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RATIONAL AND INTERACTIVE SOLUTIONS TO THE B-TYPE KADOMTSEV-PETVIASHVILI EQUATION

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Abstract In this paper, a new method to find quadratic function solutions to bilinear forms is proposed. By applying the Hirota direct method, we construct some important exact solutions to the B-type Kadomtsev-Petviashvili (BKP) equation of fourth-order. Solitons, rational solutions, lump solutions and interaction solutions are presented with the help of symbolic computations. The dynamics of some selected solutions are also studied with the aid of 3D plots.

Keywords Hirota method, the BKP equation, lump solutions, lump-kink solutions

MSC(2010) 35A25, 37K10.

1. Introduction

Searching for exact solutions to nonlinear differential equations is an extremely important task in mathematical physics. Over the years, many different kinds of exact solutions have been constructed for nonlinear differential equations through the application of powerful mathematical tools such as the inverse scattering transform [14], the Hirota method [19–21], the Wronskian and pfaffian technique [8, 24], the Bell polynomial approach [28, 64], the Darboux transformation[50] and the Bäcklund transformation [12, 16, 24, 35], Painlevé analysis [34, 68] etc. Among these tools, the Hirota method and various ansatzes with symbolic computation remain most efficient for the formulation of exact solutions and multiple collisions of solitons[7, 15, 29, 30, 36–39, 45–47, 69, 71, 72, 75, 76, 79–81, 81, 83].

The Kadomtsev-Petviashvili (KP) equation, discovered in 1970 [25], is one of the most extensively studied soliton equations in (2+1)-dimensions [27, 44, 61]. It describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate. It is also a natural extension of the classical KdV equation to two spatial dimensions. The KP equation is usually written in the form of

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3\sigma^2 u_{yy} = 0. \tag{1.1}$$

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Here the subscripts x, y and t denote partial derivatives, and $\sigma^2 = \pm 1$. When $\sigma^2 = 1$ the above equation is known as the KPII equation, and when $\sigma^2 = -1$ it is called the KPI equation.

From the viewpoint of infinite dimensional Lie algebras, the KP hierarchy, which contains the KP equation, corresponds to Lie algebras of A-type [54]. A variant of this hierarchy is the BKP hierarchy which corresponds to Lie algebras of B-type [10, 53, 54]. It is a system of nonlinear differential equations obtained as the compatibility conditions of

$$L_{KP}w(x,k) = kw(x,k), \tag{1.2}$$

$$\partial_n w(x,k) = B_n(x,\partial)w(x,k) \tag{1.3}$$

where $x = (x_1, x_3, x_5, \cdots)$, under the condition

$$B_n(x,\partial)1 = 0, \qquad n = 1, 3, 5, \cdots. \tag{1.4}$$

The first two nonlinear partial differential equations in the BKP hierarchy are [22, 23]

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0, \tag{1.5}$$

with bilinear form

$$[(D_3 - D_1^3)D_{-1} + 3D_1^2]f \cdot f = 0, \tag{1.6}$$

where $x_1 = x, x_{-1} = y, x_3 = t$, and

$$(u_t + 15uu_{xxx} + 15u_x^3 - 15u_x u_y + u_{xxxxx})_x - 5u_{xxxy} - 5u_{yy} = 0, \tag{1.7}$$

with bilinear form

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)f \cdot f = 0,$$

where $x_1 = x, x_3 = y, x_5 = t$. The latter has been studied extensively in [17, 34, 52, 67, 73, 77]. In the present paper, we mainly study equation (1.5).

The nonlinear partial differential equation (1.5) can be expressed as

$$\frac{\partial}{\partial x}\Big\{[(D_3 - D_1^3)D_{-1} + 3D_1^2]f \cdot f\Big\} = 0 \tag{1.8}$$

through the dependent variable transformation,

$$u = 2(\ln f)_x, \tag{1.9}$$

where $x_1 = x, x_{-1} = y, x_3 = t$. Therefore, whenever the function f solves the bilinear equation (1.6), the corresponding function u defined by (1.9) also solves (1.5). If we introduce the polynomial

$$P_{BKP}(t,x,y) = ty + 3x^2 - x^3 y, \tag{1.10}$$

then (1.6) can be expressed as

$$P_{BKP}(D_t, D_x, D_y)f \cdot f = 0. \tag{1.11}$$

In what follows, we shall investigate solitons and rational solutions of (1.5) in detail and also show that this equation possesses lump solutions. Moreover, we shall construct the solutions describing collisions of lumps from the soliton solutions of (1.5). For this reason, we provide a short description of soliton solutions of the BKP equation in section 2. In Section 3, we shall consider rational solutions (including lumps) of the BKP equation. Section 4 is devoted to the collisions of a soliton and a lump solution. We have comments for the results in the final section.

2. Solitons

Solitons are the most important class of solutions of integrable systems and can be derived by the IST and the Hirota method. They are usually exponentially localized solutions. However, they can be rationally localized as well. For example, soliton solutions for the Benjamin-Ono equation are given by rational functions (see [9, 49, 62] for details). For the BKP equation under discussion, a multi-soliton can also be expressed as a phaffian or in a Wronskian form [24, 26].

Solitons have a wide range of applications in nonlinear dynamics. They arise in areas such as fluid mechanics [18, 70], nonlinear optics [1], atomic physics [13], biophysics [57], biology [63], field theory [11] and in plasmas [33] and Bose-Einstein condensates [32, 58] to name but a few.

2.1. The one-soliton solution

In order to investigate solutions to the BKP equation, let us first consider the one-soliton solution. We introduce the wave function

$$\eta := kx + ly + wt + \eta^0, \tag{2.1}$$

where k, l, w and η^0 are constants. If the dispersion relation

$$P_{BKP}(w, k, l) = 0 \tag{2.2}$$

is satisfied, then we have

$$w = k^3 - \frac{3k^2}{l}. \tag{2.3}$$

According to the Hirota method, the function

$$u = 2[\ln(f_1)]_x = 2[\ln(1 + \exp{(\eta)})]_x = \frac{2k\exp{(\eta)}}{1 + \exp{(\eta)}},$$

where η is defined by (2.1), gives the one-soliton solution to the BKP equation.

2.2. The two-soliton solution

Again, according to [24], the two-soliton expression is given by

$$u = 2[\ln(f_2)]_x, \quad f_2 = 1 + \exp(\eta_1) + \exp(\eta_2) + b_{12}\exp(\eta_1 + \eta_2). \tag{2.4}$$

If we introduce parameters p_j and q_j we may rewrite η_j as

$$\eta_j = \xi_j + \hat{\xi}_j, \tag{2.5}$$
$$\xi_j = p_j^{-1}y + p_j x + p_j^3 t + \xi_j^0, \tag{2.6}$$
$$\hat{\xi}_j = q_j^{-1}y + q_j x + q_j^3 t + \hat{\xi}_j^0 \tag{2.7}$$

and thus, the dispersion relation (2.2) is automatically satisfied. Also, the phase shift term b_{12} is given by

$$b_{12} = \frac{(p_1 - p_2)(p_1 - q_2)(q_1 - p_2)(q_1 - q_2)}{(p_1 + p_2)(p_1 + q_2)(q_1 + p_2)(q_1 + q_2)}. \tag{2.8}$$

2.3. The N-soliton solution

If we put $b_{ij} = \exp(B_{ij})$, the Hirota condition is satisfied and the N-soliton solution to the BKP equation is expressed as

$$f_N = \sum \exp\left[\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} B_{ij} \mu_i \mu_j\right], \tag{2.9}$$

where $\sum$ denotes the summation over all possible combinations of $\mu_i = 0, 1$ for $i = 1, 2, \cdots, N$, and $\sum_{i<j}^{(N)}$ is the sum over all pairs $i, j (i < j)$ chosen from $\{1, 2, ..., N\}$.

3. Rational solutions

Rational solutions are the simplest solutions to nonlinear evolution equations. They have attracted a lot of interest in the past few years owing to their copious real-life applications. For example, the most commonly used mathematical model for the study of rogue waves involves rational solutions of the focusing nonlinear Schrodinger (NLS) equation [9]. Further applications include the description of vortex dynamics [3–5] and vortex solutions of the complex sine-Gordon equation [6, 55].

In what follows, we present rational function solutions to the BKP equation generated by quadratic functions.

3.1. Quadratic solutions to a bilinear form

Quadratic functions are polynomials which can be easily discussed. The transformation $u = (\ln f)_x$ makes u a rational function when f is a polynomial. We specifically look for quadratic solutions to the bilinear equation (1.11). In order to achieve this, we first introduce a general method to find all the quadratic function solutions to a bilinear form. This method can be regarded as an extension of [42, 43].

Suppose the integer $M \geq 1$ and $x \in \mathbf{R}^M$. Let $D = (D_1, D_2, \cdots, D_M)^T$, where D_j is the D-operator [24] with respect to $x_j, 1 \leq j \leq M$. We will discuss the following general bilinear form

$$P(D) f \cdot f = P(D_1, D_2, \cdots, D_M) f \cdot f = 0, \tag{3.1}$$

where P is a polynomial of M variables $x = (x_1, \cdots, x_M)$. By the property of bilinear forms, we assume that P is an even polynomial with $P(0) = 0$, i.e., $P(-x) = P(x)$. In general, we set

$$P(x) = \sum_{i,j=1}^{M} q_{ij} x_i x_j + \sum_{i,j,k,l=1}^{M} p_{ijkl} x_i x_j x_k x_l + \text{higher order terms}, \tag{3.2}$$

where q_{ij} and p_{ijkl} are coefficients of the second- and fourth-degree terms respectively, to determine quadratic function solutions. Without loss of the generality, we require $q_{ij} = q_{ji}, 1 \leq, i, j \leq M$. We denote the coefficient matrix of the second-order Hirota bilinear derivative terms by $Q = (q_{ij})_{M\times M} \in \mathbf{R}^{M\times M}$ so that it is symmetric: $Q = Q^T$. A quadratic polynomial $f : \mathbf{R}^M \to \mathbf{R}$ can always be expressed as

$$f(x) = \frac{1}{2} x^T A x + b^T x + c, \tag{3.3}$$

where $A \in \mathbf{R}^{M\times M}$ is a symmetric matrix, $b \in \mathbf{R}^M$ denotes a column vector, $c \in \mathbf{R}$ is a constant and T denotes matrix transpose. A, b, c are uniquely determined by f under the condition $A = A^T$.

We remark that, for any (2+1)-dimensional system, we can take $M = 3$ and vector $x = (t, x, y)$. Note that

$$\frac{\partial f}{\partial x_i} = \sum_{k=1}^{M} a_{ik}x_k + b_i = A_i^T x + b_i,\ \frac{\partial^2 f}{\partial x_i \partial x_j} = a_{ij},\ 1 \le i, j \le M,$$

where A_i is the ith column vector of A for $1 \le i \le M$. Now, substituting f into (3.1), we get

$$\begin{aligned}
&P(D)f\cdot f \\
&= 2\sum_{i,j=1}^{M} q_{ij}[a_{ij}f - (A_i^T x + b_i)^T(A_j^T x + b_j)] \\
&\quad +2\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) \\
&= 2c\sum_{i,j=1}^{M} a_{ij}q_{ij} - 2b^T Qb + 2\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}), \\
&\quad +2\Big[(\sum_{i,j=1}^{M} a_{ij}q_{ij})b^T - 2b^T QA\Big]x + x^T\Big[(\sum_{i,j=1}^{M} a_{ij}q_{ij})A - 2AQA\Big]x.
\end{aligned}$$

If we introduce $s = (\sum_{i,j=1}^{M} a_{ij}q_{ij})/2$, then f solves (3.1) if and only if

$$\begin{cases} 2cs - b^T Qb + \displaystyle\sum_{i,j,k,l=1}^{M} p_{ijkl}(a_{ij}a_{kl} + a_{ik}a_{jl} + a_{il}a_{jk}) = 0, \\ sb^T - b^T QA = 0, \\ sA - AQA = 0. \end{cases} \tag{3.4}$$

We can solve the second and the third equations of (3.4) to obtain A and b. The first equation of (3.4) is a scalar equation and when A, b are known, c is uniquely detremined if $s \neq 0$.

3.2. Application to the BKP equation

For the BKP equation (1.5), we take $x_1 = x, x_2 = y, x_3 = t$. Then

$$Q = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}, \qquad p_{1112} = -1, \qquad p_{ijkl} = 0, ijkl \neq 1112.$$

Let

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{bmatrix}, \quad b = \begin{bmatrix} a_7 \\ a_8 \\ a_9 \end{bmatrix}, \quad c = a_{10}. \tag{3.5}$$

It is easy to see that $s = (3a_1 + a_5)/2$. Using symbolic computation to solve (3.4), we have four classes of solutions.

Case I: The set of solutions in terms of the parameters a_1, a_6, a_7 and a_9:

$$a_2 = a_3 = 0,\ a_4 = \frac{9a_1^2}{a_6},\ a_5 = 3a_1,\ a_8 = \frac{3a_1a_9}{a_6},\ a_{10} = \frac{a_1a_9^2 + a_6a_7^2}{2a_1a_6}.$$

With these parameters and according to (3.3)–(3.5)

$$\begin{aligned} f &= \frac{a_1x^2}{2} + \frac{9a_1^2y^2}{2a_6} + \frac{a_6t^2}{2} + 3a_1ty + a_7x + \frac{3a_1a_9y}{a_6} + a_9t + \frac{a_1a_9^2 + a_6a_7^2}{2a_1a_6} \\ &= \frac{1}{2a_1a_6}[a_6(a_1x + a_7)^2 + a_1(3a_1y + a_6t + a_9)^2] \end{aligned}$$

and

$$u = \frac{4a_1a_6(a_1x + a_7)}{a_6(a_1x + a_7)^2 + a_1(3a_1y + a_6t + a_9)^2}.$$

When a_1, a_6 are all positive, the denominator of u is never zero and globe solutions of the BKP equation exist. Further more we have $\lim_{(x,y)\to\infty} u(t,x,y) = 0$ for any fixed t. Actully, we obtain lump solutions in this situation.

If $a_1a_6 < 0$, we get singular solutions, i.e., solutions are not defined for all (t, x, y). In the case where initial or boundary conditions are imposed, the solutions may blow up at some finite time. Consequently, we take parameters $a_1 = 1, a_6 = -1, a_7 = -3, a_9 = 2$, and obtain the singular solution

$$u(20, x, y) = \frac{4(x - 3)}{x^2 - 9y^2 - 6x + 108y - 315}.$$

The surface, contour and density plots of the function u when $t = 20$ are depicted in Figure 1.

Case II: The set of solutions in terms of the parameters a_2, a_5, a_8 and a_9:

$$a_1 = a_3 = a_6 = 0, a_4 = -\frac{6a_2^2}{a_5}, a_7 = \frac{a_2a_9}{a_5}, a_{10} = \frac{a_9(3a_2^2a_9 + a_5^2a_8)}{a_5^3}.$$

In this case

$$f = -\frac{6a_2^2}{a_5}y^2 + a_2xy + a_5ty + \frac{a_2a_9}{a_5}x + a_8y + a_9t + \frac{a_9(3a_2^2a_9 + a_5^2a_8)}{a_5^3}$$

and

$$u = \frac{2a_2a_5^2(a_5y + a_9)}{a_5^2(-6a_2^2y^2 + a_2a_5xy + a_5^2ty + a_2a_9x + a_5a_8y + a_5a_9t) + a_9(3a_2^2a_9 + a_5^2a_8)}.$$

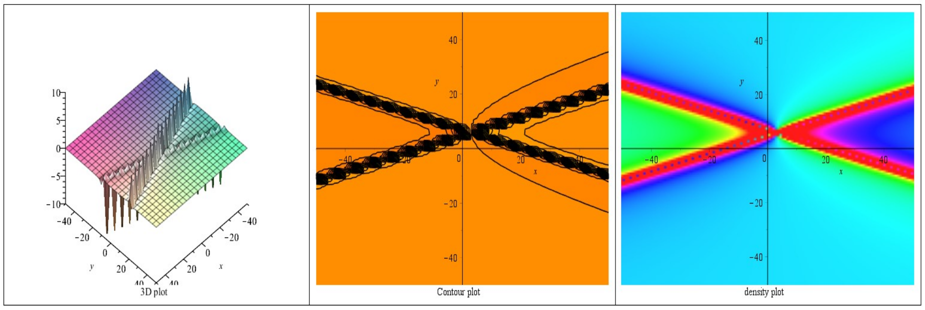

Figure 1. A rational solution to the BKP equation

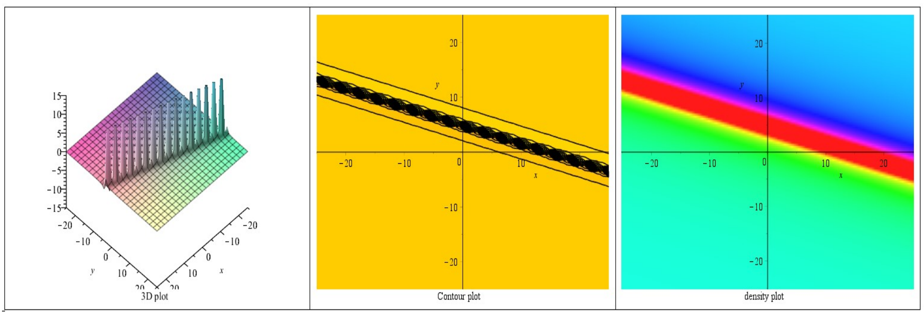

Figure 2. Another rational solution to the BKP equation

For nontrivial solutions (i.e. $a_2a_5 \neq 0$), the quadratic part of f has both positive and negative eigenvalues, and thus the solutions are always singular. A special choice of the parameters $a_2 = 2, a_5 = -2, a_8 = -3, a_9 = -2$, yields

$$u(10, x, y) = \frac{4(y+1)}{2xy + 6y^2 + 2x - 23y - 29}.$$

The surface, contour and density plots of the function u when $t = 20$ are depicted in Figure 2.

Case III: The set of solutions in terms of the parameters a_1, a_3, a_7, a_9 and a_{10}:

$$a_2 = \frac{3a_1a_3(2a_1a_{10} - a_7^2)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}, a_4 = -\frac{9a_1^2(2a_1a_{10} - a_7^2)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9},$$
$$a_5 = \frac{3(3a_1^3a_3 - a_1^2a_9^2 + 2a_1a_3^2a_{10} + 2a_1a_3a_7a_9 - 2a_3^2a_7^2)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9},$$
$$a_6 = -\frac{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}{2a_1a_{10} - a_7^2},$$
$$a_8 = \frac{-3(2a_1^2a_9a_{10} - 4a_1a_3a_7a_{10} - a_1a_7^2a_9 + 2a_3a_7^3)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}.$$

From (3.3)–(3.5) , we have

$$
\begin{aligned}
f = {} & \frac{a_1}{2}x^2 - \frac{9a_1^2(2a_1a_{10}-a_7^2)}{2(3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9)}y^2 \\
& - \frac{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}{2(2a_1a_{10}-a_7^2)}t^2 \\
& + \frac{3a_1a_3(2a_1a_{10}-a_7^2)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}xy + a_3xt \\
& + \frac{3(3a_1^3a_3 - a_1^2a_9^2 + 2a_1a_3^2a_{10} + 2a_1a_3a_7a_9 - 2a_3^2a_7^2)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}yz + a_7x \\
& - \frac{3(2a_1^2a_9a_{10} - 4a_1a_3a_7a_{10} - a_1a_7^2a_9 + 2a_3a_7^3)}{3a_1^2a_3 - a_1a_9^2 - 2a_3^2a_{10} + 2a_3a_7a_9}y + a_9t + a_{10}.
\end{aligned}
$$

and consequently,

$$u = \frac{2(a_1x + a_2y + a_3t + a_7)}{f}.$$

Case IV: The set of solutions in terms of the parameters a_1, a_3, a_6 and a_9:

$$
\begin{aligned}
& a_2 = -\frac{3a_1a_3}{a_6}, a_4 = \frac{9a_1^2}{a_6}, a_5 = \frac{3(a_1a_6 - 2a_3^2)}{a_6} \\
& a_7 = \frac{a_1r}{a_3}, a_8 = -\frac{3a_1(2r-a_9)}{a_6}, a_{10} = \frac{a_1(2ra_9 + 3a_1a_3 - a_9^2)}{2a_3^2},
\end{aligned}
$$

where r is a root of $z^2 - 2a_9z - 3a_1a_3 + a_9^2 = 0$. It is easy to see that, if $a_1a_3 \geq 0$ then r is real. In general,

$$
\begin{aligned}
f = {} & \frac{a_1}{2}x^2 - \frac{9a_1^2}{a_6}y^2 + a_6t^2 - \frac{3a_1a_3}{a_6}xy + a_3xt + \frac{3(a_1a_6 - 2a_3^2)}{a_6}yz + \frac{a_1r}{a_3}x \\
& - \frac{3a_1(2r-a_9)}{a_6}y + a_9t + \frac{a_1(2ra_9 + 3a_1a_3 - a_9^2)}{2a_3^2}
\end{aligned}
$$

and

$$u = \frac{2(a_1x + a_2y + a_3t + a_7)}{f}.$$

For example, if $a_1 = 0$ then $r = a_9$. We have

$$f(t,x,y) = a_3xt + a_6t^2 - \frac{6a_3^2}{a_6}yz + a_9t,$$

and

$$u(t,x,y) = \frac{2a_3a_6t}{a_3a_6xt + a_6^2t^2 - 6a_3^2yz + a_6a_9t}.$$

When $a_3 \neq 0$ the solution is singular.

In general, if $a_1a_6 \neq 0$, any solution, if it exists, is singular since $-\frac{9a_1^2}{a_6}$ and a_6 cannot be positive at same time.

3.3. Lump solutions

Most rational functions are singular because they have poles and therefore these functions are not defined for every (t, x, y). If a quadratic function f is positive, then $(\ln f)_x$ and $(\ln f)_{xx}$ are analytic rational functions for all independent variables. A positive quadratic function under certain conditions will generate a lump solution.

Lump solutions are a kind of analytical solutions, which are rationally localized in all directions in space. For (2+1)-dimensional systems (e.g. the KP equation), lump solutions may also be regarded as solitons [44]. The study of lump solutions has a long history [27, 61] and recently new such solutions have been presented for (2+1)-dimensional integrable equations including the KP and the BKP equations [31, 40–42, 46, 48, 65, 66, 82]. For applications of lump solutions, we refer the reader to [2, 25, 51, 56].

According to [42], the largest class of quadratic functions which generate lump solutions to a (2+1)-dimensional nonlinear PDEs can be expressed as

$$f = f_1^2 + f_2^2 + a_9, \qquad a_9 > 0, \tag{3.6}$$

with

$$f_1(x, y, t) = a_1 x + a_2 y + a_1 t + a_4, \quad f_2(x, y, t) = a_5 x + a_6 y + a_7 t + a_8, \tag{3.7}$$

where a_i, $1 \le i \le 9$ are real constants to be determined. Equation (1.11) therefore leads to one class of lump solutions which we present below. From (1.11), we obtain with the aid of a computer algebra system the following class of solutions:

$$\begin{cases} a_3 = -\dfrac{3(a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)}{a_2^2 + a_6^2}, \\ a_7 = \dfrac{3(a_1^2 a_6 - 2a_1 a_2 a_5 - a_5^2 a_6)}{a_2^2 + a_6^2}, \\ a_9 = \dfrac{(a_2^2 + a_6^2)(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}. \end{cases} \tag{3.8}$$

The parameters a_1, a_2, a_4, a_5, a_6 and a_8 are all arbitrary with a_1, a_2, a_5 and a_6 satisfying the condition

$$a_1 a_6 - a_2 a_5 \neq 0 \tag{3.9}$$

which ensures that a_3, a_7 and a_9 are well-defined. For f to be positive, we require

$$(a_1 a_2 + a_5 a_6) > 0, \tag{3.10}$$

so that f generates lump solutions to equation (1.5). Thus, f yields the following solution to equation (1.5):

$$u = \frac{4(a_1 f_1 + a_5 f_2)}{f_1^2 + f_2^2 + a_9}, \tag{3.11}$$

where,

$$f_1 = a_1 x + a_2 y - \frac{3(a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)}{a_2^2 + a_6^2} t + a_4, \tag{3.12}$$

$$f_2 = a_5 x + a_6 y + \frac{3(a_1^2 a_6 - 2a_1 a_2 a_5 - a_5^2 a_6)}{a_2^2 + a_6^2} t + a_8. \tag{3.13}$$

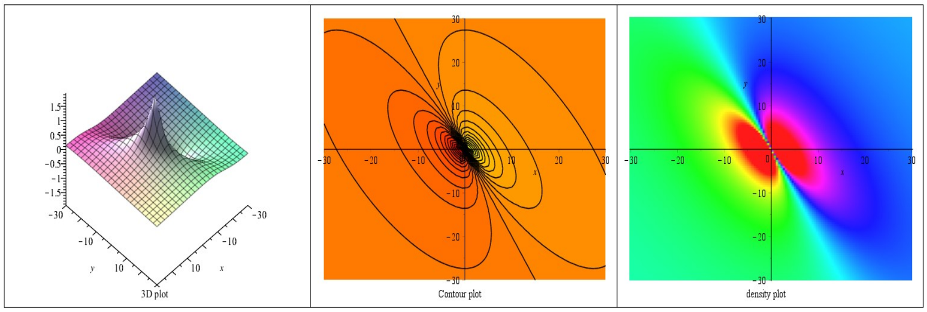

Figure 3. Lump solution to the BKP equation

Due to (3.9), the functions f_1 and f_2 are linearly independent for any fixed time t. When all the parameters a_1, a_2, a_4, a_5, a_6 and a_8 are fixed, for any given t_0, there is only one solution

$$x_0 = -\frac{t_0(a_3a_7 - a_2a_4)}{a_1a_6 - a_2a_5}, \qquad y_0 = \frac{t_0(a_3a_5 - a_1a_7)}{a_1a_6 - a_2a_5} \tag{3.14}$$

of the linear equations

$$\begin{cases} a_3t_0 + a_1x + a_2y = 0, \\ a_7t_0 + a_5x + a_6y = 0, \end{cases} \tag{3.15}$$

where a_3 and a_7 are given by (3.8). Therefore, f_1 and f_2 can be rewritten as

$$\begin{aligned} f_1 &= -\frac{3(a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2)}{a_2^2 + a_6^2}(t - t_0) + a_2(x - x_0) + a_3(y - y_0) + a_4, \\ f_2 &= \frac{3(a_1^2a_6 - 2a_1a_2a_5 - a_5^2a_6)}{a_2^2 + a_6^2}(t - t_0) + a_6(x - x_0) + a_7(y - y_0) + a_8. \end{aligned}$$

The function u is thus a solitary wave which moves with the velocity

$$\Big(-\frac{a_1a_7 - a_3a_5}{a_2a_7 - a_3a_6}, \frac{a_1a_6 - a_2a_5}{a_2a_7 - a_3a_6} \Big).$$

We remark that u is an analytic rational function and has the property that it decays in all directions for any fixed t (i.e. $\lim_{x^2+y^2\to\infty} u = 0$). It follows that u is a lump solution to equation (1.5). When we take parameters $a_1 = 1, a_2 = 2, a_4 = 0, a_5 = 3, a_6 = 1$ and $a_8 = 0$, we get

$$u(t, x, y) = \frac{4(2x + y - 6t)}{(x + y)^2 + (6t - x)^2 + 2}.$$

The surface, contour and density plots of the function u when $t = 0$ are depicted in Figure 3.

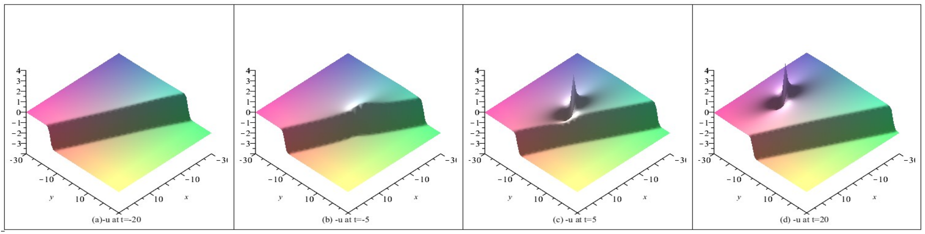

Figure 4. The graphs of the $-u$ when (a) $t=-20$, (b) $t=-5$, (c) $t=5$, (d) $t=20$.

4. Lump-kink solutions

In this section, we study the interaction of a soliton and a lump solution to the BKP equation. We assume

$$f = f_1^2 + f_2^2 + a_9 + g, \qquad a_9 > 0, \tag{4.1}$$

with

$$f_1(x,y,t) = a_1x + a_2y + a_3t + a_4, \quad f_2(x,y,t) = a_5x + a_6y + a_7t + a_8,$$
$$g(x,y,t) = a_{10}\exp(f_3), \quad f_3(x,y,t) = a_{11}x + a_{12}y + a_{13}t,$$

where a_i, $1 \le i \le 13$ are real constants to be determined. Equation (1.11) therefore leads to two classes of lump solutions which we present below. The corresponding lump-soliton solutions are given by

$$u = 2(\ln(f))_x = \frac{2a_1f_1 + 2a_5f_2 + a_{10}a_{11}\exp(f_3)}{f}.$$

Case I: We have solution

$$a_1 = -\frac{a_6a_{11}^2}{2}, a_2 = \frac{2a_5}{a_{11}^2}, a_3 = \frac{3a_5a_{11}^2}{2}, a_7 = \frac{3a_6a_{11}^4}{4}, a_9 = 0, a_{12} = \frac{2}{a_{11}}, a_{13} = -\frac{a_{11}^3}{2}.$$

Assume that a_5, a_6, a_8, a_{10} and $a_{11} \neq 0$ are arbitrary real constants. Then

$$\begin{cases} h(t,x,y) = \Big(\dfrac{3a_5a_{11}^2}{2}t - \dfrac{a_6a_{11}^2}{2}x + \dfrac{2a_5}{a_{11}^2}y + a_4\Big)^2 + \Big(\dfrac{3a_6a_{11}^4}{4}t + a_5x + a_6y + a_8\Big)^2, \\ g(t,x,y) = a_{10}\exp\Big(-\dfrac{a_{11}^3}{2}t + a_{11}x + \dfrac{2}{a_{11}}y\Big), \\ f(t,x,y) = g(t,x,y) + h(t,x,y). \end{cases}$$

We also require $a_1a_6 - a_2a_5 \neq 0$ to guarantee that f_1 and f_2 are linearly independent for any fixed t. Since for every $(t,x,y), f(t,x,y) > 0$, the solution u is bounded locally when $a_{10} > 0$. If $a_{10} \le 0$ then the solution is singular. In the case of $a_{10} = 0$, u is a proper rational function.

If we take $a_5 = 4, a_6 = 2, a_{10} = a_{11} = 1, a_4 = a_8 = 0$ then we get

$$u = \frac{2(16x + \exp(-t/2 + x + 2y))}{(3t - 2x + 4y)^2 + (3t + 2x + 4y)^2 + \exp(-t/2 + x + 2y)}.$$

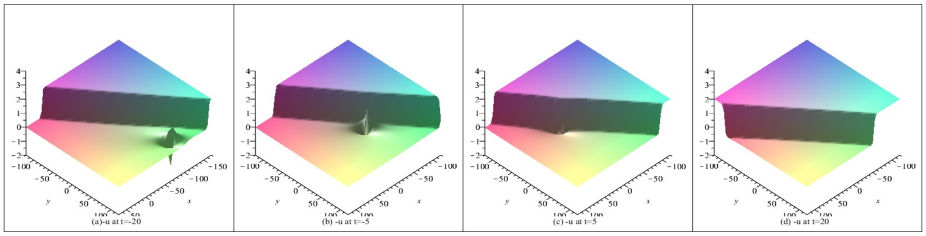

Figure 5. The graphs of the $-u$ when (a) $t = -20$, (b) $t = -5$, (c) $t = 5$, (d) $t = 20$.

The 3D plots of the function $-u$ when $t = -20, -5, 5, 20$ are depicted in Figure 4. The graphs show that initially the kink part of the solution moves steadily and at time $t = -5$, a lump appears and interacts with it. After the interaction they move away from each other.

Case II: We have solution

$$\begin{aligned}
&a_2 = 0, a_3 = -\frac{3a_1(4a_1^2 - a_6^2 a_{11}^4)}{2a_6^2 a_{11}^2}, \quad a_5 = \frac{4a_1^2 - a_6^2 a_{11}^4}{4a_6 a_{11}^2},\\
&a_7 = -\frac{3(16a_1^4 - 24a_1^2 a_6^2 a_{11}^4 + a_6^4 a_{11}^8)}{16a_6^3 a_{11}^4},\\
&a_9 = \frac{(4a_1^2 - a_6^2 a_{11}^4)(4a_1^2 + a_6^2 a_{11}^4)^2}{64a_1^2 a_6^2 a_{11}^6}, \quad a_{12} = -\frac{4a_6^2 a_{11}^3}{4a_1^2 + a_6^2 a_{11}^4},\\
&a_{13} = -\frac{12a_1^2 - a_6^2 a_{11}^4}{4a_6^2 a_{11}},
\end{aligned}$$

where $a_1, a_4, a_6, a_8, a_{10}$ and a_{11} are any real parameters such that $a_1 a_6 a_{11} \neq 0$ (a necessary condition for the solutions to be well-defined) and $4a_1^2 - a_6^2 a_{11}^4 > 0$ (which guarantees that $a_9 > 0$). Under the above conditions, the solution u given by (4.1) is bounded if $a_9 > 0$ and $a_{10} > 0$, and if $a_{10} < 0$ then the solution is singular.

Let us take the parameters $a_1 = a_6 = 2, a_{10} = 1, a_{11} = -1, a_4 = a_8 = 0$. Then we have

$$u = \frac{-\frac{225}{4}t + 25x + 12y - 2\exp(\frac{11}{4}t - x - \frac{4}{5}y)}{(-9t + 2x)^2 + (\frac{21}{8}t + \frac{3}{2}x + 2y)^2 + \frac{75}{16} + \exp(\frac{11}{4}t - x - \frac{4}{5}y)}.$$

The 3D plots of the function $-u$ when $t = -20, -5, 5, 20$ are depicted in Figure 5. The graph of the solution depicts a lump and a kink moving toward each other at first, but after interaction, the lump seems to be swallowed by the kink.

5. Concluding Remarks

In this paper, we have studied the BKP equation of fourth-order and its bilinear form. We have proposed a new method to find quadratic function solutions to bilinear equations. The proposed method worked out all the quadratic function solutions to the bilinear BKP equation. The approach works for any (N+1) nonlinear partial differential equations. We remark, however, that it is not easy to extend this method to third-order and higher-order polynomial solutions.

Furthermore, we have found kink-solitons and lumps for the BKP equation and have also depicted and discussed the dynamics of interaction solutions as well. In the first case of the interaction solutions, we observe the interaction of a lump and a kink-soliton where the lump is swallowed by the kink-soliton. The other interaction appears to be different in the sense that the two solutions move away in different directions after collision. We remark that lump-kink and lump soliton solutions to the 6th-order BKP equation were discussed in [74, 77], but such interaction behavior was not observed.

There exists several interesting types of exact solutions that are worthy of further studies. For example, it will be interesting to know whether the BKP equation enjoys multiple lumps [59] and multiple rogue wave solutions [78] as well as multiple multiple lump-soliton solutions [60]. We will investigate these in future projects.

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