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THE NUMBER OF RATIONAL SOLUTIONS OF ABEL EQUATIONS*

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Abstract In this paper, we study rational solutions of the Abel differential equations $dy/dx = f_m(x)y^2 + g_n(x)y^3$, where $f_m(x)$ and $g_n(x)$ are real polynomials of degree m and n respectively. The main result of the paper is as follows: We give a systematic upper bound on the number of the nontrivial rational solutions of such equations in all these cases. Then we prove that these upper bounds can be reached in most cases. Finally, we present some examples of Abel equations having exactly i nontrivial rational solutions, where $1 \leq i \leq 5$.

Keywords Abel differential equations, nontrivial rational solution, Liénard system, rational invariant curve.

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1. Introduction

The Abel differential equations

$$\frac{dy}{dx} = f_m(x)y^2 + g_n(x)y^3, \tag{1.1}$$

where $f_m(x)$ and $g_n(x)$ are real polynomials of degree m and n respectively, with the following explicit expressions

$$f_m(x) = \sum_{i=0}^{m} a_i x^i, \qquad g_n(x) = \sum_{i=0}^{n} b_i x^i, \qquad a_m b_n \neq 0 \tag{1.2}$$

can be found in many models of real phenomena (see [1, 10]) and have been studied intensively. Much attention has been paid to, say, the center problem (see for instance [3, 4]), the number of limit cycles (see [5, 8]), the polynomial solutions, polynomial limit cycles, and nontrivial rational limit cycles (see [9, 11]).

The study of some particular solutions (as polynomial or rational solutions) of the differential equations can be seen as an important way to understand the whole set of solutions of the system. Concerning some well-known systems, Rainville [13] in 1936 proved the existence of one or two polynomial solutions for the Riccati differential equation $y' = b_0(x) + b_1(x)y + y^2$, with $b_0(x)$ and $b_1(x)$ are polynomials.

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Behloul and Cheng [2] presented some methods to compute polynomial solutions of the differential equation

$$a(x)y' = b_0(x) + b_1(x)y + b_2(x)y^2 + \cdots + b_n(x)y^n, \tag{1.3}$$

where $a(x)$ and $b_i(x)$, with $0 \le i \le n$ are real polynomials.

As to the polynomial solutions of equation (1.3), here is a very brief survey. When $a(x) = 1$, and $b_n(x) \ne 0$, the authors in [9] proved that (1.3) has at most n polynomials solutions. When $n = 2$, the authors in [7] give an estimate of the number of polynomial solutions in terms of the degrees of all coefficient polynomials involved. In [6] the authors treated the equation (1.3) in some other special cases. Llibre and Valls [12,14] gave in detail the maximum number of polynomial solutions in the case $n = 3$, $b_3(x) \ne 0$ and $b_0(x) = b_2(x) = 0$.

Notice that when $n = 3$, $a(x) = 1$ and $b_0(x) = b_1(x) = 0$, equation (1.3) turns out to be the classical Abel equation of the form (1.1). For the Abel equation (1.1), by using ideas from [9], one can easily see that this equation can have at most two polynomial solutions, and we can found a concrete example $y' = y^2(y-1)$ with two (trivial) polymial solutions $y = 0$ and $y = 1$.

As we know beyond the polynomial solutions, the study of rational solutions is also of great importance. But to the best of our knowledge, the number of rational solutions of equation (1.1) have not been considered so far. By a rational solution of (1.1), we mean a solution of the form $y = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials of their variable. Here we shall only consider *nontrivial* rational solutions, i.e. the case $Q(x) = const$ is not within our interest, since in this case, $P(x)/Q(x)$ is nothing but a polynomial.

The importance of study of (1.1) also relies on the intrinsic relation between (1.1) and the Liénard system. In fact, by performing the change $z = 1/y$, equation (1.1) can be transformed into a Liénard system,

$$\dot{x} = z, \qquad \dot{z} = -f_m(x)z - g_n(x), \tag{1.4}$$

and a nontrivial rational solution is transformed into a rational invariant curve. Hence, the number of nontrivial rational solutions of the equation (1.1) and the number of the rational invariant curves of Liénard systems (1.4) coincide. While considering the number of invariant curves is an useful way to study the integrability of the planar polynomial differential system.

In this paper, we consider the maximum number of nontrivial rational solutions of (1.1) in all cases. For convenience of stating our results, in terms of (1.1), we call such an Abel equation of type (m, n). Notice that $y = \varphi(x)$ is a nontrivial rational solution of the equation (1.1) if and only if $\varphi(x) = 1/R(x)$ and

$$R(x)R'(x) + R(x)f_m(x) = -g_n(x), \tag{1.5}$$

where $R(x)$ is a polynomial. Therefore a crucial point to consider the maximum number of nontrivial rational solutions of the equation (1.1) will depend on the analysis of the number of polynomial solutions of equation (1.5).

By comparing the degrees of the polynomials of two sides of equation (1.5), we immediately know that if $n \le m$ or $n \ge 2m + 2$ with n even, then (1.1) of type (m, n) has no nontrivial rational solutions. For the remaining cases, our results are as follows.

Theorem 1.1. *If $m+1 \leq n \leq 2m$, or $n \geq 2m+3$ and n is odd, then (1.1) of type (m, n) has at most 2 nontrivial rational solutions, and this bound is sharp.*

Theorem 1.2. *If $n = 2m+1$ with $m \geq 1$, then the equations (1.1) of type (m, n) have at most $m+3$ nontrivial rational solutions, and this bound is sharp in the cases $m = 1$ and $m = 2$.*

At that point, we wonder whether $m+3$ is the sharp upper bound on the number of nontrivial rational solutions of equations (1.1) of type $(m, 2m+1)$ with $m \geq 3$. In fact, the answer is no in the case $m = 3$, and we shall give another method for finding the maximum number of rational solutions in this case. Our result is stated in the next theorem.

Theorem 1.3. *The equations (1.1) of type $(3, 7)$ have at most 5 nontrivial rational solutions, and this bound is sharp.*

We remark that with only one possible exception, $n = 2m+1$ and $m \geq 4$, all these upper bounds given above are sharp, and our result can also be used to study rational limit cycles.

The next two theorems give the existence of equations (1.1) of type $(m, 2m+1)$ with exactly i nontrivial rational solutions and equations (1.1) of type $(4, 9)$ with exactly 5 nontrivial rational solutions, where $1 \leq i \leq 4$ and $m \geq 1$.

Theorem 1.4. *For any integer $m \geq 1$, there exist equations (1.1) of type $(m, 2m+1)$ having exactly i nontrivial rational solutions, where $1 \leq i \leq 4$.*

Theorem 1.5. *There are equations (1.1) of type $(4, 9)$ having exactly 5 rational invariant curves.*

The rest of this paper is organized as follows. We prove Theorem 1.1 and Theorem 1.2 in section 2 while leave the proof of Theorem 1.3 to section 3. Finally, in section 4 we prove Theorem 1.4 and Theorem 1.5.

2. Proof of Theorem 1.1 and 1.2

The proof of Theorem 1.1 consists of two parts, the upper bound of the number of nontrivial rational solutions and the explicit examples having 2 nontrivial rational solutions.

Proposition 2.1. If $m+1 \leq n \leq 2m$, then the equations (1.1) of type (m, n) have at most 2 nontrivial rational solutions.

Proof. If $y = 1/R(x)$ is a nontrivial rational solution of equations (1.1), then from (1.5), we immediately know that the degree of $R(x)$ can only be $n-m$ or $m+1$.

If the degree of $R(x)$ is $n-m$, then we can assume

$$R(x) = c_{n-m}x^{n-m} + \cdots + c_1 x + c_0, \tag{2.1}$$

where $c_{n-m} \neq 0$. By substituting $R(x)$, $f_m(x)$ and $g_n(x)$ into (1.5) and comparing the coefficients of the polynomials of two sides, we can deduce that the values of $c_{n-m}, c_{n-m-1}, \cdots, c_0$ are uniquely determined. It follows that if the degree of $R(x)$ is $n-m$ then (1.1) has at most one nontrivial rational solution $y = 1/R(x)$.

If the degree of $R(x)$ is $m+1$, then we assume

$$R(x) = -\frac{a_m}{m+1}x^{m+1} - \frac{a_{m-1}}{m}x^m - \cdots - a_0 x + d_{n-m}x^{n-m} + \cdots + d_1 x + d_0, \quad (2.2)$$

where $d_{n-m} \neq 0$. By substituting the equations (2.2) and (1.2) into (1.5) and comparing the coefficients of the polynomials of two sides,we can deduce that the values $d_{n-m}, d_{n-m-1}, \cdots, d_0$ are uniquely determined.

Thus this type of equations (1.1) have at most one nontrivial rational solution $y = 1/R(x)$, with the degree of $R(x)$ is $m+1$. This completes the proof of the Proposition. □

Proposition 2.2. If $n \geq 2m+3$ and n is odd, then the equation (1.1) of type (m, n) has at most 2 nontrivial rational solutions.

Proof. We assume $n = 2k+1$, where $k \geq m+1 \in N$. If $y = 1/R(x)$ is a nontrivial rational solution of equations (1.1), then the degree of $R(x)$ can only be $k+1$. Denote

$$R(x) = e_{k+1}x^{k+1} + \cdots + e_1 x + e_0, \quad (2.3)$$

where $e_{k+1} \neq 0$. By substituting the equations (2.3) and (1.2) into (1.5) and comparing the coefficients of the highest degree terms of the equation, we obtain the following equation,

$$-b_{2k+1} = (k+1)e_{k+1}^2.$$

Thus e_{k+1} can have at most two different values.

Once we have determined the value of e_{k+1}, we can uniquely determine the values of e_i by comparing the coefficients of the polynomials of the equation (1.5), where $0 \leq i \leq k$. Thus the equations (1.1) of type (m, n) have at most 2 nontrivial rational solutions for $n \geq 2m+3$ with n is odd. The proof of the Proposition is complete. □

Below we present two explicit examples of equation (1.1) of type (m, n) having exactly 2 nontrivial rational solutions. One is for the case $m+1 \leq n \leq 2m$ and the other for the case $n \geq 2m+3$ with n is odd. Since the proof of Proposition 2.3 and Proposition 2.4 involves only tedious computation, therefore we omit the details here.

Proposition 2.3. If $m+1 \leq n \leq 2m$, then the Abel differential equation

$$\frac{dy}{dx} = -(x^m + x^{n-m-1})y^2 + \left(\frac{1}{n+1}x^n + \frac{m+1}{(n+1)^2}x^{2n-2m-1}\right)y^3 \quad (2.4)$$

has 2 nontrivial rational solutions

$$y = \frac{n+1}{x^{n-m}} \quad \text{and} \quad y = \frac{(n+1)(m+1)}{(n+1)x^{m+1} + (m+1)x^{n-m}}.$$

Proposition 2.4. If $n \geq 2m+3$ and n is odd, then the Abel differential equation

$$\frac{dy}{dx} = -(2m+n+3)x^m y^2 - \left(\frac{n+1}{2}x^n - 2(1+n)x^{2m+1}\right)y^3, \quad (2.5)$$

has 2 nontrivial rational solutions

$$y = \frac{1}{x^{\frac{n+1}{2}} + 2x^{m+1}} \quad \text{and} \quad y = \frac{-1}{x^{\frac{n+1}{2}} - 2x^{m+1}}.$$

Proof of Theorem 1.1. Proposition 2.1 and Proposition 2.2 provide the first part of the theorem, which correspond to the upper bound of the statement. Proposition 2.3 and Proposition 2.4 give concrete Abel differential equations having exactly 2 nontrivial rational solutions. These facts prove the second part of the theorem.

Now we shall prove the Theorem 1.2. We also organize the proof of the theorem in two parts, the uniform upper bound $m+3$ for $n = 2m+1$ and $m \geq 1$, and some examples having exactly 4 and 5 nontrivial rational solutions for $m = 1$ and $m = 2$, respectively.

Proposition 2.5. If $n = 2m+1$ and $m \geq 1$, then the equations (1.1) of type (m, n) have at most $m+3$ nontrivial rational solutions.

Proof. First of all, it is easy to check that if $y = 1/R(x)$ is a nontrivial rational solution of equations (1.1), then the degree of $R(x)$ can only be $m+1$. Set

$$R(x) = u_{m+1}x^{m+1} + \cdots + u_1 x + u_0, \tag{2.6}$$

where $u_{m+1} \neq 0$. By substituting (2.6) and (1.2) into (1.5) and then comparing the coefficients of the equation, we obtain the following relations:

$$\begin{aligned}
&(a_m + (m+1)u_{m+1})u_{m+1} = -b_{2m+1},\\
&(a_m + (2m+1)u_{m+1})u_m + a_{m-1}u_{m+1} = -b_{2m},\\
&\cdots = \cdots,\\
&(a_m + (m+1+j)u_{m+1})u_j + R_j(u_{j+1}, u_{j+2}, \cdots, u_{m+1}) = -b_{m+j},\\
&\cdots = \cdots,\\
&(a_0 + u_1)u_0 = -b_0,
\end{aligned}$$

where $R_j(u_{j+1}, u_{j+2}, \cdots, u_{m+1})$ is a polynomial, and $0 \leq j \leq m$. From the first equation of the above system, we know that there are at most 2 solutions of this equation, denoted by, u'_{m+1} and u''_{m+1}.

We can see that if $a_m + iu'_{m+1} \neq 0$ and $a_m + iu''_{m+1} \neq 0$ with $m+2 \leq i \leq 2m+1$, then once u_{m+1} is determined, the value of u_j with $0 \leq j \leq m$ is also uniquely determined. Hence this equation can have at most 2 nontrivial rational solutions.

If there exist u'_{m+1} such that $a_m + iu'_{m+1} = 0$, i.e, $u'_{m+1} = -a_m/i$, then by an elementary way, we know $u''_{m+1} = -(i-m-1)a_m/(im+i)$. We claim that there does not exist a natural number $k \in [m+2, 2m+1]$ such that $a_m + ku''_{m+1} = 0$. By contradiction, we assume that there exist a natural number $k \in [m+2, 2m+1]$ such that $a_m + ku''_{m+1} = 0$, then

$$\frac{i-m-1}{im+i} = \frac{1}{k}, \quad \frac{k-m-1}{km+k} = \frac{1}{i}.$$

Since $i \in [m+2, 2m+1]$ and $k \in [m+2, 2m+1]$, we obtain that

$$\frac{i-m-1}{im+i} \leq \frac{m}{im+i} < \frac{1}{i}, \quad \frac{k-m-1}{km+k} \leq \frac{m}{km+k} < \frac{1}{k}.$$

Hence

$$\frac{1}{k} < \frac{1}{i}, \quad \frac{1}{i} < \frac{1}{k},$$

a contradiction. Hence the system can have at most one nontrivial rational solution $y = 1/R(x)$, with the leading coefficient of $R(x)$ is u''_{m+1}.

On the other hand, if the leading coefficient of the term $R(x)$ of the nontrivial rational solution $y = 1/R(x)$ is u'_{m+1}, then u_{i-m-1} is an independent variable, and we may express $u_0, \cdots, u_{i-m-2}$ in terms of u_{i-m-1}. Substituting $u_0, \cdots, u_{i-m-2}$ into the remaining equations, we may then obtain equations of higher degree in u_{i-m-1}.

In order to make the equations admit more solutions, we must assume $a_m+(2m+1)u'_{m+1} = 0$. In this case, the value of u'_{m+1} is determined, and u_m is an independent variable. Note that we can express u_{m-k} in terms of u_m from the $k+2-th$ equation, where $1 \le k \le m$, namely, $u_{m-k} = \frac{c_{m-k,0}}{u'^k_{m+1}} u_m^{k+1} + \cdots + c_{m-k,k} u_m + c_{m-k,k+1}$, where $c_{m-k,0} > 0$. Furthermore, substituting $u_0, \cdots, u_{m-1}$ into the $m+3-th$ equation, we obtain an equation of $m+2$ degree since the coefficients of the highest term of this equation is d_m/u'^m_{m+1}, with $d_m > 0$. Thus there are at most $m+2$ solutions of the equations, namely, the equations (1.1) have at most $m+2$ nontrivial rational solutions in this case.

Basing on the above discussion, we know that the equations (1.1) have at most $m+3$ nontrivial rational solutions. The Proposition follows. □

In what follows we construct some explicit examples of the equations (1.1) of type $(1,3)$ and $(2,5)$ which have exactly 4 and 5 nontrivial rational solutions, respectively. Since all the computation of Proposition 2.6 and 2.7 is straightforward, therefore we omit the tedious details.

Proposition 2.6. We consider the Abel differential equation (1.1) of type $(1,3)$

$$\frac{dy}{dx} = -3xy^2 + (x^3 - x)y^3. \tag{2.7}$$

This system has the following 4 nontrivial rational solutions,

$$y = \frac{1}{x^2-1}, \quad y = \frac{2}{x^2-1}, \quad y = \frac{1}{x^2-x}, \quad y = \frac{1}{x^2+x}.$$

Proposition 2.7. We consider the Abel differential equation (1.1) of type $(2,5)$

$$\frac{dy}{dx} = -(5x^2 - 15x + 9)y^2 + (2x^5 - 15x^4 + 40x^3 - 45x^2 + 18x)y^3. \tag{2.8}$$

This system has the following 5 nontrivial rational solutions,

$$y = \frac{1}{x^3 - \frac{11}{2}x^2 + 9x - \frac{9}{2}}, \quad y = \frac{1}{x^3 - 5x^2 + 6x}$$
$$y = \frac{1}{x^3 - 4x^2 + 3x}, \quad y = \frac{1}{x^3 - \frac{7}{2}x^2 + 3x}, \quad y = \frac{1}{\frac{2}{3}x^3 - 3x^2 + 3x}.$$

Proof of Theorem 1.2. Proposition 2.5 provides the first part of the theorem, which correspond to the upper bound of the statement when $n = 2m+1$ and $m \ge 1$. Proposition 2.6 and Proposition 2.7 give concrete Abel differential equations of type $(1,3)$ and $(2,5)$ having exactly 4 and 5 nontrivial rational solutions respectively. These facts prove the second part of the theorem.

3. Proof of Theorem 1.3

In this section, we present a new method for finding the sharp upper bound on the number of nontrivial rational solutions of Abel equations (1.1). Before giving the method, we need the following two useful lemmas. Firstly, Lemma 3.1 provides a necessary condition for the existence of at least two nontrivial rational solutions of the equations (1.1).

Lemma 3.1. *Assume in equation (1.1)*

$$g_{2m+1}(x) = l(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2m+1}),$$

where $l \in N$ *and* $[\frac{m+1}{2}] \le l \le m$*. Let* $y = 1/P_1(x)$ *and* $y = 1/P_2(x)$ *be two nontrivial rational solutions of equation (1.1). Set* $P_1(x) = G(x)\widetilde{P_1}$, $P_2(x) = G(x)\widetilde{P_2}$*, where*

$$G(x) = \gcd(P_1(x), P_2(x)) = (x-\alpha_1)\cdots(x-\alpha_i),$$
$$\widetilde{P_1} = (x-\alpha_{i+1})\cdots(x-\alpha_{m+1}),\ \ \widetilde{P_2} = (x-\alpha_{m+2})\cdots(x-\alpha_{2m+2-i}),$$

with $1 \le i \le l$ *and* $\gcd(\widetilde{P_1}(x), \widetilde{P_2}(x)) = 1$*. Then if* $i = 1$*, we set* $G_2(x) = 1$*. For* $2 \le i \le l$*, let* $G_2(x) = (x-\alpha_{2m+3-i})\cdots(x-\alpha_{2m+1})$*. Finally the following equations hold:*

$$\widetilde{P_1} - \widetilde{P_2} = C(x-\alpha_{i_1})^{k_1}\cdots(x-\alpha_{i_j})^{k_j},\ \ lG_2(x) - G'(x) = G(x)\sum_{v=1}^{j}\frac{k_v}{x-\alpha_{i_v}} \quad (3.1)$$

where C *is an constant,* $1 \le i_1 < i_2 < \cdots < i_j \le i$, $k_v \ge 0$, $1 \le v \le j$ *and* $k_1 + \cdots + k_j = l - i$.

Proof. Notice that $y = 1/P_1(x)$ and $y = 1/P_2(x)$ are two nontrivial rational solutions of the equation (1.1), we know that

$$P_i(x)P_i'(x) + P_i(x)f_m(x) = -g_{2m+1}(x), \quad (3.2)$$

where $1 \le i \le 2$. Thus we can express $f_m(x)$ in terms of $P_1(x)$ and $P_2(x)$, namely,

$$\begin{aligned} f_m(x) &= \frac{P_1(x)P_1'(x) - P_2(x)P_2'(x)}{P_2(x) - P_1(x)} \\ &= -(\widetilde{P_1}(x) + \widetilde{P_2}(x))G'(x) + \frac{G(x)(\widetilde{P_1}'(x)\widetilde{P_1}(x) - \widetilde{P_2}'(x)\widetilde{P_2}(x))}{\widetilde{P_2}(x) - \widetilde{P_1}(x)} \\ &= -(\widetilde{P_1}(x) + \widetilde{P_2}(x))G'(x) - G(x)\widetilde{P_1}'(x) + \frac{G(x)\widetilde{P_2}(x)(\widetilde{P_1}'(x) - \widetilde{P_2}'(x))}{\widetilde{P_2}(x) - \widetilde{P_1}(x)}. \end{aligned}$$

Then since $f_m(x)$ is a polynomial, we deduce that a root of $\widetilde{P_2}(x) - \widetilde{P_1}(x)$ must be a root of $G(x)\widetilde{P_2}(x)$. Moreover, note that $(\widetilde{P_1}(x), \widetilde{P_2}(x)) = 1$, so a root of $\widetilde{P_2}(x) - \widetilde{P_1}(x)$ must be a root of $G(x)$, in other words, we can obtain the following equation,

$$\widetilde{P_1} - \widetilde{P_2} = C(x-\alpha_{i_1})^{k_1}\cdots(x-\alpha_{i_j})^{k_j},$$

where $1 \le i_1 < i_2 < \cdots < i_j \le i$, $k_v \ge 0$ with $1 \le v \le j$.

On the other hand, from (3.2), we obtain that

$$P_1'(x) + f_m(x) = -l\widetilde{P_2}(x)G_2(x),\ P_2'(x) + f_m(x) = -l\widetilde{P_1}(x)G_2(x).$$

It follows that

$$P_1'(x) - P_2'(x) = lG_2(x)(\widetilde{P_1}(x) - \widetilde{P_2}(x)).$$

Consequently,

$$lG_2(x) - G'(x) = G(x)\frac{\widetilde{P_1}'(x) - \widetilde{P_2}'(x)}{\widetilde{P_1}(x) - \widetilde{P_2}(x)} = G(x)\sum_{v=1}^{j}\frac{k_v}{x - \alpha_{i_v}}. \tag{3.3}$$

By comparing the leading coefficient of the equation (3.3), we obtain that $k_1 + \cdots + k_j = l - i$. Thus the proof of the lemma ends. □

Then we give a necessary condition for the existence of at least three nontrivial rational solutions $y = 1/P_i(x)$ of equations (1.1), with $P_i(0) = 0$ and $1 \leq i \leq 3$.

Lemma 3.2. *Assume an equation (1.1) with at least three nontrivial rational solutions $y = 1/P_i(x)$, with $P_i(0) = 0$ and $1 \leq i \leq 3$. Set*

$$P_1(x) = u_{1,m+1}x^{m+1} + u_{1,m}x^m + \cdots + u_{1,n_1}x^{n_1},$$

with $1 \leq n_1 \leq m$, and $u_{1,n_1} \neq 0$. Then the term $p_k(x)$ of all the other nontrivial rational solutions $y = 1/P_k(x)$ with $P_k(0) = 0$ can be express as

$$P_k(x) = u_{k,m+1}x^{m+1} + u_{k,m}x^m + \cdots + u_{k,n_1}x^{n_1},$$

where $k \geq 2$ and $u_{k,n_1} \neq 0$. Furthermore, u_{k,n_1} with $k \geq 2$ can take only two possible values:

(i) all the values are the same;

(ii) there exists a natural number k_1 such that $u_{k_1,n_1} \neq u_{1,n_1}$ and $u_{k,n_1} \equiv u_{1,n_1}$ or $u_{k,n_1} \equiv u_{k_1,n_1}$, with $k, k_1 \geq 2$ and $k \neq k_1$.

Proof. We assume without loss of generality that $y = 1/P_i(x)$, with

$$p_i(x) = u_{i,m+1}x^{m+1} + u_{i,m}x^m + \cdots + u_{i,n_i}x^{n_i},$$

$n_i \geq 1$, $u_{i,n_i} \neq 0$ and $1 \leq i \leq 3$ are three given nontrivial rational solutions of an equation (1.1) of type $(m, 2m+1)$. Hence

$$P_i'(x) + f_m(x) = -\frac{g_{2m+1}(x)}{P_i(x)}. \tag{3.4}$$

It follows that

$$\frac{(P_{i_1}' - P_{i_2}')}{(P_{i_1} - P_{i_2})} \cdot P_{i_1}P_{i_2} = g_{2m+1}(x), \tag{3.5}$$

where $1 \leq i_1 < i_2 \leq 3$.

Set the multiplicity of root 0 in the term $g_{2m+1}(x)$ is l. Then by comparing the smallest degree of the non-zero term of the polynomials on each side of equation (3.5), we get that $l = n_{i_1} + n_{i_2} - 1$. Consequently,

$$n_1 = n_2 = n_3 = \frac{l+1}{2}.$$

Finally, for any nontrivial rational solution $y = 1/P_k(x)$, with

$$P_k(x) = x^{n_k}R_k(x) = u_{k,m+1}x^{m+1} + u_{k,m}x^m + \cdots + u_{k,n_k}x^{n_k},$$

$n_k \geq 1$, $u_{k,n_k} \neq 0$ and $k \geq 4$, repeating the above progress, we deduce that $n_k = (l+1)/2 = n_1$. This complete the proof of the first part of the lemma.

Now we start to consider the coefficients of the term x^{n_1} of $P_k(x)$, with $k \geq 1$. In fact, if the coefficients of the term x^{n_1} of $P_k(x)$, with $k \geq 2$ are equal to u_{1,n_1}, then the lemma follows. Hence we only need to consider the case that there exists $k_1 \in [2, +\infty)$ such that $u_{k_1,n_1} \neq u_{1,n_1}$. In this case, from the equation (3.5), we obtain that

$$\frac{P_1' - P_{k_1}'}{P_1 - P_{k_1}} \cdot P_1 P_{k_1} = g_{2m+1}(x). \tag{3.6}$$

By comparing the coefficients of the smallest degree term of the polynomials on each side of equation (3.6), we get that

$$\frac{n_1(u_{1,n_1} - u_{k_1,n_1})}{u_{1,n_1} - u_{k_1,n_1}} \cdot u_{1,n_1} u_{k_1,n_1} = n_1 u_{1,n_1} u_{k_1,n_1} = b_{2n_1-1}.$$

Hence the coefficients of the term x^{n_1} of $P_k(x)$, with $k \geq 2$ and $k \neq k_1$ must be equal to u_{1,n_1}, or u_{k_1,n_1}.

Then we shall prove that all the coefficients of the term x^{n_1} of $P_k(x)$, with $k \geq 2$ and $k \neq k_1$ are the same. By contradiction, we assume that there exist $k_2 \geq 2$, $k_3 \geq 2$ and k_1, k_2 and k_3 are mutually different natural numbers such that $u_{k_2,n_1} \neq u_{k_3,n_1}$. Without loss of generality, we get that $u_{k_2,n_1} = u_{1,n_1}$ and $u_{k_3,n_1} = u_{k_1,n_1}$. By substituting P_{k_1}, P_{k_2}, P_{k_3} and P_1 into the equation (3.5), and comparing the coefficients of the smallest degree term of the polynomials on each side of these equations, we obtain that

$$h_1 u_{1,n_1} u_{k_2,n_1} = h_1 u_{1,n_1}^2 = b_{2n_1-1}, \quad h_2 u_{k_3,n_1} u_{k_1,n_1} = h_2 u_{k_1,n_1}^2 = b_{2n_1-1},$$

where $h_1 > n_1$, $h_2 > n_1$, and h_1 and h_2 are the smallest degree of the non-zero term of $P_1(x) - P_{k_2}(x)$ and $P_{k_1}(x) - P_{k_3}(x)$ respectively.

Hence

$$h_1 h_2 u_{1,n_1}^2 u_{k_1,n_1}^2 = n_1^2 u_{1,n_1}^2 u_{k_1,n_1}^2 = b_{2n_1-1}^2,$$

a contradiction with the fact $h_1 > n_1$ and $h_2 > n_1$. Thus all the coefficients of the term x^{n_1} of $P_k(x)$, with $k \geq 2$ and $k \neq k_1$ have the same value which is equal to u_{1,n_1}, or u_{k_1,n_1}This completes the proof of the lemma. □

Now We shall use the above two lemmas to give the sharp upper bound on the number of nontrivial rational solutions of equations (1.1) of type $(3,7)$.

Proposition 3.1. The equations (1.1) of type $(3,7)$ have at most 5 nontrivial rational solutions.

Proof. If $y = 1/R(x)$, with $R(x) = u_4 x^4 + \cdots + u_1 x + u_0$ is a nontrivial rational solution of the equations (1.1) of type $(3,7)$, then by the proof of Proposition 2.5, we deduce that u_4 can have at most two values, u_4' and u_4''. Moreover, the equation (1.1) of type $(3,7)$ have at least four nontrivial rational solutions only if $a_3 + 7u_4' = 0$ or $a_3 + 7u_4'' = 0$.

We only have to consider the case $a_3 = -7u_4'$. In fact, if $a_3 = -7u_4'$, then $u_4'' = -\frac{3}{28}a_3$ and there are at most 6 nontrivial rational solutions, $y = 1/P_i(x)$, with $1 \leq i \leq 6$, and the leading coefficient of $P_i(x)$, with $1 \leq i \leq 5$ are u_4', while the leading coefficient of $P_6(x)$ is u_4''.

To simplify the calculation, we assume that $a_3 = -7$, $b_0 = 0$ and $\alpha_1 = 0$, then $u_4' = 1$, $u_4'' = \frac{3}{4}$, and $b_7 = 3$. Let $g_7(x) = 3\prod_{i=1}^{7}(x-\alpha_i)$. We shall now give a proof of Proposition 3.1 by contradiction. Suppose otherwise, namely there exist a system (1.1) of type $(3,7)$ having five nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 5$.

Set $P_1(x) = \prod_{i=1}^{4}(x-\alpha_i)$ (it is abbreviated as $P_1(x) = \{1234\}$). Notice that changing $\alpha_k \to \alpha_{\sigma(k)}$ we can change $P_1(x), \cdots, P_j(x)$ into $R_1(x), \cdots, R_j(x)$, with $1 \le k \le 7$ and $j \ge 2$. We say that this two cases are equivalent, where $\sigma \in S_7$, and S_7 is the symmetric group on seven letters. In the following discussion, we only list a representative of each equivalent class. We shall now give all the possible cases of $P_i(x)$, where $1 \le i \le 5$.

We first use the Pigeonhole principle to give the form of $P_i(x)$, where $1 \le i \le 3$. Since there are five nontrivial rational solutions, we have 20 objectives. Moreover, there are seven boxes $\alpha_1, \cdots, \alpha_7$. By putting objectives into boxes, we obtain that there is a box with at least three objectives, namely $P_1(x)$, $P_2(x)$ and $P_3(x)$ must have a comma factor $(x-\alpha_1)$. Then use the Pigeonhole principle again, we deduce without loss of generality that $P_1(x)$ and $P_2(x)$ have a comma factor $(x-\alpha_2)$.

For the sake of simplicity, we set $G_{i_1,i_2}(x)$ is a greatest common divisor of two polynomials $P_{i_1}(x)$ and $P_{i_2}(x)$, with $1 \le i_1 < i_2 \le 4$, and denote by A the collection of $G_{1,2}(x)$, $G_{1,3}(x)$ and $G_{2,3}(x)$, namely, $A = \{G_{1,2}(x), G_{1,3}(x), G_{2,3}(x)\}$. We will distinguish three cases depending on the degree of $G_{1,2}(x)$, $G_{1,3}(x)$ and $G_{2,3}(x)$.

(I) [Case of at least two elements of A of the degree 3]: We can assume without loss of generality that the degree of $G_{1,2}(x)$ and $G_{1,3}(x)$ is 3. Hence we can choose $P_2(x) = \{1235\}$ as a representative. Then we know that $P_3(x) \in B_1$, where $B_1 = \{1236\}, \{1245\}, \{1246\}\}$. By Lemma 3.2, we obtain that the coefficients of the non-zero term of the smallest degree of $P_i(x)$ have at most two values, where $1 \le i \le 3$. Hence $P_3(x) = \{1246\}$.

When $P_3(x) = \{1246\}$, by Lemma 3.2, we deduce that $\alpha_2\alpha_4\alpha_6 = \alpha_2\alpha_3\alpha_5$. Then by submitting $P_1(x)$ and $P_2(x)$, $P_2(x)$ and $P_3(x)$ into the Lemma 3.1 respectively, we obtain the equations,

$$2\alpha_2 + 2\alpha_3 = 3\alpha_6 + 3\alpha_7, \;\; \alpha_2\alpha_3 = 3\alpha_6\alpha_7, \;\; 2\alpha_2 = 3\alpha_7, \;\; \alpha_4\alpha_6 = \alpha_3\alpha_5.$$

Hence $\alpha_2 = \alpha_7 = 0$, $\alpha_3 = \frac{3}{2}\alpha_6$, and $\alpha_4 = \frac{3}{2}\alpha_5$. Thus the nontrivial rational solution $y = 1/P_3(x)$ exists. Now we start to study the existence of the nontrivial rational solution $y = 1/P_4(x)$ in this case. Firstly, by Lemma 3.2, we deduce that the constant term of $P_4(x)$ is not zero.

It follows that $P_4(x) = \{3456\}$. By submitting $P_1(x)$ and $P_4(x)$ into Lemma 3.1, we obtain that $\alpha_3 = \alpha_6 = 0$, a contradiction. Hence the nontrivial rational solution $y = 1/P_4(x)$ doesn't exist in this case. Thus there are at most three nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 3$.

(II) [Case of one element of A of the degree 3]: We can assume without loss of generality that the degree of $G_{1,2}(x)$ is 3. Hence we can choose $P_2(x) = \{1235\}$ as a representative. By submitting $P_1(x)$ and $P_2(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_2 + 2\alpha_3 = 3\alpha_6 + 3\alpha_7, \;\; \alpha_2\alpha_3 = 3\alpha_6\alpha_7. \tag{3.7}$$

We shall now consider the equivalent class of $P_3(x)$. In fact, we deduce that $P_3(x) \in B_2$, where $B_2 = \{\{1267\}, \{1456\}, \{1467\}\}$. Finally, the proof is done with a case by case study on the form of $P_3(x)$.

(**II-i**) if $P_3(x) = \{1267\}$,then by submitting $P_1(x)$ and $P_3(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_2 - 3\alpha_5 = \beta_1, \;\; \beta_1(\alpha_6 + \alpha_7 - \alpha_3 - \alpha_4) = \alpha_6\alpha_7 - \alpha_3\alpha_4, \tag{3.8}$$

where $\beta_1 = 0$ or $\beta_1 = \alpha_2$.

We observe that if $\alpha_2 = 0$, then $\alpha_5 = 0$. By Lemma 3.2, we know that $\alpha_4 = \alpha_5 = 0$, a contradiction. Thus $\alpha_2 \neq 0$. Similarly to the above proof, we get that $\alpha_3 \neq 0$. Then by solving the equations (3.7) and (3.8), we deduce that the nontrivial rational solution $y = 1/P_3(x)$ exists.

Now we study the existence of nontrivial rational solution $y = 1/P_4(x)$. Firstly we give a list of all the possible cases of $P_4(x)$. Actually, if the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ can have at most five representatives, namely $P_4(x) \in D_1$, where $D_1 = \{\{1367\}, \{1467\}, \{1456\}, \{3456\}, \{3467\}\}$. If the degree of $G_{1,4}(x)$ is 1, then $P_4(x) = \{4567\}$.

Secondly, by Lemma 3.2, we obtain that the case $P_4(x) = \{1367\}$ and the case $P_4(x) = \{1467\}$ never happens. For the other cases, by submitting $P_1(x)$ and $P_4(x)$ into Lemma 3.1, we obtain the necessary equations for α_i, where $1 \le i \le 7$. Combing with equations (3.7) and (3.8), we obtain a system of equations for the coefficients α_i, with $1 \le i \le 7$. We reduce the study of the existence of $P_4(x)$ to the study of the existence of the solutions of the above system of equations.

In fact, if $P_4(x) = \{1456\}$, by submitting $P_1(x)$ and $P_4(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_4 - 3\alpha_7 = \beta_2, \;\; \beta_2(\alpha_5 + \alpha_6 - \alpha_2 - \alpha_3) = \alpha_5\alpha_6 - \alpha_2\alpha_3, \tag{3.9}$$

where $\beta_2 = 0$ or $\beta_2 = \alpha_4$. From the equations (3.7), (3.8) and (3.9), we get that

$$\alpha_2 = \frac{9}{2}\alpha_7, \; \alpha_3 = \frac{12}{5}\alpha_7, \; \alpha_4 = \frac{3}{2}\alpha_7, \; \alpha_5 = 3\alpha_7, \; \alpha_6 = \frac{18}{5}\alpha_7.$$

Hence this case happens.

Finally since the proof of the remaining cases are similar, we shall only discuss the case $P_4(x) = \{3456\}$ in detail. If $P_4(x) = \{3456\}$, by submitting $P_1(x)$ and $P_4(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_3 + 2\alpha_4 - 3\alpha_7 = \beta_3, \;\; \beta_3(\alpha_5 + \alpha_6 - \alpha_2) = \alpha_5\alpha_6, \tag{3.10}$$

where $\beta_3 = \alpha_3$, or $\beta_3 = \alpha_4$. From the equations (3.7), (3.8) and (3.10), we get that $\alpha_2 = 0$ or $\alpha_3 = 0$, a contradiction. Thus this case never happens. Similarly to the above proof, we obtain that the remaining cases also never happen.

In conclusion, we know that $P_4(x)$ can only be $\{1456\}$. Hence there are at most four nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

Since the structure and techniques of the discussion of these cases $P_3(x) = \{1456\}$, or $P_3(x) = \{1467\}$ are almost the same as proof of the case $P_3(x) = \{1267\}$, we only give a list of all equivalent classes of $P_4(x)$, and point out the existing cases in the following proof.

(**II-ii**) $P_3(x) = \{1456\}$. If the degree of $G_{1,4}(x)$ is 2, then $P_4(x)$ can have at most five representatives, namely $P_4(x) \in D_2 = \{\{1457\}, \{1467\}, \{2456\}, \{2457\}, \{2467\}\}$. If the degree of $G_{1,4}(x)$ is 1, then $P_4(x) = \{4567\}$. Examining all these cases, we obtain that $P_4(x)$ can only be $\{2467\}$. Hence there are at most four nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

(**II-iii**) $P_3(x) = \{1467\}$. If the degree of $G_{1,4}(x)$ is 2, then $P_4(x) = \{2467\}$. If the degree of $G_{1,4}(x)$ is 1, then $P_4(x)$ can have at most there representatives, namely $P_4(x) \in D_3 = \{\{1567\}, \{2567\}, \{4567\}\}$. Examining all these cases, we obtain that these four cases never happen. Hence there are at most three nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 3$.

(III) [Case of none of A of the degree 3]: Firstly, the degree of $G_{1,2}(x)$ can only be 2, i.e, $P_2(x) = \{1256\}$. By submitting $P_1(x)$ and $P_2(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_2 - 3\alpha_7 = \beta_5, \;\; \beta_5(\alpha_5 + \alpha_6 - \alpha_3 - \alpha_4) = \alpha_5\alpha_6 - \alpha_3\alpha_4, \tag{3.11}$$

where $\beta_5 = 0$, or $\beta_5 = \alpha_2$. Then we deduce that $P_3(x)$ can have only one representative, namely $P_3(x) = \{1357\}$.

When $P_3(x) = \{1357\}$. By submitting $P_1(x)$ and $P_3(x)$ into Lemma 3.1, we obtain the equations

$$2\alpha_3 - 3\alpha_6 = \beta_6, \;\; \beta_6(\alpha_5 + \alpha_7 - \alpha_2 - \alpha_4) = \alpha_5\alpha_7 - \alpha_2\alpha_4, \tag{3.12}$$

where $\beta_6 = 0$, or $\beta_6 = \alpha_3$. Similarly to the proof of case (II-i), we obtain that $\alpha_2 \neq 0$. Solving the equations (3.11) and (3.12), we know that the nontrivial rational solution $y = 1/P_3(x)$ exists. Then the proof is done with a case by case study on the existence of the nontrivial rational solution $y = 1/P_4(x)$. Firstly, we know that the degree of $G_{1,4}(x)$ can only be 2. Hence $P_4(x)$ can have at most three representatives, namely $P_4(x) \in D_4 = \{\{1467\}, \{2367\}, \{2467\}\}$.

Similarly to the proof of case (II-i), we obtain that the case $P_4(x) = \{1467\}$ never happens. For cases $P_4(x) = \{2367\}$ and $P_4(x) = \{2467\}$, by submitting $P_1(x)$ and $P_4(x)$ into Lemma 3.1, we obtain the equations,

$$2\alpha_2 + 2\alpha_3 - 3\alpha_5 = \beta_7, \;\; \beta_7(\alpha_6 + \alpha_7 - \alpha_4) = \alpha_6\alpha_7, \tag{3.13}$$

and

$$2\alpha_2 + 2\alpha_4 - 3\alpha_5 = \beta_8, \;\; \beta_8(\alpha_6 + \alpha_7 - \alpha_3) = \alpha_6\alpha_7, \tag{3.14}$$

where $\beta_7 = \alpha_2$, or $\beta_7 = \alpha_3$, and $\beta_8 = \alpha_2$, or $\beta_8 = \alpha_3$. Solving the system of the equations (3.11), (3.12) and (3.13), we obtain three solutions. Hence the nontrivial rational solution $y = 1/P_4(x)$ with $P_4(x) = \{2367\}$ exists. On the other hand, solving the system of equations (3.11), (3.12) and (3.14), we obtain another three solutions. Hence the nontrivial rational solution $y = 1/P_4(x)$ with $P_4(x) = \{2467\}$ exists. But these two curves cannot exist at the same time, because $\forall \sigma \in S_7$, changing $\alpha_k \to \alpha_{\sigma(k)}$ with $1 \le k \le 7$ we cannot change the solutions of the system of equations (3.11), (3.12) and (3.13) into the solution of (3.14). Hence there are at most four nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$.

Therefore we have shown that in all these cases the systems (1.1) of type $(3, 7)$ have at most 4 nontrivial rational solutions $y = 1/P_i(x)$, with the leading coefficient of $P_i(x)$ is 1, and $1 \le i \le 4$, a contradiction. Furthermore, by the proof of Proposition 2.5, we know that there are at most one nontrivial rational solution $y = 1/Q(x)$, with the leading coefficient of $Q(x)$ is $3/4$. Hence the the Proposition follows. $\square$

The next result gives the equations (1.1) of type $(3, 7)$ with exactly 5 nontrivial rational solutions. Since an easy computation can proof Proposition 3.2, we omit the proof here.

Proposition 3.2. We consider the equation (1.1) of type $(3, 7)$

$$\frac{dy}{dx} = -(7x^3 + \frac{7}{3}x^2 - \frac{7}{3}x - \frac{1}{3})y^2 + (3x^7 + \frac{7}{3}x^6 - \frac{28}{9}x^5 - \frac{70}{27}x^4 + \frac{7}{81}x^3 + \frac{7}{27}x^2 + \frac{2}{81}x)y^3,$$

this equation has the following 5 nontrivial rational solutions,

$$y = \frac{1}{x^4 + \frac{1}{3}x^3 - \frac{11}{9}x^2 - \frac{1}{3}x + \frac{2}{9}},\ y = \frac{1}{x^4 + \frac{1}{9}x^3 - x^2 - \frac{1}{9}x},$$
$$y = \frac{1}{x^4 + x^3 - \frac{1}{9}x^2 - \frac{1}{9}x},\ y = \frac{1}{x^4 - \frac{7}{9}x^2 - \frac{2}{9}x},\ y = \frac{1}{\frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{13}{18}x^2 - \frac{1}{3}x - \frac{1}{36}}.$$

Proof of Theorem 1.3. The proof of the theorem follows from collecting the above two propositions.

4. Proof of Theorem 1.4 and 1.5

The strategy of the proof of Theorem 1.4 and 1.5 are rather straightforward. Namely, we construct equations (1.1) of type $(m, 2m + 1)$ having exactly i nontrivial rational solutions, and the equation (1.1) of type $(4, 9)$ with exactly 5 nontrivial rational solutions, where $m \ge 1$ and $1 \le i \le 4$.

Proposition 4.1. For $m \ge 1$, consider the eqaution (1.1) of type $(m, 2m + 1)$

$$\frac{dy}{dx} = -(2m + 2)x^m y^2 + (m + 1)x^{2m+1}y^3,$$

and the equation

$$\frac{dy}{dx} = 2x^m y^2 + \frac{3}{4(m + 1)}x^{2m+1}y^3.$$

Then these two equations have exactly 1 and 2 nontrivial rational solutions, respectively.

Proof. For the first equation, if $y = 1/P(x)$ is a nontrivial rational solution of this equation, then we have

$$P(x)(P'(x) + f_m(x)) = -g_{2m+1}(x).$$

Since $g_{2m+1}(x) = (m + 1)x^{2m+1}$, and a root of $P(x)$ must be a root of $g_{2m+1}(x)$, we get that

$$P(x) = a_1 x^{m+1}.$$

Substituting $P(x)$ into the above equation, we have $a_1 = 1$. Hence this equation have exactly one nontrivial rational solution.

Similar to the above discussion, we can deduce that the nontrivial rational solution $y = 1/Q(x)$ of the second equation must be of the form $Q(x) = b_1 x^{m+1}$.

Substituting $Q(x)$ into the above equation, we get that

$$b_1 = -\frac{1}{2(m+1)}, \text{ or } b_1 = -\frac{3}{2(m+1)}.$$

Hence this equation have exactly two nontrivial rational solutions. □

Proposition 4.2. If $m = 1$, then the following equation (1.1) of type $(1,3)$,

$$\frac{dy}{dx} = -(3x-7)y^2 + (x^3 - 7x^2 + 6x)y^3,$$

have exactly 3 nontrivial rational solutions. For $m \geq 2$, consider the equation (1.1) of type $(m, 2m+1)$,

$$\begin{aligned}\frac{dy}{dx} = &-\Big((2m+1)x^m - \frac{4m(2m-1)}{m-1}x^{m-1} + \frac{3m(2m-1)}{m-1}x^{m-2}\Big)y^2 \\ &+ mx^{2m-3}\Big(x - \frac{m}{m-1}\Big)\Big(x - \frac{3m}{m-1}\Big)(x-3)(x-1)y^3.\end{aligned}$$

Then these equations have exactly 3 nontrivial rational solutions,

$$y = \frac{1}{P_1(x)}, \quad y = \frac{1}{P_2(x)}, \quad y = \frac{1}{P_3(x)},$$

with

$$\begin{aligned}P_1(x) &= x^{m-1}\Big(x - \frac{m}{m-1}\Big)(x-3), \\ P_2(x) &= x^{m-1}\Big(x - \frac{3m}{m-1}\Big)(x-1), \\ P_3(x) &= x^{m-1}\Big(x - \frac{m}{m-1}\Big)\Big(x - \frac{3m}{m-1}\Big).\end{aligned}$$

Proof. Firstly, for $1 \leq m \leq 3$, an easy computation can proof the Proposition, so we omit the proof here. When $m \geq 4$, with a tedious computation we know that $y = 1/P_1(x)$, $y = 1/P_2(x)$, and $y = 1/P_3(x)$ are three nontrivial rational solutions of the Abel equation. Now we shall prove that there are no other nontrivial rational solutions of the Abel equation.

By contradiction, we assume $y = 1/P_4(x)$ is another nontrivial rational solution, then the leading coefficient of $P_4(x)$ can only be 1 and $m/(m+1)$. If the leading coefficient of $P_4(x)$ is 1, by Lemma 3.2, we obtain that $P_4(x)$ can have at most there representatives, namely$P_4(x) \in E_1$, where

$$E_1 = \Big\{x^{m-1}\Big(x - \frac{m}{m-1}\Big)(x-1),\ x^{m-1}\Big(x - \frac{3m}{m-1}\Big)(x-3),\ x^{m-1}(x-1)(x-3)\Big\}.$$

With a tedious computation we get that

$$P_4(x)(P_4'(x) + f_m(x)) \neq -g_{2m+1}(x)$$

when $P_4(x) \in E_1$. Thus this case never happens.

Now we obtain that the leading coefficient of $P_4(x)$ can only be $m/(m+1)$. Set

$$P_4(x) = \frac{m}{m+1}x^{m+1} + u_m x^m + \cdots + u_0.$$

It follows from the equation

$$P_4(x)(P_4'(x) + f_m(x)) = -g_{2m+1}(x)$$

that $u_m = -\frac{4m(2m-1)}{(2m+1)(m-1)}$. Moreover, since a root of $P_4(x)$ must be a root of $g_{2m+1}(x)$, and the multiplicity of root 0 in the polynomial $P_4(x)$ is $m-1$, we know that $u_m \in E_2$, where

$$\begin{aligned}E_2 =&\{-\frac{4m^2}{(m-1)(m+1)}, -\frac{(2m-1)m}{(m-1)(m+1)}, -\frac{(4m-3)m}{(m-1)(m+1)}, -\frac{(4m-1)m}{(m-1)(m+1)},\\ &-\frac{(6m-3)m}{(m-1)(m+1)}, -\frac{4m}{m+1}\}.\end{aligned}$$

Since for $1 \le m \in N$, $-\frac{4m(2m-1)}{(2m+1)(m-1)} \notin E_2$, we deduce that the nontrivial rational solution $y = \frac{1}{P_4(x)}$ with the leading coefficient of $P_4(x)$ is $m/(m+1)$ doesn't exist. Hence the Proposition follows. $\square$

Proposition 4.3. If $m = 2$, then the following equation (1.1) of type $(2,5)$,

$$\frac{dy}{dx} = -(5x^2 - \frac{50}{3}x + \frac{32}{3})y^2 + (2x^5 - \frac{50}{3}x^4 + \frac{140}{3}x^3 - \frac{160}{3}x^2 + \frac{64}{3}x)y^3,$$

have exactly 4 nontrivial rational solutions. For $m \ge 3$, consider the system (1.1) of type $(m, 2m+1)$,

$$\begin{aligned}\frac{dy}{dx} =& -((2m+1)x^m - \frac{6m^2-5m+1}{m}x^{m-1} + \frac{24m^4-76m^3+82m^2-37m+6}{4m^2(m-1)}x^{m-2}\\ &+\frac{-8m^4+36m^3-54m^2+31m-6}{4m^2(m-1)}x^{m-3})y^2 + mx^{2m-5}(x-\frac{m-1}{m})(x-\frac{2m-1}{2m})\\ &\times(x-1)(x-\frac{2m-1}{2(m-1)})(x-\frac{(2m-1)(m-2)}{2m^2})(x-\frac{(2m-1)(m-2)}{2m(m-1)})y^3.\end{aligned}$$

Then these equations have exactly 4 nontrivial rational solutions,

$$y = \frac{1}{P_1(x)}, \;\; y = \frac{1}{P_2(x)}, \;\; y = \frac{1}{P_3(x)}, \;\; y = \frac{1}{P_4(x)}$$

with

$$\begin{aligned}P_1(x) &= x^{m-2}(x - \frac{2m-1}{2(m-1)})(x - \frac{m-1}{m})(x - \frac{2m-1}{2m}),\\ P_2(x) &= x^{m-2}(x - \frac{2m-1}{2m})(x-1)(x - \frac{(2m-1)(m-2)}{2m(m-1)}),\\ P_3(x) &= x^{m-2}(x - \frac{2m-1}{2(m-1)})(x-1)(x - \frac{(2m-1)(m-2)}{2m^2}),\\ P_4(x) &= x^{m-2}(x - \frac{2m-1}{2(m-1)})(x - \frac{m-1}{m})(x - \frac{(2m-1)(m-2)}{2m(m-1)}).\end{aligned}$$

Proof. Firstly, for $2 \leq m \leq 5$, a tedious computation can proof the Proposition, so we omit the proof here. When $m \geq 6$, with an easy computation we know that $y = 1/P_1(x)$, $y = 1/P_2(x)$, $y = 1/P_3(x)$ and $y = 1/P_4(x)$ are four nontrivial rational solutions of the Abel equation. Now we shall prove that there are no other nontrivial rational solutions of the Abel equation.

By contradiction, we assume $y = 1/P_5(x)$ is another nontrivial rational solution, then the leading coefficient of $P_5(x)$ can only be 1 and $m/(m+1)$. If the leading coefficient of $P_5(x)$ is 1, by Lemma 3.2,we get that the multiplicity of root 0 in the polynomial $P_5(x)$ must be $m-2$.

For convenience of stating our proof, we set

$$\begin{aligned} &Y_1 = x - \frac{2m-1}{2(m-1)}, \quad Y_2 = x - \frac{m-1}{m}, \quad Y_3 = x - \frac{2m-1}{2m}, \\ &Y_4 = x-1, \quad Y_5 = x - \frac{(2m-1)(m-2)}{2m^2}, \quad Y_6 = x - \frac{(2m-1)(m-2)}{2m(m-1)}. \end{aligned}$$

Since the multiplicity of root 0 in the polynomial $P_5(x)$ must be $m-2$, we get that $P_4(x)$ can have at most sixteen representatives. Namely, $P_5(x) \in F_1$, where

$$\begin{aligned} F_1 = \{ & x^{m-2}Y_1Y_2Y_4, \ x^{m-2}Y_1Y_2Y_5, \ x^{m-2}Y_1Y_3Y_4, \ x^{m-2}Y_1Y_3Y_5, \ x^{m-2}Y_1Y_3Y_6, \\ & x^{m-2}Y_1Y_4Y_6, \ x^{m-2}Y_1Y_5Y_6, \ x^{m-2}Y_2Y_3Y_4, \ x^{m-2}Y_2Y_3Y_5, \ x^{m-2}Y_2Y_3Y_6, \\ & x^{m-2}Y_2Y_4Y_5, \ x^{m-2}Y_2Y_4Y_6, \ x^{m-2}Y_2Y_5Y_6, \ x^{m-2}Y_3Y_4Y_5, \ x^{m-2}Y_3Y_5Y_6, \\ & x^{m-2}Y_4Y_5Y_6 \}. \end{aligned}$$

Since $y = 1/P_i(x)$ and $y = 1/P_5(x)$ are nontrivial rational solutions of this equation, we obtain that

$$\frac{(P_i' - P_5')}{(P_i - P_5)} \cdot P_i P_5 = g_{2m+1}(x), \tag{4.1}$$

where $1 \leq i \leq 4$. If the degree of the common factor of the polynomials P_i and P_5 is m, i.e,

$$P_i(x) = x^{m-2}Y_{i_1}Y_{i_2}Y_k, \ \ P_5(x) = x^{m-2}Y_{i_1}Y_{i_2}Y_j, \tag{4.2}$$

where $1 \leq i_1 < i_2 \leq 6$, and $Y_k \neq Y_j$. By substituting the equations (4.2) into (4.1), we deduce that

$$(m-2)Y_{i_1}Y_{i_2} + xY_{i_1} + xY_{i_2} = m\frac{\prod_{i=1}^{6} Y_i}{Y_{i_1}Y_{i_2}Y_jY_k}. \tag{4.3}$$

Basing on the equation (4.3), with a simple computation, we obtain that all these sixteen cases never happen.

Now we obtain that the leading coefficient of $P_5(x)$ can only be $m/(m+1)$. Set

$$P_5(x) = \frac{m}{m+1}x^{m+1} + u_m x^m + \cdots + u_0.$$

It follows from the equation $P_5(x)(P_5'(x) + f_m(x)) = -g_{2m+1}(x)$ that

$$u_m = -\frac{(3m-1)(2m-1)}{(2m+1)m}.$$

Moreover, since a root of $P_5(x)$ must be a root of $g_{2m+1}(x)$, and the multiplicity of root 0 in the polynomial $P_5(x)$ is $m-2$, we know that $u_m \in F_2$, where

$$F_2 = \{\sum_{1\le j_1<j_2<j_3\le 6} -\frac{m}{m+1}(u_{j_1}+u_{j_2}+u_{j_3})\},$$

where

$$u_1 = \frac{2m-1}{2(m-1)},\quad u_2 = \frac{m-1}{m},\quad u_3 = \frac{2m-1}{2m},\quad u_4 = 1,$$
$$u_5 = \frac{(2m-1)(m-2)}{2m^2},\quad u_6 = \frac{(2m-1)(m-2)}{2m(m-1)}.$$

Since for $1 \le m \in N$, $-\frac{(3m-1)(2m-1)}{(2m+1)m} \notin F_2$, we deduce that the nontrivial rational solution $y = 1/P_5(x)$ with the leading coefficient of $P_5(x)$ is $m/(m+1)$ doesn't exist. Hence the Proposition follows. $\square$

Proof of Theorem 1.4. The proof of the theorem follows from collecting the above three propositions.

Proof of Theorem 1.5. Consider the equation (1.1) of type $(4,9)$

$$\frac{dy}{dx} = -(9x^4 - \frac{252}{5}x^3 + \frac{576}{5}x^2 - \frac{576}{5}x + \frac{192}{5})y^2 + (4x^9 - \frac{252}{5}x^8 + \frac{7368}{25}x^7$$
$$-\frac{25536}{25}x^6 + \frac{56448}{25}x^5 - \frac{16128}{5}x^4 + \frac{72192}{25}x^3 - \frac{36864}{25}x^2 + \frac{8192}{25}x)y^3,$$

this equation has the following 5 nontrivial rational solutions,

$$y = \frac{1}{\frac{4}{5}x^5 - \frac{28}{5}x^4 + \frac{88}{5}x^3 - \frac{128}{5}x^2 + \frac{64}{5}x},$$
$$y = \frac{1}{x^5 - \frac{36}{5}x^4 + 24x^3 - \frac{192}{5}x^2 + \frac{128}{5}x},$$
$$y = \frac{1}{x^5 - \frac{37}{5}x^4 + \frac{128}{5}x^3 - \frac{224}{5}x^2 + \frac{192}{5}x - \frac{64}{5}},$$
$$y = \frac{1}{x^5 + (-\frac{34}{5} + \frac{4}{5}I)\,x^4 + (\frac{96}{5} - \frac{16}{5}I)\,x^3 + (-\frac{128}{5} + \frac{16}{5}I)\,x^2 + \frac{64}{5}x},$$
$$y = \frac{1}{x^5 + (-\frac{34}{5} - \frac{4}{5}I)\,x^4 + (\frac{96}{5} + \frac{16}{5}I)\,x^3 + (-\frac{128}{5} - \frac{16}{5}I)\,x^2 + \frac{64}{5}x}.$$

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