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PROPERTIES AND UNIQUE POSITIVE SOLUTION FOR FRACTIONAL BOUNDARY VALUE PROBLEM WITH TWO PARAMETERS ON THE HALF-LINE*

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Abstract Based on the theory of cone and operators, this paper concerned with the existence of unique positive solution for a class of nonlinear fractional boundary value problem with two parameters (one is called an eigenvalue parameter and another is a disturbance parameter) on the half-line. More important, the solutions dependence on two parameters was discussed, which shows that different parameters have different effects on the properties of positive solutions, and the results reflect an interesting fact different from our inference. Some examples are given to illustrate the main results.

Keywords Boundary value problem, positive solution, half-line, parameter.

MSC(2010) 34B18, 34B15.

1. Introduction

Fractional boundary value problems (FBVP for short) on infinite intervals have important applications in some fields of mathematics and physics such as the problems of radially symmetric solutions of nonlinear elliptic equations, velocity of the unsteady flow of a gas, electrostatic measurement of solid-propellant rockets, and so forth [1]. For this reason, in recent years, many researchers studied fractional boundary value problems on infinite intervals, see [2, 4, 6, 8–10, 14, 15, 17, 19–22, 26] and references therein, and many of these works focused on fractional boundary value problems on infinite intervals with parameters, for example, authors of [12] investigated the following fractional boundary value problem on infinite interval with a disturbance parameter λ in the boundary conditions

$$\begin{cases} D_{0^+}^{\alpha} x(t) + a(t) f(t, x(t)) = 0, t \in (0, \infty), \\ x(0) = D_{0^+}^{\alpha-1} x(\infty) = 0, D_{0^+}^{\alpha-2} x(0) = \sum\limits_{i=1}^{+\infty} g(\xi_i) D_{0^+}^{\alpha-1} x(\xi_i) + \lambda, \end{cases} \tag{1.1}$$

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where $2 < \alpha < 3$, and obtained the existence and multiplicity results of positive solutions for (1.1) by the method of upper and lower solutions, fixed point index theory, and the Schauder fixed point theorem.

In addition, Zhai et. al [25] considered the fractional boundary value problems on half line with a parameter as follows

$$\begin{cases} D_{0+}^{\alpha}x(t) + \mu a(t)f(t, x(t)) = 0, t \in (0, \infty), \\ x(0) = x'(0) = 0, D_{0+}^{\alpha-1}x(\infty) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.2}$$

where $2 < \alpha < 3$, $a : [0, +\infty) \to [0, +\infty)$ and $f : [0, +\infty) \times [0, +\infty) \to [0, \infty)$ are continuous. When the nonlinearity $f(t, x)$ is increasing and $\varphi-$ concave in x, the authors obtained some sufficient conditions of the existence of unique positive solution for (1.2) and properties depend on the parameter μ. In order to distinguish the disturbance parameter, which involved in boundary value condition of a boundary value problem, we call the parameter μ an eigenvalue parameter which is involved in equation of the problem.

By discussions of (1.1) and (1.2), an interesting question was raised: for a fractional boundary value problem with both an eigenvalue parameter and a disturbance parameter, whether the solution still has the similar properties of dependence on these two parameters as those of the problem with an eigenvalue parameter, if the nonlinearity conditions keep the same. To our knowledge, there is no any answer to this question, which inspire us to study the following FBVP on infinite intervals with two parameters

$$\begin{cases} D_{0+}^{\alpha}x(t) + \mu\big(f(t, x(t)) + q(t)g(x(t))\big) = 0, \ t \in (0, \infty), \\ x(0) = 0, \quad x'(0) = 0, \\ D_{0+}^{\alpha-1}x(\infty) = \beta \int_0^{\eta} x(s)ds + \lambda, \end{cases} \tag{1.3}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $2 < \alpha < 3$, $\beta, \eta > 0$ and $\Gamma(\alpha + 1) > \beta\eta^{\alpha}$; $\mu, \lambda \geq 0$ are so called a eigenvalue parameter and disturbance parameter respectively; $R^+ = [0, \infty)$, $q : R^+ \to R^+$ and $\int_0^{+\infty} q(s)ds > 0$; $f : R^+ \times R^+ \to R^+$ is measurable in t for every $x \in R^+$, and continuous in x for a.e.$t \in R^+$, in addition f is φ-concave(or convex) in x; $g : R^+ \to R^+$ is continuous and subhomogeneous (or hyperhomogeneous).

In this paper, we discuss the existence of a unique positive solution for the problem (1.3), more important, we consider the dependence properties on these two parameters of the positive solution by means of the operator theory and analytical technique. On the other hand, it is clear that the problem

$$\begin{cases} D_{0+}^{\alpha}x(t) + \mu f(t, x(t)) = 0, t \in [0, \infty), \\ x(0) = x'(0) = 0, \quad D_{0+}^{\alpha-1}x(\infty) = \beta \int_0^{\eta} x(s)ds. \end{cases} \tag{1.4}$$

is the special case of (1.3) with $\lambda = 0$ and $g(x) \equiv 0$ for $x \in R^+$, however, under the same condition on $f(t, x)$ of (1.3), we can not obtain directly the corollary for (1.4) from the result of (1.3), which is also interesting and encourage us to do this work.

2. Preliminaries and Lemmas

In this section, we shall introduce some definitions and lemmas.

A function $f: R^+ \times R^+ \to R^+$ is said to be φ-concave (or φ-convex) in x, if for any $r \in (0,1)$ there exists $\varphi(r) \in (r,1)$ such that

$$f(t,rx) \geq \varphi(r)f(t,x) \ \big(\text{or } f(t,rx) \leq \frac{1}{\varphi(r)}f(t,x)\ \big),\ t \in R^+, x \in R^+. \tag{2.1}$$

A function $g: R^+ \to R^+$ is called subhomogeneous (or hyperhomogeneous), if

$$g(rx) \geq rg(x) \ \big(\text{or } g(rx) \leq r^{-1}g(x)\ \big),\ r \in (0,1), x \in R^+. \tag{2.2}$$

Remark 2.1. It is clear that (2.1) is equivalent to

$$f(t,sx) \leq \frac{1}{\varphi(s^{-1})}f(t,x) \ (\text{or } f(t,sx) \geq \varphi(s^{-1})f(t,x)\ ), s>1, t \in R^+, x \in R^+;$$

and (2.2) is equivalent to

$$g(sx) \leq sg(x) \ (\text{or } g(sx) \geq s^{-1}g(x)),\ s>1, x \in R^+.$$

In addition, if f is φ-concave (or φ-convex) in x, then f is subhomogeneous(or hyperhomogeneous) in x, but not vice versa.

Remark 2.2. The nonlinearity function $f(t,x)+q(t)g(x)$ in this paper may not have any φ- concave property, even if f is some a φ_0-concave in x and g is subhomogeneous. For example, $f(t,x) = e^{-t}x^{\frac{1}{2}}, g(x) = x$ and $q(t) = e^{-t}$, it is clear that f is φ_0-concave where $\varphi_0(r) = r^{\frac{1}{2}}$, and g is the subhomogeneous, but for any $\varphi:(0,1) \to (0,1)$ with $\varphi(r) \in (r,1)$,

$$f(t,rx)+q(t)g(rx) \ngeq \varphi(r)\big(f(t,x)+q(t)g(x)\big),\ r \in (0,1), t \in R^+, x \in R^+;$$

in addition, the function $f(t,x)+q(t)g(x)$ may not be monotonic, even if f and g are monotonic. That is to say, the properties of nonlinearity function for (1.3) are different from the one of [25].

Basic notations and related results on Riemann-Liouville fractional integral and fractional derivative can be found in [11, 16, 18].

For $y \in L[0,+\infty)$, consider the linear fractional boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}x(t)+y(t)=0, \quad 0<t<+\infty, \\ x(0)=0,\ x'(0)=0, \\ D_{0^+}^{\alpha-1}x(\infty)=\beta\int_0^{\eta}x(s)ds+\lambda, \end{cases} \tag{2.3}$$

where $2<\alpha<3$, $\beta,\eta>0$ and $\lambda \geq 0$.

Similarly, by Lemma 3.1 in [15], we can easily obtain the following results.

Lemma 2.1. *Suppose that $y \in L[0,+\infty)$, then FBVP(2.3) has a unique solution*

$$x(t) = \int_0^{+\infty} G(t,s)y(s)ds + \lambda\kappa t^{\alpha-1},\ t \in R^+,$$

where

$$\kappa = \frac{\alpha}{\Gamma(\alpha+1)-\beta\eta^{\alpha}},$$

and

$$G(t,s) = G_1(t,s) + \kappa\beta t^{\alpha-1} G_2(\eta,s),\ t,s\in R^+,$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, & 0\le s\le t<+\infty,\\ t^{\alpha-1}, & 0\le t\le s<+\infty,\end{cases}$$

$$G_2(t,s) = \frac{1}{\Gamma(\alpha+1)}\begin{cases} t^{\alpha} - (t-s)^{\alpha}, & 0\le s\le t<+\infty,\\ t^{\alpha}, & 0\le t\le s<+\infty.\end{cases}$$

Lemma 2.2. *The functions $G(t,s), G_1(t,s)$ and $G_2(t,s)$ defined by Lemma 2.1 satisfy the following properties:*

(i) $G(t,s)$ is continuous on $R^+\times R^+$;

(ii) $0\le G_1(t,s)\le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $0\le G_2(t,s)\le \frac{1}{\Gamma(\alpha+1)}t^{\alpha}$ for $t,s\in R^+$, moreover, $0\le G(t,s)\le \kappa t^{\alpha-1}$ for $t,s\in R^+$.

Following definitions and known results can be found from [5,7,23,24].

Let E be a real Banach space which is partially ordered by a cone $P\subset E$, that is, $x\le y$ iff $y-x\in P$. If $x\le y$ and $x\ne y$, then we mean $x<y$ or $y>x$. By θ we denote the zero element of E. A cone P is said to be normal if there exists a positive number N such that $\theta\le x\le y$ implies $\|x\|\le N\|y\|$. For $u,v\in E, u\le v$, denote $[u,v]=\{x\in E | u\le x\le v\}$. Given $e>0$ (i.e., $e\in P$ and $e\ne\theta$), set

$$P_e = \{x\in E \mid \text{there exist } l_1(x), l_2(x) > 0 \text{ such that } l_1(x)e\le x\le l_2(x)e\}. \tag{2.4}$$

Let $D\subset E$. An operator $T: D\to E$ is said to be increasing (decreasing) if $x,y\in D, x\le y \Rightarrow Tx\le Ty (Tx\ge Ty)$. An element $x^*\in D$ is called a fixed point of T if $Tx^*=x^*$; An operator $T: D\times D\to E$ is said to be mixed monotone, if $x_i,y_i\in D (i=1,2), x_1\le y_1, x_2\ge y_2 \Rightarrow T(x_1,x_2)\le T(y_1,y_2)$. An element $x^*\in D$ is called a fixed point of T if $T(x^*,x^*)=x^*$.

Lemma 2.3 (Theorem 2.1, [23])**.** *Let P be a normal cone in E, and $T: P_e\to P_e$ satisfy one of the following conditions:*

(L1) T is an increasing operator, and for any $r\in(0,1)$, there exists $\alpha(r)\in(0,1)$ such that $T(rx)\ge r^{\alpha(r)}Tx$ for $x\in P_e$ and $r\in(0,1)$;

(L2) T is a decreasing operator, and for any $r\in(0,1)$, there exists $\alpha(r)\in(0,1)$ such that $T(rx)\le r^{-\alpha(r)}Tx$ for $x\in P_e$ and $r\in(0,1)$.

Then T has a unique fixed point x^* in P_e. Moreover, for any initial value $u_0\in P_e$ and a sequence $u_n=Tu_{n-1} (n=1,2,\cdots)$, one has $\lim\limits_{n\to+\infty}\|u_n-x^*\|=0$.

Lemma 2.4 (Theorem 2.1, [24])**.** *Let P be a normal cone in E and $e\in P$ with $e\ne\theta$, $C: P\times P\to P$ be a mixed monotone operator. Suppose that*

(L3) $C(e,e)\in P_e$;

(L4) for every $r\in(0,1)$, there exists $\varphi(r)\in(r,1]$ such that

$$C(rx,\frac{1}{r}y)\ge\varphi(r)C(x,y),\ x,y\in P.$$

Then C has a unique fixed point x^* in P_e. Moreover, for any initial value $x_0,y_0\in P_e$, constructing successively the sequence

$$x_n = C(x_{n-1},y_{n-1}),\quad y_n = C(y_{n-1},x_{n-1}),\ n=1,2,\cdots,$$

one has $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to +\infty$.

Lemma 2.5 (Theorem 2.3, [24])**.** *Suppose that operator* C *satisfies the conditions of Lemma 2.4. Let* $x_\lambda(\lambda > 0)$ *denote the unique solution of operator equation* $T(x,x) = \lambda x$ *in* P_e*. If* $\varphi(r) > r^{\frac{1}{2}}$*, then* x_λ *is strictly decreasing in* λ.

In this paper, we always set

$$X =: C_{\alpha-1}[0,+\infty) = \{x \in C[0,+\infty) | \sup_{t\in[0,+\infty)} \frac{|x(t)|}{1+t^{\alpha-1}} < +\infty\},$$

then X is a Banach space with the norm $\|x\| = \sup\limits_{t\in[0,+\infty)} \frac{|x(t)|}{1+t^{\alpha-1}}$. Denote

$$P = \{x \in X \mid x(t) \geq 0,\ t \in [0,+\infty)\},$$

then P is a normal cone in X, and the normal constant is 1, see [25]. Let

$$e(t) = t^{\alpha-1},\ t \in [0,+\infty),$$

and define P_e as (2.4).

In this paper, the following hypotheses will be used.

(H1) $f(t,x)$ is increasing and φ- concave in $x \in [0,+\infty)$, and $f(t, 1+t^{\alpha-1}) \in L^1[0,+\infty)$.

(H2) $f(t,x)$ is decreasing and φ- convex in $x \in [0,+\infty)$, and $f(t,0) \in L^1[0,+\infty)$.

(H3) $g(x)$ is increasing and subhomogeneous on $[0,+\infty)$, and $g(\infty) = \lim\limits_{x\to+\infty} g(x) < +\infty$.

(H4) $g(x)$ is decreasing and hyperhomogeneous on $[0,+\infty)$.

If f satisfies one of (H1) and (H2), and g satisfies (H3) or (H4), then the solution of FBVP(1.3) is equivalent to the solution of the following integral equation

$$x(t) = \mu \int_0^{+\infty} G(t,s)\Big(f(s,x(s)) + q(s)g(x(s))\Big)ds + \lambda\kappa t^{\alpha-1},\ x \in X. \tag{2.5}$$

Define the operator $T_{(\mu,\lambda)}$ by

$$T_{(\mu,\lambda)}x = \mu Ax + B_{(\mu,\lambda)}x,\ x \in P, \tag{2.6}$$

where

$$(Ax)(t) = \int_0^{+\infty} G(t,s)f(s,x(s))ds,\ x \in P,$$

and

$$(B_{(\mu,\lambda)}x)(t) = \mu \int_0^{+\infty} G(t,s)q(s)g(x(s))ds + \lambda\kappa t^{\alpha-1},\ x \in P.$$

Then, $T_{(\mu,\lambda)}(P) \subset P$, which together with (2.5) implies that x is a positive solution of FBVP(1.3) if and only if x is a non-zero solution of $T_{(\mu,\lambda)}x = x$ in P.

For convenience, we introduce the following binary operator

$$C_{(\mu,\lambda)}(x,y) = \mu Ax + B_{(\mu,\lambda)}y,\ x,y \in P, \tag{2.7}$$

and

$$C^*_{(\mu,\lambda)}(x,y) = C_{(\mu,\lambda)}(y,x) = \mu Ay + B_{(\mu,\lambda)}x,\ x,y \in P. \tag{2.8}$$

Then

$$T_{(\mu,\lambda)}x = C_{(\mu,\lambda)}(x,x) = C^*_{(\mu,\lambda)}(x,x),\ x \in P. \tag{2.9}$$

Lemma 2.6 (Lemma 2.2, [13]). *Let W be a bounded subset of X. If the following conditions holds:*

(i) $\{\frac{x(t)}{1+t^{\alpha-1}} \mid x \in W\}$ *is equicontinuous on any compact interval of* $[0, +\infty)$*;*

(ii) $\{\frac{x(t)}{1+t^{\alpha-1}} \mid x \in W\}$ *is equiconvergent at infinity.*

Then W is relatively compact in X.

3. Existence and Uniqueness

In this section, we shall discuss the existence of unique positive solution for BVP(1.3).

Theorem 3.1. *Assume that one of the following four assumptions is satisfied.*

(i) (H1) and (H3) hold; (ii) (H1) and (H4)hold;

(iii) (H2) and (H4) hold; (iv) (H2) and (H3) hold.

Then FBVP(1.3) has a unique positive solution $x_{(\mu,\lambda)}$ for any $\mu \geq 0$ and $\lambda > 0$. Moreover, for any $x_0 \in P$, set

$$x_n(t) = \mu \int_0^{+\infty} G(t,s)\Big(f(s, x_{n-1}(s)) + q(s)g(x_{n-1}(s))\Big)ds + \lambda\kappa t^{\alpha-1}, \tag{3.1}$$

$n = 1, 2, \cdots$ *, we have*

$$\lim_{n\to\infty} \sup_{t\in[0,+\infty)} \frac{|x_n(t) - x_{(\mu,\lambda)}(t)|}{1 + t^{\alpha-1}} = 0. \tag{3.2}$$

Proof. It is clear that $\lambda\kappa t^{\alpha-1}$ is the unique positive solution of FBVP(1.3) for $\mu = 0$ and $\lambda > 0$.

In the sequel, consider FBVP(1.3) for $\mu > 0$ and $\lambda > 0$ under different conditions (i),(ii),(iii) and (iv), respectively.

(i) If (H1) and (H3) hold, $T_{(\mu,\lambda)} : P \to P$ is an increasing operator, and

$$\begin{aligned}\lambda\kappa t^{\alpha-1} \leq (T_{(\mu,\lambda)}x)(t) &= \mu \int_0^{+\infty} G(t,s)\Big(f(s, x(s)) + q(s)g(x(s))\Big)ds + \lambda\kappa t^{\alpha-1} \\ &\leq \kappa\Big(\mu \int_0^{+\infty} \Big((1 + \|x\|)f(s, 1 + s^{\alpha-1}) + g(\infty)q(s)\Big)ds + \lambda\Big)t^{\alpha-1}, x \in P,\end{aligned}$$

which implies that

$$T_{(\mu,\lambda)}(P) \subset P_e. \tag{3.3}$$

In order to discuss the properties of operator $T_{(\mu,\lambda)}$, we first prove that the operator $B_{(\mu,\lambda)}$ satisfies

$$B_{(\mu,\lambda)}(rx) \geq \phi(\mu, \lambda, r)B_{(\mu,\lambda)}x,\ r \in (0, 1), x \in P_e, \tag{3.4}$$

where

$$\phi(\mu, \lambda, r) = \frac{\lambda\varphi(r) + r\mu g(\infty)\int_0^{+\infty} q(s)ds}{\lambda + \mu g(\infty)\int_0^{+\infty} q(s)ds},\ r \in (0, 1). \tag{3.5}$$

Indeed, by (3.5) we have $r < \phi(\mu, \lambda, r) \leq \varphi(r) < 1$, and

$$\Big(\phi(\mu, \lambda, r) - r\Big)\mu g(\infty) \int_0^{+\infty} q(s)ds = \Big(\varphi(r) - \phi(\mu, \lambda, r)\Big)\lambda < \Big(1 - \phi(\mu, \lambda, r)\Big)\lambda.$$

Note that

$$\int_0^{+\infty} G(t,s)q(s)g(x(s))ds \le \kappa t^{\alpha-1} g(\infty)\int_0^{+\infty} q(s)ds,\ x\in P_e,$$

then

$$\Big(\phi(\mu,\lambda,r)-r\Big)\mu\int_0^{+\infty} G(t,s)q(s)g(x(s))ds \le \Big(1-\phi(\mu,\lambda,r)\Big)\lambda\kappa t^{\alpha-1},\ x\in P_e.$$

Therefore,

$$\begin{aligned}B_{(\mu,\lambda)}(rx)(t) \ge& r\mu\int_0^{+\infty} G(t,s)q(s)g(x(s))ds+\lambda\kappa t^{\alpha-1}\\ \ge&\phi(\mu,\lambda,r)\Big(\mu\int_0^{+\infty} G(t,s)q(s)g(x(s))ds+\lambda\kappa t^{\alpha-1}\Big)\\ =&\phi(\mu,\lambda,r)(B_{(\mu,\lambda)}x)(t),\ t\in R^+, x\in P_e,\end{aligned}$$

which means that (3.4) holds.

Set $a(\mu,\lambda,r)=\frac{\ln\phi(\mu,\lambda,r)}{\ln r}$, then $a(\mu,\lambda,r)\in(0,1)$. By (3.4) and (H1),

$$\begin{aligned}T_{(\mu,\lambda)}(rx) &= \mu A(rx)+B_{(\mu,\lambda)}(rx) \ge \varphi(r)\mu Ax+\phi(\mu,\lambda,r)B_{(\mu,\lambda)}x\\ &\ge \phi(\mu,\lambda,r)\Big(\mu Ax+B_{(\mu,\lambda)}x\Big)\\ &= r^{a(\mu,\lambda,r)}T_{(\mu,\lambda)}x,\ r\in(0,1), x\in P_e.\end{aligned}$$

The application of Lemma 2.3 can show that $T_{(\mu,\lambda)}$ has a unique fixed point $x_{(\mu,\lambda)}$ in P_e, furthermore for any $x_0\in P_e$, setting $x_n=T_{(\mu,\lambda)}x_{n-1}, n=1,2,\cdots$, then $\lim\limits_{n\to\infty}\|x_n-x_{(\mu,\lambda)}\|=0$. By (3.3), the fixed point $x_{(\mu,\lambda)}$ is a unique positive solution of FBVP(1.3), and (3.1) and (3.2) are satisfied.

(ii) Consider the operator $C_{(\mu,\lambda)}$ defined by (2.7). If (H1) and (H4) hold, then $C_{(\mu,\lambda)}: P\times P\to P$ is a mixed monotone operator. In addition, for any $x,y\in P$ we have

$$\begin{aligned}\lambda\kappa t^{\alpha-1}\le (C_{(\mu,\lambda)}(x,y))(t) &= \mu\int_0^{+\infty} G(t,s)\big(f(s,x(s))+q(s)g(y(s))\big)ds+\lambda\kappa t^{\alpha-1}\\ &\le \kappa\Big(\mu\int_0^{+\infty}\Big((1+\|x\|)f(s,1+s^{\alpha-1})+g(0)q(s)\Big)ds+\lambda\Big)t^{\alpha-1}, t\in R^+,\end{aligned}$$

which means that

$$C_{(\mu,\lambda)}(P\times P)\subset P_e. \tag{3.6}$$

For the operator $B_{(\mu,\lambda)}$, by using (H4) and arguments similar to (3.4), we can prove that

$$B_{(\mu,\lambda)}(rx)\le \frac{1}{\psi(\mu,\lambda,r)}B_{(\mu,\lambda)}x,\ r\in(0,1), x\in P_e, \tag{3.7}$$

where

$$\psi(\mu,\lambda,r)=\frac{\mu g(0)\int_0^{+\infty} q(s)ds+\lambda}{\frac{\mu}{r}g(0)\int_0^{+\infty} q(s)ds+\frac{\lambda}{\varphi(r)}},\ r\in(0,1). \tag{3.8}$$

It is clear that

$$r < \psi(\mu,\lambda,r) \leq \varphi(r),\ r \in (0,1). \tag{3.9}$$

Thus, it follows from (H1),(3.7) and (3.9) that

$$\begin{aligned} C_{(\mu,\lambda)}(rx,\frac{1}{r}y) &= \mu A(rx) + B_{(\mu,\lambda)}(\frac{1}{r}y) \geq \varphi(r)\mu Ax + \psi(\mu,\lambda,r)B_{(\mu,\lambda)}y \\ &\geq \psi(\mu,\lambda,r)C_{(\mu,\lambda)}(x,y),\ r \in (0,1),\ x,y \in P_e. \end{aligned}$$

Therefore, Lemma 2.4 together with (3.6) tells us $C_{(\mu,\lambda)}$ has a unique fixed point $x_{(\mu,\lambda)}$ in $P\backslash\{\theta\}$ which is a unique positive solution of (1.3). Moreover, for any $x_0, y_0 \in P$, set

$$x_n = C_{(\mu,\lambda)}(x_{n-1}, y_{n-1}), \quad y_n = C_{(\mu,\lambda)}(y_{n-1}, x_{n-1}), \quad n = 1,2,\cdots,$$

then

$$\lim_{n\to\infty} \|x_n - x_{(\mu,\lambda)}\| = 0, \quad \lim_{n\to\infty} \|y_n - x_{(\mu,\lambda)}\| = 0.$$

Taking $x_0 = y_0$, then $x_n = y_n, n = 1,2,\cdots$, so (3.1) and (3.2) are satisfied.

(iii) When (H2) and (H4) hold, we still consider the operator $T_{(\mu,\lambda)}$ defined by (2.6). Obviously, $T_{(\mu,\lambda)} : P \to P$ is decreasing, and $T_{(\mu,\lambda)}(P) \subset P_e$. In addition, by the proof of the above (ii) we know that $B_{(\mu,\lambda)}$ satisfies (3.7). Taking $\bar{a}(\mu,\lambda,r) = \frac{\ln\psi(\mu,\lambda,r)}{\ln r}$, then $\bar{a}(\mu,\lambda,r) \in (0,1)$, and

$$\begin{aligned} T_{(\mu,\lambda)}(rx) &= \mu A(rx) + B_{(\mu,\lambda)}(rx) \leq \frac{\mu}{\varphi(r)}Ax + \frac{1}{\psi(\mu,\lambda,r)}B_{(\mu,\lambda)}x \\ &\leq \frac{1}{\psi(\mu,\lambda,r)}(T_{(\mu,\lambda)}x) = r^{-\bar{a}(\mu,\lambda,r)}T_{(\mu,\lambda)}x,\ r \in (0,1), x \in P_e. \end{aligned}$$

Similarly to the proof of (i), the application of Lemma 2.3 finishes the proof of this part.

(iv) When (H2) and (H3) hold, we consider the operator $C^*_{(\mu,\lambda)}(x,y) = \mu Ay + B_{(\mu,\lambda)}x$ defined as (2.8). It is evident that $C^*_{(\mu,\lambda)} : P \times P \to P$ is a mixed monotone operator, and $C^*_{(\mu,\lambda)}(P \times P) \subset P_e$.

By the above proof (i), $B_{(\mu,\lambda)}$ satisfies (3.4) where φ is given by (H2), and

$$\begin{aligned} C^*_{(\mu,\lambda)}(rx,\frac{1}{r}y) &= \mu A(\frac{1}{r}y) + B_{(\mu,\lambda)}(rx) \geq \varphi(r)\mu Ay + \phi(\mu,\lambda,r)B_{(\mu,\lambda)}x \\ &\geq \phi(\mu,\lambda,r)C^*_{(\mu,\lambda)}(x,y),\ r \in (0,1), x,y \in P_e. \end{aligned}$$

Using similar arguments as the proof of (ii) and Lemma 2.4, the proof is finished. □

Remark 3.1. If $\lambda = 0$, then Theorem 3.1 may not be true. Indeed, when $\mu = \lambda = 0$, it is easy to check that FBVP(1.3) has a unique zero solution, but no positive solution; when $\lambda = 0$ and $\mu > 0$, we can not guarantee the existence of unique positive solution of FBVP(1.3) under the conditions of Theorem 3.1, but we can give some sufficient conditions of existence of unique positive solution for FBVP(1.3) with $\lambda = 0$ and $g \equiv 0$ (see Theorem 5.1). We use the following Table 1 to show this concisely.

Table 1. The effect of parameters on the existence of unique positive solution

	Parameters		Condition of g	Condition of f	Unique solution
Existence	$\lambda > 0$	$\mu \geq 0$	(H3) or (H4)	(H1) or (H2)	Yes
	$\lambda = 0$	$\mu > 0$	$g \not\equiv 0$; (H3) or (H4)	(H1) or (H2)	Uncertain
		$\mu > 0$	$g \equiv 0$	(H1) or (H2)	Yes
	$\lambda = 0$	$\mu = 0$	Arbitrary	Arbitrary	No

4. Dependence of solution on parameters

For any given $\mu \geq 0$ and $\lambda > 0$, by Theorem 3.1, FBVP(1.3) has a unique positive solution under some conditions. Based on this fact, we regard the unique positive solution as a function of parameter pair of (μ, λ) denoted by $x_{(\mu,\lambda)}$, and discuss the monotonic and continuous properties of $x_{(\mu,\lambda)}$ with respect to (μ, λ).

Set

$$w_0(t) = \kappa e(t) = \kappa t^{\alpha-1}, \quad t \in R^+.$$

Theorem 4.1. *Under the condition (i) of Theorem 3.1, the following results hold.*

(a) for any fixed $\mu \geq 0$, $x_{(\mu,\lambda)}$ is increasing and continuous with respect to λ for $\lambda > 0$, and $\lim\limits_{\lambda\to\infty} \|x_{(\mu,\lambda)}\| = +\infty$;

(b) for any fixed $\lambda > 0$, $x_{(\mu,\lambda)}$ is increasing and continuous with respect to μ for $\mu \geq 0$, and $\lim\limits_{\mu\to 0+} \|x_{(\mu,\lambda)} - \lambda w_0\| = 0$, in addition, $\lim\limits_{\mu\to+\infty} \|x_{(\mu,\lambda)}\| = +\infty$ if $\int_1^{+\infty} G_2(\eta, s) f(s, 1) ds > 0$;

(c) for $\mu = \nu\lambda$, $\nu \geq 0$, $\lambda > 0$, $x_{(\mu,\lambda)}$ is increasing and continuous with respect to λ for $\lambda > 0$, $\lim\limits_{\lambda\to+\infty} \|x_{(\mu,\lambda)}\| = +\infty$ and $\lim\limits_{\lambda\to 0+} \|x_{(\mu,\lambda)}\| = 0$.

Proof. (a) For any fixed $\mu \geq 0$, it is easy to see that $T_{(\mu,\lambda)}x$ is increasing with respect to λ and x for $\lambda > 0$ and $x \in P$. Next to show $x_{(\mu,\lambda)}$ is increasing with respect to λ for $\lambda > 0$. Let $\lambda_1, \lambda_2 \in (0, +\infty)$ with $\lambda_1 \leq \lambda_2$ and

$$u_0 = \lambda_1 w_0, \; u_n = T_{(\mu,\lambda_1)} u_{n-1}, \; n = 1, 2, \cdots, \tag{4.1}$$

then $u_0 \leq \lambda_2 w_0 \leq x_{(\mu,\lambda_2)}$. Moreover, we have

$$T_{(\mu,\lambda_1)} u_0 \geq u_0, \quad T_{(\mu,\lambda_1)} x_{(\mu,\lambda_2)} \leq T_{(\mu,\lambda_2)} x_{(\mu,\lambda_2)} = x_{(\mu,\lambda_2)},$$

which, together with (4.1), leads to

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq x_{(\mu,\lambda_2)}, \; n = 1, 2, \cdots. \tag{4.2}$$

Note that $x_{(\mu,\lambda_1)}$ is the unique positive fixed point of $T_{(\mu,\lambda_1)}$, then it follows from Theorem 3.1 and (4.2) that $\|u_n - x_{(\mu,\lambda_1)}\| \to 0$ as $n \to +\infty$ and $x_{(\mu,\lambda_1)} \in [u_0, x_{(\mu,\lambda_2)}]$, which means that $x_{(\mu,\lambda_1)} \leq x_{(\mu,\lambda_2)}$, that is, $x_{(\mu,\lambda)}$ is increasing with respect to λ for $\lambda > 0$.

Next to prove that $x_{(\mu,\lambda)}$ is continuous with respect to λ for $\lambda > 0$, we take $\lambda_0 \in (0, +\infty)$ and any sequence $\{\lambda_n\}$ satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \leq \lambda_0 \;\text{ and }\; \lim_{n\to+\infty} \lambda_n = \lambda_0.$$

Set

$$D = \{x_{(\mu,\lambda_n)} | x_{(\mu,\lambda_n)} = T_{(\mu,\lambda_n)} x_{(\mu,\lambda_n)}, n = 1, 2, \cdots\},$$

then $D \subset P_e$ and

$$x_{(\mu,\lambda_1)} \le x_{(\mu,\lambda_2)} \le \cdots \le x_{(\mu,\lambda_n)} \le \cdots \le x_{(\mu,\lambda_0)}. \tag{4.3}$$

The normality of the cone P implies that D is a bounded subset. In addition, by (H1),(H3) and Lemma 2.2, $\{\frac{x(t)}{1+t^{\alpha-1}} \mid x \in D\}$ is equicontinuous on any compact interval of $[0, +\infty)$, and equiconvergent at infinity. Hence, it follows from Lemma 2.6 that D is a relatively compact. This, together with (4.3), implies that there exists $x^* \in [x_{(\mu,\lambda_1)}, x_{(\mu,\lambda_0)}] \subset P_e$ such that $\lim\limits_{n\to+\infty} \|x_{(\mu,\lambda_n)} - x^*\| = 0$. Note that

$$x_{(\mu,\lambda_n)}(t) = \mu \int_0^{+\infty} G(t,s)\Big(f(s, x_{(\mu,\lambda_n)}(s)) + q(s)g(x_{(\mu,\lambda_n)}(s))\Big)ds + \kappa\lambda_n t^{\alpha-1},$$

by the Lebesgue dominated convergence theorem, we have

$$x^*(t) = \mu \int_0^{+\infty} G(t,s)\Big(f(s, x^*(s)) + q(s)g(x^*(s))\Big)ds + \kappa\lambda_0 t^{\alpha-1},$$

which means that x^* is a fixed point of $T_{(\mu,\lambda_0)}$ in P_e. The uniqueness of the fixed point of $T_{(\mu,\lambda_0)}$ implies $x^* = x_{(\mu,\lambda_0)}$, so $\lim\limits_{\lambda\to\lambda_0^-} \|x_{(\mu,\lambda)} - x_{(\mu,\lambda_0)}\| = 0$. Similar argument can show $\lim\limits_{\lambda\to\lambda_0^+} \|x_{(\mu,\lambda)} - x_{(\mu,\lambda_0)}\| = 0$. Therefore, $x_{(\mu,\lambda)}$ is continuous with respect to λ for $\lambda > 0$.

In addition, noticing that $x_{(\mu,\lambda)}(t) = \big(T_{(\mu,\lambda)} x_{(\mu,\lambda)}\big)(t) \ge \lambda\kappa t^{\alpha-1}$ for $t \in R^+$, we have $\|x_{(\mu,\lambda)}\| \ge \lambda\kappa$ which implies that $\lim\limits_{\lambda\to+\infty} \|x_{(\mu,\lambda)}\| = +\infty$.

(b) For any fixed $\lambda > 0$, since $T_{(\mu,\lambda)}x$ is increasing in μ and x for $\mu \ge 0$ and $x \in P$. Similarly to the proof of above (a) we can show that $x_{(\mu,\lambda)}$ is increasing and continuous with respect to μ for $\mu > 0$.

Note that $x_{(\mu,\lambda)} \le x_{(1,\lambda)}$ for any $\mu \in (0,1)$, then for any $\mu \in (0,1)$,

$$\begin{aligned} 0 \le & x_{(\mu,\lambda)}(t) - \lambda w_0(t) \le \mu\kappa t^{\alpha-1} \int_0^{+\infty} \Big(f(s, x_{(1,\lambda)}(s)) + q(s)g(\infty)\Big)ds \\ \le & \mu\kappa t^{\alpha-1} \int_0^{+\infty} \Big((1 + \|x_{(1,\lambda)}\|) f(s, 1 + s^{\alpha-1}) + q(s)g(\infty)\Big)ds, \ t \in R^+, \end{aligned}$$

which implies that $\lim\limits_{\mu\to0+} \|x_{(\mu,\lambda)} - \lambda w_0\| = 0$.

Now, we take $\mu_0 \ge 1$. Because of $x_{(\mu_0,\lambda)} \in P_e$, there exists a number $l \in (0,1)$ such that

$$x_{(\mu,\lambda)}(t) \ge x_{(\mu_0,\lambda)}(t) \ge le(t) = lt^{\alpha-1}, \ t \in R^+, \mu \ge \mu_0.$$

Moreover, it follows from (H1) and (H3) that

$$\begin{aligned} x_{(\mu,\lambda)}(t) =& T_{(\mu,\lambda)} x_{(\mu,\lambda)}(t) \ge \mu\beta\kappa t^{\alpha-1} \int_0^{+\infty} G_2(\eta, s) f(s, ls^{\alpha-1})ds \\ \ge & \mu\beta\kappa l t^{\alpha-1} \int_1^{+\infty} G_2(\eta, s) f(s, 1)ds, \ t \in R^+, \mu \ge \mu_0, \end{aligned}$$

which together with $\int_1^{+\infty} G_2(\eta,s)f(s,1)ds > 0$ leads to $\lim\limits_{\mu\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$.

(c) For $\mu = \nu\lambda$, $\nu \geq 0, \lambda > 0$, the unique positive solution $x_{(\mu,\lambda)}$ is denoted by x_λ. Noticing that

$$x_\lambda(t) = \lambda\Big(\nu\int_0^{+\infty} G(t,s)\big(f(s,x_\lambda(s)) + q(s)g(x_\lambda(s))\big)ds + w_0(t)\Big),$$

the conclusion (c) can be present similarly. The proof is complete. □

Similarly to the above proofs we can obtain the following result.

Theorem 4.2. *Under the condition (ii) in Theorem 3.1, we have the following results:*

(a) for any fixed $\mu \geq 0$, $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$;

(b) for any fixed $\lambda > 0$, $\lim\limits_{\mu\to 0+}\|x_{(\mu,\lambda)} - \lambda w_0\| = 0$, in addition, $\lim\limits_{\mu\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ if $\int_1^{+\infty} G_2(\eta,s)f(s,1)ds > 0$;

(c) for $\mu = \nu\lambda$, $\nu \geq 0, \lambda > 0$, $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ and $\lim\limits_{\lambda\to 0+}\|x_{(\mu,\lambda)}\| = 0$.

Theorem 4.3. *Under the condition (iii) of Theorem 3.1, we have the following results:*

(a) for any fixed $\mu \geq 0$, $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$;

(b) for any fixed $\lambda > 0$, $x_{(\mu,\lambda)}$ is increasing and continuous with respect to μ for $\mu \geq 0$, and $\lim\limits_{\mu\to 0+}\|x_{(\mu,\lambda)} - \lambda w_0\| = 0$, in addition, $\lim\limits_{\mu\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ if $\int_0^1 G_2(\eta,s)f(s,1)ds > 0$;

(c) for $\mu = \nu\lambda$, $\nu \geq 0, \lambda > 0$, $x_{(\mu,\lambda)}$ is increasing and continuous in λ for $\lambda > 0$, $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ and $\lim\limits_{\lambda\to 0+}\|x_{(\mu,\lambda)}\| = 0$.

Proof. By (H2) and (H4), $T_{(\mu,\lambda)}x$ is increasing with respect to μ and λ for $\mu \geq 0$ and $\lambda > 0$, and decreasing with respect to x for $x \in P$. The conclusion (a) is easy to be proved. Next we prove (b). Any given $\lambda > 0$, in order to show that $x_{(\mu,\lambda)}$ is increasing with respect to μ for $\mu \geq 0$, we let $\mu_1, \mu_2 \in [0,+\infty)$, $\mu_1 \leq \mu_2$. If $\mu_1 = 0$, then $x_{(\mu_1,\lambda)} = \lambda w_0 \leq x_{(\mu_2,\lambda)}$. Thus, we only need to prove $x_{(\mu_1,\lambda)} \leq x_{(\mu_2,\lambda)}$ for $\mu_1 > 0$. Noting that $x_{(\mu_1,\lambda)}, x_{(\mu_2,\lambda)} \in P_e$, there exists $\epsilon > 0$ such that $\epsilon x_{(\mu_1,\lambda)} \leq x_{(\mu_2,\lambda)}$. Set

$$r_0 = \sup\{r > 0 \mid r x_{(\mu_1,\lambda)} \leq x_{(\mu_2,\lambda)}\},$$

then $0 < r_0 < +\infty$ and $r_0 x_{(\mu_1,\lambda)} \leq x_{(\mu_2,\lambda)}$. We assert that $r_0 \geq 1$. Suppose, to the contrary, that $0 < r_0 < 1$. By (3.7) we have

$$x_{(\mu_2,\lambda)} = T_{(\mu_2,\lambda)}x_{(\mu_2,\lambda)} \leq T_{(\mu_2,\lambda)}(r_0 x_{(\mu_1,\lambda)}) \leq \frac{1}{\psi(\mu_2,\lambda,r_0)}T_{(\mu_2,\lambda)}x_{(\mu_1,\lambda)},$$

which, together with

$$\begin{aligned}(T_{(\mu_2,\lambda)}x_{(\mu_1,\lambda)})(t) &\leq \frac{\mu_2}{\mu_1}\Big[\mu_1\int_0^{+\infty} G(t,s)\Big(f(s,x_{(\mu_1,\lambda)}(s)) + q(s)g(x_{(\mu_1,\lambda)}(s))\Big)ds \\ &\quad + \lambda\kappa t^{\alpha-1}\Big] \\ &= \frac{\mu_2}{\mu_1}(T_{(\mu_1,\lambda)}x_{(\mu_1,\lambda)})(t) = \frac{\mu_2}{\mu_1}x_{(\mu_1,\lambda)}(t),\ t \in R^+,\end{aligned}$$

implies that

$$\frac{\mu_1\psi(\mu_2,\lambda,r_0)}{\mu_2}x_{(\mu_2,\lambda)} \le x_{(\mu_1,\lambda)}.$$

Since $0 < \frac{\mu_1\psi(\mu_2,\lambda,r_0)}{\mu_2} < 1$, it follows from (H2) and (H4) that

$$\begin{aligned}
x_{(\mu_1,\lambda)}(t) &= T_{(\mu_1,\lambda)}x_{(\mu_1,\lambda)}(t) \le T_{(\mu_1,\lambda)}\big(\frac{\mu_1\psi(\mu_2,\lambda,r_0)}{\mu_2}x_{(\mu_2,\lambda)}\big)(t)\\
&= \mu_1\int_0^{+\infty} G(t,s)\Big(f\big(s,\frac{\mu_1\psi(\mu_2,\lambda,r_0)}{\mu_2}x_{(\mu_2,\lambda)}(s)\big)\\
&\quad + q(s)g\big(\frac{\mu_1\psi(\mu_2,\lambda,r_0)}{\mu_2}x_{(\mu_2,\lambda)}\big)\Big)ds + \kappa\lambda t^{\alpha-1}\\
&\le \frac{\mu_2}{\psi(\mu_2,\lambda,r_0)}\int_0^{+\infty} G(t,s)\Big(f(s,x_{(\mu_2,\lambda)}(s))\\
&\quad + q(s)g(x_{(\mu_2,\lambda)}(s))\Big)ds + \kappa\lambda t^{\alpha-1}\\
&\le \frac{1}{\psi(\mu_2,\lambda,r_0)}(T_{(\mu_2,\lambda)}x_{(\mu_2,\lambda)})(t)\\
&= \frac{1}{\psi(\mu_2,\lambda,r_0)}x_{(\mu_2,\lambda)}(t),\ t\in R^+.
\end{aligned}$$

So, by the definition of r_0 and (3.9) we have $r_0 < \psi(\mu_2,\lambda,r_0) \le r_0$, which is a contradiction. Hence $r_0 \ge 1$. Moreover, $x_{(\mu_1,\lambda)} \le x_{(\mu_2,\lambda)}$.

In addition, arguing similarly to Theorem 4.1, other results in the conclusion (b) can be proved.

The proof of the conclusion (c) can be finished by the similar way as the above conclusion (b). This completes the proof. □

Theorem 4.4. *Under the condition (iv) of Theorem 3.1, the following results hold.*

(a) for any fixed $\mu \ge 0$, $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$;

(b) for any fixed $\lambda > 0$, $\lim\limits_{\mu\to 0+}\|x_{(\mu,\lambda)} - \lambda w_0\| = 0$, in addition, $\lim\limits_{\mu\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ if $\int_0^1 G_2(\eta,s)f(s,1)ds > 0$;

(c) for $\mu = \nu\lambda$, $\nu \ge 0, \lambda > 0$, (c-1) $\lim\limits_{\lambda\to+\infty}\|x_{(\mu,\lambda)}\| = +\infty$ and $\lim\limits_{\lambda\to 0+}\|x_{(\mu,\lambda)}\| = 0$; (c-2) $x_{(\mu,\lambda)}$ is increasing and continuous with respect to λ for $\lambda > 0$ if $\nu = 0$ or $0 < \nu g(\infty)\int_0^{+\infty} q(s)ds < 1$ and $\varphi(r) > r^{\frac{1}{2}}$ for $r \in (0,1)$.

Proof. We only prove the conclusion (c-2), other conclusions can be proved similarly to Theorem 4.1.

For $\mu = \nu\lambda$, $\nu \ge 0, \lambda > 0$, we consider two cases. If $\nu = 0$ then $\mu = 0$, thus $x_{(\mu,\lambda)}(t) = \lambda\kappa t^{\alpha-1}$, and (c-2) is obvious. If $\nu > 0$, we consider the operator $C^*_{(\mu,\lambda)}$ defined as (2.8), and have

$$C^*_{(\mu,\lambda)}(x,y) = \nu\lambda Ay + B_{(\nu\lambda,\lambda)}x = \lambda(\nu Ay + B_{(\nu,1)}x) =: C^*_\lambda(x,y),\ x,y\in P.$$

It is clear that $C^*_\lambda : P\times P \to P_e$ is a mixed monotone operator. In addition, we claim that the operator $B_{(\nu,1)}$ satisfies the following property

$$B_{(\nu,1)}(rx) \ge r^\delta B_{(\nu,1)}x,\ r\in(0,1), x\in P_e, \tag{4.4}$$

where

$$\delta = \frac{\nu g(\infty)\int_0^{+\infty} q(s)ds}{1+\nu g(\infty)\int_0^{+\infty} q(s)ds}.$$

Indeed, it is clear that $0 < \delta < \frac{1}{2}$. By straightforward calculations, for any $r \in (0,1)$ we have

$$\frac{1-r^\delta}{r^\delta - r} > \lim_{r\to 1-} \frac{1-r^\delta}{r^\delta - r} = \frac{\delta}{1-\delta} \geq \nu g(\infty)\int_0^{+\infty} q(s)ds.$$

Hence,

$$r^\delta\Big(\nu g(\infty)\int_0^{+\infty} q(s)ds + 1\Big) < r\nu g(\infty)\int_0^{+\infty} q(s)ds + 1,$$

which implies that

$$\begin{aligned} B_{(\nu,1)}(rx)(t) \geq& r\nu\int_0^{+\infty} G(t,s)q(s)g(x(s))ds + \kappa t^{\alpha-1} \\ \geq& r^\delta\Big(\nu\int_0^{+\infty} G(t,s)q(s)g(x(s))ds + \kappa t^{\alpha-1}\Big) \\ =& r^\delta (B_{(\nu,1)}x)(t),\ r \in (0,1), x \in P_e, t \in R^+. \end{aligned}$$

This show that $B_{(\nu,1)}$ satisfies (4.4). Moreover, let

$$\overline{\varphi}(r) = \min\{\varphi(r), r^\delta\},\ r \in (0,1),$$

then

$$C^*_\lambda(rx, \frac{1}{r}y) \geq \lambda\Big(\nu\varphi(r)Ay + r^\delta B_{(\nu,1)}x\Big) \geq \overline{\varphi}(r)C^*_\lambda(x,y),\ x,y \in P_e, r \in (0,1).$$

Note that $0 < \delta < \frac{1}{2}$ and $\varphi(r) > r^{\frac{1}{2}}$, then $\overline{\varphi}(r) > r^{\frac{1}{2}}$ for $r \in (0,1)$. Applying Lemma 2.5 we obtain that $x_{(\mu,\lambda)}$ is increasing with respect to λ for $\lambda > 0$. Moreover, the continuity of $x_{(\mu,\lambda)}$ with respect to λ can be proved similarly to Theorem 4.1. This ends the proof. □

Remark 4.1. In order to clearly show the dependence properties of solution on parameters in different cases, we give a conclusion in following Table 2. In this table, "inc." means "increasing ", and "cont." means "continuous".

Table 2. Continuous dependence of the solution on two parameters

Parameters	Dependence	(H1),(H3)	(H1),(H4)	(H2),(H4)	(H2),(H3)
Fixed μ	inc. and cont. in λ	yes	uncertain	uncertain	uncertain
Fixed λ	inc. and cont. in μ	yes	uncertain	yes	uncertain
$\mu = \nu\lambda$	inc. and cont. in λ	yes	uncertain	yes	yes
Other cases	cont. in (λ,μ)	uncertain	uncertain	uncertain	uncertain

5. The boundary condition without disturbance parameter

For the problem (1.4), although we can not derive the corollary from Theorem 3.1, we can obtain the different result. In section, we shall consider FBVP(1.4). To be clear, we present it again

$$\begin{cases} D_{0^+}^{\alpha} x(t) + \mu f(t, x(t)) = 0, t \in [0, \infty), \\ x(0) = x'(0) = 0, \quad D_{0^+}^{\alpha-1} x(\infty) = \beta \int_0^{\eta} x(s)ds. \end{cases}$$

It is clear that x is a positive solution of FBVP(1.4) if and only if x is a fixed point of the operator μA in $P \backslash \{\theta\}$.

By the proofs of Theorem 3.1, Theorem 4.1 and Theorem 4.3, we can obtain the following result.

Theorem 5.1. *Assume that one of the following assumptions conditions is satisfied.*

(F1) (H1) holds and $\int_1^{+\infty} G_2(\eta, s) f(s, 1)ds > 0$*;*

(F2) (H2) holds and $\int_0^1 G_2(\eta, s) f(s, 1)ds > 0$.

Then (1.4) has a unique positive solution x_μ *in* P_e *for any* $\mu > 0$. *Furthermore, for any* $x_0 \in P_0$*, constructing successively the sequence*

$$x_n(t) = \mu \int_0^{+\infty} G(t, s) f(s, x_{n-1}(s))ds, \; n = 1, 2, \cdots,$$

then

$$\lim_{n\to\infty} \sup_{t\in[0,+\infty)} \frac{|x_n(t) - x_\mu(t)|}{1 + t^{\alpha-1}} = 0.$$

In addition, such a positive solution x_μ *is increasing and continuous in* μ *for* $\mu > 0$, $\lim\limits_{\mu\to 0+} \|x_\mu\| = 0$, *and* $\lim\limits_{\mu\to+\infty} \|x_\mu\| = +\infty$.

6. Examples

Consider the fractional boundary value problem on half line

$$\begin{cases} D_{0^+}^{\frac{8}{3}} x(t) + \mu\Big(\frac{e^{-t^{\frac{3}{2}}}\left(1+t^{\frac{5}{3}}+\sqrt{x(t)}\right)^p}{\left(1+t^{\frac{5}{3}}\right)^{4+p}} + \frac{e^{-t}(1+ax(t))}{1+bx(t)} \Big) = 0, \; t \in [0, \infty), \\ x(0) = 0, \quad x'(0) = 0, \\ D_{0^+}^{\frac{5}{3}} x(\infty) = \frac{1}{951} \int_0^{22} x(s)ds + \lambda, \end{cases} \tag{6.1}$$

where $0 < |p| < 2$ and $a, b > 0, a \neq b$. That is, in FBVP(1.3), $\alpha = \frac{8}{3}, \beta = \frac{1}{951}, \eta = 22$, $q(t) = e^{-t}$, $f(t, x) = \frac{e^{-t^{\frac{3}{2}}}\left(1+t^{\frac{5}{3}}+\sqrt{x}\right)^p}{\left(1+t^{\frac{5}{3}}\right)^{4+p}}$ and $g(x) = \frac{1+ax}{1+bx}$. Then

$$\beta\eta^\alpha = \frac{\sqrt[3]{22^8}}{951} \approx 3.9959 < 4.0124 \approx \frac{80}{27}\Gamma(\frac{2}{3}) = \Gamma(\alpha + 1), \quad \kappa \approx 161.6162,$$

and $\int_0^{+\infty} q(s)ds = 1 > 0$.

It is easy to check that $g(\infty)=\frac{a}{b}$, and $g:R^+\to R^+$ is not only continuous but also increasing for $a>b$, and decreasing for $a<b$. In addition, for any $r\in(0,1)$,

$$g(rx)=\frac{r(\frac{1}{r}+ax)}{1+brx}\geq rg(x),\quad \text{and } g(rx)=\frac{1+arx}{r(\frac{1}{r}+bx)}\leq\frac{1}{r}g(x).$$

So, (H3) holds for $a>b$ and (H4) holds for $a<b$.

It is clear that $f(t,x)$ is continuous. Two cases are considered on p.

Case 1 $p\in(0,2)$. In this case, $f(t,x)$ is increasing in x, moreover,

$$f(t,1+t^{\alpha-1})=\frac{e^{-t^{\frac{3}{2}}}\big(1+t^{\frac{5}{3}}+\sqrt{1+t^{\frac{5}{3}}}\big)^p}{\big(1+t^{\frac{5}{3}}\big)^{4+p}}\leq 2^pe^{-t^{\frac{3}{2}}}\big(1+t^{\frac{5}{3}}\big)^{-4}\leq 2^pe^{-t^{\frac{3}{2}}},$$

which implies that $f(t,1+t^{\frac{5}{3}})\in L^1[0,+\infty)$.

In addition,

$$f(t,rx)=\frac{r^{\frac{p}{2}}e^{-t^{\frac{3}{2}}}\big(\frac{1+t^{\frac{5}{3}}}{\sqrt{r}}+\sqrt{x}\big)^p}{\big(1+t^{\frac{5}{3}}\big)^{4+p}}\geq r^{\frac{p}{2}}f(t,x),\quad x\geq 0,\ r\in(0,1),$$

and

$$f(t,1)=\frac{e^{-t^{\frac{3}{2}}}\big(2+t^{\frac{5}{3}}\big)^p}{\big(1+t^{\frac{5}{3}}\big)^{4+p}}\geq e^{-t^{\frac{3}{2}}}\big(1+t^{\frac{5}{3}}\big)^{-4}.$$

Thus, (H1) is satisfied and $\int_1^{+\infty}G_2(\eta,s)f(s,1)ds>0$

Case 2 $p\in(-2,0)$. In this case, $f(t,x)$ is decreasing in x, and

$$f(t,0)=e^{-t^{\frac{3}{2}}}\big(1+t^{\frac{5}{3}}\big)^{-4}\leq e^{-t^{\frac{3}{2}}},$$

which implies that $f(t,0)\in L^1[0,+\infty)$.

In addition,

$$f(t,rx)=\frac{1}{r^{-\frac{p}{2}}e^{t^{\frac{3}{2}}}\big(1+t^{\frac{5}{3}}\big)^{4+p}\big(\frac{1+t^{\frac{5}{3}}}{\sqrt{r}}+\sqrt{x}\big)^{-p}}\leq\frac{1}{r^{\frac{|p|}{2}}}f(t,x),\quad x\geq 0,\ r\in(0,1),$$

and

$$f(t,1)\geq e^{-t^{\frac{3}{2}}}\big(2+t^{\frac{5}{3}}\big)^{-4}.$$

Hence, (H2) is satisfied and $\int_0^1G_2(\eta,s)f(s,1)ds>0$.

Theorem 3.1 shows that FBVP(6.1) has a unique positive solution $x_{(\mu,\lambda)}$ for any $\mu\geq 0$ and $\lambda>0$, and for any $x_0\in P$, set

$$x_n(t)=\mu\int_0^{+\infty}G(t,s)\Big(\frac{e^{-s^{\frac{3}{2}}}\big(1+s^{\frac{5}{3}}+\sqrt{x_{n-1}(s)}\big)^p}{\big(1+s^{\frac{5}{3}}\big)^{4+p}}+\frac{e^{-s}(1+ax_{n-1}(s))}{1+bx_{n-1}(s)}\Big)ds+\lambda\kappa t^{\frac{5}{3}}$$

for $n=1,2,\cdots$, we have

$$\lim_{n\to\infty}\sup_{t\in[0,+\infty)}\frac{|x_n(t)-x_{(\mu,\lambda)}(t)|}{1+t^{\alpha-1}}=0.$$

Next to discuss the dependence of such a solution $x_{(\mu,\lambda)}$ on parameters μ and λ in four cases.

Case 1. If $p \in (0, 2)$ and $a > b$, then all conditions of Theorem 4.1 are satisfied. Hence, the positive solution $x_{(\mu,\lambda)}$ satisfies all conclusions in Theorem 4.1.

Case 2. If $p \in (0, 2)$ and $a < b$, then all conditions of Theorem 4.2 are satisfied, so we can obtain all conclusions in Theorem 4.2 for the positive solution $x_{(\mu,\lambda)}$.

Case 3. If $p \in (-2, 0)$ and $a < b$, then all conditions of Theorem 4.3 are satisfied, So, all conclusions in Theorem 4.3 hold for the positive solution $x_{(\mu,\lambda)}$.

Case 4. If $p \in (-2, 0)$ and $a > b$, then (H2) and (H3) are satisfied and $\int_0^1 G_2(\eta, s)f(s, 1)ds > 0$. So, we can obtain conclusions (a),(b) and (c-1) in Theorem 4.4 for the positive solution $x_{(\mu,\lambda)}$. In addition, we can conclude by the conclusion (c-2) in Theorem 4.4 that the positive solution $x_{(\mu,\lambda)}$ is continuous and increasing in λ for $0 \leq \nu < \frac{b}{a}$, $\mu = \nu\lambda$, $\lambda > 0$ and $-1 < p < 0$.

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