, *Analytical solutions of the one-dimensional convective-dispersive solute transport equation*, Technical Bulletins 157268, United States Department of Agriculture, Economic Research Service, 1982.

[25] G. Hwang, *The modified Korteweg-de Vries equation on the quarter plane with t-periodic data*, J. Nonlinear Math. Phys., 2017, 24, 620–634.

[26] G. Hwang, *Initial-boundary value problems for the one-dimensional linear advection-dispersion equation with decay*, Z. Naturforshc. A., 2020, 75, 713–725.

[27] G. Hwang, *A unified approach to two-dimensional linear advection-dispersion equation in cylindrical coordinates on a finite domain*, Int. J. Heat Mass Transf., 2021, 164, 120569.

[28] G. Hwang and A. S. Fokas, *The modified Korteweg-de Vries equation on the half-line with a sine-wave as Dirichlet datum*, J. Nonlinear Math. Phys., 2013, 20, 135–157.

[29] D. Mantzavinos and A. S. Fokas, *The unified method for the heat equation: I. non-separable boundary conditions and non-local constraints in one dimension*, Eur. J. App. Math., 2013, 24, 857–886.

[30] M. Massabó, R. Cianci and O. Paladino, *Some analytical solutions for two-dimensional convection-dispersion equation in cylindrical geometry*, Environ. Model. Softw., 2006, 21, 681–688.

[31] M. Massabó, R. Cianci and O. Paladino, *An analytical solution of the advection dispersion equation in a bounded domain and its application to laboratory experiments*, J. Appl. Math., 2011, 2011, 493014.

[32] P. D. Miller and D. A. Smith, *The diffusion equation with nonlocal data, J. Math. Anal. Appl.*, 2018, 466, 1119–114.

[33] A. Mojtabi and M. O. Deville, *One-dimensional linear advection-diffusion equation: Analytical and finite element solutions*, Comput. Fluids., 2015, 107, 189–195.

[34] B. Moon and G. Hwang, *The Korteweg-de Vries equation on the quarter plane with asymptotically t-periodic data via the Fokas method*, Asy. Anal., 2018, 107, 115–133.

[35] B. Moon and G. Hwang, *Discrete linear evolution equations in a finite lattice*, J. Differ. Equ. Appl., 2019, 25, 630–646.

[36] A. Moranda, R. Cianci and O. Paladino, *Analytical solutions of one-dimensional contaminant transport in soils with source production-decay*, Soil Systems, 2018, 2(40), 1–16.

[37] *K. W. Morton and D. F. Mayers*, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, 2005.

[38] B. Pelloni and D. A. Pinotsis, *The elliptic sine-Gordon equation in a half plane*, *Nonlinearity*, 2010, 23, 77–88.

[39] B. Pelloni and D. A. Smith, *Nonlocal and multipoint boundary value problems for linear evolution equations*, Stud. Appl. Math., 2018, 141, 46–88.

[40] D. Plümacher, M. Oberlack, Y. Wang and M. Smuda *On a non-linear droplet oscillation theory via the unified method*, Phys. Fluid., 2020, 32, 067104.

[41] N. E. Sheils and B. Deconinck, *Heat conduction on the ring: interface problems with periodic boundary conditions*, App. Math. Lett., 2014, 37, 107–111.

[42] D. Tang, E. O. Frind and E. A. Sudicky, Contaminant transport in fractured porous media: analytical solution for a single fracture, Water Resources Research, 1981, 17, 555–564.