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A DELAYED SEMILINEAR PARABOLIC PREDATOR-PREY SYSTEM WITH HABITAT COMPLEXITY AND HARVESTING EFFECTS

Haicheng Liu[1], Bin Ge[1,†], Qiyuan Liang[1] and Jiaqi Chen[1]

Abstract In this paper, we propose a delayed reaction-diffusive system with habitat complexity and harvesting effects, and study dynamic behaviors of the system. Firstly, for the system without time delay, the stability of equilibria is studied. It is found that when habitat complexity reaches a certain critical value, the positive equilibrium will change from unstable to locally asymptotically stable. Secondly, time delay effect on the dynamic behaviors of diffusion system is studied. The existence conditions of Hopf bifurcation are given, and the properties of bifurcating periodic solutions are studied by using the center manifold and normal form theories, including the direction of Hopf bifurcation, the stability of bifurcating periodic solutions and the period. Finally, the corresponding numerical simulations and biological interpretation are made to verify the results of theoretical analysis.

Keywords Predator-prey system, habitat complexity effect, harvesting effect, time delay, diffusion term.

MSC(2010) 34K18, 35B32.

1. Introduction

1.1. Development of the model

Since American mathematician Lotka and Italian mathematician Volterra proposed the population dynamic models [16], in the predator-prey system, functional response is an essential factor, it reflects predator's ability to prey, representing the quantity of prey taken by a single predator per unit time. After the concept "functional response" was put forward, the study of predator-prey system with Holling type functional response has become a mainstream direction of biomathematics [6, 10, 18], the theory and method of dynamic system are applied more and more widely in biomathematics, which has attracted the attention of many scholars [1, 2, 5, 13, 15, 20, 23, 24, 26, 28, 29]. Meanwhile, more and more biological effects are interpreted and applied to the predator-prey system, such as habitat complexity effect [3], shelter effect [7], delay effect [17] and harvesting effect [4]. Studying on the predator-prey systems with biological effects is better consistent with and explains some natural phenomena.

Time-delay systems often exhibit complex dynamic behaviors, such as stability switch, periodic solution phenomenon, bifurcation and chaos, etc [12, 14, 21].

†The corresponding author. Email:gebin791025@hrbeu.edu.cn(B. Ge)
[1]College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, Heilongjiang, China

Time-delay effect occurs in almost all ecosystems, which is the key factor of population dynamic change. More and more scholars introduce delay effect into the predator-prey system, and make a comprehensive research on the corresponding dynamic system. From the perspective of ecology and economics, an important and interesting problem is how to find a reasonable harvest strategy. In order to find reasonable control measures, we must first understand the impact of harvesting effect on resources, therefore, the research on predator-prey system with harvesting effect attracts the attention of lots of scholars [8, 11, 22].

1.2. Model Building

In [19], the following predator-prey system with Holling type functional response is established:

$$\begin{cases} \frac{dx}{dt} = rx(1-\frac{x(t-\tau)}{K}) - \frac{c(1-\beta)x^n y}{1+ch(1-\beta)x^n}, \\ \frac{dy}{dt} = \frac{ec(1-\beta)x^n y}{1+ch(1-\beta)x^n} - dy, \\ x(\xi) = \phi(\xi) > 0, y(\xi) = \psi(\xi) > 0,\ \xi \in (-\tau, 0], \end{cases} \tag{1.1}$$

where, $x(t)$ and $y(t)$ represent prey and predator densities at time t respectively, the other parameters are positive. τ is production delay of prey, r is the intrinsic growth rate of prey, K is the maximum environmental capacity of prey, $\frac{c(1-\beta)x^n}{1+ch(1-\beta)x^n}, n \geq 1$ represents Holling function response, c is the attack rate of predator on prey, h indicates the handling time, $e(0 < e < 1)$ is the conversion efficiency, and d is the mortality of predator, that is, the death number of predators per unit time, $\beta(0 < \beta < 1)$ indicates the intensity of habitat complexity effect.

In order to make system (1.1) more consistent with biological significance, we introduce the linear harvesting effect of prey, and establish a delayed predator-prey system with habitat complexity and linear harvesting effects. Meanwhile, considering that the state of predator-prey system depends not only on time but also on space, we introduce the reaction-diffusion term and establish a delayed reaction-diffusion predator-prey model with habitat complexity and linear harvesting effects:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru(1-\frac{u(x,t-\tau)}{K}) - \frac{c(1-\beta)u^n v}{1+ch(1-\beta)u^n} - qEu, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \frac{ec(1-\beta)u^n v}{1+ch(1-\beta)u^n} - dv, \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi,t) = v_x(l\pi,t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in \Omega = (0, l\pi). \end{cases} \tag{1.2}$$

Where q represents capture coefficient and E represents harvesting intensity.

1.3. Existence of the steady state solutions

In the following, we discuss the conditions which ensure the existence of equilibria and biological significance. By calculation, we can obtain three equilibria of system (1.2):

$$P_0 = (0,0), P_1 = \left(K\left(1-\frac{qE}{r}\right), 0\right), P^* = (u_0, v_0),$$

where

$$u_0 = (\frac{1}{1-\beta})^{\frac{1}{n}} (\frac{d}{c\,(e-dh)})^{\frac{1}{n}}, v_0 = \frac{e}{d} u_0 \left[r(1-\frac{u_0}{K}) - qE \right],$$

u_0 can be regarded as function of β, suppose $u_0 = u(\beta)$. For convenience, denote $\beta^* = 1 - \frac{dr^n}{c(e-dh)K^n(r-qE)^n}$, and make the following assumptions:

Assumption 1.1. (A_0) *$h < e/d$ and $n \geq 1, r > qE$.*

Assumption 1.2. (A_1) *$\beta < \beta^*$.*

Theorem 1.1. *Suppose that* (A_0) *and* (A_1) *hold, then system* (1.2) *has only one positive equilibrium.*

2. Stability of diffusion system without delay

When $\tau = 0$, system (1.2) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru(1-\frac{u}{K}) - \frac{c(1-\beta)u^n v}{1+ch(1-\beta)u^n} - qEu, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \frac{ec(1-\beta)u^n v}{1+ch(1-\beta)u^n} - dv, \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi,t) = v_x(l\pi,t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in \Omega = (0, l\pi). \end{cases} \tag{2.1}$$

Define the real-valued Sobolev space

$$X := \left\{ (u,v)^T \left| u, v \in H^2\left(0, l\pi\right), \left(u_x, v_x\right)\right|_{x=0,l\pi} = (0,0) \right\},$$

and let the complexification of X be

$$X_C := X \oplus iX = \left\{ x_1 + ix_2 \left| x_1, x_2 \in X \right. \right\}.$$

Let

$$U = (u,v) \in H^2\left(0, l\pi\right), D = diag\left(d_1, d_2\right), F\left(\eta, U\right) = \left(f, g\right),$$

then system (2.1) can be abstracted as

$$\dot{U}\left(t\right) = D\Delta U\left(t\right) + F\left(\eta, U\right).$$

Use $J\left(F\right)$ to represent the Jacobian matrix of F , then the linearized operator of the steady-state system corresponding to system (2.1) at $(\eta, 0, 0)$ is

$$L\left(\eta\right) = D\frac{\partial^2}{\partial x^2} + J\left(F\right)|_{U\equiv 0} = \begin{pmatrix} a_{11} + d_1\frac{\partial^2}{\partial x^2} & a_{12} \\ a_{21} & a_{22} + d_2\frac{\partial^2}{\partial x^2} \end{pmatrix}.$$

Use $\mu_n = \frac{n^2}{l^2}, n \in \mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$ to represent the nth eigenvalue of $-\varphi_{xx} = \mu\varphi, \varphi_x\left|_{x=0,l\pi}\right. = 0$, define the linear operator

$$L_n\left(\eta\right) = \begin{pmatrix} a_{11} - d_1\mu_n & a_{12} \\ a_{21} & a_{22} - d_2\mu_n \end{pmatrix},$$

then the eigenvalue of $L(\eta)$ can be given by the eigenvalue of $L_n(\eta)$, and the characteristic equation of $L_n(\eta)$ is

$$\lambda^2 + E_n(\eta)\lambda + F_n(\eta) = 0, \tag{2.2}$$

where

$$E_n(\eta) = -tr(L_n(\eta)) = -(a_{11} + a_{22}) + (d_1 + d_2)\mu_n,$$
$$F_n(\eta) = |L_n(\eta)| = d_1 d_2 {\mu_n}^2 - (a_{11}d_2 + a_{22}d_1)\mu_n + a_{11}a_{22} - a_{12}a_{21}.$$

2.1. Stability of the positive steady state

The Jacobian matrix of system (2.1) at the positive equilibrium $P^* = (u_0, v_0)$ is

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$a_{11} = \left(r - \frac{2ru_0}{K}\right) + n\left(1 - \frac{dh}{e}\right)\left(r - \frac{ru_0}{K}\right) - qE,$$
$$a_{12} = -\frac{c(1-\beta){u_0}^n}{1 + ch(1-\beta){u_0}^n} = -\frac{d}{e},$$
$$a_{21} = nr\left(1 - \frac{u_0}{K}\right)(e - dh), a_{22} = 0,$$
$$L_n = \begin{pmatrix} \left(r - \frac{2ru_0}{K}\right) + n\left(1 - \frac{dh}{e}\right)\left(r - \frac{ru_0}{K}\right) - qE - d_1\mu_n & -\frac{d}{e} \\ nr\left(1 - \frac{u_0}{K}\right)(e - dh) & -d_2\mu_n \end{pmatrix},$$
$$E_n = -tr(L_n) = -\left(\left(r - \frac{2ru_0}{K}\right) + n\left(1 - \frac{dh}{e}\right)\left(r - \frac{ru_0}{K}\right) - qE\right) + (d_1 + d_2)\mu_n,$$
$$F_n = |L_n| = d_1 d_2 {\mu_n}^2 - \left(\left(r - \frac{2ru_0}{K}\right) + n\left(1 - \frac{dh}{e}\right)\left(r - \frac{ru_0}{K}\right) - qE\right) d_2\mu_n$$
$$+ \frac{d}{e} nr\left(1 - \frac{u_0}{K}\right)(e - dh).$$

The characteristic roots of (2.2) are

$$\lambda_{1,2}^{(n)} = \frac{-E_n \pm \sqrt{{E_n}^2 - 4F_n}}{2}, n \in \mathbb{N}_0.$$

Lemma 2.1. *Suppose that* (A_0) *and* (A_1) *hold, and* $dh < e < \frac{n}{n-1}dh$ *satisfies, then we have the following conclusions.*

(i) If $1 - \frac{dr^n[(n-2)e - ndh]^n}{c(e-dh)K^n(r-qE)^n[(n-1)e-ndh]^n} < \beta < \beta^*$*, then* $E_n > 0, F_n > 0$*, thus all the roots of Eq.* (2.2) *have negative real parts;*

(ii) If $\beta < 1 - \frac{dr^n[(n-2)e-ndh]^n}{c(e-dh)K^n(r-qE)^n[(n-1)e-ndh]^n}$*, then* $E_0 < 0$*, thus Eq.* (2.2) *has at least one root with positive real part.*

Theorem 2.1. *Suppose that* (A_0) *and* (A_1) *hold, and* $dh < e < \frac{n}{n-1}dh$ *satisfies, then we have the following conclusions.*

(i) If $1-\frac{dr^n[(n-2)e-ndh]^n}{c(e-dh)K^n(r-qE)^n[(n-1)e-ndh]^n}<\beta<\beta^*$*, then the steady state* $P^*=(u_0,v_0)$ *is locally asymptotically stable;*

(ii) If $\beta<1-\frac{dr^n[(n-2)e-ndh]^n}{c(e-dh)K^n(r-qE)^n[(n-1)e-ndh]^n}$*, then the steady state* $P^*=(u_0,v_0)$ *is unstable.*

2.2. Stability of the boundary equilibria

The corresponding characteristic roots at $P_0=(0,0)$ are

$$\lambda_{01}^n=r-qE-d_1\mu_n,\quad \lambda_{02}^n=-d-d_2\mu_n<0,\quad n\in\mathbb{N}_0.$$

The corresponding characteristic roots at $P_1=\left(K\left(1-\frac{qE}{r}\right),0\right)$ are

$$\lambda_{11}^n=-\left(r-qE\right)-d_1\mu_n<0,\quad \lambda_{12}^n=\frac{ec\left(1-\beta\right)K^n}{1+ch\left(1-\beta\right)K^n}-d-d_2\mu_n,\quad n\in\mathbb{N}_0.$$

Theorem 2.2. *For system* (2.1)*, the following results are true.*

(i) The trivial steady state $P_0=(0,0)$ *is unstable;*

(ii) Suppose that (A_0) *holds, if* $\beta>\beta^*$*, then the semi-trivial steady state* $P_1=\left(K\left(1-\frac{qE}{r}\right),0\right)$ *is locally asymptotically stable; otherwise, it is unstable.*

Theorem 2.3. *For system* (2.1)*, if* $\beta>\beta^*$*, then the semi-trivial steady state* $P_1=\left(K\left(1-\frac{qE}{r}\right),0\right)$ *is globally asymptotically stable.*

Proof. According to the first equation,

$$\frac{\partial u}{\partial t}-d_1\Delta u=ru(1-\frac{u}{K})-\frac{c(1-\beta)u^nv}{1+ch(1-\beta)u^n}-qEu\le u\left[r(1-\frac{u}{K})-qE\right].$$

Using the comparison principle, we have $\lim\limits_{t\to\infty}\max\limits_{x\in[0,l\pi]}u\left(x,t\right)\le K\left(1-\frac{qE}{r}\right)$.

According to the second equation,

$$\begin{aligned}\frac{\partial v}{\partial t}-d_2\Delta v&=\frac{ec(1-\beta)u^nv}{1+ch(1-\beta)u^n}-dv(t)\\&=v(t)(\frac{ec(1-\beta)}{1/u^n+ch(1-\beta)}-d)\\&\le v(t)(\frac{ec(1-\beta)}{1/K^n+ch(1-\beta)}-d)<0,\end{aligned}$$

hence $(\frac{ec(1-\beta)u^n}{1+ch(1-\beta)u^n}-d)v<0$. Therefore, for any $\varepsilon>0$, there exist $T>0, v\left(x,t\right)\le\varepsilon$, we can obtain

$$\begin{aligned}\frac{\partial u}{\partial t}-d_1\Delta u&=ru(1-\frac{u}{K})-\frac{c(1-\beta)u^nv}{1+ch(1-\beta)u^n}-qEu\\&\ge u[r(1-\frac{u}{K})-c(1-\beta)u^{n-1}\varepsilon-qE]\\&\ge u[r(1-\frac{u}{K})-c(1-\beta)\left(K\left(1-\frac{qE}{r}\right)\right)^{n-1}\varepsilon-qE].\end{aligned}$$

Applying the comparison principle again,

$$u(x,t) \geq K\left(1 - \frac{c(1-\beta)\left(K\left(1-\frac{qE}{r}\right)\right)^n \varepsilon + qE}{r}\right), t > T, x \in [0, l\pi],$$

thereby, $\lim_{t\to\infty} \max_{x\in[0,l\pi]} u(x,t) = K\left(1-\frac{qE}{r}\right)$, that is, the semi-trivial steady state $P_1 = \left(K\left(1-\frac{qE}{r}\right), 0\right)$ is globally asymptotically stable. □

2.3. Existence of Hopf bifurcation at the positive coexistence

System (2.1) has a unique positive equilibrium $P^* = (u_0, v_0)$, let $\delta := u_0 = (\frac{1}{1-\beta})^{\frac{1}{n}}(\frac{d}{c(e-dh)})^{\frac{1}{n}}, v_\delta := v_0 = \frac{e}{d}\delta\left[r(1-\frac{\delta}{K}) - qE\right]$. Next, we select δ as bifurcation parameter, the diffusion terms $d_i, i = 1, 2$ can be regard as function of δ, we can get some results about diffusion effect on dynamics of the system. The Jacobian matrix of (2.1) at $P^* = (u_0, v_0)$ is as follows:

$$\begin{pmatrix} a_{11}(\delta) & a_{12}(\delta) \\ a_{21}(\delta) & a_{22}(\delta) \end{pmatrix},$$

with

$$\begin{aligned}
a_{11}(\delta) &= r\left(1-\frac{2\delta}{K}\right) + nr\left(1-\frac{dh}{e}\right)\left(1-\frac{\delta}{K}\right) - qE,\\
a_{12}(\delta) &= -\frac{c(1-\beta)\delta^n}{1+ch(1-\beta)\delta^n} = -\frac{d}{e},\\
a_{21} &= nr\left(1-\frac{\delta}{K}\right)(e-dh), a_{22} = 0,\\
L_n(\delta) &= \begin{pmatrix} a_{11}(\delta) - d_1\mu_n & -\frac{d}{e} \\ nr\left(1-\frac{\delta}{K}\right)(e-dh) & -d_2\mu_n \end{pmatrix},\\
E_n(\delta) &= -tr(L_n) = -a_{11}(\delta) + (d_1+d_2)\mu_n,\\
F_n(\delta) &= |L_n(\delta)| = d_1 d_2 {\mu_n}^2 - a_{11}(\delta) d_2\mu_n + \frac{d}{e} nr\left(1-\frac{\delta}{K}\right)(e-dh).
\end{aligned}$$

The characteristic roots of Eq. (2.2) are

$$\lambda_{1,2}^{(n)} = \frac{-E_n(\delta) \pm \sqrt{{E_n}^2(\delta) - 4F_n(\delta)}}{2}, n \in \mathbb{N}_0.$$

According to [27], we have the following lemma.

Lemma 2.2. *At some critical point δ_0, the sufficient conditions for system* (2.1) *to generate Hopf bifurcation are as follows:*

(i) There exists $n \in \mathbb{N}_0$ such that $E_n(\delta_0) = 0, F_n(\delta_0) > 0, E_j(\delta_0) \neq 0, F_j(\delta_0) \neq 0, j \neq n$;

(ii) Denote $\alpha(\eta) \pm i\omega(\eta)$ as a pair of complex characteristic roots near pure imaginary roots, then we have $\alpha(\delta_0) = 0, \omega(\delta_0) \neq 0, \alpha'(\delta_0) \neq 0$.

According to Theorem 2.1, if $K\frac{(r-qE)e+nr(e-dh)}{2re+nr(e-dh)} < \delta < K\left(1-\frac{qE}{r}\right)$, then the system is locally asymptotically stable at $P^* = (u_0, v_0)$, therefore, any possible Hopf bifurcation point must be within $\left(0, \frac{(r-qE)e+nr(e-dh)}{2re+nr(e-dh)}K\right]$. For any $\delta_0 \in \left(0, \frac{(r-qE)e+nr(e-dh)}{2re+nr(e-dh)}K\right]$, let $\alpha(\delta) \pm i\omega(\delta)$ be the eigenvalues of $L_n(\delta)$, then

$$\alpha(\delta) = \frac{a_{11}(\delta)}{2} - \frac{(d_1+d_2)}{2}\mu_n,\ \omega(\delta) = \sqrt{F_n(\delta) - \alpha^2(\delta)}.$$

By calculation, we have

$$\alpha'(\delta_0) = \frac{{a'}_{11}(\delta_0)}{2} = -\frac{r}{2K}\left[2 + n\left(1 - \frac{dh}{e}\right)\right] < 0,$$

so the transversality condition holds. Through the above analysis, studying Hopf bifurcation points can be translated into studying δ_0 which satisfies $E_i(\delta_0) = 0, F_i(\delta_0) > 0, E_j(\delta_0) \neq 0, F_j(\delta_0) \neq 0, j \neq i$ in the following set

$$\Delta := \left\{ \delta |\, \delta \in \left(0, \frac{(r-qE)\,e + nr(e-dh)}{2re + nr(e-dh)}K\right]\right\}.$$

Let $\delta_0^H := \frac{(r-qE)e+nr(e-dh)}{2re+nr(e-dh)}K$, clearly $\delta_0^H \in \Delta$. And because for any $j \geq 1$, $E_0\left(\delta_0^H\right) = 0, E_j\left(\delta_0^H\right) > 0$; for any $i \in \mathbb{N}_0$, $F_i\left(\delta_0^H\right) > 0$, then δ_0^H is the bifurcation point where the system produces spatially homogeneous periodic solutions.

In the following, we discuss spatially inhomogeneous periodic solutions generated by the system when $i \geq 1$. Because $a_{11}(\delta_0^H) = 0$ and $a_{11}(\delta)$ is decreasing in $\left(0, \delta_0^H\right)$, we have $a_{11}(\delta) > 0$. Define $l_i = i\sqrt{\frac{d_1+d_2}{M}}, i \in \mathbb{N}$, where $M = r - qE + nr\left(1 - \frac{dh}{e}\right)$. For $l_i < l < l_{i+1}$ and $j \in \mathbb{N}$, let δ_j^H be a solution of $a_{11}(\delta) = \frac{(d_1+d_2)j^2}{l^2}$, where $0 < \delta_j^H < \delta_0^H$, these points satisfy $0 < \delta_1^H < \delta_2^H < \delta_3^H < \cdots < \delta_{n-1}^H < \delta_n^H < \delta_0^H$, for $i \neq j$, $E_j(\delta_j^H) = 0$, $E_i(\delta_j^H) \neq 0$. We only need to verify that when $i \in \mathbb{N}_0$, $F_i(\delta_j^H) \neq 0$, specifically, $F_i(\delta_j^H) > 0$. Next, we discuss the conditions of $F_i(\delta) > 0$ for all $\delta \in \left(0, \delta_0^H\right]$. We know that the following inequality is true:

$$F_i(\delta) \geq d_1 d_2 {\mu_n}^2 - M d_2 \mu_n + dn\frac{(r+qE)\,(e-dh)}{2e + n(e-dh)}.$$

To make $g(y) = d_1 d_2 y^2 - M d_2 y + dn\frac{(r+qE)(e-dh)}{2e+n(e-dh)}$ to be positive, we only need to guarantee $\frac{d_1}{d_2} > \frac{M^2}{4dn(r+qE)}\left(\frac{2e}{e-dh} + n\right)$ holds. Make the following hypothesis:

Assumption 2.1. (A_2) $\quad \frac{d_1}{d_2} > \frac{M^2}{4dn(r+qE)}\left(\frac{2e}{e-dh} + n\right)$.

Theorem 2.4. *Suppose that* $(\mathrm{A}_0) - (\mathrm{A}_2)$ *hold, for any* $l_i < l \leq l_{i+1}, i \in \mathbb{N}$, *there exist* i *bifurcation points* $(0 < \delta_1^H < \delta_2^H < \delta_3^H < \cdots < \delta_{i-1}^H < \delta_i^H < \delta_0^H)$ *which make Hopf bifurcation occur at* $\delta = \delta_j^H$ *and* $\delta = \delta_0^H$. *When* $\delta = \delta_0^H$, *the bifurcating periodic solutions are spatially homogeneous; when* $\delta = \delta_j^H$, *the bifurcating periodic solutions are spatially non-homogeneous.*

3. Hopf bifurcation properties of the system with time delay

In this section, we shall study time delay effect on the dynamic properties of diffusion system (1.2).

3.1. Existence of Hopf bifurcation induced by delay

Assume that (A_0) and (A_1) are true, system (1.2) has a unique positive equilibrium $P^* = (u_0, v_0)$. For convenience, we make the transformations $\hat{u} = u - u_0, \hat{v} = v - v_0$ to move $P^* = (u_0, v_0)$ to $(0, 0)$. Here we still use u, v to represent $\hat{u}, \hat{v}$, then system (1.2) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + r(u+u_0)(1 - \frac{u(x,t-\tau)+u_0}{K}) - \frac{c(1-\beta)(u+u_0)^n(v+v_0)}{1+ch(1-\beta)(u+u_0)^n} - qE(u+u_0), \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \frac{ec(1-\beta)(u+u_0)^n(v+v_0)}{1+ch(1-\beta)(u+u_0)^n} - d(v+v_0), \\ u_x(0,t) = v_x(0,t) = 0, u_x(l\pi,t) = v_x(l\pi,t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in \Omega = (0, l\pi). \end{cases} \tag{3.1}$$

Let

$$u_1(t) = u(\cdot,t), u_2(t) = v(\cdot,t), U = (u_1,u_2)^T, X = C\left([0,l\pi], \mathbb{R}^2\right),$$

in phase space $\mathbb{C}_\tau = C([-\tau, 0], X)$, (3.1) can be abstracted as

$$\dot{U}(t) = D\Delta U(t) + L(U_t) + F(U_t), \tag{3.2}$$

where $\varphi = (\varphi_1, \varphi_2)^T, D = diag(d_1, d_2), L : \mathbb{C}_\tau \to X, F : \mathbb{C}_\tau \to X$ are defined as follows:

$$L(\varphi) = \begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(-\tau) \\ \varphi_2(-\tau) \end{pmatrix}, F(\phi) = \begin{pmatrix} F_1(\phi) \\ F_2(\phi) \end{pmatrix},$$

with

$$\begin{aligned} F_1(\phi) &= r(\phi_1(0)+u_0)\left(1 - \frac{\phi_1(-\tau)+u_0}{K}\right) - \frac{c(1-\beta)(\phi_1(0)+u_0)^n(\phi_2(0)+v_0)}{1+ch(1-\beta)(\phi_1(0)+u_0)^n} \\ &\quad - qE(\phi_1(0)+u_0) - a_1\phi_1(0) - a_2\phi_2(0) - c_1\phi_1(-\tau), \\ F_2(\phi) &= \frac{ec(1-\beta)(\phi_1(0)+u_0)^n(\phi_2(0)+v_0)}{1+ch(1-\beta)(\phi_1(0)+u_0)^n} - d(\phi_2(0)+v_0) + a_3\phi_2(0), \\ a_1 &= n\left(1 - \frac{dh}{e}\right)\left[r\left(1 - \frac{u_0}{K}\right) - qE\right]u_0, \quad a_2 = -\frac{d}{e}, \\ a_3 &= rn\left(1 - \frac{u_0}{K}\right)(e - dh), \quad c_1 = -r\frac{u_0}{K}. \end{aligned}$$

Then, the linearized equation of (3.1) at $(0, 0)$ is

$$\dot{U}(t) = D\Delta U(t) + L(U_t), \tag{3.3}$$

where

$$L(U_t) = L_1 U + L_2 U_t, L_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $-\varphi'' = \mu\varphi, x \in (0, l\pi), \varphi'(0) = \varphi'(l\pi) = 0$, $b_n = \cos\frac{n\pi}{l}, n \in \mathbb{N}_0$ are the eigenvectors corresponding to the eigenvalues $\mu_n = n^2/l^2$ $, n \in \mathbb{N}_0$. λ is the eigenvalue of (3.3). Substitute $y = \sum_{n=0}^{\infty} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} \cos\frac{n\pi}{l}$ into $\lambda y - d\Delta y - L(e^{\lambda} y) = 0$, we can obtain

$$\begin{pmatrix} a_1 + c_1 e^{-\lambda\tau} - d_1\mu_n & a_2 \\ a_3 & -d_2\mu_n \end{pmatrix} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \lambda \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}.$$

The corresponding characteristic equation is

$$\det\left(\lambda I + \mu_n D - L_1 - L_2 e^{-\lambda\tau}\right) = 0.$$

So the characteristic equation is equivalent to

$$\lambda^2 + A_n\lambda + B_n + C_n e^{-\lambda\tau} = 0, \tag{3.4}$$

where

$$\begin{aligned} A_n &= (d_1 + d_2)\mu_n - a_1, \\ B_n &= d_1 d_2 {\mu_n}^2 - a_1 d_2 \mu_n - a_2 a_3, \\ C_n &= -c_1(\lambda + d_2\mu_n). \end{aligned}$$

Make the following assumptions:

Assumption 3.1. (A_3) $a_1 < 0$.

Assumption 3.2. (A_4) $c_1 < a_1$.

Assumption 3.3. (A_5) ${a_1}^2 + 2a_2a_3 - {c_1}^2 > 0$.

Lemma 3.1. *If* $(\mathrm{A}_0) - (\mathrm{A}_3)$ *are true, the following conclusions can be drawn for* $n \in \mathbb{N}_0$.

(i) When $\tau = 0$, all the characteristic roots of Eq. (3.4) *have negative real parts, system* (3.1) *is locally asymptotically stable at $P^* = (u_0, v_0)$;*

(ii) $\lambda = 0$ is not the root of Eq. (3.4).

Lemma 3.2. *Suppose that* (A_3) *holds, when $\tau \neq 0$, we have the following results.*

(i) If (A_4) *holds, then Eq.* (3.4) *has a pair of pure imaginary roots $\pm i\omega_n^+$ at $\tau = \tau_n^{j,+}$ for $N_1 \le n \le \min\{N_2, N_3\}$;*

(ii) If (A_4) *holds, then Eq.* (3.4) *has a pair of pure imaginary roots $\pm i\omega_n^+$ at $\tau = \tau_n^{j,+}$ for $\max\{N_1, N_3\} < n < N_2$;*

(iii) If (A_4) *holds, then when $0 \le n \le \min\{N_1, N_3\}$ or $N_2 < n < N_3$, Eq.* (3.4) *has two pairs of pure imaginary roots $\pm i\omega_n^{\pm}$ at $\tau_n^{j,\pm}$;*

(iv) If (A_4) *is true, then when $n > \max\{N_2, N_3\}$ or $N_3 < n < N_1$, Eq.* (3.4) *has no pure imaginary roots;*

(v) If (A_5) *is true, then Eq.* (3.4) *has no pure imaginary roots for* $n \geq 0$,

where

$$N_1 = \begin{cases} \left[\hat{N} = l\sqrt{\frac{1}{2d_1d_2}\left[(a_1-c_1)d_2 - \sqrt{((a_1-c_1)d_2)^2 + 4d_1d_2a_2a_3}\right]}\right], \ \hat{N} \notin \mathbb{N}, \\ \left[\hat{N} = l\sqrt{\frac{1}{2d_1d_2}\left[(a_1-c_1)d_2 - \sqrt{((a_1-c_1)d_2)^2 + 4d_1d_2a_2a_3}\right]}\right] - 1, \ \hat{N} \in \mathbb{N}, \end{cases}$$

$$N_2 = \begin{cases} \left[\bar{N} = l\sqrt{\frac{1}{2d_1d_2}\left[(a_1-c_1)d_2 + \sqrt{((a_1-c_1)d_2)^2 + 4d_1d_2a_2a_3}\right]}\right], \ \bar{N} \notin \mathbb{N}, \\ \left[\bar{N} = l\sqrt{\frac{1}{2d_1d_2}\left[(a_1-c_1)d_2 + \sqrt{((a_1-c_1)d_2)^2 + 4d_1d_2a_2a_3}\right]}\right] - 1, \ \bar{N} \in \mathbb{N}, \end{cases}$$

$$N_3 = \begin{cases} \left[\tilde{N} = l\sqrt{\frac{1}{(d_1^2+d_2^2)}\left[a_1d_1 + \sqrt{d_1^2{a_1}^2 - (d_1^2+d_2^2)\left({a_1}^2 + 2a_2a_3 - {c_1}^2\right)}\right]}\right], \ \tilde{N} \notin \mathbb{N}, \\ \left[\tilde{N} = l\sqrt{\frac{1}{(d_1^2+d_2^2)}\left[a_1d_1 + \sqrt{d_1^2{a_1}^2 - (d_1^2+d_2^2)\left({a_1}^2 + 2a_2a_3 - {c_1}^2\right)}\right]}\right] - 1, \tilde{N} \in \mathbb{N}, \end{cases}$$

$$\tau_n^{j,\pm} = \frac{1}{\omega_n^{\pm}} \arccos \frac{(D_n + c_1A_n)\left(\omega_n^{\pm}\right)^2 - D_nB_n}{{D_n}^2 + {c_1}^2\left(\omega_n^{\pm}\right)^2} + \frac{2j\pi}{\omega_n^{\pm}}, j \in \mathbb{N}_0,$$

$$\omega_n^{\pm} = \sqrt{\frac{-({A_n}^2 - 2B_n - {c_1}^2) \pm \sqrt{({A_n}^2 - 2B_n - {c_1}^2)^2 - 4({B_n}^2 - {D_n}^2)}}{2}}.$$

Proof. Let $\lambda = i\omega\,(\omega > 0)$ be a solution of Eq. (3.4), for some $n \in \mathbb{N}_0$, ω satisfies

$$-\omega^2 + i\omega A_n + B_n + c_1(i\omega + d_2\mu_n)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

Then we have

$$\begin{cases} c_1\omega\sin\omega\tau + c_1d_2\mu_n\cos\omega\tau = \omega^2 - B_n, \\ c_1d_2\mu_n\sin\omega\tau - c_1\omega\cos\omega\tau = A_n\omega. \end{cases} \tag{3.5}$$

Let $D_n = c_1d_2\mu_n$, then

$$\omega^4 + ({A_n}^2 - 2B_n - {c_1}^2)\omega^2 + {B_n}^2 - {D_n}^2 = 0. \tag{3.6}$$

Let $z = \omega^2$, then (3.6) can be changed into

$$z^2 + ({A_n}^2 - 2B_n - {c_1}^2)z + {B_n}^2 - {D_n}^2 = 0. \tag{3.7}$$

By direct computation,

$$\begin{aligned} &B_n - D_n = d_1d_2{\mu_n}^2 - (c_1 + a_1)d_2\mu_n - a_2a_3 > 0, \\ &B_n + D_n = d_1d_2{\mu_n}^2 + (c_1 - a_1)d_2\mu_n - a_2a_3, \\ &{A_n}^2 - 2B_n - {c_1}^2 = ({d_1}^2 + {d_2}^2){\mu_n}^2 - 2a_1d_1\mu_n + {a_1}^2 + 2a_2a_3 - {c_1}^2. \end{aligned}$$

Under (A_4), when $N_1 \leq n \leq N_2$, $B_n + D_n < 0$, so ${B_n}^2 - {D_n}^2 < 0$. When $n > N_2$ or $0 < n \leq N_1$, $B_n + D_n > 0$, then ${B_n}^2 - {D_n}^2 > 0$. We can obtain that ${A_n}^2 - 2B_n - {c_1}^2 < 0$ for $0 \leq n \leq N_3$; ${A_n}^2 - 2B_n - {c_1}^2 \geq 0$ for $n > N_3$.

If (A_5) is true, ${A_n}^2 - 2B_n - {c_1}^2$ increases monotonically with respect to n, then for any $n \geq 0$, ${A_n}^2 - 2B_n - {c_1}^2 > 0$, and $B_n + D_n > 0$, so ${B_n}^2 - {D_n}^2 > 0$.

In conclusion, the conclusions are true, and the roots of Eq.(3.7) are

$$z^{\pm} = \frac{-({A_n}^2 - 2B_n - {c_1}^2) \pm \sqrt{({A_n}^2 - 2B_n - {c_1}^2)^2 - 4({B_n}^2 - {D_n}^2)}}{2}.$$

Then Eq. (3.6) has at least one positive root $\omega_n^+ = \sqrt{z_n^+}$. □

For convenience, we consider cases (i) and (ii) in Lemma 3.2. Denote $\tau_n^{j,+}$ as τ_n^j, setting $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ to be the roots of Eq. (3.4) which satisfies $\alpha\left(\tau_n^j\right) = 0, \beta\left(\tau_n^j\right) = \omega_n$ when τ is sufficiently close to τ_n^j, then we have the following transversality condition.

Lemma 3.3. *Suppose* (A_3) *holds, then* $\alpha'\left(\tau_n^j\right) = \frac{d\lambda}{d\tau}\Big|_{\tau=\tau_n^j} > 0$.

Proof. Differentiating (3.4) with respect to τ, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + A_n)e^{\lambda\tau} + c_1}{c_1\lambda(\lambda + d_2\mu_n)} - \frac{\tau}{\lambda},$$

then

$$\begin{aligned}
&sign\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_n^j}\right)^{-1}\right\} \\
=&sign\left\{\mathrm{Re}\left(\frac{(2\lambda + A_n)e^{\lambda\tau} + c_1}{c_1\lambda(\lambda + d_2\mu_n)} - \frac{\tau}{\lambda}\right)\right\}_{\tau=\tau_n^j} \\
=&sign\left\{\frac{2\omega^2 - 2B_n + {A_n}^2 - {c_1}^2}{{c_1}^2\omega^2 + {D_n}^2}\right\} \\
=&sign\left\{\frac{\sqrt{({A_n}^2 - 2B_n - c_1)^2 - 4({B_n}^2 - {D_n}^2)}}{{c_1}^2\omega^2 + {D_n}^2}\right\} > 0.
\end{aligned}$$

Therefore, when $\tau = \tau_n^j$, the transversality condition $\alpha'\left(\tau_n^j\right) = \frac{d\lambda}{d\tau}\Big|_{\tau=\tau_n^j} > 0$ holds. □

Obviously, $\tau_n^0 = \min\limits_{j \in \mathbb{N}_0}\left\{\tau_n^j\right\}$, let $\tau_*^0 = \min\limits_{N_1 \leq n \leq \min\{N_2, N_3\}}\left\{\tau_n^0\right\}$, we have the following theorem.

Theorem 3.1. *Suppose that* (A_0) $-$ (A_3) *hold, if* (A_4)(*or*(A_5))*satisfies, for system* (3.1)*, the following results are true.*

- *(i) When* $\tau \in [0, \tau_*^0)$*, the equilibrium* $P^* = (u_0, v_0)$ *is locally asymptotically stable;*
- *(ii) When* $\tau > \tau_*^0$*, the equilibrium* $P^* = (u_0, v_0)$ *is unstable;*
- *(iii) When* $\tau = \tau_0^j, j \in \mathbb{N}_0$*, the system undergoes Hopf bifurcation at* $P^* = (u_0, v_0)$*, and the bifurcating periodic solutions are homogeneous; When* $\tau \in \{\tau_n^j : \tau_n^j \neq \tau_m^i, m \neq n, N_1 \leq n, m \leq \min\{N_2, N_3\}, j, i \in \mathbb{N}_0\}/\{\tau_0^k \,|k \in \mathbb{N}_0\}$*, the system undergoes Hopf bifurcation at* $P^* = (u_0, v_0)$*, and the bifurcating periodic solutions are inhomogeneous.*

3.2. Direction and periodic solution of Hopf bifurcation

In this section, based on the method of Hassard et al. [9], we shall apply the central manifold theorem and normal form principle to discuss the direction of Hopf bifurcation and the stability of bifurcating periodic solutions. Fix $j \in \mathbb{N}_0, N_1 \leq n \leq \min\{N_2, N_3\}$, denote $\tilde{\tau} = \tau_n^j$, setting $\bar{u}(x,t) = u(x,\tau t) - u_0, \bar{v}(x,t) = v(x,\tau t) - v_0, \tau = \tilde{\tau} + \mu, u_1(t) = u(\cdot,t), u_1(t) = u(\cdot,t), u_2(t) = v(\cdot,t), U = (u_1,u_2)^T$, omitting "-", system (2.1) can be rewritten as

$$\begin{cases} \frac{\partial u}{\partial t} = \tau \left[d_1 \Delta u + r(u+u_0)\left(1 - \frac{u(t-\tau)+u_0}{k}\right) - \frac{c(1-\beta)(u+u_0)^n(v+v_0)}{1+ch(1-\beta)(u+u_0)^n} - qE(u+u_0) \right], \\ \frac{\partial v}{\partial t} = \tau \left[d_2 \Delta v + \frac{ec(1-\beta)(u+u_0)^n(v+v_0)}{1+ch(1-\beta)(u+u_0)^n} - d(v+v_0) \right]. \end{cases} \tag{3.8}$$

Then system (3.8) can be written as an abstract form in the phase space $\ell_1 := C([-1,0],X)$:

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \mu), \tag{3.9}$$

where $L_\mu(\phi)$ and $F(\phi,\mu)$ are defined by

$$L_\mu(\phi) = \mu \begin{pmatrix} a_1\phi_1(0) + a_2\phi_2(0) + c_1\phi_1(-1) \\ a_3\phi_2(0) \end{pmatrix}, \tag{3.10}$$

$$\begin{aligned} F(\phi,\mu) &= \mu D \Delta \phi + L_\mu(\phi) + f(\phi,\mu), \\ f(\phi,\mu) &= (\tilde{\tau}+\mu)(F_1(\phi,\mu), F_2(\phi,\mu))^T, \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} F_1(\phi,\mu) =& r(\phi_1(0)+u_0)\left(1 - \frac{\phi_1(-1)+u_0}{K}\right) - \frac{c(1-\beta)(\phi_1(0)+u_0)^n(\phi_2(0)+v_0)}{1+ch(1-\beta)(\phi_1(0)+u_0)^n} \\ & - qE(\phi_1(0)+u_0) - a_1\phi_1(0) - a_2\phi_2(0) - c_1\phi_1(-1), \\ F_2(\phi,\mu) =& \frac{ec(1-\beta)(\phi_1(0)+u_0)^n(\phi_2(0)+v_0)}{1+ch(1-\beta)(\phi_1(0)+u_0)^n} - d(\phi_2(0)+v_0) - a_3\phi_2(0). \end{aligned}$$

The linearized equation of Eq. (3.8) is

$$L_\mu(U_t) = K_1 U + K_2 U_t, \tag{3.12}$$

where

$$\begin{aligned} & K_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U = (u,v)^T, \quad U_t = (u_t, v_t)^T, \\ & a_1 = f_u = n\left(1 - \frac{dh}{e}\right)\left[r\left(1 - \frac{u_0}{K}\right) - qE\right]u_0, \quad a_2 = f_v = -\frac{d}{e}, \\ & a_3 = g_u = rn\left(1 - \frac{u_0}{K}\right)(e - dh), \quad c_1 = f_{u_t} = -r\frac{u_0}{K}. \end{aligned}$$

The characteristic Eq. (3.12) has a pair of pure imaginary eigenvalues $\Lambda_n = \{i\omega_n\tilde{\tau}, -i\omega_n\tilde{\tau}\}$, consider

$$\frac{dU(t)}{dt} = -\tilde{\tau} D \frac{n^2}{l^2} U_t + L_{\tilde{\tau}}(U_t), \tag{3.13}$$

by Risze theorem, there exists $\eta_k(\tilde{\tau},\theta)\,(-1\le\theta\le 0)$ such that for any $\phi\in C,\psi\in C^*=C\left([0,1],\mathbb{R}^2\right)$, $-\tilde{\tau}D\varphi(0)+L_{\tilde{\tau}}(\varphi)=\int_{-1}^{0}d\eta_k(\tilde{\tau},\theta)\varphi(\theta)$.
Select

$$\eta_k(\tilde{\tau},\theta)=\begin{cases}-\tilde{\tau}K_2,\ \theta=-1,\\ 0,\ \theta\in(-1,0),\\ \tilde{\tau}(K_1-\mu_k D),\ \theta=0.\end{cases}\tag{3.14}$$

Let $A(\tilde{\tau})$ be the infinitesimal generators of the solution semigroup of (3.13), define the bilinear paring

$$\begin{aligned}(\psi(s),\phi(\theta))&=\psi(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi\\&=\psi(0)\phi(0)+\tilde{\tau}\int_{-1}^{0}\psi(\xi+1)\begin{pmatrix}0&0\\a_{21}&0\end{pmatrix}\phi(\xi)d\xi.\end{aligned}\tag{3.15}$$

Under the bilinear pairing, $A(\tilde{\tau})$ is the adjoint operator of A^*. We know $\pm i\omega_n\tilde{\tau}$ are the eigenvalues of $A(\tilde{\tau})$ and A^*. Denote P and P^* as the eigenspaces of $A(\tilde{\tau})$ and A^* corresponding to Λ_n, then P^* and P are conjugate, $\dim P=\dim P^*=2$. And $q(\theta)=q(0)e^{i\omega_n\tilde{\tau}\theta}(-1\le\theta\le 0),\hat{q}^*(s)=\hat{q}^*(0)e^{i\omega_n\tilde{\tau}s}(0\le s\le 1)$, in which, $q(0)=\begin{pmatrix}1\\a\end{pmatrix},\hat{q}^*(0)=\begin{pmatrix}b\\1\end{pmatrix}$. $q(\theta)$ is the eigenvector of operator $A(\tilde{\tau})$ corresponding to eigenvalue $i\omega_n\tilde{\tau}$, $\hat{q}^*(s)$ is the eigenvector of operator A^* corresponding to eigenvalue $-i\omega_n\tilde{\tau}$. namely,

$$\begin{aligned}&\left(i\omega_n I+\mu_k D-K_1-K_2e^{-i\omega_n\tilde{\tau}}\right)q(0)=0,\\&\left(-i\omega_n I+\mu_k D-K_1^T-K_2^Te^{i\omega_n\tilde{\tau}}\right)\hat{q}^*(0)=0.\end{aligned}$$

Then $b=\frac{d_1\mu_n-i\omega_n}{a_2},a=\frac{a_3}{d_2\mu_n+i\omega_n}$. According to $q^*=M\hat{q}^*$ and $(q^*,q)=1$, we have

$$M=\frac{1}{(\overline{\hat{q}^*},\bar{q})}=\left(\frac{d_1c_1\mu_ne^{-i\omega_n\tilde{\tau}}+d_1\mu_n-i\omega_nc_1e^{-i\omega_n\tilde{\tau}}-i\omega_n}{a_2}+\frac{a_3\left(d_2\mu_n+i\omega_n\right)}{\left(d_2\mu_n\right)^2+{\omega_n}^2}\right)^{-1}.$$

Let $\Phi=(\Phi_1,\Phi_2)$ and $\Psi^*=(\Psi_1^*,\Psi_2^*)^T$, then

$$\Phi_1(\theta)=\frac{q(\theta)+\overline{q(\theta)}}{2}=\begin{pmatrix}\mathrm{Re}\left(e^{i\omega_n\tilde{\tau}\theta}\right)\\\mathrm{Re}\left(Me^{i\omega_n\tilde{\tau}\theta}\right)\end{pmatrix},$$

$$\Phi_2(\theta)=\frac{q(\theta)-\overline{q(\theta)}}{2i}=\begin{pmatrix}\mathrm{Im}\left(e^{i\omega_n\tilde{\tau}\theta}\right)\\\mathrm{Im}\left(Me^{i\omega_n\tilde{\tau}\theta}\right)\end{pmatrix},\theta\in(-1,0),$$

$$\Psi_1^*(s)=\frac{q^*(s)+\overline{q^*(s)}}{2}=\begin{pmatrix}\mathrm{Re}\left(e^{-i\omega_n\tilde{\tau}s}\right)\\\mathrm{Re}\left(Ne^{-i\omega_n\tilde{\tau}s}\right)\end{pmatrix},$$

$$\Psi_2^*(s)=\frac{q^*(s)-\overline{q^*(s)}}{2i}=\begin{pmatrix}\mathrm{Im}\left(e^{-i\omega_n\tilde{\tau}s}\right)\\\mathrm{Im}\left(Ne^{-i\omega_n\tilde{\tau}s}\right)\end{pmatrix},s\in(0,1).$$

Define

$$(\Psi^*, \Phi) = \begin{pmatrix} (\Psi_1^*, \Phi_1) & (\Psi_1^*, \Phi_2) \\ (\Psi_2^*, \Phi_1) & (\Psi_2^*, \Phi_2) \end{pmatrix},$$

construct a basis Ψ of P^*, $\Psi = (\Psi_1 \Psi_2)^T = (\Psi^*, \Phi)^{-1}\Psi^*$, then $(\Psi, \Phi) = I_2$. Define $f_n = (\varphi_n^1, \varphi_n^2)$ and

$$\alpha \cdot f_n = \alpha_1 \varphi_n^1 + \alpha_2 \varphi_n^2, \alpha = (\alpha_1, \alpha_2)^T \in C.$$

In addition, in Hilbert space X_C, define the inner product $\langle \cdot, \cdot \rangle$ of the complex value L^2: for any $U_1 = (u_1, u_2), U_2 = (v_1, v_2) \in X_C$,

$$\langle U_1, U_2 \rangle = \frac{1}{l\pi} \int_0^{l\pi} (u_1 \bar{v}_1 + u_2 \bar{v}_2) dx,$$

and for $\phi \in C([-1, 0], X)$, $\langle \phi, f_1 \rangle = (\langle \phi, \beta_1^1 \rangle, \langle \phi, \beta_1^2 \rangle)$. So when $\alpha = 0$, the central subspace of (3.12) is $P_{CN}\mathbb{C}$, and

$$P_{CN}\mathbb{C}(\phi) = \Phi(\Psi, \langle \phi, f_1 \rangle) \cdot f_1, \phi \in \mathbb{C},$$
$$P_S\mathbb{C} = \{(q(\theta)z + \bar{q}(\theta)\bar{z}) \cdot f_1, z \in \mathbb{C}\}.$$

Decompose $\mathbb{C}$ into $\mathbb{C} = P_{CN}\mathbb{C} \oplus P_S\mathbb{C}$, where $P_S\mathbb{C}$ is the complementary subspace of $P_{CN}\mathbb{C}$ in $\mathbb{C}$. Let $A_{\tilde{\tau}}$ be the infinitesimal generators of semigroup included by the solutions of (3.12), then Eq. (3.9) can be written in abstract form

$$\frac{dU(t)}{dt} = A_{\tilde{\tau}} U_t + X_0 F(U_t, \mu), \tag{3.16}$$

where $X_0(\theta) = \begin{cases} 0, -1 \le \theta < 0, \\ I, \quad \theta = 0. \end{cases}$ Then the solution of (3.16) is

$$U_t = \Phi(\Psi, < U_t, f_n >) f_n + h(x_1, x_2, \mu),$$
$$U(t) = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1, x_2, \mu), \tag{3.17}$$

in which,

$$h(x_1, x_2, \mu) \in P_s C, h(0,0,0) = 0, Dh(0,0,0) = 0.$$

Therefore, on the central manifold, the solution of Eq. (3.9) is

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \tag{3.18}$$

Let $z = x_1 - \mathrm{i}x_2$ and $p_1 = \Phi_1 + \mathrm{i}\Phi_2$, then we have

$$U_t = \frac{1}{2}(p_1 z + \overline{p_1} z) f_n + h\left(\frac{z + \bar{z}}{2}, \frac{\mathrm{i}(z - \bar{z})}{2}, 0\right) = \frac{1}{2}(p_1 z + \overline{p_1}\bar{z}) f_n + W(z, \bar{z}). \tag{3.19}$$

By [25], z satisfies

$$\dot{z} = i\omega_n \tilde{\tau} z + g(z, \bar{z}), \tag{3.20}$$

where

$$g(z, \bar{z}) = (\Psi_1(0) - i\Psi_2(0)) \langle F(U_t, 0), f_n \rangle, \tag{3.21}$$

Set

$$W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11} z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.22}$$

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{3.23}$$

Comparing coefficients, we can obtain

$$\begin{aligned}
g_{20} = \bar{M} \Bigg\{ & \bar{b} \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right) \left(1 - \frac{dh}{e}\right) \left(n - 1 - 2n\frac{dh}{e}\right) \right. \\
& \left. -2nd\frac{e-dh}{e^2}\frac{1}{u_0}a - 2\frac{r}{K}e^{-i\omega_n\tilde{\tau}} \right] \\
& + \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh) \left(n - 1 - 2n\frac{dh}{e}\right) + 2nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}a \right] \Bigg\}, \\
g_{11} = \bar{M} \Bigg\{ & \bar{b} \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right) \left(1 - \frac{dh}{e}\right) \left(n - 1 - 2n\frac{dh}{e}\right) \right. \\
& \left. -nd\frac{e-dh}{e^2}\frac{1}{u_0}(a + \bar{a}) - \frac{r}{K}\left(e^{i\omega_n\tilde{\tau}} + e^{-i\omega_n\tilde{\tau}}\right) \right] \\
& + \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh) \left(n - 1 - 2n\frac{dh}{e}\right) + nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}(a + \bar{a}) \right] \Bigg\}, \\
g_{02} = \bar{M} \Bigg\{ & \bar{b} \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right) \left(1 - \frac{dh}{e}\right) \left(n - 1 - 2n\frac{dh}{e}\right) - 2nd\frac{e-dh}{e^2}\frac{1}{u_0}\bar{a} - 2\frac{r}{K}e^{i\omega_n\tilde{\tau}} \right] \\
& + \left[rn \left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh) \left(n - 1 - 2n\frac{dh}{e}\right) + 2nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}\bar{a} \right] \Bigg\}, \\
g_{21} = {} & \frac{3}{8}\bar{M}\left(\bar{b}Q_1 + Q_2\right) + \bar{M}\left(\bar{b}\int_\Omega Q_3 b_k^2 dx + \int_\Omega Q_4 b_k^2 dx\right), b_k = \cos\frac{k\pi}{l}, k \in \mathbb{N}_0,
\end{aligned}$$

where

$$\begin{aligned}
Q_1 = {} & rn\left(1 - \frac{dh}{e}\right)\left(\frac{1}{{u_0}^2} - \frac{1}{u_0 K}\right)\left(n - 2 - 2n\frac{dh}{e} - 2n^2 dh\frac{e-dh}{e^2}\right) \\
& + \frac{1}{{u_0}^2}nd\left(1 - \frac{dh}{e}\right)[e(n-1) - 2nd](2a + \bar{a}), \\
Q_2 = {} & rn(e - dh)\left(\frac{1}{{u_0}^2} - \frac{1}{u_0 K}\right)\left(n - 2 - 2n\frac{dh}{e} - 2n^2 dh\frac{e-dh}{e^2}\right) \\
& + \frac{1}{{u_0}^2}nd(e - dh)[e(n-1) - 2nd](2a + \bar{a}), \\
Q_3 = {} & rn\left(\frac{1}{u_0} - \frac{1}{K}\right)\left(1 - \frac{dh}{e}\right)\left(n - 1 - 2n\frac{dh}{e}\right)\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) \\
& - 2nd\frac{e-dh}{e^2}\frac{1}{u_0}\left(\frac{1}{2}W_{20}^{(2)}(0) + \frac{\bar{a}}{2}W_{20}^{(1)}(0) + aW_{11}^{(1)}(0) + W_{11}^{(2)}(0)\right)
\end{aligned}$$

$$
\begin{aligned}
&- 2\frac{r}{K}\left(\frac{1}{2}W_{20}^{(1)}(-1) + \frac{1}{2}W_{20}^{(1)}(0)e^{i\omega_n\tilde{\tau}} + W_{11}^{(1)}(-1) + W_{11}^{(1)}(0)e^{-i\omega_n\tilde{\tau}}\right),\\
Q_4 &= rn\left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh)\left(n - 1 - 2n\frac{dh}{e}\right)\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right)\\
&- 2nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}\left(\frac{1}{2}W_{20}^{(2)}(0) + \frac{\bar{a}}{2}W_{20}^{(1)}(0) + aW_{11}^{(1)}(0) + W_{11}^{(2)}(0)\right).
\end{aligned}
$$

Because g_{21} depends on $W_{20}(\theta)$ and $W_{11}(\theta)$, so we calculate $W_{20}(\theta)$ and $W_{11}(\theta), \theta \in [-1, 0]$ below. By [25], we have

$$
\dot{W}(z,\bar{z}) = W_{20}(\theta)z\dot{z} + W_{11}(\theta)\dot{z}\bar{z} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\bar{z}\dot{z} + \cdots, \tag{3.24}
$$

$$
A_{\tilde{\tau}}W = A_{\tilde{\tau}}W_{20}(\theta)\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}(\theta)z\bar{z} + A_{\tilde{\tau}}W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.25}
$$

We know that $\dot{W}(z,\bar{z})$ satisfies $\dot{W} = A_{\tilde{\tau}}W + H(z,\bar{z},\theta)$, and

$$
\dot{W} = \dot{u}_t - \dot{z}qb_k - \bar{z}\,\bar{q}b_k = \begin{cases} W - 2\,\mathrm{Re}\{g(z,\bar{z})q(\theta)\}b_k, & \theta \in [-1, 0),\\ AW - 2\,\mathrm{Re}\{g(z,\bar{z})q(\theta)\}b_k + \tilde{F}, & \theta = 0,\end{cases} \tag{3.26}
$$

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.
$$

Clearly,

$$
H_{20}(\theta) = \begin{cases} -g_{20}q(\theta)b_k - \bar{g}_{02}\bar{q}(\theta)b_k, & \theta \in [-1, 0),\\ -g_{20}q(0)b_k - \bar{g}_{02}\bar{q}(0)b_k + \tilde{F}''_{zz}, & \theta = 0,\end{cases}
$$

$$
H_{11}(\theta) = \begin{cases} -g_{11}q(\theta)b_k - \bar{g}_{11}\bar{q}(\theta)b_k, & \theta \in [-1, 0),\\ -g_{11}q(0)b_k - \bar{g}_{11}\bar{q}(0)b_k + \tilde{F}''_{z\bar{z}}, & \theta = 0.\end{cases}
$$

According to Eqs. (3.24) and (3.26),

$$
(A_{\tilde{\tau}} - 2i\omega_n\tilde{\tau})W_{20}(\theta) = -H_{20}(\theta), \quad A_{\tilde{\tau}}W_{11}(\theta) = -H_{11}(\theta), \cdots. \tag{3.27}
$$

Through calculation, we have

$$
\begin{aligned}
W_{20}(\theta) &= -\frac{g_{20}}{i\omega_0\tilde{\tau}}q(0)e^{i\omega_n\tilde{\tau}\theta}b_k - \frac{\bar{g}_{02}}{3i\omega_0\tilde{\tau}}\bar{q}(0)e^{-i\omega_n\tilde{\tau}\theta}b_k + E_1e^{2i\omega_n\tilde{\tau}\theta},\\
W_{11}(\theta) &= \frac{g_{11}}{i\omega_n\tilde{\tau}}q(0)e^{i\omega_n\tilde{\tau}\theta}b_k - \frac{\bar{g}_{11}}{i\omega_n\tilde{\tau}}\bar{q}(0)e^{-i\omega_n\tilde{\tau}\theta}b_k + E_2.
\end{aligned} \tag{3.28}
$$

When $\theta = 0$, from (3.27) and (3.28), we have

$$
(2i\omega_n\tilde{\tau} - A_{\tilde{\tau}})E_1e^{2i\omega_n\tilde{\tau}\theta}\big|_{\theta=0} = F_{20}b_k^2,\ A_{\tilde{\tau}}E_2|_{\theta=0} = -F_{11}b_k^2,
$$

where $F_{20} = \left(F_{20}^{(1)}, F_{20}^{(2)}\right)^T, F_{11} = \left(F_{11}^{(1)}, F_{11}^{(2)}\right)^T$, with

$$
\begin{aligned}
F_{20}^{(1)} &= rn\left(\frac{1}{u_0} - \frac{1}{K}\right)\left(1 - \frac{dh}{e}\right)\left(n - 1 - 2n\frac{dh}{e}\right) - 2nd\frac{e-dh}{e^2}\frac{1}{u_0}a - 2\frac{r}{K}e^{-i\omega_n\tilde{\tau}}, \\
F_{20}^{(2)} &= rn\left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh)\left(n - 1 - 2n\frac{dh}{e}\right) - 2nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}a, \\
F_{11}^{(1)} &= rn\left(\frac{1}{u_0} - \frac{1}{K}\right)\left(1 - \frac{dh}{e}\right)\left(n - 1 - 2n\frac{dh}{e}\right) \\
&\quad - nd\frac{e-dh}{e^2}\frac{1}{u_0}(a + \bar{a}) - \frac{r}{K}\left(e^{i\omega_n\tilde{\tau}} + e^{-i\omega_n\tilde{\tau}}\right), \\
F_{11}^{(2)} &= rn\left(\frac{1}{u_0} - \frac{1}{K}\right)(e - dh)\left(n - 1 - 2n\frac{dh}{e}\right) + nd\left(1 - \frac{dh}{e}\right)\frac{1}{u_0}(a + \bar{a}).
\end{aligned}
$$

Suppose that $b_k^2 = \sum\limits_{k=1}^{\infty} c_k b_k$, where c_k is the coordinate, we have

$$
\begin{aligned}
E_1 &= \sum_{k=1}^{\infty}\left(2i\omega_n + \mu_k D - K_1 - K_2 e^{-2i\omega_n\tilde{\tau}}\right)^{-1} F_{20} c_k b_k, \\
E_2 &= \sum_{k=1}^{\infty}(\mu_k D - K_1 - K_2)^{-1} F_{11} c_k b_k,
\end{aligned}
$$

where

$$
\begin{aligned}
&\left(2i\omega_n + \mu_k D - K_1 - K_2 e^{-2i\omega_n\tilde{\tau}}\right)^{-1} \\
=&\frac{1}{\alpha_1^k}\begin{pmatrix} 2i\omega_n + d_2\mu_k & a_2 \\ a_3 & 2i\omega_n + d_1\mu_k - a_1 - c_1 e^{-2i\omega_n\tilde{\tau}} \end{pmatrix}, \\
&(\mu_k D - K_1 - K_2)^{-1} = \frac{1}{\alpha_2^k}\begin{pmatrix} d_2\mu_k & a_2 \\ a_3 & d_1\mu_k - a_1 - c_1 \end{pmatrix}, \\
&\alpha_1^k = -4{\omega_n}^2 - a_2 a_3 - \left[d_2\left(a_1 + c_1 e^{-2i\omega_n\tilde{\tau}}\right)\right]\mu_k + d_1 d_2 \mu_k^2 + 2i\omega_n(d_1 + d_2)\mu_k \\
&\qquad - 2i\omega_n\left(a_1 + c_1 e^{-2i\omega_n\tilde{\tau}}\right), \\
&\alpha_2^k = -a_2 a_3 - \left[d_2(a_1 + c_1)\right]\mu_k + d_1 d_2 \mu_k^2.
\end{aligned}
$$

So far, all the unknown terms in (3.21) are obtained, so that its norm form coefficients can be calculated and the following quantities can be calculated:

$$
\begin{cases}
C_1(0) = \frac{i}{2\omega_n\tilde{\tau}}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{1}{2}g_{21}, \\
\mu_2 = -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\lambda'(\tilde{\tau}))}, \\
\beta_2 = 2\,\mathrm{Re}\left(C_1(0)\right), \\
T_2 = -\frac{\mathrm{Im}(C_1(0)) + \mu_2\,\mathrm{Im}\left(\lambda'(\tilde{\tau})\right)}{\omega_n\tilde{\tau}}.
\end{cases}
$$

Then we have the following theorem.

Theorem 3.2. *For the critical value τ_n^j, $\mu_2 > 0(resp.\mu_2 < 0)$, the Hopf bifurcation is forward(resp. backward); $\beta_2 < 0(resp.\beta_2 > 0)$, the bifurcating periodic solutions are stable(resp. unstable); $T_2 > 0(resp.T_2 < 0)$, the period increases(resp. decreases).*

4. Biological significance

Owing to limited resources and uneven spatial distribution of the population, organisms will search for food everywhere in order to survive, and then migration and diffusion will occur. Population diffusion is a manifestation of biological adaptability, when the population density is too high, it can expand its distribution area. Different populations can find new environment and food in different seasons, adapt to environmental change, and prevent the adverse consequences of inbreeding through individual exchange within and between populations. Controlling the habitat complexity and the intensity of harvesting effect can predict the change of prey and predator, and protect the cubs from predation during lactation. When the population quantity is small, the complexity of habitat reduces the encounter rate between predator and prey, thus reducing the predation rate. Therefore, the habitat complexity effect on the interaction between predator and prey can not be ignored.

$P_0 = (0, 0)$ means that both predator and prey are extinct, which indicates that when the intensity of habitat complexity effect is low, the prey is quickly eaten by the predator, resulting in a sharp reduction of the prey to extinction, and ultimately leading to the extinction of predator without food. $P_1 = \left(K\left(1 - \frac{qE}{r}\right), 0\right)$ means the extinction of predator, which shows that when the intensity of habitat complexity effect is high, the predator cannot get food, the mortality rate of predator is higher than the growth rate, and the predator eventually die. The prey is absolutely safe and the number eventually stabilizes at the maximum carrying capacity of environment. The coexistence equilibrium $P^* = (u_0, v_0)$ means that when the intensity of habitat complexity effect is low, if the predator's predation ability is low and production delay is low, then predator and prey can coexist in time and space, and the population quantity will remain near the stable value.

Hopf bifurcation is an important dynamic bifurcation to describe periodic phenomena. When the system parameter passes a certain critical value τ_n^j, the local stability of the equilibrium changes, and the small amplitude periodic solution is generated on one side of the critical point. Diffusion term and production delay cause Hopf bifurcation at $P^* = (u_0, v_0)$, the system has spatially homogeneous or inhomogeneous periodic solutions, that is, if the production delay is close to Hopf bifurcation value, the system may have stable periodic solutions, at this time, predator and prey can coexist, but the population quantity will have stable periodic solutions.

5. Numerical simulations

We study the dynamic behaviors of systems with and without time delay, and analyze the effects of habitat complexity effect β and production delay τ on the stability of equilibrium. In the following, we shall verify the reliability of theoretical results by numerical simulations, here we only consider the case $n = 1$.

1.Stability of semilinear parabolic equation without delay
In system (2.1), select parameters as

$$r = 0.9, K = 300, c = 0.46, e = 0.058, h = 0.053, d = 0.6, q = 0.2, E = 0.5.$$

By Theorem 2.1, when $0 < \beta < 0.4726$, system (2.1) is unstable at $P^* = (u_0, v_0)$, when $0.4726 < \beta < 0.8134$, system (2.1) is locally asymptotically stable at $P^* = (u_0, v_0)$. When $\beta = 0.5, d_1 = 1, d_2 = 0.5$, by calculation, $P^* = (99.568, 4.825)$, system (2.1) is locally asymptotically stable at $P^* = (99.568, 4.825)$(see Fig.1). When $\beta = 0.4, d_1 = 1, d_2 = 0.5$, by calculation, $P^* = (82.974, 4.419)$, system (2.1) is unstable at $P^* = (82.974, 4.419)$, and the periodic solutions appear near the equilibrium(see Fig.2). By Theorem 2.4, when $\frac{d_1}{d_2} > 0.37378$, select $d_1 = 1, d_2 = 2, \beta = 0.203$, we have $\delta = \delta_0^H = 61.71$, the system produces spatially homogeneous periodic solutions(see Fig.3).

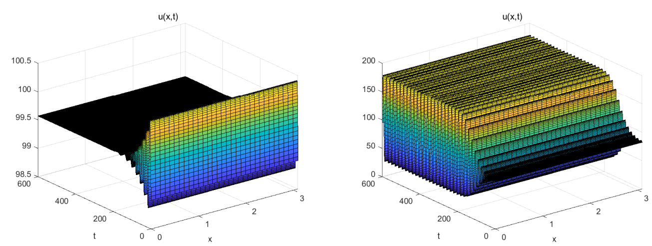

Figure 1. $P^* = (99.568, 4.825)$ is locally asymptotically stable, and the initial value is $(99.5, 4.8)$.

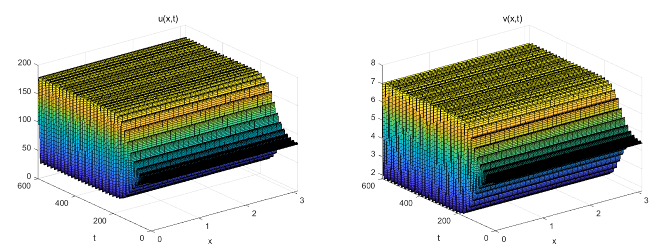

Figure 2. The system produces periodic solutions, and the initial value is $(82.9, 4.4)$.

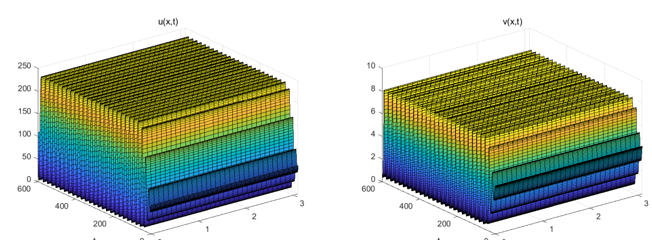

Figure 3. The system produces spatially homogeneous periodic solutions, and the initial value is $(62.2, 4.2)$.

2. Stability of semilinear parabolic equation with time delay

In system (2.1), select parameters:

$$\begin{aligned} &d_1 = 1, d_2 = 0.5, r = 0.9, K = 300, c = 0.46, \\ &e = 0.058, h = 0.053, d = 0.6, q = 0.2, E = 0.5. \end{aligned}$$

When $n = 1$, setting $\beta = 0.5$, then $P^* = (99.568, 4.825), \tau_0^0 \approx 1.141, \omega_0 = 0.346$, the initial values are $u_0(x) = 99.568 + 0.1 * \sin x, v_0(x) = 4.825 + 0.1 * \cos x$. By theorem, when $\tau \in (0, \tau_0^0]$, $P^* = (u_0, v_0)$ is locally asymptotically stable(see Fig.4). When τ crosses τ_0^0, $P^* = (u_0, v_0)$ loses stability, Hopf bifurcation occurs(see Fig.5). By Theorem 3.2, $c_1(\tau_0^0) = -9.018 - 69.366i$, $\lambda'(\tau_0^0) = 1.235 + 2.1523i$, thus we have

$$\begin{aligned} &\operatorname{Re}\left(c_1\left(\tau_0^0\right)\right) \approx -9.018, \quad \operatorname{Im}\left(c_1\left(\tau_0^0\right)\right) \approx 69.366, \\ &\operatorname{Re}\left(\lambda'\left(\tau_0^0\right)\right) \approx 1.235, \quad \operatorname{Im}\left(\lambda'\left(\tau_0^0\right)\right) \approx 2.1523, \\ &\mu_2 \approx 7.302 > 0, \quad \beta_2 \approx -215.515 < 0, \quad T_2 \approx -18.036 < 0. \end{aligned}$$

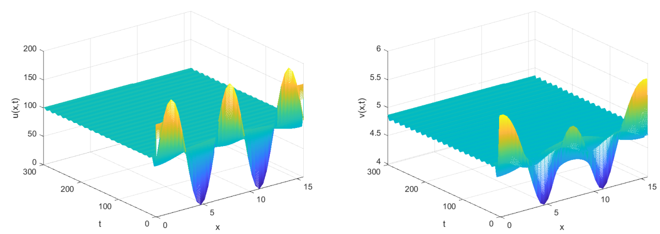

Figure 4. $\tau = 1 < \tau_0$, the system is locally asymptotically stable at $P^* = (u_0, v_0)$.

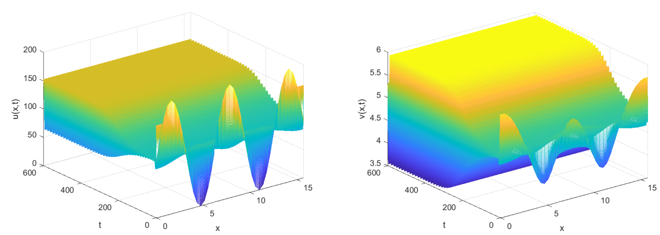

Figure 5. $\tau = 1.2 > \tau_0$, the system produces periodic solutions at $P^* = (u_0, v_0)$.

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