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POSITIVE SOLUTIONS FOR A FRACTIONAL MAGNETIC SCHRÖDINGER EQUATIONS WITH SINGULAR NONLINEARITY AND STEEP POTENTIAL*

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Abstract The paper deals with the following magnetic Schrödinger equation with singular nonlinearity and steep potential

$$\begin{cases} (-\Delta)^s_A u + V_\lambda(x)u = \mu f(x)u^{-\gamma} + g(x)u^{p-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $(-\Delta)^s_A$ is the fractional magnetic Laplacian operator with $0 < s < 1$, and $0 < \gamma < 1$, $2 < p < 2^*_s$ $\left(2^*_s = \frac{2N}{N-2s} \text{ for } N > 2s\right)$, the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$, $\lambda, \mu > 0$ are parameters, $f \in L^{\frac{p}{p+\gamma-1}}(\mathbb{R}^N)$ is a positive weight, while $g \in L^\infty(\mathbb{R}^N)$ is a sign-changing function. By applying the Nehari manifold and fibering map, we obtain the existence of at least two positive solutions, where some new estimates will be established. Recent some results from the literature are extended.

Keywords Fractional magnetic operators, singular nonlinearity, steep potential, Nehari manifold.

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1. Introduction and main results

In this work, we study the multiplicity of solutions to the following fractional magnetic Schrödinger equation

$$\begin{cases} (-\Delta)^s_A u + V_\lambda(x)u = \mu f(x)u^{-\gamma} + g(x)u^{p-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $(-\Delta)^s_A$ is the fractional magnetic Laplacian operator with $s \in (0, 1)$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a $C^{0,\alpha}$ magnetic potential of exponent $\alpha \in (0, 1]$, the parameters

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$\lambda, \mu > 0$, $0 < \gamma < 1$, $2 < p < 2_s^*$ $\left(2_s^* = \frac{2N}{N-2s} \text{ for } N > 2s\right)$, the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$. We first assume that $V(x)$ satisfy the following conditions:

$(V1)$ V^+ is a continuous function on $\mathbb{R}^N$ and $V^- \in L^{N/2}(\mathbb{R}^N)$.

$(V2)$ there exists $\kappa > 0$ such that the set $\{V^+ < \kappa\} = \{x \in \mathbb{R}^N : V^+(x) < \kappa\}$ is nonempty and has finite measure.

$(V3)$ $\Omega = \mathrm{int}\{x \in \mathbb{R}^N : V^+(x) = 0\}$ is nonempty and has a smooth boundary with $\bar{\Omega} = \mathrm{int}\{x \in \mathbb{R}^N : V^+(x) = 0\}$.

$(V4)$ there exists a constant $\mu_0 > 1$ such that

$$\mu_1(\lambda) := \inf_{u \in H_A^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}} dxdy + \int_{\mathbb{R}^N} \lambda V^+ u^2 dx}{\int_{\mathbb{R}^N} V^- u^2 dx} \geq \mu_0,$$

for all $\lambda > 0$, where $H_A^s(\mathbb{R}^N, \mathbb{C})$ is the Hilbert space related to magnetic field A (see Section 2).

This type of assumptions was first introduced by Bartsch and Wang [9] in the study of the nonlinear Schrödinger equations, imply that λV^+ represents a potential well whose depth is controlled by λ. The potential V_λ with V satisfies $(V1) - (V3)$ is called as the steep well potential. For more details about steep well potential, we refer to [17, 24, 27].

The operator $(-\Delta)_A^s$ is the fractional magnetic Laplacian and it is defined for $u \in C_0^\infty(\mathbb{R}^N, \mathbb{C})$ by

$$(-\Delta)_A^s u(x) = 2 \lim_{r \to 0^+} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)}{|x-y|^{N+2s}} dy, \tag{1.2}$$

where $B_r(x) = \{y \in \mathbb{R}^N : |x-y| < r\}$ with $r > 0$. This nonlocal operator has been defined in [14] as a fractional extension (for an arbitrary $s \in (0,1)$) of the magnetic pseudo-relativistic operator, or Weyl pseudo-differential operator defined with mid-point prescription, introduced in [19] by Ichinose and Tamura. As stated in [28], when $s \to 1$, the operator $(-\Delta)_A^s$ reduces to the well-known magnetic Laplacian $-(\nabla - iA)^2$, which has been widely investigated by many authors; see [4, 5, 8, 20] for more details.

More in general, nonlocal and fractional operators have received a considerable attention from many mathematicians and physical phenomena, such as finance, phase transition phenomena, minimal surfaces, as they are the infinitesimal generators of Lévy stable diffusion processes, see [6, 12, 13] and the references therein. For more work on nonlocal fractional operators and their applications, interested readers are referred to [10, 23] and references therein.

In absence of the magnetic field, i.e. $A = 0$, the operator $(-\Delta)_A^s$ reduces to the celebrated fractional Laplacian $(-\Delta)^s$. There are also some interesting results are obtained by using some different approaches under various hypotheses on the potential and the nonlinearity. For instance, Zhang et al. [35] investigated the periodic and asymptotically periodic fractional Schrödinger equation, and they obtained the existence of solutions by variational methods, similar problems have also been considered in [16, 18]. Cui and Sun [11] studied the existence and multiplicity results under the assumptions that the potential V is indefinite. In [21], the authors established the multiplicity of sign-changing solutions for fractional Schrödinger equations involving critical or supercritical exponent. Moreover, there is a wide literature concerning the study of the existence of solutions for fractional Schrödinger equation with critical growth, see for example [7, 26, 30, 34] for the recent advances in this direction.

On the other hand, in last decade, great attention have been paid on the study of the classical magnetic nonlinear Schrödinger equations, see for instance [1, 2, 15, 22, 31]. More precisely, Xiang et al. [29] considered the following fractional Schrödinger-Kirchhoff problem

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x,|u|)u, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $s \in (0,1)$, $N > 2s$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential, the nonlinearity f satisfies the subcritical growth. Using variational methods, the authors obtained several existence results for problem (1.3). Not long after, Zhang et al. [33] studied singularly perturbed fractional Schrödinger equations involving critical frequency and critical growth in the presence of a magnetic field.

Subsequently, Yang et al. [32] studied the following degenerate magnetic fractional problem involving critical Sobolev-Hardy nonlinearities

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = \lambda f(x,|u|)u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha}, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $s \in (0,1)$, $N > 2s$, $2_s^*(\alpha) = \frac{2(N-\alpha)}{N-2s}$ is the fractional Hardy-Sobolev critical exponent with $\alpha \in [0,2s)$, λ is a positive parameter and $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function. Under some conditions on V and by using the new version of symmetric mountain pass theorem of Kajikiya, the authors proved that the problem (1.4) admits infinitely many solutions for the suitable value of λ.

Most recently, Mao and Xia [25] investigated the following fractional nonlinear Schrödinger equation

$$(-\Delta)_A^s u + V_\lambda(x)u = f(x)|u|^{q-2} + g(x)|u|^{p-1}, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $0 < s < 1$, $N > 2s$, $1 < q < 2 < p < 2_s^*$ with $2_s^* = 2N/(N-2s)$, the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$, $\lambda > 0$ is a parameter. When λ is sufficiently large, combining variational approach with the Nehari manifold, they obtained the existence and multiplicity of non-trivial solutions for problem (1.5).

Motivated by the mentioned works, our goal in this paper is to establish the existence and multiplicity of solutions for problem (1.1) with steep well potential and singular nonlinearity. To the best of our knowledge, no similar results are obtained on such questions in current literature.

In this context, the presence of the nonlocal operator (1.2) makes our analysis more complicated and intriguing, and new techniques are needed to overcome the difficulties that appear.

Consider the functions $f(x)$ and $g(x)$, we make the following hypotheses:

(F) $f \in L^{\frac{p}{p+\gamma-1}}(\mathbb{R}^N)$ is a positive continuous function.

(G) $g \in L^\infty(\mathbb{R}^N)$ is a sign-changing function such that $\|g^+\|_{L^\infty(\mathbb{R}^N)} > 0$, where $g^+ = \max\{g(x), 0\}$.

Our main result is described as follows.

Theorem 1.1. *Let $0 < \gamma < 1$ and $2 < p < 2_s^*$. Assume f, g and V satisfy the assumptions (F), (G) and $(V1) - (V4)$, then there exists $\lambda^* > 0$ and $\mu^* > 0$ such that for all $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, problem (1.1) has at least two positive solutions.*

Turing to layout of the article, in Section 2, we recall some basic notations and preliminary results which are crucial in proving our main results. The last Section is devoted to prove Theorem 1.1. Also throughout this paper, we shall denote by C and C_i $(i = 0, 1, 2, \cdots)$ for various positive constants.

2. Preliminaries and functional setting

To prove our main results, we need to do some preparatory work. Let $L^2(\mathbb{R}^N, \mathbb{C})$ be the Lebesgue space of complex-valued functions with summable square endowed with the real scalar product

$$\langle u, v\rangle_{L^2} := \Re(\int_{\mathbb{R}^N} u\bar{v}dx),$$

for all $u, v \in L^2(\mathbb{R}^N, \mathbb{C})$, and $A : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function. We consider the magnetic Gagliardo semi-norm defined by

$$[u]_{s,A}^2 := \iint_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{N+2s}}dxdy,$$

endowed with the norm $\|u\|_{s,A} := ([u]_{s,A}^2 + \|u\|_{L^2}^2)^{1/2}$. We take the space $\mathcal{H}$ of measurable functions $u : \mathbb{R}^N \to \mathbb{C}$ such that $\|u\|_{s,A} < \infty$, then $(\mathcal{H}, \langle\cdot,\cdot\rangle_{s,A})$ is a real Hilbert space. We define $H_A^s(\mathbb{R}^N, \mathbb{C})$ as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ in $\mathcal{H}$, $H_A^s(\mathbb{R}^N, \mathbb{C})$ is a real Hilbert space. Moreover, the space $H_A^s(\mathbb{R}^N, \mathbb{C})$ is continuously embedded in $L^r(\mathbb{R}^N, \mathbb{C})$ for every $r \in [2, 2_s^*]$ and compactly embedded in $L^r(K, \mathbb{C})$ for every $r \in [1, 2_s^*)$ and any compact $K \subset \mathbb{R}^N$; see [14].

Next, we establish the variational framework to deal with the problem (1.1). Define the work space $X_\lambda = \{u \in H_A^s(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} \lambda V^+ u^2 dx < \infty\}$ with the inner product

$$\begin{aligned}\langle u, v\rangle_\lambda : = &\Re \iint_{\mathbb{R}^N} \frac{(u(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}u(y))\overline{(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}dxdy \\ &+\lambda\Re\int_{\mathbb{R}^N} V^+ u\bar{v}dx\end{aligned}$$

and the corresponding norm denoted by

$$\|u\|_\lambda := \langle u, u\rangle_\lambda^{1/2}.$$

For simplicity, we let $\|u\|_{\lambda,V}^2 := [u]_{s,A}^2 + \int_{\mathbb{R}^N} V_\lambda u^2 dx$, by condition (V4),

$$\|u\|_\lambda^2 \geq \|u\|_{\lambda,V}^2 \geq \frac{\mu_0 - 1}{\mu_0}\|u\|_\lambda^2, \quad \text{for all} \quad \lambda \geq 0. \tag{2.1}$$

Hence, $\|u\|_{\lambda,V}$ and $\|u\|_\lambda$ are equivalent in X_λ. As shown in [14, 25], there exists a constant $\mathcal{M}_{s,A} > 0$ such that

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)} \leq \mathcal{M}_{s,A}^{-1}[u]_{s,A}. \tag{2.2}$$

Let λ^* the constant given by

$$\lambda^* := \frac{\mathcal{M}_{s,A}^2}{\kappa}|\{V^+ < \kappa\}|^{-\frac{2_s^*-2}{2_s^*}}.$$

Then, by the conditions (V1) and (V2), and the Hölder and Sobolev inequalities again, we have

$$\int_{\mathbb{R}^N} |u|^p dx \leq |\{V^+ < \kappa\}|^{\frac{2_s^*-p}{2_s^*}} \mathcal{M}_{s,A}^{-p}\|u\|_\lambda^p, \tag{2.3}$$

for $p \in [2, 2_s^*)$ and $\lambda \geq \lambda^*$. And also, combining this with (F), one has

$$\begin{aligned}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx &\leq \|f\|_{L^{\frac{p}{p+\gamma-1}}}(\int_{\mathbb{R}^N} |u|^p dx)^{\frac{1-\gamma}{p}} \\ &\leq \|f\|_{L^{\frac{p}{p+\gamma-1}}} |\{V^+ < \kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^* p}} \mathcal{M}_{s,A}^{\gamma-1}\|u\|_\lambda^{1-\gamma}. \end{aligned}\tag{2.4}$$

The energy functional corresponding to problem (1.1) given by

$$\Phi_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^- u^2 dx - \frac{\mu}{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - \frac{1}{p}\int_{\mathbb{R}^N} g|u|^p dx. \tag{2.5}$$

It is clearly that $\Phi_{\lambda,\mu}$ is a C^1 functional. Since the energy functional $\Phi_{\lambda,\mu}$ is not bounded below on X_λ, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{u \in X_\lambda \setminus \{0\} : \langle \Phi'_{\lambda,\mu}(u), u\rangle = 0\}.$$

We analyze $\mathcal{N}_{\lambda,\mu}$ in terms of the stationary points of fibering maps $\phi_u : (0,\infty) \to \mathbb{R}$ given by

$$\phi_u(t) = \Phi_{\lambda,\mu}(tu), \quad \text{for } \ t > 0.$$

Then for each $u \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned}\phi'_u(t) &= t\|u\|_{\lambda,V}^2 - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx, \\ \phi''_u(t) &= \|u\|_{\lambda,V}^2 + \mu\gamma t^{-\gamma-1}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - (p-1)t^{p-2}\int_{\mathbb{R}^N} g|u|^p dx.\end{aligned}$$

It is easy to see that

$$t\phi'_u(t) = t^2\|u\|_{\lambda,V}^2 - \mu t^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^p\int_{\mathbb{R}^N} g|u|^p dx,$$

and so, for $u \in X_\lambda \setminus \{0\}$ and $t > 0$, $\phi'_u(t) = 0$ if and only if $tu \in \mathcal{N}_{\lambda,\mu}$, that is, positive critical points of ϕ_u correspond to points on the Nehari manifold. In particular, $\phi'_u(1) = 0$ if and only if $u \in \mathcal{N}_{\lambda,\mu}$. Thus, it is nature to divide $\mathcal{N}_{\lambda,\mu}$ into three parts as

$$\begin{aligned}\mathcal{N}_{\lambda,\mu}^+ &= \{u \in \mathcal{N}_{\lambda,\mu} : \phi''_u(1) > 0\}, \\ \mathcal{N}_{\lambda,\mu}^0 &= \{u \in \mathcal{N}_{\lambda,\mu} : \phi''_u(1) = 0\}, \\ \mathcal{N}_{\lambda,\mu}^- &= \{u \in \mathcal{N}_{\lambda,\mu} : \phi''_u(1) < 0\}.\end{aligned}$$

The existence of solutions to the problem (1.1) can be studied by considering the existence of minimizers to functional $\Phi_{\lambda,\mu}$ on manifold $\mathcal{N}_{\lambda,\mu}$. Furthermore, for each $u \in \mathcal{N}_{\lambda,\mu}$, we know that

$$\begin{aligned}\phi_u''(1) &= \|u\|_{\lambda,V}^2 + \mu\gamma \int_{\mathbb{R}^N} f|u|^{1-\gamma} dx - (p-1) \int_{\mathbb{R}^N} g|u|^p dx \\ &= (1+\gamma)\|u\|_{\lambda,V}^2 - (p+\gamma-1) \int_{\mathbb{R}^N} g|u|^p dx \\ &= (2-p)\|u\|_{\lambda,V}^2 + \mu(p+\gamma-1) \int_{\mathbb{R}^N} f|u|^{1-\gamma} dx.\end{aligned} \tag{2.6}$$

Lemma 2.1. *The energy functional $\Phi_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $u \in \mathcal{N}_{\lambda,\mu}$, then we have

$$\|u\|_{\lambda,V}^2 - \mu \int_{\mathbb{R}^N} f|u|^{1-\gamma} dx - \int_{\mathbb{R}^N} g|u|^p dx = 0.$$

Therefore, by (2.1), (2.4) and (2.5), we obtain

$$\begin{aligned}\Phi_{\lambda,\mu}(u) &= \frac{p-2}{2p}\|u\|_{\lambda,V}^2 - \frac{\mu(p+\gamma-1)}{p(1-\gamma)} \int_{\mathbb{R}^N} f|u|^{1-\gamma} dx \\ &\geq \frac{(p-2)(\mu_0-1)}{2p\mu_0}\|u\|_\lambda^2 \\ &\quad - \frac{\mu(p+\gamma-1)}{p(1-\gamma)}\|f\|_{L^{\frac{p}{p+\gamma-1}}} |\{V^+ < \kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^* p}} M_{s,A}^{\gamma-1}\|u\|_\lambda^{1-\gamma}.\end{aligned}$$

Since $0 < \gamma < 1$, we conclude that $\Phi_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$. $\square$

Before the proof of the following lemma, we define

$$\begin{aligned}\mu^* = &\frac{(\mu_0-1)(p-2)\mathcal{M}_{s,A}^{1-\gamma}}{\mu_0(p+\gamma-1)\|f\|_{L^{\frac{p}{p-1+\gamma}}} |\{V^+ < \kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^* p}}} \\ &\times \left(\frac{(\mu_0-1)(1+\gamma)\mathcal{M}_{s,A}^p}{\mu_0(p+\gamma-1)\|g^+\|_\infty |\{V^+ < \kappa\}|^{\frac{2_s^*-p}{2_s^*}}} \right)^{\frac{1+\gamma}{p-2}}.\end{aligned}$$

Then we have the following result.

Lemma 2.2. *Suppose that the functions f, g and V satisfy the conditions (F), (G) and $(V1) - (V4)$. Then the set $\mathcal{N}_{\lambda,\mu}^0$ is empty for $(\lambda,\mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$.*

Proof. If $\mathcal{N}_{\lambda,\mu}^0 \neq \emptyset$, then for every $u \in \mathcal{N}_{\lambda,\mu}^0$, by (2.6), we have

$$(1+\gamma)\|u\|_{\lambda,V}^2 - (p+\gamma-1) \int_{\mathbb{R}^N} g|u|^p dx = 0$$

and

$$(2-p)\|u\|_{\lambda,V}^2 + \mu(p+\gamma-1) \int_{\mathbb{R}^N} f|u|^{1-\gamma} dx = 0.$$

It follows that, by (2.1), (2.3), (2.4) and the Hölder inequality, we get

$$\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2 \leq \frac{p+\gamma-1}{1+\gamma}\int_{\mathbb{R}^N} g|u|^p \leq \frac{p+\gamma-1}{1+\gamma}\|g^+\|_\infty |\{V^+<\kappa\}|^{\frac{2_s^*-p}{2_s^*}}\mathcal{M}_{s,A}^{-p}\|u\|_\lambda^p$$

and

$$\begin{aligned}\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2 &\leq \frac{\mu(p+\gamma-1)}{p-2}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx \\ &\leq \frac{\mu(p+\gamma-1)}{p-2}\|f\|_{L^{\frac{p}{p+\gamma-1}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^* p}} M_{s,A}^{\gamma-1}\|u\|_\lambda^{1-\gamma}.\end{aligned}$$

That is

$$\|u\|_\lambda \geq \left(\frac{(\mu_0-1)(1+\gamma)\mathcal{M}_{s,A}^p}{\mu_0(p+\gamma-1)\|g^+\|_\infty |\{V^+<\kappa\}|^{\frac{2_s^*-p}{2_s^*}}}\right)^{\frac{1}{p-2}}$$

and

$$\|u\|_\lambda \leq \left(\frac{\mu_0\mu(p+\gamma-1)}{(\mu_0-1)(p-2)}\|f\|_{L^{\frac{p}{p+\gamma-1}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^* p}}\mathcal{M}_{s,A}^{\gamma-1}\right)^{\frac{1}{1+\gamma}}.$$

Hence, we obtain $\mu \geq \mu^*$ which is impossible. Thus $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$. □

In the following result, we show that the decompositions of the Nehari manifold are non-empty.

Lemma 2.3. *Suppose (F), (G) and $(V1)-(V4)$ hold. Then for $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$ and $u\in X_\lambda\setminus\{0\}$, we have the following results.*
(i) if $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, then exists a unique $0<t^+<t_{\max}$ such that $t^+u\in\mathcal{N}_{\lambda,\mu}^+$ and

$$\Phi_{\lambda,\mu}(t^+u) = \inf_{t>0}\Phi_{\lambda,\mu}(tu).$$

(ii) if $\int_{\mathbb{R}^N} g|u|^p dx > 0$, then there just have two positive numbers $t^+>0$ and $t^->0$, with $0<t^+<t_{\max}<t^-$, such that $t^+u\in\mathcal{N}_{\lambda,\mu}^+$, $t^-u\in\mathcal{N}_{\lambda,\mu}^-$ and

$$\Phi_{\lambda,\mu}(t^+u) = \inf_{0<t\leq t_{\max}}\Phi_{\lambda,\mu}(tu), \qquad \Phi_{\lambda,\mu}(t^-u) = \sup_{t\geq t_{\max}}\Phi_{\lambda,\mu}(tu).$$

Proof. Fix $u\in X_\lambda\setminus\{0\}$ with $\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx>0$. Note that

$$\phi_u'(t) = t\|u\|_{\lambda,V}^2 - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx.$$

Define

$$G(t) := t^{2-p}\|u\|_{\lambda,V}^2 - \mu t^{1-\gamma-p}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx,$$

for all $t>0$. Note that for $t>0$, $tu\in\mathcal{N}_{\lambda,\mu}$ if and only if t is a solution of the equation

$$G(t) = \int_{\mathbb{R}^N} g|u|^p dx.$$

A simple calculation yields that $G(t) \to -\infty$ as $t \to 0^+$, $G(t) \to 0$ as $t \to \infty$. While, since

$$G'(t) = (2-p)t^{1-p}\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)t^{-\gamma-p}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx.$$

Then $G(t)$ possesses a unique maximum point $t_{\max} > 0$, which is given by

$$t_{\max} = \left(\frac{\mu(p+\gamma-1)\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx}{(p-2)\|u\|^2_{\lambda,V}}\right)^{\frac{1}{\gamma+1}}.$$

Moreover, we have $G(t)$ is increasing on $(0, t_{\max})$ and decreasing on $(t_{\max}, \infty)$. Thus

$$\begin{aligned}
G(t_{\max}) &= \left[\left(\frac{\mu(p+\gamma-1)}{p-2}\right)^{\frac{2-p}{\gamma+1}} - \mu\left(\frac{\mu(p+\gamma-1)}{p-2}\right)^{\frac{1-\gamma-p}{\gamma+1}}\right]\frac{(\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx)^{\frac{2-p}{\gamma+1}}}{\|u\|_{\lambda,V}^{\frac{2(1-\gamma-p)}{\gamma+1}}} \\
&= \mu^{\frac{2-p}{\gamma+1}}\|u\|^p_{\lambda,V}\frac{\gamma+1}{p-2}\left(\frac{p+\gamma-1}{p-2}\right)^{\frac{1-\gamma-p}{\gamma+1}}\left(\frac{\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx}{\|u\|^{1-\gamma}_{\lambda,V}}\right)^{\frac{2-p}{\gamma+1}} \\
&\geq \mu^{\frac{2-p}{\gamma+1}}\|u\|^p_{\lambda,V}\frac{\gamma+1}{p-2}\left(\frac{p+\gamma-1}{p-2}\right)^{\frac{1-\gamma-p}{\gamma+1}} \\
&\quad\times\left((\frac{\mu_0}{\mu_0-1})^{\frac{1-\gamma}{2}}\|f\|_{L^{\frac{p}{p+\gamma-1}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2^*_s-p)}{2^*_s p}}\mathcal{M}^{\gamma-1}_{s,A}\right)^{\frac{2-p}{\gamma+1}}. \qquad (2.7)
\end{aligned}$$

(i) If $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, then there is a unique $0 < t^+ < t_{\max}$ such that

$$G(t^+) = \int_{\mathbb{R}^N} g|u|^p dx, \quad \text{and} \quad G'(t^+) > 0.$$

Thus, $t^+u \in \mathcal{N}_{\lambda,\mu}$, and we have

$$\begin{aligned}
\phi''_{t^+u}(1) &= (2-p)(t^+)^2\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)(t^+)^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx \\
&= t^{1+p}G'(t^+) \\
&> 0.
\end{aligned}$$

Therefore, $t^+u \in \mathcal{N}^+_{\lambda,\mu}$. Since for $0 < t < t_{\max}$, one has

$$\frac{d}{dt}\Phi_{\lambda,\mu}(tu) = t\|u\|^2_{\lambda,V} - \mu t^{-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx - t^{p-1}\int_{\mathbb{R}^N} g|u|^p dx = 0$$

and

$$\frac{d^2}{dt^2}\Phi_{\lambda,\mu}(tu) = (2-p)t^2\|u\|^2_{\lambda,V} + \mu(p+\gamma-1)t^{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx > 0,$$

for $t = t^+$. Thus, $\Phi_{\lambda,\mu}(t^+u) = \inf_{t>0}\Phi_{\lambda,\mu}(tu)$ holds.

(ii) If $\int_{\mathbb{R}^N} g|u|^p dx > 0$, by (2.4), (2.7) and $\mu \in (0, \mu^*)$, we deduce that

$$0 < \int_{\mathbb{R}^N} g|u|^p dx \leq (\frac{\mu_0}{\mu_0-1})^{\frac{p}{2}}\|g^+\|_\infty|\{V^+<\kappa\}|^{\frac{2^*_s-p}{2^*_s}}\mathcal{M}^{-p}_{s,A}\|u\|^p_{\lambda,V}$$

$$
\begin{aligned}
&= (\mu^*)^{\frac{2-p}{\gamma+1}}\|u\|^p_{\lambda,V}\frac{1+\gamma}{p+\gamma-1}\left(\frac{p-2}{p+\gamma-1}\right)^{\frac{p-2}{1+\gamma}}\\
&\quad\times\left((\frac{\mu_0}{\mu_0-1})^{\frac{1-\gamma}{2}}\|f\|_{L^{\frac{p}{p+\gamma-1}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2^*_s-p)}{2^*_s p}}\mathcal{M}^{\gamma-1}_{s,A}\right)^{\frac{2-p}{\gamma+1}}\\
&< G(t_{\max}).
\end{aligned}
$$

There are t^+ and t^- such that $0<t^+<t_{\max}<t^-$,

$$G(t^+)=\int_{\mathbb{R}^N} g|u|^p dx = G(t^-)$$

and

$$G'(t^-)<0<G'(t^+).$$

Again, as in the case (i), we have $t^+u\in\mathcal{N}^+_{\lambda,\mu}$, $t^-u\in\mathcal{N}^-_{\lambda,\mu}$, and $\Phi_{\lambda,\mu}(t^-u)\ge\Phi_{\lambda,\mu}(tu)\ge\Phi_{\lambda,\mu}(t^+u)$ for each $t\in[t^+,t^-]$ and $\Phi_{\lambda,\mu}(t^+u)=\inf\limits_{0<t\le t_{\max}}\Phi_{\lambda,\mu}(tu)$, $\Phi_{\lambda,\mu}(t^-u)=\sup\limits_{t\ge t_{\max}}\Phi_{\lambda,\mu}(tu)$. Therefore, conclusion (ii) holds. □

We remark that from Lemma 2.2 and Lemma 2.3, one has $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\bigcup\mathcal{N}^-_{\lambda,\mu}$ for all $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$. Since $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$ are non-empty, thus, by Lemma 2.3, we may define

$$c^+_{\lambda,\mu}=\inf_{u\in\mathcal{N}^+_{\lambda,\mu}}\Phi_{\lambda,\mu}(u)\quad\text{and}\quad c^-_{\lambda,\mu}=\inf_{u\in\mathcal{N}^-_{\lambda,\mu}}\Phi_{\lambda,\mu}(u).$$

Then we have the following result.

Lemma 2.4. *Suppose that the functions f, g and V satisfy the conditions (F), (G) and $(V1)-(V4)$. Then, for $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$, there exists a positive constant C such that $c^+_{\lambda,\mu}<0<C<c^-_{\lambda,\mu}$.*

Proof. (i) Let $u\in\mathcal{N}^+_{\lambda,\mu}\subset\mathcal{N}_{\lambda,\mu}$, then we have

$$(1+\gamma)\|u\|^2_{\lambda,V}-(p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx>0.$$

It follows that

$$
\begin{aligned}
\Phi_{\lambda,\mu}(u)&=\frac{1}{2}\|u\|^2_{\lambda,V}-\frac{\mu}{1-\gamma}\int_{\mathbb{R}^N} f|u|^{1-\gamma}dx-\frac{1}{p}\int_{\mathbb{R}^N} g|u|^p dx\\
&=-\frac{1+\gamma}{2(1-\gamma)}\|u\|^2_{\lambda,V}+\frac{p+\gamma-1}{p(1-\gamma)}\int_{\mathbb{R}^N} g|u|^p dx\\
&<-\frac{p-2}{2p}\frac{1+\gamma}{1-\gamma}\|u\|^2_{\lambda,V}<0.
\end{aligned}
$$

Therefore, $c^+_{\lambda,\mu}<0$.

(ii) Let $u\in\mathcal{N}^-_{\lambda,\mu}$, then we have

$$(1+\gamma)\|u\|^2_{\lambda,V}-(p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx<0.$$

According to (2.1) and (2.3), we get

$$\begin{aligned}\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2 \le \|u\|_{\lambda,V}^2 < \frac{p+\gamma-1}{1+\gamma}\int_{\mathbb{R}^N} g|u|^p dx \\ \le \frac{p+\gamma-1}{1+\gamma}\|g^+\|_\infty |\{V^+<\kappa\}|^{\frac{2_s^*-p}{2_s^*}}\mathcal{M}_{s,A}^{-p}\|u\|_\lambda^p.\end{aligned}$$

Therefore, we can show that

$$\|u\|_\lambda > \left(\frac{(\mu_0-1)(1+\gamma)}{\mu_0(p+\gamma-1)}\frac{\mathcal{M}_{s,A}^p}{\|g^+\|_\infty |\{V^+<\kappa\}|^{\frac{2_s^*-p}{2_s^*}}}\right)^{\frac{1}{p-2}} := C_0.$$

Then, we know

$$\begin{aligned}\Phi_{\lambda,\mu}(u) \ge\ & \frac{(p-2)(\mu_0-1)}{2p\mu_0}\|u\|_\lambda^2 \\ & -\frac{\mu(p-1+\gamma)}{p(1-\gamma)}\|f^+\|_{L^{\frac{p}{p-1+\gamma}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^*p}}M_{s,A}^{\gamma-1}\|u\|_\lambda^{1-\gamma} \\ >\ & C_0^{1-\gamma}\Bigg[\frac{(p-2)(\mu_0-1)}{2p\mu_0}C_0^{1+\gamma} \\ & -\frac{\mu(p-1+\gamma)}{p(1-\gamma)}\|f^+\|_{L^{\frac{p}{p-1+\gamma}}}|\{V^+<\kappa\}|^{\frac{(1-\gamma)(2_s^*-p)}{2_s^*p}}M_{s,A}^{\gamma-1}\Bigg] \\ :=\ & C.\end{aligned}$$

Since $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$, we can verify that $C>0$. Hence $\Phi_{\lambda,\mu}(u)>C>0$ for all $u\in\mathcal{N}_{\lambda,\mu}^-$ and the proof is completed. □

Lemma 2.5. *Suppose that the functions f, g and V satisfy the conditions (F), (G) and $(V1)-(V4)$. Then $\mathcal{N}_{\lambda,\mu}^-$ is a closed subset in X_λ for $(\lambda,\mu)\in[\lambda^*,+\infty)\times(0,\mu^*)$.*

Proof. In order to prove $\mathcal{N}_{\lambda,\mu}^-$ is a closed subset in X_λ, let us consider a sequence $\{u_n\}\subset\mathcal{N}_{\lambda,\mu}^-$ such that $u_n\to u$ in X_λ. It is obvious that $\langle\Phi'_{\lambda,\mu}(u),u\rangle=0$. By the proof of Lemma 2.4, we have

$$\|u\|_\lambda = \lim_{n\to\infty}\|u_n\|_\lambda \ge C_0>0.$$

Thus, $u\in\mathcal{N}_{\lambda,\mu}$. By the definition of $\mathcal{N}_{\lambda,\mu}^-$, it holds

$$(1+\gamma)\|u_n\|_{\lambda,V}^2-(p+\gamma-1)\int_{\mathbb{R}^N} g|u_n|^p dx<0.$$

This, together with (2.3), lead to

$$(1+\gamma)\|u\|_{\lambda,V}^2-(p+\gamma-1)\int_{\mathbb{R}^N} g|u|^p dx\le 0.$$

which implies that $u\in\mathcal{N}_{\lambda,\mu}^-\bigcup\mathcal{N}_{\lambda,\mu}^0$. By Lemma 2.2, we know $\mathcal{N}_{\lambda,\mu}^0=\emptyset$. Therefore, $u\in\mathcal{N}_{\lambda,\mu}^-$. Then $\mathcal{N}_{\lambda,\mu}^-$ is a closed subset in X_λ. □

Lemma 2.6. *Suppose $u \in \mathcal{N}^+_{\lambda,\mu}$ and $v \in \mathcal{N}^-_{\lambda,\mu}$ are minimizers of $\Phi_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$, respectively. Then for every nonegative $\omega \in X_\lambda$, we have*

(i) there exists $\varepsilon_0 > 0$ such that $\Phi_{\lambda,\mu}(u+\varepsilon\omega) \geq \Phi_{\lambda,\mu}(u)$ for all $0 \leq \varepsilon \leq \varepsilon_0$.

(ii) $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$, where for each $\varepsilon \geq 0$, t_ε is the unique positive real number satisfying $t_\varepsilon(v+\varepsilon\omega) \in \mathcal{N}^-_{\lambda,\mu}$.

Proof. (i) Let $\omega \geq 0$ and for each $\varepsilon \geq 0$, set

$$\sigma(\varepsilon) = \|u+\varepsilon\omega\|^2_{\lambda,V} + \mu\gamma \int_{\mathbb{R}^N} f|u+\varepsilon\omega|^{1-\gamma}dx - (p-1)\int_{\mathbb{R}^N} |u+\varepsilon\omega|^p dx.$$

Then by using continuity of σ and $\sigma(0) = \phi''_u(1) > 0$, there exists $\varepsilon_0 > 0$ such that $\sigma(\varepsilon) > 0$ for all $0 \leq \varepsilon \leq \varepsilon_0$. Since for each $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that $s_\varepsilon(u+\varepsilon\omega) \in \mathcal{N}^+_{\lambda,\mu}$, for each $\varepsilon \in [0,\varepsilon_0]$, we have

$$\Phi_{\lambda,\mu}(u+\varepsilon\omega) \geq \Phi_{\lambda,\mu}(s_\omega(u+\varepsilon\omega)) \geq \Phi_{\lambda,\mu}(u).$$

(ii) For each $v \in \mathcal{N}^-_{\lambda,\mu}$, we define $H : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ by

$$H(t,l_1,l_2,l_3) = l_1 t - \mu l_2 t^{-\gamma} - l_3 t^{p-1},$$

for $(t,l_1,l_2,l_3) \in (0,\infty)\times\mathbb{R}^3$. Since $v \in \mathcal{N}^-_{\lambda,\mu}$, we obtain

$$\frac{\partial H}{\partial t}(1, \|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|u|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx) = \phi''_v(1) < 0$$

and for each $\varepsilon > 0$,

$$H(t_\varepsilon, \|v+\varepsilon\omega\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v+\varepsilon\omega|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v+\varepsilon\omega|^p dx) = 0.$$

Moreover,

$$H(1, \|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx) = \phi'_u(1) = 0.$$

Applying the implicit function theorem, there exists an open neighbourhood $A \subset (0,\infty)$ and $B \subset \mathbb{R}^3$ containing 1 and

$$(\|v\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v|^p dx)$$

respectively such that for all $H(t,y) = 0$ has a unique solution $t = h(y)$ with $h : B \to A$ being a smooth function. Consequently, we get

$$(\|v+\varepsilon\omega\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v+\varepsilon\omega|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v+\varepsilon\omega|^p dx) \in B$$

and

$$h(\|v+\varepsilon\omega\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v+\varepsilon\omega|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v+\varepsilon\omega|^p dx) = t_\varepsilon.$$

Since

$$H(t_\varepsilon, \|v+\varepsilon\omega\|^2_{\lambda,V}, \int_{\mathbb{R}^N} f|v+\varepsilon\omega|^{1-\gamma}dx, \int_{\mathbb{R}^N} g|v+\varepsilon\omega|^p dx) = 0.$$

Thus, by continuity of g, we get $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$. $\square$

Lemma 2.7. *Suppose $u \in \mathcal{N}^+_{\lambda,\mu}$ and $v \in \mathcal{N}^-_{\lambda,\mu}$ are minimizers of $\Phi_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$, respectively. Then for each nonnegative $\omega \in X_\lambda$, we have*

$$\langle u, \omega\rangle_{\lambda,V} - \mu\int_{\mathbb{R}^N} fu^{-\gamma}\omega dx - \int_{\mathbb{R}^N} gu^{p-1}\omega dx \geq 0,$$
$$\langle v, \omega\rangle_{\lambda,V} - \mu\int_{\mathbb{R}^N} fv^{-\gamma}\omega dx - \int_{\mathbb{R}^N} gv^{p-1}\omega dx \geq 0.$$

Proof. Let $\omega \in X_\lambda$ be nonnegative function, then by Lemma 2.6, for each $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned} 0 &\leq \frac{\Phi_{\lambda,\mu}(u+\varepsilon\omega) - \Phi_{\lambda,\mu}(u)}{\varepsilon} \\ &= \frac{1}{2\varepsilon}(\|u+\varepsilon\omega\|^2_{\lambda,V} - \|\omega\|^2_{\lambda,V}) - \frac{\mu}{(1-\gamma)}\int_{\mathbb{R}^N} f\frac{(u+\varepsilon\omega)^{1-\gamma} - u^{1-\gamma}}{\varepsilon}dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^N} g\frac{(u+\varepsilon\omega)^p - u^p}{\varepsilon}dx. \end{aligned} \tag{2.8}$$

It can be easily verified that, as $\varepsilon \to 0^+$

$$\frac{1}{2\varepsilon}(\|u+\varepsilon\omega\|^2_{\lambda,V} - \|w\|^2_{\lambda,V}) \to \langle u, \omega\rangle_{\lambda,V}.$$

By (G) and the Lebesgue dominate convergence theorem, one has

$$\lim_{\varepsilon\to 0^+}\frac{1}{p}\int_{\mathbb{R}^N} g\frac{(u+\varepsilon\omega)^p - u^p}{\varepsilon}dx = \int_{\mathbb{R}^N} gu^{p-1}\omega dx.$$

Due to $0 < \gamma < 1$ and f is a positive continuous function, we have

$$f((u+\varepsilon\omega)^{1-\gamma} - u^{1-\gamma}) \geq 0.$$

It follows from (2.8) that

$$\liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} f\frac{(u+\varepsilon\omega)^{1-\gamma} - u^{1-\gamma}}{\varepsilon}dx < \infty.$$

Then, by (2.8) and Fatou's lemma, we can deduce that

$$\begin{aligned} \mu\int_{\mathbb{R}^N} fu^{-\gamma}\omega dx &\leq \frac{\mu}{1-\gamma}\liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}^N} f\frac{(u+\varepsilon\omega)^{1-\gamma} - u^{1-\gamma}}{\varepsilon}dx \\ &\leq \langle u, \omega\rangle_{\lambda,V} - \int_{\mathbb{R}^N} gu^{p-1}\omega dx. \end{aligned}$$

Consequently, for each nonnegative $\omega \in X_\lambda$, we have

$$\langle u, \omega\rangle_{\lambda,V} - \mu\int_{\mathbb{R}^N} fu^{-\gamma}\omega dx - \int_{\mathbb{R}^N} gu^{p-1}\omega dx \geq 0.$$

Next, we will show that these properties are also held for $v \in \mathcal{N}^-_{\lambda,\mu}$. For each $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon(v + \varepsilon\omega) \in \mathcal{N}^-_{\lambda,\mu}$. By Lemma 2.6, for sufficiently small $\varepsilon > 0$, one has

$$\Phi_{\lambda,\mu}(t_\varepsilon(v + \varepsilon\omega)) \geq \Phi_{\lambda,\mu}(v) \geq \Phi_{\lambda,\mu}(t_\varepsilon v),$$

which implies $\Phi_{\lambda,\mu}(t_\varepsilon(v + \varepsilon\omega)) - \Phi_{\lambda,\mu}(v) \geq 0$. Thus, we have

$$\begin{aligned} &\frac{\mu t_\varepsilon^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^N} f\frac{(v+\varepsilon\omega)^{1-\gamma} - v^{1-\gamma}}{\varepsilon}dx \\ &\leq \frac{t_\varepsilon^2}{2\varepsilon}(\|v+\varepsilon\omega\|^2_{\lambda,V} - \|v\|^2_{\lambda,V}) - \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N} g\frac{(v+\varepsilon\omega)^p - v^p}{\varepsilon}dx. \end{aligned}$$

Since as $\varepsilon \to 0^+$, $t_\varepsilon \to 1$, using similar argument as in the previous case, we obtain

$$\langle v, \omega\rangle_{\lambda,V} - \mu\int_{\mathbb{R}^N} f v^{-\gamma}\omega dx - \int_{\mathbb{R}^N} g v^{p-1}\omega dx \geq 0.$$

3. Proof of Theorem 1.1

Since $\Phi_{\lambda,\mu}(u) = \Phi_{\lambda,\mu}(|u|)$, we can assume that $u \geq 0$ for all $u \in X_\lambda$. Now, we prove the following propositions.

Proposition 3.1. *Suppose that* $0 < \gamma < 1$, $2 < p < 2^*_s$, *and the conditions* (F), (G) *and* $(V1) - (V4)$ *are satisfied. Then, for* $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, *the functional* $\Phi_{\lambda,\mu}$ *has a minimizer* u_0 *in* $\mathcal{N}^+_{\lambda,\mu}$ *such that* $\Phi_{\lambda,\mu}(u_0) = c^+_{\lambda,\mu}$.

Proof. We apply the Ekeland's variational principle (see [3] for the details) to consider a minimizing sequence $\{u_n\} \subset \mathcal{N}^+_{\lambda,\mu}$ satisfying

(*i*) $c^+_{\lambda,\mu} < \Phi_{\lambda,\mu}(u_n) < c^+_{\lambda,\mu} + \frac{1}{n}$,

(*ii*) $\Phi_{\lambda,\mu}(u) \geq \Phi_{\lambda,\mu}(u_n) - \frac{1}{n}\|u_n - u\|$.

Moreover, by Lemma 2.1, we can deduce that $\{u_n\}$ is a bounded sequence in X_λ. Therefore, there exists a subsequence of $\{u_n\}$ (we still denotes $\{u_n\}$) such that

$$\begin{aligned} &u_n \rightharpoonup u_0, \quad \text{in } X_\lambda, \\ &u_n \to u_0, \quad \text{in } L^q(\mathbb{R}^N), \quad q \in [2, 2^*_s), \end{aligned}$$

with $u_0 \geq 0$. Since $0 < \gamma < 1$, $f \in L^{\frac{p}{p+\gamma-1}}(\mathbb{R}^N)$ is a positive continuous function, by Vitali's convergence theorem, one can prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f|u_n|^{1-\gamma}dx = \int_{\mathbb{R}^N} f|u_0|^{1-\gamma}dx.$$

We divide the proof into two steps.

Step 1. $u_n \to u_0$ **in** X_λ **and** $u_0 \in \mathcal{N}^+_{\lambda,\mu}$**.**

First, we show that $u_0 \neq 0$. Using the weak lower semi-continnity norm, we have

$$\Phi_{\lambda,\mu}(u_0) \leq \liminf_{n\to\infty}\Phi_{\lambda,\mu}(u_n) = c^+_{\lambda,\mu} < 0.$$

If $u_0 = 0$, then $\Phi_{\lambda,\mu}(u_0) = 0$, which is a contradiction.

Next, we prove that $u_n \to u_0$ strongly in X_λ as $n \to \infty$. Suppose the contrary, by (2.1), we get

$$\|u_0\|^2_{\lambda,V} < \liminf_{n\to\infty} \|u_n\|^2_{\lambda,V}.$$

Combining this with $\{u_n\} \subset \mathcal{N}^+_{\lambda,\mu}$, one has

$$\begin{aligned}&\|u_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|u_0|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|u_0|^p dx\\ &< \liminf_{n\to\infty} \left[\|u_n\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|u_n|^{1-\gamma}dx - \int_{\mathbb{R}^N} g|u_n|^p dx\right] = 0. \end{aligned}\tag{3.1}$$

Now, we show that for u_0, there exists $0 < t^+ \neq 1$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

If $\int_{\mathbb{R}^N} g|u|^p dx \leq 0$, then by Lemma 2.3 (i), there exists $t^+ > 0$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $\Phi'_{\lambda,\mu}(t^+u_0) = 0$. By (3.1), we known that $\Phi'_{\lambda,\mu}(u_0) \neq 0$. Hence, $t^+ \neq 1$.

If $\int_{\mathbb{R}^N} g|u|^p dx > 0$, by Lemma 2.3 (ii), then there exists $0 < t^+ \neq 1$ such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

Since t^+u_0 is a minimizer of $\Phi_{\lambda,\mu}$ in X_λ. Then,

$$\Phi_{\lambda,\mu}(t^+u_0) < \Phi_{\lambda,\mu}(u_0) = \lim_{n\to\infty} \Phi_{\lambda,\mu}(u_n) = c^+_{\lambda,\mu},$$

which contradicts $c^+_{\lambda,\mu} = \inf\limits_{u\in\mathcal{N}^+_{\lambda,\mu}} \Phi_{\lambda,\mu}(u)$. Therefore, we obtain $u_n \to u_0$ in X_λ.

Finally, we claim that $u_0 \in \mathcal{N}^+_{\lambda,\mu}$. On the contrary, assume that $u_0 \in \mathcal{N}^-_{\lambda,\mu}$ ($\mathcal{N}^0_{\lambda,\mu} = \emptyset$ for $(\lambda,\mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$). It follows from (2.6) and $u_0 \in \mathcal{N}^-_{\lambda,\mu}$ that

$$\int_{\mathbb{R}^N} g|u_0|^p dx > 0.$$

Then, by Lemma 2.3 (ii), there exist unique $t^+ > 0$, $t^- > 0$, with $t^- > t^+ > 0$, such that $t^+u_0 \in \mathcal{N}^+_{\lambda,\mu}$, $t^-u_0 \in \mathcal{N}^-_{\lambda,\mu}$ and

$$\Phi_{\lambda,\mu}(t^+u_0) = \inf_{0<t\leq t_{\max}} \Phi_{\lambda,\mu}(tu_0), \qquad \Phi_{\lambda,\mu}(t^-u_0) = \sup_{t\geq t_{\max}} \Phi_{\lambda,\mu}(tu_0).$$

Since $u_0 \in \mathcal{N}^-_{\lambda,\mu}$, it suffices to prove that

$$\frac{d}{dt}\Phi_{\lambda,\mu}(u_0) = 0, \qquad \frac{d^2}{dt^2}\Phi_{\lambda,\mu}(u_0) < 0.$$

This indicates that $t^- = 1$. Also, since

$$\frac{d}{dt}\Phi_{\lambda,\mu}(t^+u_0) = 0, \qquad \frac{d^2}{dt^2}\Phi_{\lambda,\mu}(t^+u_0) > 0.$$

Then, there exists $t \in (t^+, 1]$ such that

$$c^+_{\lambda,\mu} \leq \Phi_{\lambda,\mu}(t^+u_0) < \Phi_{\lambda,\mu}(tu_0) \leq \Phi_{\lambda,\mu}(u_0) = c^+_{\lambda,\mu},$$

this is a contradiction. So $u_0 \in \mathcal{N}^+_{\lambda,\mu}$.

Step 2. u_0 is a solution of system (1.1).

In what follows, we show that the solution u_0 is a weak solution of problem (1.1). Let $v \in X_\lambda$ and $\varepsilon > 0$. Put

$$\psi = (u_0 + \varepsilon v)^+, \quad \text{and} \quad \varphi = u_0 + \varepsilon v < 0.$$

Set $\Omega_+ = \{x \in \mathbb{R}^N : u_0 + \varepsilon v \geq 0\}$ and $\Omega_- = \{x \in \mathbb{R}^N : u_0 + \varepsilon v < 0\}$, then by lemma 2.7, we can obtain that

$$\begin{aligned}
0 &\leq \langle u_0, \psi\rangle_{\lambda,V} - \mu \int_{\Omega_+} f {u_0}^{-\gamma}(u_0 + \varepsilon v)dx - \int_{\Omega_+} g {u_0}^{p-1}(u_0 + \varepsilon v)dx \\
&= \|u_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f {u_0}^{1-\gamma} dx - \int_{\mathbb{R}^N} g {u_0}^{p} dx \\
&\quad + \varepsilon \left(\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f {u_0}^{-\gamma} v dx - \int_{\mathbb{R}^N} g {u_0}^{p-1} v dx \right) \\
&\quad - \left(\langle u_0, \varphi\rangle_{\lambda,V} - \mu \int_{\Omega_-} f {u_0}^{-\gamma}(u_0 + \varepsilon v)dx - \int_{\Omega_-} g {u_0}^{p-1}(u_0 + \varepsilon v)dx \right).
\end{aligned}$$

Then using the fact $u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $f(x)$ is a positive continuous function, we have

$$\begin{aligned}
0 \leq& \varepsilon \left(\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f {u_0}^{-\gamma} v dx - \int_{\mathbb{R}^N} g {u_0}^{p-1} v dx \right) \\
&- \varepsilon \Re \int_{\Omega_-} \left[\frac{(u_0(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_0(y)) \overline{(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} v(y))}}{|x-y|^{N+2s}} + V_\lambda u_0 \bar{v} \right] dxdy \\
&+ \int_{\Omega_-} g {u_0}^{p-1}(u_0 + \varepsilon v)dx.
\end{aligned} \tag{3.2}$$

Since the measure of the domain of integration $\Omega_- = \{x \in \mathbb{R}^N : u_0 + \varepsilon v < 0\}$ tends to 0 as $\varepsilon \to 0^+$, it follows that

$$\int_{\Omega_-} \left[\frac{(u_0(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u_0(y)) \overline{(v(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})} v(y))}}{|x-y|^{N+2s}} + V_\lambda u_0 \bar{v} \right] dxdy \to 0.$$

Moreover, by (G) and (2.3), when $\varepsilon \to 0^+$, we have

$$\left| \int_{\Omega_-} g {u_0}^{p-1}(u_0 + \varepsilon v)dx \right| \leq \|g\|_\infty \int_{\Omega_-} |u_0|^p dx + \varepsilon \|g\|_\infty \left| \int_{\Omega_-} |u_0|^{p-1} v dx \right| \to 0.$$

Dividing by ε and letting $\varepsilon \to 0$ in (3.2), we obtain

$$\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f {u_0}^{-\gamma} v dx - \int_{\mathbb{R}^N} g {u_0}^{p-1} v dx \geq 0.$$

Since v was arbitrary, this holds for $-v$ also. Hence, for all $v \in X_\lambda$, one has

$$\langle u_0, v\rangle_{\lambda,V} - \mu \int_{\mathbb{R}^N} f {u_0}^{-\gamma} v dx - \int_{\mathbb{R}^N} g {u_0}^{p-1} v dx = 0.$$

Then u_0 is a positive solution of problem (1.1).

Proposition 3.2. *Suppose that* $0 < \gamma < 1$, $2 < p < 2_s^*$, *and the conditions* (F), (G) *and* $(V1)-(V4)$ *are satisfied. Then, for* $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$, *the functional* $\Phi_{\lambda,\mu}$ *has a minimizer* v_0 *in* $\mathcal{N}^-_{\lambda,\mu}$ *such that* $\Phi_{\lambda,\mu}(v_0) = c^-_{\lambda,\mu}$.

Proof. On account of $\Phi_{\lambda,\mu}$ is also coercive on $\mathcal{N}^-_{\lambda,\mu}$, we apply the Ekeland's variational principle to the minimization problem $c^-_{\lambda,\mu} = \inf\limits_{u \in \mathcal{N}^-_{\lambda,\mu}} \Phi_{\lambda,\mu}(u)$, there exists a minimizing sequence $\{v_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ of $\Phi_{\lambda,\mu}$ with the following properties

(i) $c^-_{\lambda,\mu} < \Phi_{\lambda,\mu}(v_n) < c^-_{\lambda,\mu} + \frac{1}{n}$,

(ii) $\Phi_{\lambda,\mu}(v) \geq \Phi_{\lambda,\mu}(v_n) - \frac{1}{n}\|v_n - v\|$.

Moreover, $\{v_n\}$ is bounded in X_λ, up to a subsequence if necessary, there exists $v_0 \in X_\lambda$ such that

$$\begin{aligned} &v_n \rightharpoonup v_0, \quad \text{in } \ X_\lambda, \\ &v_n \to v_0, \quad \text{in } \ L^q(\mathbb{R}^N), \quad q \in [2, 2_s^*), \end{aligned}$$

with $v_0 \geq 0$. Then, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f|v_n|^{1-\gamma} dx = \int_{\mathbb{R}^N} f|v_0|^{1-\gamma} dx \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} g|v_n|^p dx = \int_{\mathbb{R}^N} g|v_0|^p dx.$$

We will show that $v_0 \neq 0$. If $v_0 = 0$, then v_n converges to 0 strongly in X_λ, which contradicts Lemma 2.4. Next, we prove that $v_n \to v_0$ in X_λ. If $v_n \nrightarrow v_0$ in X_λ then

$$\begin{aligned} &\|v_0\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|v_0|^{1-\gamma} dx - \int_{\mathbb{R}^N} g|v_0|^p dx \\ &< \liminf_{n\to\infty} \left[\|v_n\|^2_{\lambda,V} - \mu \int_{\mathbb{R}^N} f|v_n|^{1-\gamma} dx - \int_{\mathbb{R}^N} g|v_n|^p dx \right] = 0. \end{aligned} \tag{3.3}$$

Since $\{v_n\} \subset \mathcal{N}^-_{\lambda,\mu}$, we deduce from (2.6) that

$$\mu(\gamma + 1) \int_{\mathbb{R}^N} f|v_0|^{1-\gamma} dx + (2 - p) \int_{\mathbb{R}^N} g|v_0|^p dx \leq 0.$$

Consequently, we have $\int_{\mathbb{R}^N} g|v_0|^p dx > 0$. Then by Lemma 2.4 (ii), there exists a $t^- > 0$ such that $\Phi'_{\lambda,\mu}(t^- v_0) = 0$ and $t^- v_0 \in \mathcal{N}^-_{\lambda,\mu}$. Note that $\Phi'_{\lambda,\mu}(v_0) \neq 0$ by (3.3). Thus, $t^- \neq 1$. Since $t^- v_n \rightharpoonup t^- v_0$ and $t^- v_n \nrightarrow t^- v_0$ in X_λ. Hence,

$$\Phi_{\lambda,\mu}(t^- v_0) < \liminf_{n\to\infty} \Phi_{\lambda,\mu}(t^- v_n).$$

Observe that the function $\Phi_{\lambda,\mu}(t v_n)$ attains its maximum at $t = 1$. Thus, we have

$$\Phi_{\lambda,\mu}(t^- v_0) < \liminf_{n\to\infty} \Phi_{\lambda,\mu}(t^- v_n) \leq \lim_{n\to\infty} \Phi_{\lambda,\mu}(v_n) = c^+_{\lambda,\mu}.$$

which is absurd. Therefore, we obtain that $v_n \to v_0$ in X_λ.

Since $\mathcal{N}^-_{\lambda,\mu}$ is closed by Lemma 2.5, it follows that $v_0 \in \mathcal{N}^-_{\lambda,\mu}$.

By Lemma 2.6 and 2.7, similar to Proposition 3.1, we get that v_0 is also a positive solution of problem (1.1). □

Proof of Theorem 1.1. Combining Proposition 3.1 and Proposition 3.2, for $(\lambda, \mu) \in [\lambda^*, +\infty) \times (0, \mu^*)$ we know that problem (1.1) admits at least two positive solutions $u_0 \in \mathcal{N}^+_{\lambda,\mu}$ and $v_0 \in \mathcal{N}^-_{\lambda,\mu}$ in X_λ. Since $\mathcal{N}^+_{\lambda,\mu} \bigcap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, the two solutions are distinct. This finishes the proof. □

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