Journal of Applied Analysis and Computation
Volume 11, Number 5, October 2021, 2601–2610
Website:http://www.jaac-online.com
DOI:10.11948/20210043

MAXIMAL AND MINIMAL NONDECREASING BOUNDED SOLUTIONS OF A SECOND ORDER ITERATIVE FUNCTIONAL DIFFERENTIAL EQUATION*

Hou Yu Zhao† and Jing Chen

Abstract In this paper, we use the method of lower and upper solutions to study the maximal and minimal nondecreasing bounded solutions of a second order iterative functional differential equation.

Keywords Iterative differential equations, lower and upper solutions, maximal and minimal nondecreasing bounded solutions.

MSC(2010) 39B12, 39B82.

1. Introduction

Let $x = x(t)$ be a real function and $x^{[i]}(t)$ denote its i-th iterate, i.e., $x^{[i]}(t) = x(x^{[i-1]}(t)), x^0(t) = t$. Iterative functional differential equations as an important class of functional differential equations with state-dependent delays, modeled extensively in many fields such as classical electrodynamics, commodity price fluctuations, populations. Many papers concerned with the first order equations ([1–6, 8, 12, 15–18]), there are only few results about second order iterative functional differential equations. Petahov [11] gave the existence and uniqueness of solutions with a boundary value condition for the second order equation

$$x''(t) = x(x(t)).$$

Latter, Si and his collaborators in [13, 14] further discussed the analytic solutions of equation

$$x''(t) = x^{[m]}(t)$$

and

$$x''(t) = f\Big(\sum_{i=0}^{m} c_i x^{[i]}(t)\Big) + G(t).$$

†The corresponding author. Email address: houyu19@gmail.com(H. Y. Zhao)
School of mathematics, Chongqing Normal University, Chongqing, 401331, China

*This research was supported by the Natural Science Foundation of Chongqing(Grant No. cstc2020jcyj-msxmX0857, cstc2018jcyjAX0418), Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN201900525), Foundation of youth talent of Chongqing Normal University (02030307-00039).

In 2018, using Schauder fixed point theorem, Kaufmann [10] considered the boundary-value problem of

$$x''(t) = f(t, x(t), x(x(t))).$$

In this paper, using the method in [7] which studied the bounded solutions, we consider the existence of maximal and minimal nondecreasing bounded solutions of

$$\alpha x''(t) + \beta x'(t) = g(t, x(t), x^{[2]}(t), \ldots, x^{[n]}(t)), \quad \forall t \in \mathbb{R}. \tag{1.1}$$

For convenience, we use $C(\mathbb{R}, \mathbb{R})$ to denote the set of all real valued continuous functions from $\mathbb{R}$ into $\mathbb{R}$, endowed with the usual metric $d(f, g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m}$ for $\|f - g\|_m = \max_{t\in[-m,m]} |f(t) - g(t)|$, so the topology on $C(\mathbb{R}, \mathbb{R})$ is the uniform convergence on each compact intervals of $\mathbb{R}$. We also consider the set $BC(\mathbb{R}, \mathbb{R})$ of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the norm $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$, so the topology on $BC(\mathbb{R}, \mathbb{R})$ is the uniform convergence on $\mathbb{R}$. For $M, L > 0$, define

$$B_C(M, L) = \Big\{\varphi \in C(\mathbb{R}, \mathbb{R}) \Big| |\varphi(t)| \leq M, |\varphi(t_2) - \varphi(t_1)| \leq L|t_2 - t_1|, \text{ for all } t, t_1, t_2 \in \mathbb{R}\Big\}.$$

By the Arzelá-Ascoli theorem, the subset $B_C(M, L)$ is compact in $C(\mathbb{R}, \mathbb{R})$.

In order to study (1.1) by using the method of lower and upper solutions, we recall a definition as in [9].

Definition 1.1. $\varphi_0 \in C^2(\mathbb{R}, \mathbb{R})$ is called a lower solution of (1.1) if it satisfies the following condition

$$\alpha\varphi_0''(t) + \beta\varphi_0'(t) \leq g(t, \varphi_0^{[1]}(t), \varphi_0^{[2]}(t), \ldots, \varphi_0^{[n]}(t)), \ \forall t \in \mathbb{R},$$

and $\psi_0 \in C^2(\mathbb{R}, \mathbb{R})$ is called an upper solution of (1.1) if it satisfies the following condition

$$\alpha\psi_0''(t) + \beta\psi_0'(t) \geq g(t, \psi_0^{[1]}(t), \psi_0^{[2]}(t), \ldots, \psi_0^{[n]}(t)), \ \forall t \in \mathbb{R}.$$

We wish to find a nondecreasing $x \in BC(\mathbb{R}, \mathbb{R})$ satisfying $\varphi_0(t) \leq x(t) \leq \psi_0(t)$ and (1.1) on $\mathbb{R}$, where φ_0 and ψ_0 are defined as in Definition 1.1. Our method is based on a monotone iteration approach. This paper is organized as follows. In Section 2, we establish the existence of maximal and minimal nondecreasing bounded solutions for (1.1). In Section 3, we give some examples to illustrate our result.

2. Existence of maximal and minimal nondecreasing bounded solutions

In this section, the existence of maximal and minimal nondecreasing bounded solutions of equation (1.1) is proved. We will assume that the following conditions hold

(H1)

$$G = \sup_{t\in\mathbb{R},\, \tilde{\varphi}_i \leq x_i \leq \tilde{\psi}_i, \quad i=1,2,\cdots,n} |g(t, x_1, x_2, \cdots, x_n)| < \infty.$$

(H2) There is a constant $\gamma = \frac{\beta^2}{4\alpha} > 0$ such that

$$g(t_1, x_1, x_2, \cdots, x_n) - g(t_2, y_1, y_2, \cdots, y_n) \leq -\gamma(x_1 - y_1)$$

for all $t_1, t_2, x_i, y_i \in \mathbb{R}$ such that

$$t_1 \leq t_2, \quad \tilde{\varphi}_i \leq x_i \leq y_i \leq \tilde{\psi}_i, \quad i = 1, 2, \cdots, n.$$

We begin with the following lemma.

Lemma 2.1. *Suppose that $\varphi \in BC(\mathbb{R}, \mathbb{R})$, $h \in BC(\mathbb{R}^{n+1}, \mathbb{R})$ and $\alpha, \beta, \gamma > 0$ (or $\alpha, \beta, \gamma < 0$) are given. Then $x \in BC(\mathbb{R}, \mathbb{R})$ is a solution of equation*

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = h(t, \varphi(t), \varphi^{[2]}(t), \ldots, \varphi^{[n]}(t)), \tag{2.1}$$

if and only if

$$x(t) = \frac{1}{\alpha} \int_{-\infty}^{t} e^{\frac{\overline{\beta}}{\alpha}(u-t)} \int_{-\infty}^{u} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) e^{\frac{\widetilde{\beta}}{\alpha}(s-u)} ds du. \tag{2.2}$$

where $\widetilde{\beta} + \overline{\beta} = \beta$, $\widetilde{\beta}\overline{\beta} = \alpha\gamma$, with $\alpha, \widetilde{\beta}, \overline{\beta}, \gamma > 0$ or $\alpha, \widetilde{\beta}, \overline{\beta}, \gamma < 0, \forall t \in \mathbb{R}$.

Proof. By direct calculation, we can see that (2.2) is a solution of (2.1).

Suppose $x \in BC(\mathbb{R}, \mathbb{R})$ is a solution of (2.1), then it is easy to find Eq (2.1) can be written in the form of

$$\begin{aligned} & x''(t) e^{\frac{\widetilde{\beta}}{\alpha} t} + \frac{\widetilde{\beta}}{\alpha} x'(t) e^{\frac{\widetilde{\beta}}{\alpha} t} + \frac{\overline{\beta}}{\alpha} x'(t) e^{\frac{\widetilde{\beta}}{\alpha} t} + \frac{\gamma}{\alpha} x(t) e^{\frac{\widetilde{\beta}}{\alpha} t} \\ = & \frac{1}{\alpha} h(t, \varphi(t), \varphi^{[2]}(t), \ldots, \varphi^{[n]}(t)) e^{\frac{\widetilde{\beta}}{\alpha} t}, \end{aligned}$$

or

$$\left(x'(t) e^{\frac{\widetilde{\beta}}{\alpha} t}\right)' + \frac{\overline{\beta}}{\alpha} \left(x(t) e^{\frac{\widetilde{\beta}}{\alpha} t}\right)' = \frac{1}{\alpha} h(t, \varphi(t), \varphi^{[2]}(t), \ldots, \varphi^{[n]}(t)) e^{\frac{\widetilde{\beta}}{\alpha} t}. \tag{2.3}$$

Integrating (2.3) from $-\infty$ to t and using $\alpha\widetilde{\beta} > 0$ we obtain

$$x'(t) e^{\frac{\widetilde{\beta}}{\alpha} t} + \frac{\overline{\beta}}{\alpha} x(t) e^{\frac{\widetilde{\beta}}{\alpha} t} = \frac{1}{\alpha} \int_{-\infty}^{t} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) e^{\frac{\widetilde{\beta}}{\alpha} s} ds,$$

i.e.,

$$x'(t) + \frac{\overline{\beta}}{\alpha} x(t) = \frac{1}{\alpha} \int_{-\infty}^{t} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) e^{\frac{\widetilde{\beta}}{\alpha}(s-t)} ds, \tag{2.4}$$

then we have

$$x'(t) e^{\frac{\overline{\beta}}{\alpha} t} + \frac{\overline{\beta}}{\alpha} x(t) e^{\frac{\overline{\beta}}{\alpha} t} = \frac{1}{\alpha} e^{\frac{\overline{\beta}}{\alpha} t} \int_{-\infty}^{t} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) e^{\frac{\widetilde{\beta}}{\alpha}(s-t)} ds,$$

i.e.,

$$\left(x(t) e^{\frac{\overline{\beta}}{\alpha} t}\right)' = \frac{1}{\alpha} e^{\frac{\overline{\beta}}{\alpha} t} \int_{-\infty}^{t} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s)) e^{\frac{\widetilde{\beta}}{\alpha}(s-t)} ds,$$

Integrating it from $-\infty$ to t and using the fact $\alpha\overline{\beta} > 0$ we get

$$x(t) = \frac{1}{\alpha}\int_{-\infty}^{t} e^{\frac{\overline{\beta}}{\alpha}(u-t)}\int_{-\infty}^{u} h(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s))e^{\frac{\widetilde{\beta}}{\alpha}(s-u)}dsdu.$$

This completes the proof. □

Remark 2.1. If $\beta^2 = 4\alpha\gamma$, taking $\widetilde{\beta} = \overline{\beta} = \frac{\beta}{2} = \sqrt{\alpha\gamma}$ in (2.2), then $x \in BC(\mathbb{R},\mathbb{R})$ is a solution of equation (2.1) can be written by

$$x(t) = \frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u} h(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}dsdu. \tag{2.5}$$

Theorem 2.1. *Assume that* (**H1**), (**H2**) *hold,* $\alpha,\beta > 0$ *and* $g(t,x_1,x_2,\cdots,x_n)$ *is a continuous function on* $\mathbb{R}^{n+1}$. *Suppose that* (1.1) *has a lower solution* $\varphi_0(t)$ *and an upper solution* $\psi_0(t)$ *with* $\varphi_0,\psi_0 \in BC(\mathbb{R},\mathbb{R})$ *and*

$$\begin{aligned}&\varphi_0(t) \le \psi_0(t), \quad \forall\ t \in \mathbb{R},\\ &\varphi_0(t) \text{ and } \psi_0(t) \text{ are nondeacresing on } \mathbb{R}.\end{aligned} \tag{2.6}$$

Moreover, setting

$$\tilde{\varphi}_i = \inf_{t\in\mathbb{R}} \varphi_0^{[i]}(t), \quad \tilde{\psi}_i = \sup_{t\in\mathbb{R}} \psi_0^{[i]}(t), \quad i = 1,2,\cdots,n.$$

Then (1.1) *has a minimal nondecreasing bounded solution* $\varphi_*(t)$ *and a maximal nondecreasing bounded solution* $\psi_*(t)$. *Moreover,*

$$\varphi_0 \le \varphi_* \le \psi_* \le \psi_0.$$

Furthermore, set

$$\varphi_k = A\varphi_{k-1}, \quad \psi_k = A\psi_{k-1} \tag{2.7}$$

for $k \in \mathbb{N}$. *Then* $\{\varphi_k\}_{k=1}^{\infty}$ *and* $\{\psi_k\}_{k=1}^{\infty}$ *are monotonically convergent to* φ_* *and* ψ_* *in* $C(\mathbb{R},\mathbb{R})$, *respectively, and any nondecreasing bounded solution* $x(t)$ *of* (1.1) *in* $[\varphi_0,\psi_0]$ *belongs to* $[\varphi_*,\psi_*]$.

Proof. For $\varphi \in BC(\mathbb{R},\mathbb{R})$, we consider an auxiliary equation

$$\alpha x''(t) + \beta x'(t) + \gamma x(t) = h(t,\varphi(t),\varphi^{[2]}(t),\ldots,\varphi^{[n]}(t)), \tag{2.8}$$

where

$$h(t,x_1,x_2,\ldots,x_n) = g(t,x_1,x_2,\ldots,x_n) + \gamma x_1$$

and $\gamma = \frac{\beta^2}{4\alpha}$. From remark 2.1 and (**H1**), we know that (2.8) has exactly one solution

$$x_\varphi(t) = \frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u} h(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}dsdu \tag{2.9}$$

in $BC(\mathbb{R},\mathbb{R})$.

Following (2.9), we consider a map $A : BC(\mathbb{R},\mathbb{R}) \to BC(\mathbb{R},\mathbb{R})$ defined as follows:

$$(A\varphi)(t) = \frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u} h(s,\varphi(s),\varphi^{[2]}(s),\ldots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-u)}dsdu. \tag{2.10}$$

We take

$$D(M,L)=\{x\in B_C(M,L):\varphi_0\le x\le\psi_0,\ x(t)\text{ is nondeacresing on }\mathbb{R}\}$$

with

$$L=\frac{\beta}{2\alpha}M+\frac{2}{\beta}(G+\gamma\max\{\|\varphi_0\|,\|\psi_0\|\}),\tag{2.11}$$
$$M=\max\left\{\max\{\|\varphi_0\|,\|\psi_0\|\},\frac{G+\gamma\max\{\|\varphi_0\|,\|\psi_0\|\}}{\gamma}\right\},\ \gamma=\frac{\beta^2}{4\alpha}.$$

Note $\varphi_0,\psi_0\in D(M,L)$. First, we show that

$$A:D(M,L)\to D(M,L).\tag{2.12}$$

Indeed, if $\varphi\in D(M,L)$, then $\varphi_0(t)\le\varphi(t)\le\psi_0(t)$ for all $t\in\mathbb{R}$, so

$$\|\varphi\|\le\max\{\|\varphi_0\|,\|\psi_0\|\}.$$

Since φ_0, ψ_0 and φ are nondecreasing, we have

$$\tilde{\varphi}_i\le\varphi_0^{[i]}(t)\le\varphi^{[i]}(t)\le\psi_0^{[i]}(t)\le\tilde{\psi}_i,\quad t\in\mathbb{R},\quad i=1,2,\cdots,n.\tag{2.13}$$

Then (**H1**) implies

$$\begin{aligned}|h(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s))|&\le|g(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s)|+\gamma|\varphi(s)|\\&\le G+\gamma\max\{\|\varphi_0\|,\|\psi_0\|\}.\end{aligned}$$

Thus

$$\begin{aligned}\Big|(A\varphi)(t)\Big|&\le\left|\frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u}h(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}dsdu\right|\\&\le\frac{G+\gamma\max\{\|\varphi_0\|,\|\psi_0\|\}}{\gamma}\le M,\quad\gamma=\frac{\beta^2}{4\alpha}.\end{aligned}$$

Hence $\|A\varphi\|\le M$. Next, recalling

$$(A\varphi)'(t)=-\frac{\beta}{2\alpha}(A\varphi)(t)+\frac{1}{\alpha}\int_{-\infty}^{t}h(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}ds,\tag{2.14}$$

we derive

$$\begin{aligned}|(A\varphi)'(t)|&\le\frac{\beta}{2\alpha}|(A\varphi)(t)|+\frac{1}{\alpha}\left|\int_{-\infty}^{t}h(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}ds\right|\\&\le\frac{\beta}{2\alpha}M+\frac{2}{\beta}(G+|\gamma|\max\{\|\varphi_0\|,\|\psi_0\|\})\\&=L.\end{aligned}$$

Consequently, we arrive at

$$A\varphi\in B_C(M,L).\tag{2.15}$$

Next, we show

$$\varphi_0(t)\le(A\varphi_0)(t),\ \forall t\in\mathbb{R}.\tag{2.16}$$

Let $m(t) = \varphi_1(t) - \varphi_0(t)$, where $\varphi_1 = A\varphi_0$. Then

$$\varphi_1'(t) = -\frac{\beta}{2\alpha}\varphi_1(t) + \frac{1}{\alpha}\int_{-\infty}^{t} h(s, \varphi_0(s), \varphi_0^{[2]}(s), \ldots, \varphi_0^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}ds$$

and noting $\varphi_0(t)$ is a lower solution for (1.1), we have

$$m'(t) = \varphi_1'(t) - \varphi_0'(t) \geq -\frac{\beta}{2\alpha}\Big(\varphi_1(t) - \varphi_0(t)\Big) = -\frac{\beta}{2\alpha}m(t). \tag{2.17}$$

Then $m(t) \geq 0$ for any $t \in \mathbb{R}$. Suppose to contrary that there exists $t_0 \in \mathbb{R}$ such that $m(t_0) < 0$, then from (2.17), $m'(t_0) \geq -\frac{\beta}{2\alpha}m(t_0) > 0$. Thus $m(t) < m(t_0) < 0$ for any $t < t_0$ near t_0. Then we have $m(t) < 0$ for any $t \in (-\infty, t_0)$. In fact, if there exists $-\infty < a < t_0$ such that $m(t) < 0, \forall t \in (a, t_0]$ and $m(a) \geq 0$. Then $m(a) = 0$ and from (2.17),

$$m'(t) \geq -\frac{\beta}{2\alpha}m(t) \geq 0, \ \forall t \in [a, t_0],$$

and thus $m(t) \geq m(a) = 0$ for all $t \in [a, t_0]$, which is a contradiction. So $a = -\infty$ and $m'(t) \geq -\frac{\beta}{2\alpha}m(t_0) > 0$ for any $t \in (-\infty, t_0)$. This implies

$$m(t) = m(t_0) - \int_t^{t_0} m'(s)ds \leq m(t_0) + \frac{\beta}{2\alpha}m(t_0)(t_0 - t) \to -\infty$$

as $t \to -\infty$. But $\|m\| \leq \|\varphi_0\| + \|\psi_0\| < \infty$, which is again a contradiction. Thus $m(t) \geq 0$, i.e., $A\varphi_0 \geq \varphi_0$. So (2.16) is shown. Similarly, we can prove $A\psi_0 \leq \psi_0$.

Next, (**H2**) and (2.13) give

$$\begin{aligned} h(t, \varphi_0(t), \varphi_0^{[2]}(t), \ldots, \varphi_0^{[n]}(t)) &\leq h(t, \varphi(t), \varphi^{[2]}(t), \ldots, \varphi^{[n]}(t)) \\ &\leq h(t, \psi_0(t), \psi_0^{[2]}(t), \ldots, \psi_0^{[n]}(t)), \end{aligned}$$

and thus

$$\varphi_0(t) \leq (A\varphi_0)(t) \leq (A\varphi)(t) \leq (A\psi_0)(t) \leq \psi_0(t), \quad t \in \mathbb{R}. \tag{2.18}$$

Furthermore, since all $\varphi^{[i]}(t)$, $i = 1, 2, \cdots, n$ are nondecreasing and using (**H2**), we derive

$$\begin{aligned} (A\varphi)(t) &= \frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}dsdu \\ &\leq \frac{1}{\alpha}\int_{-\infty}^{t}\int_{-\infty}^{u} h(u, \varphi(u), \varphi^{[2]}(u), \ldots, \varphi^{[n]}(u))e^{\frac{\beta}{2\alpha}(s-t)}dsdu \\ &= \frac{2}{\beta}\int_{-\infty}^{t} h(u, \varphi(u), \varphi^{[2]}(u), \ldots, \varphi^{[n]}(u))e^{\frac{\beta}{2\alpha}(u-t)}du, \end{aligned}$$

which by (2.14) implies

$$\begin{aligned} (A\varphi)'(t) &= -\frac{\beta}{2\alpha}(A\varphi)(t) + \frac{1}{\alpha}\int_{-\infty}^{t} h(s, \varphi(s), \varphi^{[2]}(s), \ldots, \varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}ds \\ &\geq -\frac{1}{\alpha}\int_{-\infty}^{t} h(u, \varphi(u), \varphi^{[2]}(u), \ldots, \varphi^{[n]}(u))e^{\frac{\beta}{2\alpha}(u-t)}du \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\alpha}\int_{-\infty}^{t} h(s,\varphi(s),\varphi^{[2]}(s),\dots,\varphi^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-t)}ds \\
&= 0.
\end{aligned}
\tag{2.19}
$$

Summarizing, (2.15), (2.18) and (2.19) implies (2.12).

Moreover, arguments leading to (2.18) show that A is nondecreasing. So sequences $\{\varphi_k\}_{k=1}^{\infty} \subset B_C(M,L)$ and $\{\psi_k\}_{k=1}^{\infty} \subset B_C(M,L)$ monotonically and pointwisely converge to functions φ_* and ψ_* on $\mathbb{R}$, respectively. But we already know that $B_C(M,L) \subset C(\mathbb{R},\mathbb{R})$ is compact. So there are subsequences $\{\varphi_{k_i}\}_{i=1}^{\infty}$ and $\{\psi_{k_i}\}_{i=1}^{\infty}$ converging to φ_* and ψ_* in $C(\mathbb{R},\mathbb{R})$. But this implies that $\{\varphi_k\}_{k=1}^{\infty}$ and $\{\psi_k\}_{k=1}^{\infty}$ converge to φ_* and ψ_* in $C(\mathbb{R},\mathbb{R})$. Clearly $\varphi_* \le \psi_*$ by (2.6).

Next, we show that $A \in C(B_C(M,L), C(\mathbb{R},\mathbb{R}))$. Let $\varphi_j \to \varphi_*$ as $j \to \infty$ for $\varphi_j \in B_C(M,L)$, $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ uniformly on any compact interval $[-m,m], m \in \mathbb{N}$ of $\mathbb{R}$. Set

$$h_j(s) = h(s,\varphi_j(s),\varphi_j^{[2]}(s),\dots,\varphi_j^{[n]}(s)), \quad j \in \mathbb{N}_0.$$

Then $h_j \to h_* = h(s,\varphi_*(s),\varphi_*^{[2]}(s),\dots,\varphi_*^{[n]}(s))$ uniformly on $[-m,m]$. Next we have

$$\left| h(s,\varphi_j(s),\varphi_j^{[2]}(s),\dots,\varphi_j^{[n]}(s))e^{\frac{\beta}{2\alpha}(s-(-m))} \right| \le \|h\|_m e^{\frac{\beta}{2\alpha}(s+m)}, \quad s \in (-\infty,-m).$$

Since

$$\int_{-\infty}^{-m}\int_{-\infty}^{u} e^{\frac{\beta}{2\alpha}(s+m)}dsdu = \frac{4\alpha^2}{\beta^2},$$

we can apply the Lebesgue dominated convergence theorem to obtain $A\varphi_j(-m) \to A\varphi_*(-m)$. From (2.4),

$$(x_j'(t) - x_*'(t)) + \frac{\beta}{2\alpha}(x_j(t) - x_*(t)) = \frac{1}{\alpha}\int_{-\infty}^{t} (h_j(s) - h_*(s))e^{\frac{\beta}{2\alpha}(s-t)}ds. \tag{2.20}$$

Integrating the both sides of (2.20) from $-m$ to t, we have

$$
\begin{aligned}
&(x_j(t) - x_*(t)) + \frac{\beta}{2\alpha}\int_{-m}^{t}(x_j(s) - x_*(s))ds \\
&= (x_j(-m) - x_*(-m)) + \frac{1}{\alpha}\int_{-m}^{t}\int_{-\infty}^{u}(h_j(s) - h_*(s))e^{\frac{\beta}{2\alpha}(s-u)}ds,
\end{aligned}
$$

and

$$|x_j(t)-x_*(t)| \le |x_j(-m)-x_*(-m)| + \frac{4m}{\beta}\|h_j - h_*\|_m + \frac{\beta}{2\alpha}\int_{-m}^{t}|x_j(s)-x_*(s)|ds,$$

for any $t \in [-m,m]$. Then Gronwall's inequality implies

$$\|x_j - x_*\|_m \le e^{\frac{\beta}{\alpha}}\Big(|x_j(-m) - x_*(-m)| + \frac{4m}{\beta}\|h_j - h_*\|_m\Big),$$

which means

$$\|A\varphi_j - A\varphi_*\|_m \le e^{\frac{\beta}{\alpha}}\Big(|A\varphi_j(-m) - A\varphi_*(-m)| + \frac{4m}{\beta}\|h_j - h_*\|_m\Big).$$

Hence $A\varphi_j(t) \to A\varphi_*(t)$ uniformly on $t \in [-m, m]$. Since $m \in \mathbb{N}$ is arbitrarily, we get $A\varphi_j \to A\varphi_*$ in $C(\mathbb{R}, \mathbb{R})$, i.e., $A : B_C(M, L) \to C(\mathbb{R}, \mathbb{R})$ is continuous. This proves the continuity of A.

Using

$$\varphi_* \leftarrow \varphi_{k+1} = A\varphi_k \to A\varphi_*, \quad \psi_* \leftarrow \psi_{k+1} = A\psi_k \to A\psi_*,$$

we obtain

$$A\varphi_* = \varphi_*, \quad A\psi_* = \psi_*.$$

Finally, if $x(t)$ is a nondecreasing bounded solution of (1.1) in $[\varphi_0, \psi_0]$, then

$$\varphi_k \le x \le \psi_k,$$

so

$$\varphi_* \le x \le \psi_*.$$

This completes the proof. □

3. Examples

In this section, two examples are given to illustrate that the assumptions of Theorem 2.1 do not self-contradict.

Example 3.1. Consider the following equation:

$$x''(t) + 6x'(t) = \Big(\frac{t}{1+|t|} - 7\Big)x(t) + (2 + \arctan(t))(x(x(t)))^2 + 1 + \tanh(t), \quad (3.1)$$

where $\alpha = 1, \beta = 6, g(t, x_1, x_2) = \Big(\frac{t}{1+|t|} - 7\Big)x_1 + (2 + \arctan(t))x_2^2 + 1 + \tanh(t)$. Taking $\gamma = \frac{\beta^2}{4\alpha} = 9$, we get

$$\begin{aligned} h(t, x_1, x_2) &= g(t, x_1, x_2) + \gamma x_1 \\ &= \Big(\frac{t}{1+|t|} + 2\Big)x_1 + (2 + \arctan(t))x_2^2 + 1 + \tanh(t). \end{aligned}$$

In order to simplify the calculation, let us choose $\varphi_0 = 0, \psi_0 = 1$. Then

$$\begin{aligned} \varphi_0''(t) + 6\varphi_0'(t) &= 0 \le 1 + \tanh(t) \\ &= \Big(\frac{t}{1+|t|} - 7\Big)\varphi_0(t) + (2 + \arctan(t))(\varphi_0(\varphi_0(t)))^2 + 1 + \tanh(t), \end{aligned}$$

and

$$\begin{aligned} \psi_0''(t) + 6\psi_0'(t) &= 0 \ge \frac{t}{1+|t|} - 7 + 2 + \arctan(t) + 1 + \tanh(t) \\ &= \Big(\frac{t}{1+|t|} - 7\Big)\psi_0(t) + (2 + \arctan(t))(\psi_0(\psi_0(t)))^2 + 1 + \tanh(t), \end{aligned}$$

where

$$\frac{t}{1+|t|} + \arctan(t) + \tanh(t) \le 2 + \frac{\pi}{2} \doteq 3.5707964, \quad t \in \mathbb{R}.$$

A simple calculation yields

$$|g(t, x_1, x_2)| \leq 8x_1 + (2 + \frac{\pi}{2})x_2^2 + 2.$$

Noting

$$\widetilde{\varphi}_0 = \widetilde{\varphi}_1 = 0,\ \widetilde{\psi}_0 = \widetilde{\psi}_1 = 1,$$

so

$$G = 12 + \frac{\pi}{2}.$$

We see $h(t, x_1, x_2)$ is nondecreasing in its arguments $t \in \mathbb{R}$ and $x_1, x_2 \in [0, 1]$. Therefore, (**H1**) and (**H2**) are satisfied. By Theorem 2.1, Eq. (3.1) has a has a minimal solution $\varphi_*(t)$ and a maximal solution $\psi_*(t)$ in $BC(\mathbb{R}, \mathbb{R})$ which are nondecreasing and $0 \leq \varphi_*(t) \leq \psi_*(t) \leq 1$. Moreover, they are given by iteration schemas (2.7).

Example 3.2. Consider

$$x''(t) + \lambda x'(t) = \Big(\frac{t}{1 + |t|} - 7\Big)x(t) + (2 + \arctan(t))(x(x(t)))^2 + 1 + \tanh(t), \quad (3.2)$$

where $\lambda > 0$ is a parameter. Then as in Example 3.1, $\alpha = 1, \beta = \lambda, g(t, x_1, x_2) = \Big(\frac{t}{1+|t|} - 7\Big)x_1 + (2 + \arctan(t))x_2^2 + 1 + \tanh(t)$. Taking $\gamma = \frac{\lambda^2}{4}$ and

$$\begin{aligned} h(t, x_1, x_2) &= g(t, x_1, x_2) + \gamma x_1 \\ &= \Big(\frac{t}{1 + |t|} + \gamma - 7\Big)x_1 + (2 + \arctan(t))x_2^2 + 1 + \tanh(t). \end{aligned}$$

As in Example 3.1, we have an lower solution $\varphi_0(t) = 0$ and an upper solution $\psi_0(t) = 1$, and (**H1**) holds,

$$\widetilde{\varphi}_0 = \widetilde{\varphi}_1 = 0,\ \widetilde{\psi}_0 = \widetilde{\psi}_1 = 1.$$

If $\gamma \geq 7$, i.e., $\lambda \geq 2\sqrt{7}$, we have $h(t, x_1, x_2)$ is nondecreasing for $t \in \mathbb{R}$ and $x_1, x_2 \in [0, 1]$, (**H2**) holds. Then Theorem 2.1 implies that Eq. (3.2) has a minimal solution $\varphi_*(t)$ and a maximal solution $\psi_*(t)$ in $BC(\mathbb{R}, \mathbb{R})$ which are nondecreasing and $0 \leq \varphi_*(t) \leq \psi_*(t) \leq 1$. Moreover, they are given by iteration schemas (2.7). Clearly, $\beta = 6$ in Example 3.1 satisfies the condition $6 > 2\sqrt{7}$.

References

[1] R. Bellman and K. Cooke, *Differential-Difference Equations*, Acadmic Press, Santa Monica, 1963.

[2] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, New York, 2006.

[3] S. Chow, *Existence of periodic solutions of autonomous functional differential equations*, J. Diff. Eqs., 1974, 15(2), 350–378.

[4] K. L. Cooke, *Functional differential systems: some models and perturbation problems, in: Proceedings of the International Symposium on Differential Equations and Dynamical Systems*, Academic Press, New York, 1967.

[5] E. Eder, *The functional differential equation $x'(t) = x(x(t))$*, J. Diff. Eqs., 1984, 54(3), 390–400.

[6] M. Fečkan, *On a certain type of functional differential equations*, Math. Slovaca., 1993, 43(1), 39–43.

[7] M. Fečkan, J. Wang and H. Zhao, *Maximal and minimal nondecreasing bounded solutions of iterative functional differential equations*, Appl. Math. Lett., 2021. DOI:/10.1016/j.aml.2020.106886.

[8] J. Hale, *Theory of Functional Differential Equations*, Springer Verlag, New York, 1977.

[9] S. Kang, B. Shi and G. Wang, *Existence of maximal and minimal periodic solutions for first-order functional differential equations*, Appl. Math. Lett., 2010, 23(1), 22–25.

[10] E. R. Kaufmann, *Existence and uniqueness of solutions for a second-order iterative boundary-value problem*, Electron. J. Differential Equations, 2018, 150, 1–6.

[11] V. R. Petahov, *On a boundary value problem*, Trudy Sem. Teor. Diff. Uravnenii Otklon. Argument, Univ. Druzby Narodov Patrisa Lumumby, 1965, 3, 252–255.

[12] B. Shi and Z. Li, *Existence of solutions and bifurcation of a class of first-order functional differential equations*, Acta Math. Appl. Sinica., 1995, 18(1), 83–89.

[13] J. Si and X. Wang, *Analytic solutions of a second-order iterative functional differential equation*, Comput. Math. Appl., 2002, 43(1), 81–90.

[14] J. Si and W. Zhang, *Analytic solutions of a second-order nonautonomous iterative functional differential equation*, J. Math. Anal. Appl., 2005, 306(2), 398–412.

[15] J. Si and M. Ma, *Local invertible analytic solution of a functional differential equation with deviating arguments depending on the state derivative*, J. Math. Anal. Appl., 2007, 327(1), 723–734.

[16] J. Si, W. Zhang and G. Kim, *Analytic solutions of an iterative functional differential equation*, Appl. Math. Comput., 2004, 150(3), 647–659.

[17] B. H. Stephen, *On the existence of periodic solutions of $z'(t) = -az(t - r + \mu k(t, z(t))) + F(t)$*, J. Diff. Eqs., 1969, 6(3), 408–419.

[18] Y. Zeng, P. Zhang, T. Lu, and W. Zhang, *Existence of solutions for a mixed type differential equation with state-dependence*, J. Math. Anal. Appl., 2017, 453(1), 629–644.