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PEAKON AND PSEUDO-PEAKON IN A GENERALIZED CAMASSA-HOLM TYPE EQUATION*

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Abstract This paper studies traveling wave solutions of a nonlinear generalization of the Camassa-Holm equation introduced by Anco et al. in 2015 and 2019. Under given parameter conditions, the corresponding traveling system is a singular system of the first class defined in [8]. The bifurcations of traveling wave solutions in the parameter space are investigated from the perspective of dynamical systems. The existence of solitary wave solution, periodic peakon solutions and peakon, pseudo-peakon are proved. Possible exact explicit parametric representations of various solutions are given.

Keywords Solitary wave, peakon, pseudo-peakon, periodic peakon, bifurcation, Camassa-Holm type equation.

MSC(2010) 34C37, 34C23, 74J30, 58Z05.

1. Introduction

In 2015 and 2019, as a nonlinear generalization of the Camassa-Holm equation with peakon solutions, Anco et al., [1–3] suggested to study the following equation ($gCH-$equation):

$$u_t - u_{xxt} = \frac{1}{2}(p+1)(p+2)u^p u_x - \frac{1}{2}p(p-1)u^{p-2}u_x^3 - 2pu^{p-1}u_x u_{xx} - u^p u_{xxx}, \quad p \neq 0. \tag{1.1}$$

This equation reduces to the $CH-$equation when $p = 1$ and shares one of the Hamiltonian structures of $CH-$equation. For all $p > 0$, the authors of [1] proved that system (1.1) admits a peakon solution. They stated that it is worth to further study $gCH-$equation and understand how its nonlinearity affects properties of its solutions compared to the $CH-$ equation. We notice that these authors did not study the bifurcations and possible exact solutions for the corresponding traveling wave systems of equation (1.1). In this paper, we consider these problems for the solutions of the corresponding traveling wave systems of equation (1.1) depending on the parameters.

To study the traveling wave solutions of equation (1.1), we set $u(x,t) = \phi(x+ct) \equiv \phi(\xi)$, where $\xi = x + ct$ and c is the wave speed. We always assume that $p > 0, c > 0$ in this paper. Substituting $u(x,t) = \phi(x+ct) \equiv \phi(\xi)$ into (1.1) and

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integrating the obtained equation once, we obtain

$$(\phi^p - c)\phi'' = -\frac{1}{2}p\phi^{p-1}(\phi')^2 + \left(\frac{1}{2}p + 1\right)\phi^{p+1} - c\phi + g, \tag{1.2}$$

where g is an integral constant, and the prime stands for the derivative with respect to ξ. Equation (1.2) is equivalent to the following planar dynamical system with three-parameter group (p, c, g):

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2}p\phi^{p-1}y^2 + \left(\frac{1}{2}p + 1\right)\phi^{p+1} - c\phi + g}{\phi^p - c}. \tag{1.3}$$

System (1.3) has a first integral as follows:

$$H(\phi, y) = y^2(\phi^p - c) - (\phi^{p+2} - c\phi^2 + 2g\phi) = h. \tag{1.4}$$

Clearly, on the straight lines $\phi = c^{\frac{1}{p}}$, system (1.3) is discontinuous. Such a system is called a singular traveling wave system of the first class defined by Li and Chen in [8] and Li in [7].

It is interesting to find that the singular traveling system has peakon, pseudo-peakon, periodic peakon and compacton solution family. Periodic peakon is a classical solution with two time-scales of a singular traveling system. Peakon is a limit solution of a family of periodic peakons or a limit solution of a family of pseudo-peakons under two classes of limit senses (see [9–11]). Compacton family is a solution family of a singular system, for which all solutions $\phi(\xi)$ have finite sets of support, i.e., the defined region of every $\phi(\xi)$ with respect to ξ is finite and the value region of ϕ is bounded. Corresponding to different types of phase orbits, in Li and Chen [8] and Li [7], a classification for different wave profiles of $\phi(\xi)$ was given.

In this paper, the above-mentioned theory of singular traveling wave systems is used to analyze the wave profiles of the wave function $\phi(\xi)$ in the solutions of systems (1.3). By considering the dynamics of the traveling wave solutions determined by the travelling wave system (1.3), all possible exact explicit parametric representations for the traveling wave solutions of equation (1.1) will be given under different parameter conditions.

Our main result is given below.

Theorem 1.1. *(i) For a given fixed parameter pair* (c, p)*, when* g *is varied, system* (1.3) *has the bifurcations of phase portraits shown in Fig.1 and Fig.2.*

(ii) For any $p > 0$*, equation* (1.1) *has a peakon solution of Camassa-Holm type given by* (3.3)*. While when* p *is an even number, equation* (1.1) *has a peakon solution and an anti-peakon solution of Camassa-Holm type given by* (3.3) *and* (3.4)*.*

(iii) When $p = 2, 0 < g < \frac{2c}{3}\sqrt{\frac{c}{3}}$*, equation* (1.1) *has an exact solitary solution given by* (3.6) *and an exact periodic peakon solution given by* (3.8)*.*

The proof of this theorem is given in next sections.

This paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of system (1.3) depending on the changes of parameter g when c and p are fixed. In section 3, we investigate exact solitary wave solution, peakon, periodic peakon, and give possible exact explicit parametric representations for these solutions.

2. Bifurcations of phase portraits of system (1.3)

We first consider all possible phase portraits of system (1.3). It is known that system (1.3) has the same invariant curve solutions as the associated regular system:

$$\frac{d\phi}{d\zeta} = y(\phi^p - c), \quad \frac{dy}{d\zeta} = -\frac{1}{2}p\phi^{p-1}y^2 + \left(\frac{1}{2}p + 1\right)\phi^{p+1} - c\phi + g, \tag{2.1}$$

where $d\xi = (\phi^p - c)d\zeta$, for $\phi^p - c \neq 0$.

To find the equilibrium points of system (2.1), we write that $f(\phi) = (p+2)\phi^{p+1} - 2c\phi + 2g, f'(\phi) = (p+1)(p+2)\phi^p - 2c$. Obviously, when $\phi = \tilde{\phi}_0 = \left(\frac{2c}{(p+1)(p+2)}\right)^{\frac{1}{p}}, f'(\tilde{\phi}_0) = 0$. And $f(\tilde{\phi}_0) = 2g - \frac{2cp\tilde{\phi}_0}{p+1}$. When p is not an even number, since $f(\mp\infty) = \infty$, function $f(\phi)$ has two zeros ϕ_1 and ϕ_2 for $g < \frac{cp\tilde{\phi}_0}{p+1}$. Correspondingly, system (2.1) has two equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_2, 0)$ on the $\phi-$axis. While when p is an even number, because $f'(\pm\tilde{\phi}_0) = 0, f(-\infty) = -\infty, f(\infty) = \infty$, system (2.1) has three equilibrium points $E_j(\phi_j, 0), j = 1, 2, 3$ on the $\phi-$axis for $g < |\frac{cp\tilde{\phi}_0}{p+1}|$.

When p is not an even number and $g > g_1 = -\frac{1}{2}pc^{1+\frac{1}{p}}$, on the straight line $\phi = \phi_s = c^{\frac{1}{p}}$, system (2.1) has two equilibrium points $S^+_\mp(\phi_s, \mp y_s)$, where $y_s = \sqrt{Y_s}, Y_s = \frac{(p+2)\phi_s^{p+1} - 2c\phi_s + 2g}{p\phi_s^{p-1}}$. In addition, when p is an even number and $g < -g_1 = \frac{1}{2}pc^{1+\frac{1}{p}}$, on the straight line $\phi = -\phi_s$, system (2.1) has two equilibrium points $S^-_\mp(-\phi_s, \mp y_{s1})$, where $y_{s1} = \sqrt{Y_{s1}}, Y_{s1} = \frac{(p+2)(-\phi_s)^{p+1} - 2c(-\phi_s) + 2g}{p(-\phi_s)^{p-1}}$.

Let $M(\phi_j, 0)$ be the coefficient matrix of the linearized system of (2.1) at the equilibrium point $E_j(\phi_j, 0)$. We have

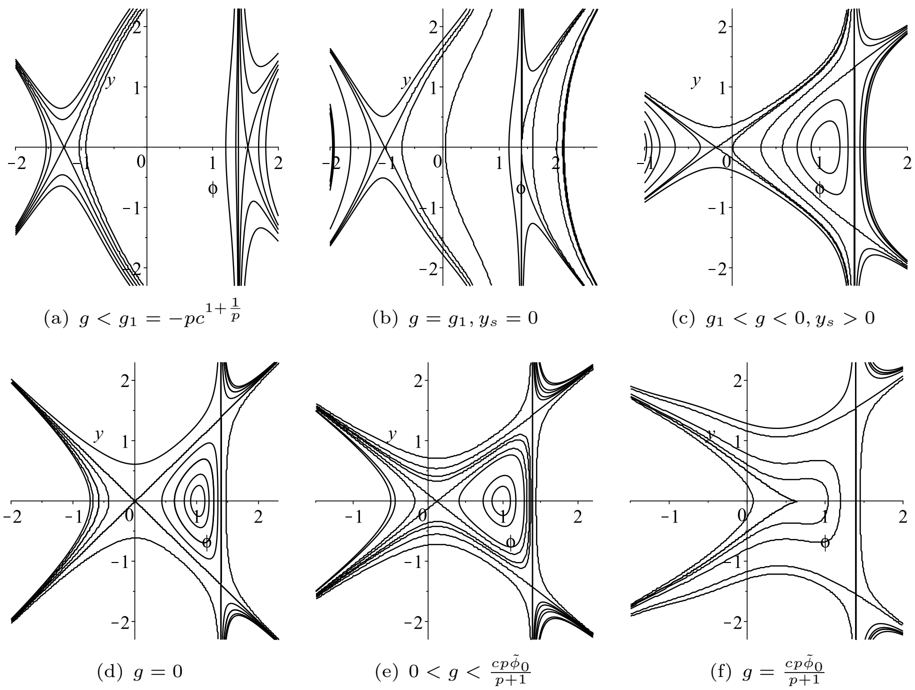
$$\begin{aligned} J(\phi_j, 0) &= \det M(\phi_j, 0) = -\tfrac{1}{2}(\phi_j^p - c)f'(\phi_j), \\ J(\phi_s, y_s) &= \det M(\phi_s, y_s) = -p^2 y_s^2 \phi_s^{2p-2} < 0, \\ J(-\phi_s, y_{s1}) &= \det M(-\phi_s, y_{s1}) = -p^2 y_{s1}^2 \phi_s^{2p-2} < 0. \end{aligned} \tag{2.2}$$

By the theory of planar dynamical systems (see [7]), for an equilibrium point of a planar integrable system, if $J < 0$, the equilibrium point is a saddle point; if $J > 0$ and $(\text{Trace}M)^2 - 4J < 0\ (> 0)$, the equilibrium point is a center point (a node point); if $J = 0$ and the Poincaré index of the equilibrium point is 0, this equilibrium point is a cusp.

We write that $h_j = H(\phi_j, 0), h_s = H(\phi_s, y_s), h_{s1} = H(\phi_{s1}, y_{s1})$, where H is given by (1.4).

By the above discussion, for a fixed parameter pair (c, p), when p is not even number and $p > 0$, we have the bifurcations of phase portraits of system (1.3) shown in Fig.1 by varying the value of g.

When p is an even number, for a fixed parameter group (c, p), we obtain the bifurcations of phase portraits of system (3.1) shown in Fig.2 by changing the value of g.



(a) $g < g_1 = -pc^{1+\frac{1}{p}}$ (b) $g = g_1, y_s = 0$ (c) $g_1 < g < 0, y_s > 0$

(d) $g = 0$ (e) $0 < g < \frac{cp\tilde{\phi}_0}{p+1}$ (f) $g = \frac{cp\tilde{\phi}_0}{p+1}$

Figure 1. The bifurcations of phase portraits of system (1.3) when p is not an even number

3. Exact peakon solutions and solitary wave solutions determined by the orbits of system (1.3)

We see from (1.4) that $y^2 = \frac{h+2g\phi-c\phi^2+\phi^{p+2}}{(\phi^p-c)}$. By using the first equation of (1.3), we obtan

$$\xi = \int_{\phi_0}^{\phi} \frac{(c-\phi^p)d\phi}{\sqrt{-(c-\phi^p)(h+2g\phi-c\phi^2+\phi^{p+2})}}. \tag{3.1}$$

(i) Exact explicit peakon solution

Supposing $g = 0$, we consider the heteroclinic triangles in Fig.1 (d) and Fig.2 (f) defined by the level curve of $H(\phi, y) = 0$. Then (3.1) becomes

$$\xi = \int_{\phi}^{\phi_s} \frac{(c-\phi^p)d\phi}{\phi\sqrt{(c-\phi^p)^2}} = \int_{\phi}^{\phi_s} \frac{d\phi}{\phi}. \tag{3.2}$$

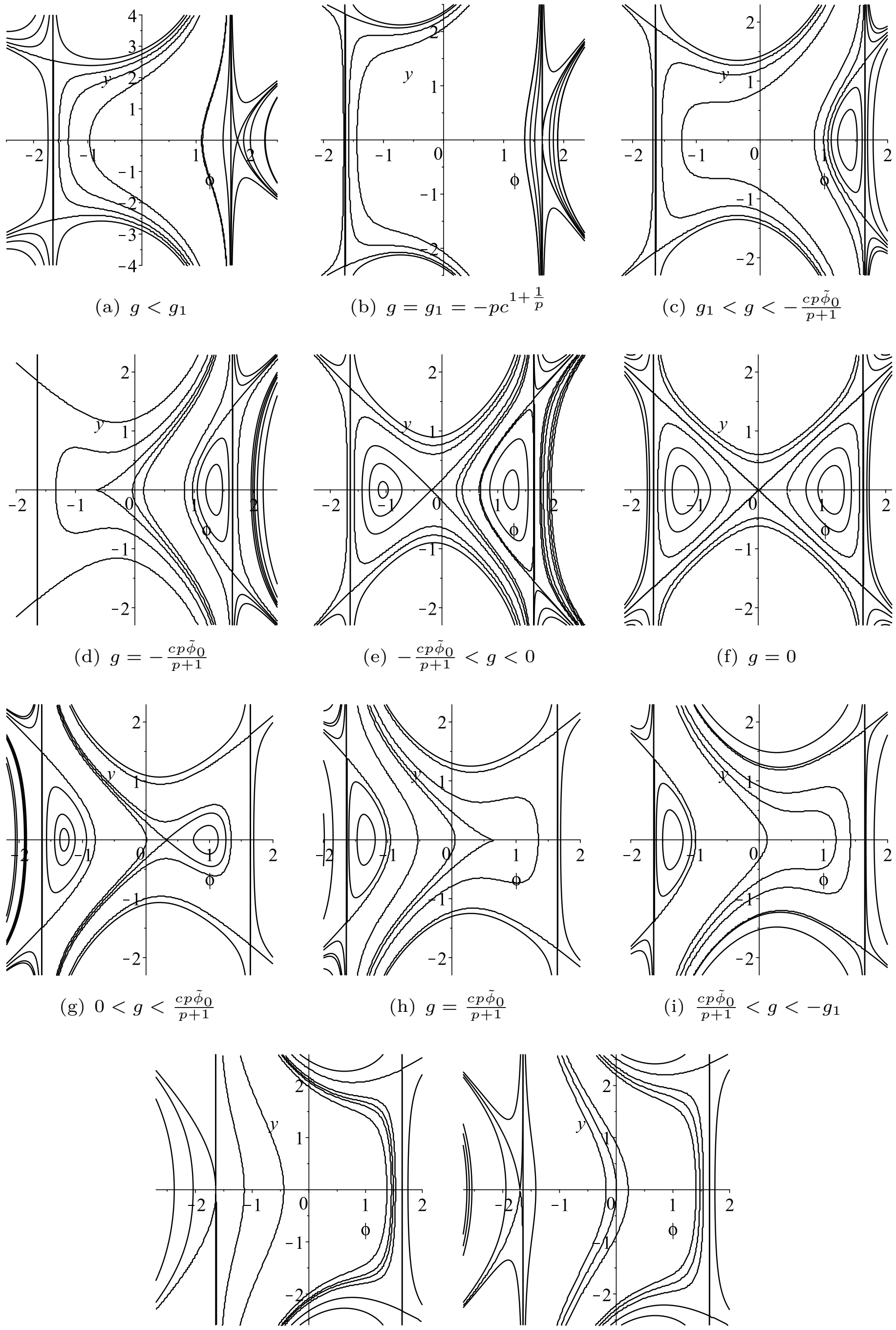
From (3.2), the peakon solution of Camassa-Holm type equation (1.3) follows (see Camassa, et al., [5,6]):

$$\phi(\xi) = \phi_s e^{-|\xi|}. \tag{3.3}$$

Fig.3 (a) shows the profile of this peakon solution.

When p is an even number, there are two heteroclinic triangles in Fig.2 (f). Besides the peakon solution (3.3), we also have an anti-peakon solution:

$$\phi(\xi) = -\phi_s e^{-|\xi|}. \tag{3.4}$$



(a) $g < g_1$ (b) $g = g_1 = -pc^{1+\frac{1}{p}}$ (c) $g_1 < g < -\frac{cp\tilde{\phi}_0}{p+1}$

(d) $g = -\frac{cp\tilde{\phi}_0}{p+1}$ (e) $-\frac{cp\tilde{\phi}_0}{p+1} < g < 0$ (f) $g = 0$

(g) $0 < g < \frac{cp\tilde{\phi}_0}{p+1}$ (h) $g = \frac{cp\tilde{\phi}_0}{p+1}$ (i) $\frac{cp\tilde{\phi}_0}{p+1} < g < -g_1$

(j) $g = -g_1 = pc^{1+\frac{1}{p}}$ (k) $g > -g_1$

Figure 2. The bifurcations of phase portraits of system (1.3) when p is an even number

(ii) Exact explicit pseudo-peakon solutions or solitary wave solutions

for $p = 2$

We consider the homoclinic orbit in Fig.2 (g) to the equilibrium point $E_2(\phi_2, 0)$, which is defined by $H(\phi, y) = h_2$ when $p = 2$. Now, (3.1) can be written as

$$\begin{aligned}\xi &= \int_{\phi}^{\phi_M} \frac{(c-\phi^2)d\phi}{\sqrt{(c-\phi^2)(h_2+2g\phi-c\phi^2+\phi^4)}} \\ &= \int_{\phi}^{\phi_M} \frac{(c-\phi^2)d\phi}{(\phi-\phi_2)\sqrt{(\sqrt{c}-\phi)(\phi_M-\phi)(\phi-\phi_l)(\phi+\sqrt{c})}} \\ &= -\int_{\phi}^{\phi_M} \frac{(\phi_2+\phi)d\phi}{\sqrt{(\sqrt{c}-\phi)(\phi_M-\phi)(\phi-\phi_l)(\phi+\sqrt{c})}} \\ &\quad + (c-\phi_2^2)\int_{\phi}^{\phi_M} \frac{d\phi}{(\phi-\phi_2)\sqrt{(\sqrt{c}-\phi)(\phi_M-\phi)(\phi-\phi_l)(\phi+\sqrt{c})}}.\end{aligned} \tag{3.5}$$

Thus, (3.5) gives rise to the following exact solitary wave and pseudo-peakon solution (when $|h - h_s| \ll 1$) of equation (1.3) (see Fig.3 (b)):

$$\begin{aligned}\phi(\chi) &= \sqrt{c} - \frac{\sqrt{c}-\phi_M}{1-\hat{\alpha}_1^2\mathrm{sn}^2(\chi,k)},\ \chi \in (-\chi_{01}, \chi_{01}), \\ \xi(\chi) &= \frac{2}{\sqrt{(\sqrt{c}-\phi_l)(\sqrt{c}+\phi_M)}}\left[\left(\frac{(c-\phi_2^2)(\phi_M-\phi_l)}{(\phi_M-\phi_2)(\sqrt{c}-\phi_l)} - \phi_2 - \sqrt{c}\right)\chi\right. \\ &\quad + (\sqrt{c}-\phi_M)\Pi(\arcsin(\mathrm{sn}(\chi,k)), \hat{\alpha}_1^2, k) \\ &\quad \left. + \frac{(\sqrt{c}+\phi_2)(\sqrt{c}-\phi_M)}{(\phi_M-\phi_2)}\Pi(\arcsin(\mathrm{sn}(\chi,k)), \hat{\alpha}_2^2, k)\right],\end{aligned} \tag{3.6}$$

where $\hat{\alpha}_1^2 = \frac{\phi_M-\phi_l}{\sqrt{c}-\phi_l}, \hat{\alpha}_2^2 = \frac{\hat{\alpha}_1^2(\sqrt{c}-\phi_2)}{\phi_M-\phi_2}, k^2 = \frac{2\hat{\alpha}_1^2\sqrt{c}}{\phi_M-\sqrt{c}}, \chi_{01} = \mathrm{sn}^{-1}\sqrt{\frac{\phi_M-\phi_2}{\hat{\alpha}_1^2(\sqrt{c}-\phi_2)}}$, $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$ are Jacobin elliptic functions, and $\Pi(\cdot, \cdot, k)$ is the elliptic integral of the third kind (see [4]).

(iii) Exact explicit periodic peakon solutions for $p = 2$

We now consider the the arch orbit in Fig.2 (g) connecting the equilibrium points $S_{\mp}^{-}(-\phi_s, \mp y_{s1})$, which is defined by $H(\phi, y) = h_{s1}$ when $p = 2, \phi_s = \sqrt{c}$. Now, (3.1) can be written as

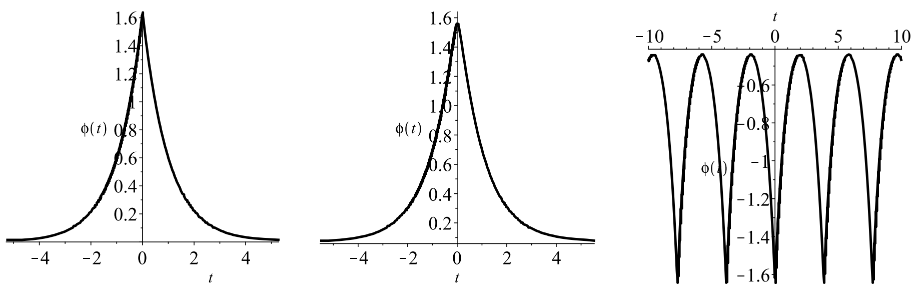
$$\begin{aligned}\xi &= \int_{-\sqrt{c}}^{\phi} \frac{(\phi^2-c)d\phi}{\sqrt{(\phi^2-c)(h_{s1}+2g\phi-c\phi^2+\phi^4)}} \\ &= \int_{-\sqrt{c}}^{\phi} \frac{(\phi+\sqrt{c})d\phi}{\sqrt{(\phi-\phi_M)(\phi+\sqrt{c})[(\phi-b_1)^2+a_1^2]}} \\ &= \sqrt{c}\int_{-\sqrt{c}}^{\phi} \frac{d\phi}{\sqrt{(\phi-\phi_M)(\phi+\sqrt{c})[(\phi-b_1)^2+a_1^2]}} \\ &\quad + \int_{-\sqrt{c}}^{\phi} \frac{\phi d\phi}{\sqrt{(\phi-\phi_M)(\phi+\sqrt{c})[(\phi-b_1)^2+a_1^2]}}.\end{aligned} \tag{3.7}$$

By using (3.7), we obtain the following periodic peakon solution (see Fig.3 (c)):

$$\phi(\chi) = A_1 + \frac{B_1}{1+\tilde{\alpha}\mathrm{cn}(\chi,k)},\ \chi \in (-\chi_{02}, \chi_{02}),$$

$$\xi(\chi) = \frac{1}{\sqrt{AB}}\Bigg[-\left(\frac{B(\phi_M+\sqrt{c})}{B-A}\right)\sqrt{c}\chi - \left(\frac{(\sqrt{c}A+\phi_M B)(\hat{\alpha}-\hat{\alpha}_2)}{(A-B)}\right) \times \int_0^{\chi}\frac{d\chi}{1+\hat{\alpha}\mathrm{cn}(\chi,k)}\Bigg], \tag{3.8}$$

where $A^2 = (\phi_M - b_1)^2 + a_1^2$, $B^2 = (\sqrt{c} + b_1)^2 + a_1^2$, $A_1 = \frac{\phi_M B + \sqrt{c}A}{A-B}$, $B_1 = \frac{2(A^2\sqrt{c}+B^2\phi_M)}{B^2-A^2}$, $k^2 = \frac{(\phi_M+\sqrt{c})^2-(A-B)^2}{4AB}$, $\hat{\alpha} = \frac{A-B}{A+B}$, $\hat{\alpha}_2 = -\frac{\sqrt{c}A+\phi_M B}{\phi_M B - \sqrt{c}A}$, $\chi_{01} = \mathrm{cn}^{-1}\left(\frac{A_1+B_1-\phi_M}{\tilde{\alpha}(\phi_M - A_1)}\right)$.



(a) Peakon (b) Solitary wave (c) Periodic peakon

Figure 3. Profiles of traveling waves of equation (1.1)

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