Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 2687–2700
Website:http://www.jaac-online.com
DOI:10.11948/20200299

NEW EXISTENCE RESULTS FOR NONLINEAR FRACTIONAL JERK EQUATIONS WITH INITIAL-BOUNDARY VALUE CONDITIONS AT RESONANCE

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Abstract In this paper, a novel jerk system involving fractional-order-derivatives is proposed. We obtain the existence of solutions of nonlinear fractional jerk differential equations with initial-boundary value conditions at resonance by coincidence degree theory. This paper enriches some existing literatures. Finally, we present an example to illustrate our main results.

Keywords Existence, jerk system, initial-boundary value conditions, resonance

MSC(2010) 26A33, 34B15.

1. Introduction

A special third-order autonomous dynamical system written by

$$\begin{cases} \dot{x}(t) = y, \\ \dot{y}(t) = z, \\ \dot{z}(t) = f(x, y, z), \end{cases}$$

can be expressed as

$$\dddot{x}(t) = f(x, \dot{x}, \ddot{x}).$$

Because it involves a third derivative of x, which is a rate of change of acceleration in a mechanical system, it is called jerk equation. In 1978, Schot [23] first presented the definition of jerk equation and discussed its applications in geometry and plane motion. During the past few years, the study of nonlinear jerk equations is an interesting issue in the realm of nonlinear dynamics and the applications of jerk equations in the various fields of science and engineering dealing with dynamical systems have been increasing. Scholars paid much attention to the solutions to jerk equation and many related achievements have been made in this field, see [1,3,4,12,19,21,23].

In [16], by using homotopy perturbation method, Ma *et al.* obtained high-order analytic approximate periods and periodic solutions to the following nonlinear jerk

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equations with initial conditions:

$$\begin{cases} \dddot{x}(t) = f(x, \dot{x}, \ddot{x}), \\ x(0) = 0, \dot{x}(0) = B, \ddot{x}(0) = 0, \end{cases}$$

where $B \in \mathbb{R}$. Furthermore, the authors compared the analytic approximate periods and periodic solutions with the results obtained by the harmonic balance method.

In 2020, Liu *et al.* [13] developed two new iterative algorithms and determined the periodic solutions of the following nonlinear jerk equation:

$$\begin{cases} \dddot{x}(t) = f(x, \dot{x}, \ddot{x}), \\ x(0) = x_0, \ \dot{x}(0) = \dot{x}_0, \ \ddot{x}(0) = \ddot{x}_0, \end{cases}$$

where the initial values $x_0, \dot{x}_0$ and $\ddot{x}_0$ for the periodic solution of the above jerk system are unknown (or given) and the period is denoted by $T > 0$.

At the same time, we notice that the subject of fractional jerk equations has gained popularity and importance because its varied applications in many fields of science and engineering, such as it has recently proved to be valuable tools for the modeling of many physical phenomena.

In 2018, Prakash *et al.* [18] determined a new fractional jerk system which does not have equilibrium point

$$\begin{cases} D^{\alpha}_{0+} x(t) = y(t), \\ D^{\beta}_{0+} y(t) = z(t), \\ D^{\gamma}_{0+} z(t) = f(x, y, z), \end{cases}$$

where $f = -y(t) + 3y^2(t) - x^2(t) - x(t)z(t) + \beta - W(x(t))y(t)$, α, β, γ are fractional order and $0 < \alpha, \beta, \gamma \leq 1$. The authors successfully designed a fractional-order backstepping controller to stabilise the chaos in the above fractional jerk equations.

In 2020, Echenausía-Monroy *et al.* [2] considered a multi-scroll generator system based on fractional jerk equations:

$$\begin{cases} D^{q_x}_{0+} x(t) = y(t), \\ D^{q_y}_{0+} y(t) = z(t), \\ D^{q_z}_{0+} z(t) = F(x, y, z), \end{cases}$$

where $F(x, y, z) = -\alpha[x + y + z - f(x)]$, $D^{q_i}_{0+}$ denote the Caputo fractional derivatives $i = \{x, y, z\}$. The control parameter α is a constant, which defines the eigenvalues of the system, where $\alpha \in \mathbb{R}$. They also gave a further study of the effects of fractional integration-orders in a jerk system and provided a physical interpretation based on statistical analysis.

Actually, the fractional differential equations (FDEs for short) have been developed over the years. FDEs and fractional boundary value problems are a very active area of research that attracted a considerable interest for their various applications in physics, chemistry, biology, economy, engineering, etc. For a detailed depiction of the origination of fractional calculus, FDEs and their applications, we refer the readers to the famous books and research articles [5–11, 14, 17, 20, 22, 25–30]. As far as fractional jerk equations be concerned, the related theoretical research is just at the beginning. Until now, there have been limited research papers that were published on existence of solutions for fractional jerk equations, such as [2,18]. Furthermore, in term of resonant boundary value problems, it is much less known for the research of the solutions of fractional jerk equations and also lack of the theory instruction. Then a natural question arises "How to find the solutions?" Thus, it is interesting and meaningful for us to determine the solutions of fractional jerk equation under resonant conditions.

Motivated by the mentioned papers and the thought, the following resonant boundary value problems of fractional jerk equations are considered:

$$\begin{cases} D_{0+}^{\alpha}u(t) = y(t), \\ D_{0+}^{\beta}y(t) = z(t), \\ D_{0+}^{\gamma}z(t) = f\big(t, u(t), u'(t), u''(t)\big), \end{cases} \tag{1.1}$$

subject to infinite-point boundary conditions

$$u(0) = D_{0+}^{\alpha+\beta-1}u(0) = 0, D_{0+}^{\alpha+\beta}u(1) = \sum_{i=1}^{\infty} \lambda_i D_{0+}^{\alpha+\beta}u(\xi_i),$$

where $t \in (0,1)$, $0 < \alpha, \beta, \gamma \leq 1$, $2 < \alpha+\beta+\gamma \leq 3$, $0 < \xi_i \leq 1$, $i = 1, 2, \cdots, \infty$, $\sum_{i=1}^{\infty} \lambda_i \xi_i^{\gamma-1} = 1$, D_{0+}^{α}, D_{0+}^{β}, D_{0+}^{γ} denote the standard Riemann-Liouville fractional derivatives and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous.

Equivalently, BVP (1.1) can be rewritten in the fractional jerk form as follows:

$$\begin{cases} \big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t) = f\big(t, u(t), u'(t), u''(t)\big), \\ u(0) = D_{0+}^{\alpha+\beta-1}u(0) = 0, D_{0+}^{\alpha+\beta}u(1) = \displaystyle\sum_{i=1}^{\infty} \lambda_i D_{0+}^{\alpha+\beta}u(\xi_i). \end{cases} \tag{1.2}$$

In general, BVP (1.2) is called resonance if the associated linear homogeneous equation has a nontrivial solution. According to the infinite-point boundary conditions, the corresponding homogeneous BVP:

$$\begin{cases} \big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t) = 0, \\ u(0) = D_{0+}^{\alpha+\beta-1}u(0) = 0, \\ D_{0+}^{\alpha+\beta}u(1) = \displaystyle\sum_{i=1}^{\infty} \lambda_i D_{0+}^{\alpha+\beta}u(\xi_i), \end{cases}$$

has a nontrivial solution $u(t) = ct^{\alpha+\beta+\gamma-1}$, $c \in \mathbb{R}$.

First, we should also take the necessity of $2 < \alpha+\beta+\gamma \leq 3$ into account. Generally speaking,

$$\big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t) \neq D_{0^+}^{\alpha+\beta+\gamma}u(t).$$

But if $u(t)$ is enough "strong", then $\big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t)$ can be equal to $D_{0^+}^{\alpha+\beta+\gamma}u(t)$. So, (1.1) can be rewritten by

$$D_{0^+}^{\alpha+\beta+\gamma}u(t) = f\big(t, u(t), u'(t), u''(t)\big)$$

which is real generalization of standard integral order jerk equation $\dddot{u}(t) = f(t, u, \dot{u}, \ddot{u})$ under the condition: $\alpha+\beta+\gamma > 2$. Thus, the condition $\alpha+\beta+\gamma > 2$ is not artificial but necessary. Second, we can see if $\alpha = \beta = \gamma = 1$, the equation (1.2) can be rewritten as

$$\begin{cases} \dddot{u}(t) = f(t, u, \dot{u}, \ddot{u}), \\ u(0) = \dot{u}(0) = 0, \\ \ddot{u}(1) = \sum_{i=1}^{\infty} \lambda_i \ddot{u}(\xi_i), \end{cases}$$

which is an initial-boundary value problem of standard jerk equation. Therefore, the results of this paper enrich and extend the existing literatures, such as [2,13,18].

The remainder of the paper is arranged as follows. Section 2 introduces some necessary notations, definitions and lemmas. In Section 3, the existence of solutions of BVP (1.2) is investigated by applying the coincidence degree theory due to Mawhin [15]. Finally, an example is given to illustrate our results in section 4.

2. Preliminaries

As for prerequisites, we briefly present some necessary definitions and lemmas from fractional calculus theory that will be used to prove our main theorems.

Definition 2.1 ([9]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([9]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([9]). *Let $n - 1 < \alpha \le n$, $u \in C(0,1) \cap L^1(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

Lemma 2.2 ([9]). *If $\beta > 0$, $\alpha + \beta > 0$, then the equation*

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(x) = I_{0+}^{\alpha+\beta} f(x),$$

is satisfied for continuous function f.

Firstly, we briefly recall some definitions on the coincidence degree theory. For more details, see [15].

Let Y, Z be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that

$$\operatorname{Ker} L = \operatorname{Im} P, \ \operatorname{Im} L = \operatorname{Ker} Q, \ Y = \operatorname{Ker} L \oplus \operatorname{Ker} P, \ Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that
$$L|_{\operatorname{dom} L\cap \operatorname{Ker}P} : \operatorname{dom} L \cap \operatorname{Ker}P \to \operatorname{Im} L$$
is invertible. We denote the inverse of this map by K_P. If Ω is an open bounded subset of Y, the map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Theorem 2.1 ([15]). *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$ for each $(x,\lambda) \in [(\operatorname{dom} L\backslash \operatorname{Ker}L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ for each $x \in \operatorname{Ker}L \cap \partial\Omega$;

(3) $\deg(JQN|_{\operatorname{Ker}L},\ \Omega\cap\operatorname{Ker}L,\ 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \operatorname{Ker}Q$ and $J : \operatorname{Im} Q \to \operatorname{Ker}L$ is any isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

3. Main results

In this section, we will discuss the existence of solutions to equation (1.2).

Denote by E the Banach space $E = C[0,1]$ with the norm $\|u\|_\infty = \max_{0\leq t\leq 1} |u(t)|$. We denote a Banach space $X = \{u(t) : u(t), u'(t), u''(t) \in E\}$ with the norm $\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|t^{3-\alpha-\beta-\gamma}u''(t)\|_\infty$.

Define
$$L : \operatorname{dom}L \to E, u \mapsto \big(D^\gamma_{0+}(D^\beta_{0+}(D^\alpha_{0+}u))\big)(t), \tag{3.1}$$
$$N : X \to E, u \mapsto f\big(t, u(t), u'(t), u''(t)\big), \tag{3.2}$$

where
$$\operatorname{dom}L = \Big\{u \in X :\ u(0) = D^{\alpha+\beta-1}_{0+}u(0) = 0, D^{\alpha+\beta}_{0+}u(1) = \sum_{i=1}^\infty \lambda_i D^{\alpha+\beta}_{0+}u(\xi_i)\Big\}.$$

Then BVP (1.2) is equivalent to the operator equation $Lx = Nx, x \in \operatorname{dom}L$.

Lemma 3.1. *L is defined as (3.1), then*
$$\operatorname{Ker}L = \{u \in X :\ u = ct^{\alpha+\beta+\gamma-1},\ c \in \mathbb{R}\}, \tag{3.3}$$
$$\operatorname{Im}L = \left\{x \in E :\ I^\gamma_{0+}x(1) = \sum_{i=1}^\infty \lambda_i I^\gamma_{0+}x(\xi_i)\right\}. \tag{3.4}$$

Proof. By $Lu = 0$ and Lemmas 2.1, we have $\big(D^\beta_{0+}(D^\alpha_{0+}u)\big)(t) = c_0t^{\gamma-1}, c_0 \in \mathbb{R}$. By Lemmas 2.1 again, one has
$$(D^\alpha_{0+}u)(t) = I^\beta_{0+}c_0t^{\gamma-1} + c_1t^{\beta-1}, c_0, c_1 \in \mathbb{R}$$
and
$$u(t) = I^{\alpha+\beta}_{0+}c_0t^{\gamma-1} + I^\alpha_{0+}c_1t^{\beta-1} + c_2t^{\alpha-1},\ c_0, c_1, c_2 \in \mathbb{R}.$$
For $0 < \alpha, \beta, \gamma \leq 1$, $\alpha + \beta + \gamma > 2$ implies $1 < \alpha + \beta \leq 2$. Obviously, $\alpha + \beta \leq 2$. If $\alpha + \beta < 1$, then by $0 < \gamma \leq 1$, we have $\alpha + \beta + \gamma \leq 2$ which is contradict to $\alpha + \beta + \gamma > 2$. Therefore, $1 < \alpha + \beta \leq 2$.

According to $0<\alpha\le 1$ and $u(0)=0$, we have $c_2=0$. By $D_{0+}^{\alpha+\beta-1}u(0)=0$, we have $c_1=0$. Then, we can show that

$$u(t)=I_{0+}^{\alpha+\beta}c_0t^{\gamma-1}=c_0\frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}t^{\alpha+\beta+\gamma-1}:=ct^{\alpha+\beta+\gamma-1},$$

where $c=c_0\frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}$. It is a nontrivial solution which satisfies $D_{0+}^{\alpha+\beta}u(1)=\sum_{i=1}^{\infty}\lambda_iD_{0+}^{\alpha+\beta}u(\xi_i)$. Then, (3.3) holds.

Next we prove (3.4) holds. Let $x\in \mathrm{Im}L$, so there exists $u\in \mathrm{dom}L$ such that $x(t)=\big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t)$. By Lemma 2.1 and the definition of dom$L$, we have

$$u(t)=I_{0+}^{\alpha+\beta+\gamma}x(t)+I_{0+}^{\alpha+\beta}c_0t^{\gamma-1}+I_{0+}^{\alpha}c_1t^{\beta-1}+c_2t^{\alpha-1},c_0,c_1,c_2\in\mathbb{R}.$$

In view of $u(0)=D_{0+}^{\alpha+\beta-1}u(0)=0$, we get $c_1=c_2=0$. Hence,

$$u(t)=I_{0+}^{\alpha+\beta+\gamma}x(t)+I_{0+}^{\alpha+\beta}c_0t^{\gamma-1}.$$

According to $D_{0+}^{\alpha+\beta}u(1)=\sum_{i=1}^{\infty}\lambda_iD_{0+}^{\alpha+\beta}u(\xi_i)$ and $\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma-1}=1$, we have

$$I_{0+}^{\gamma}x(1)=\sum_{i=1}^{\infty}\lambda_iI_{0+}^{\gamma}x(\xi_i).$$

On the other hand, suppose x satisfies the above equations. Let $u(t)=I_{0+}^{\alpha+\beta+\gamma}x(t)$, we can prove $u(t)\in\mathrm{dom}L$ and $Lu(t)=x$. Then, (3.4) holds. □

In the following, for simplicity, let

$$p=\frac{1+(\alpha+\beta+\gamma)^2}{\Gamma(1+\alpha+\beta+\gamma)},\quad q=\frac{1}{\Gamma(\alpha+\beta+\gamma)}[(\alpha+\beta+\gamma)^2-2(\alpha+\beta+\gamma)+2].$$

Lemma 3.2. *The mapping $L:domL\subset Y\to Z$ is a Fredholm operator of index zero.*

Proof. The linear continuous projector operator P can be defined as

$$Pu=\frac{1}{\Gamma(\alpha+\beta+\gamma)}D_{0+}^{\alpha+\beta+\gamma-1}u(0)t^{\alpha+\beta+\gamma-1}.$$

Obviously, $P^2=P$. It is clear that

$$\mathrm{Ker}P=\big\{u:D_{0+}^{\alpha+\beta+\gamma-1}u(0)=0\big\}.$$

It follows from $u=u-Pu+Pu$ that $Y=\mathrm{Ker}P+\mathrm{Ker}L$. For $u\in\mathrm{Ker}L\cap\mathrm{Ker}P$, then $u=ct^{\alpha+\beta+\gamma-1},c\in\mathbb{R}$. Furthermore, by the definition of Ker$P$, we have $c=0$. Thus,

$$Y=\mathrm{Ker}L\oplus\mathrm{Ker}P.$$

For $\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma}=1$, then we have $\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma-1}\neq 1$. If $\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma-1}=1$, then $\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma-1}=\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma}$. So, we have $\sum_{i=1}^{\infty}\lambda_i(\xi_i^{\gamma}-\xi_i^{\gamma-1})=0$ which is contradict to $\xi_i^{\gamma}<\xi_i^{\gamma-1}$, $i=1,2,\cdots,\infty$.

Thus, the linear operator Q can be well-defined by

$$Qx(t)=\frac{\Gamma(1+\gamma)}{1-\sum_{i=1}^{\infty}\lambda_i\xi_i^{\gamma}}\Big[I_{0+}^{\gamma}x(1)-\sum_{i=1}^{\infty}\lambda_iI_{0+}^{\gamma}x(\xi_i)\Big].$$

For $x(t) \in E$, we have

$$Q\big(Qx(t)\big) = Qx(t) \cdot \frac{\Gamma(1+\gamma)}{1-\sum_{i=1}^{\infty}\lambda_i \xi_i^{\gamma}}\left[(I_{0+}^{\gamma}1)_{t=1} - \sum_{i=1}^{\infty}\lambda_i (I_{0+}^{\gamma}1)_{t=\xi_i}\right] = Qx(t).$$

So, the operator Q is idempotent. It follows from $Qx \cong \mathbb{R}$ and $x = x - Qx + Qx$ that $E = \mathrm{Im}L + \mathrm{Im}Q$. Moreover, by $\mathrm{Ker}Q = \mathrm{Im}L$ and $Q^2 = Q$, we get $\mathrm{Im}L \cap \mathrm{Im}Q = \{0\}$. Hence,

$$E = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Now, $\mathrm{Ind}\, L = \dim \mathrm{Ker}\, L - \mathrm{codim}\, \mathrm{Im}\, L = 0$, and so L is a Fredholm mapping of index zero.

For every $u \in X$, we have

$$\begin{aligned}
\|Pu\|_X &= \|Pu\|_\infty + \|(Pu)'\|_\infty + \|t^{3-\alpha-\beta-\gamma}(Pu)''(t)\|_\infty \\
&= \|Pu\|_\infty + \|(Pu)'\|_\infty + \|t^{3-\alpha-\beta-\gamma}(Pu)''(t)\|_\infty \\
&= \frac{|D_{0+}^{\alpha+\beta+\gamma-1}u(0)|}{\Gamma(\alpha+\beta+\gamma)} \cdot \Big[\|t^{\alpha+\beta+\gamma-1}\|_\infty + \|(\alpha+\beta+\gamma-1)t^{\alpha+\beta+\gamma-2}\|_\infty \\
&\quad + \|(\alpha+\beta+\gamma-1)(\alpha+\beta+\gamma-2)t^{3-\alpha-\beta-\gamma}t^{\alpha+\beta+\gamma-3}\|_\infty\Big] \\
&= \frac{|D_{0+}^{\alpha+\beta+\gamma-1}u(0)|}{\Gamma(\alpha+\beta+\gamma)}\big[(\alpha+\beta+\gamma)^2 - 2(\alpha+\beta+\gamma) + 2\big] \\
&= q|D_{0+}^{\alpha+\beta+\gamma-1}u(0)|.
\end{aligned} \tag{3.5}$$

Furthermore, the operator $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \mathrm{Ker}P$ can be defined by

$$K_P x(t) = I_{0+}^{\alpha+\beta+\gamma}x(t).$$

For $x(t) \in \mathrm{Im}L$, we have

$$LK_P x(t) = LI_{0+}^{\alpha+\beta+\gamma}x(t) = \big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha}I_{0+}^{\alpha+\beta+\gamma}x))\big)(t) = x(t). \tag{3.6}$$

On the other hand, for $u \in \mathrm{dom}L \cap \mathrm{Ker}P$, according to Lemma 2.1 and the definitions of $\mathrm{dom}L$ and $\mathrm{Ker}P$, we have

$$I_{0+}^{\alpha+\beta+\gamma}Lu(t) = I_{0+}^{\alpha+\beta+\gamma}(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha}u)))(t) = u(t). \tag{3.7}$$

Combining (3.6) and (3.7), we have $K_P = (L_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$.

For $x \in \mathrm{Im}L$, we have

$$\begin{aligned}
\|K_P x\|_X &= \|I_{0+}^{\alpha+\beta+\gamma}x\|_X \\
&= \|I_{0+}^{\alpha+\beta+\gamma}x\|_\infty + |(I_{0+}^{\alpha+\beta+\gamma}x)^{'}\|_\infty + \|t^{3-\alpha-\beta-\gamma}(I_{0+}^{\alpha+\beta+\gamma}x)^{''}\|_\infty \\
&= \left[\frac{1}{\Gamma(1+\alpha+\beta+\gamma)} + \frac{1}{\Gamma(\alpha+\beta+\gamma)} + \frac{1}{\Gamma(\alpha+\beta+\gamma-1)}\right]\|x\|_\infty \\
&= \frac{1+(\alpha+\beta+\gamma)^2}{\Gamma(1+\alpha+\beta+\gamma)} \cdot \|x\|_\infty \\
&= p\|x\|_\infty.
\end{aligned} \tag{3.8}$$

Again for $u \in \Omega_1$, $u \in \mathrm{dom}(L)\backslash\mathrm{Ker}(L)$, then $(I-P)u \in \mathrm{dom}L \cap \mathrm{Ker}P$ and $LPu = 0$, thus from (3.8), we have

$$\|(I-P)u\|_X = \|K_P L(I-P)u\|_X = \|K_P Lu\|_X = p\|Nu\|_\infty. \tag{3.9}$$

□

Lemma 3.3. *$K_P(I-Q)N : Y \to Y$ is completely continuous.*

Proof. Assume $\Omega \subset X$ is an open bounded subset. By the continuity of f, we can get that $QN(\bar{\Omega})$ and $K_P(I-Q)N(\bar{\Omega})$ are bounded. So, in view of of the Arzelà-Ascoli theorem, we need only prove that $K_P(I-Q)N(\bar{\Omega})$ is equicontinuous.

From the continuity of f, there exists a constant $A > 0$ such that $|(I-Q)Nx| < A$, for $\forall x \in \bar{\Omega}, t \in [0,1]$. For $0 \le t_1 < t_2 \le 1$, $u \in \overline{\Omega}$, we have

$$\begin{aligned}
&|K_{P,Q}Nu(t_2) - K_{P,Q}Nu(t_1)| \\
&= |K_P(I-Q)Nu(t_2) - K_P(I-Q)Nu(t_1)| \\
&\le \frac{1}{\Gamma(\alpha+\beta+\gamma)}\Big|\int_0^{t_2}(t_2-s)^{\alpha+\beta+\gamma-1}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha+\beta+\gamma-1}(I-Q)Nu(s)ds\Big| \\
&\le \frac{A}{\Gamma(\alpha+\beta+\gamma)}\Big[\int_0^{t_1}(t_2-s)^{\alpha+\beta+\gamma-1} - (t_1-s)^{\alpha+\beta+\gamma-1}ds \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta+\gamma-1}ds\Big] \\
&\le \frac{A}{\Gamma(\alpha+\beta+\gamma+1)}\big(t_2^{\alpha+\beta+\gamma} - t_1^{\alpha+\beta+\gamma}\big)
\end{aligned}$$

and

$$\begin{aligned}
&|(K_{P,Q}Nu)'(t_2) - (K_{P,Q}Nu)'(t_1)| \\
&= |(K_P(I-Q)Nu)'(t_2) - (K_P(I-Q)Nu)'(t_1)| \\
&\le \frac{1}{\Gamma(\alpha+\beta+\gamma-1)}\Big|\int_0^{t_2}(t_2-s)^{\alpha+\beta+\gamma-2}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha+\beta+\gamma-2}(I-Q)Nu(s)ds\Big| \\
&\le \frac{A}{\Gamma(\alpha+\beta+\gamma-1)}\Big[\int_0^{t_1}(t_2-s)^{\alpha+\beta+\gamma-2} - (t_1-s)^{\alpha+\beta+\gamma-2}ds \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta+\gamma-2}ds\Big] \\
&\le \frac{A}{\Gamma(\alpha+\beta+\gamma)}\big(t_2^{\alpha+\beta+\gamma-1} - t_1^{\alpha+\beta+\gamma-1}\big).
\end{aligned}$$

In addition, we have

$$\begin{aligned}
&\big|t_2^{3-\alpha-\beta-\gamma}(K_{P,Q}Nu)''(t_2) - t_1^{3-\alpha-\beta-\gamma}(K_{P,Q}Nu)''(t_1)\big| \\
&= \big|t_2^{3-\alpha-\beta-\gamma}(K_P(I-Q)Nu)''(t_2) - t_1^{3-\alpha-\beta-\gamma}(K_P(I-Q)Nu)''(t_1)\big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\bigg| \int_0^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1} t_1^{3-\alpha-\beta-\gamma}(t_1-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds\bigg| \\
&\leq \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\bigg| \int_0^{t_1} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds \\
&\quad + \int_{t_1}^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1} t_1^{3-\alpha-\beta-\gamma}(t_1-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds\bigg| \\
&\leq \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\bigg| \int_0^{t_1} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds \\
&\quad - \int_0^{t_1} t_1^{3-\alpha-\beta-\gamma}(t_1-s)^{\alpha+\beta+\gamma-3}(I-Q)Nu(s)ds\bigg| \\
&\quad + \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\int_{t_1}^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}|(I-Q)Nu(s)|ds \\
&\leq \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\int_0^{t_1} \bigg| t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3} \\
&\quad - t_1^{3-\alpha-\beta-\gamma}(t_1-s)^{\alpha+\beta+\gamma-3}\bigg| |(I-Q)Nu(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta+\gamma-2)}\int_{t_1}^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}|(I-Q)Nu(s)|ds \\
&\leq \frac{A}{\Gamma(\alpha+\beta+\gamma-2)}\int_0^{t_1}\bigg| t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3} - t_1^{3-\alpha-\beta-\gamma}(t_1-s)^{\alpha+\beta+\gamma-3}\bigg| ds \\
&\quad + \frac{A}{\Gamma(\alpha+\beta+\gamma-2)}\int_{t_1}^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}ds \\
&\leq \frac{A}{\Gamma(\alpha+\beta+\gamma-2)}\int_0^{t_1}\left| \left(1-\frac{s}{t_1}\right)^{\alpha+\beta+\gamma-3} - \left(1-\frac{s}{t_2}\right)^{\alpha+\beta+\gamma-3}\right| ds \\
&\quad + \frac{A}{\Gamma(\alpha+\beta+\gamma-2)}\int_{t_1}^{t_2} t_2^{3-\alpha-\beta-\gamma}(t_2-s)^{\alpha+\beta+\gamma-3}ds \\
&\leq \frac{A}{\Gamma(\alpha+\beta+\gamma-1)}\big[(t_2-t_1) + 2t_2^{3-\alpha-\beta-\gamma}(t_2-t_1)^{\alpha+\beta+\gamma-2}\big].
\end{aligned}
$$

Since t, $t^{\alpha+\beta+\gamma}$, $t^{\alpha+\beta+\gamma-1}$ are uniformly continuous on $[0,1]$, we can get that $K_{P,Q}N(\bar{\Omega})$, $(K_{P,Q}N)'(\bar{\Omega})$ and $t^{3-\alpha-\beta-\gamma}(K_{P,Q}N)''(\bar{\Omega})$ are equicontinuous on $[0,1]$. Thus, we obtain that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. □

Theorem 3.1. *Assume the following conditions hold:*
(H_1) *There exist nonnegative functions* $\psi(t), \varphi_0(t), \varphi_1(t), \varphi_2(t) \in E$*, such that for all* $t \in [0,1], (u_1,u_2,u_3) \in \mathbb{R}^3$*, one has*

$$|f(t,u_1,u_2,u_3)| \leq \psi(t) + \varphi_0(t)|u_1| + \varphi_1(t)|u_2| + \varphi_2(t)|t^{3-\alpha-\beta-\gamma}u_3|.$$

(H_2) *There exists a positive constant A such that $|D_{0+}^{\alpha+\beta+\gamma-1}u(t)| > A$, one has*

$$QN(u) \neq 0.$$

(H_3) *There exists $k > 0$ such that if $|c| > k, c \in \mathbb{R}$, one has either*

$$cQN(ct^{\alpha+\beta+\gamma-1}) > 0$$

or

$$cQN(ct^{\alpha+\beta+\gamma-1}) < 0.$$

Then, BVP (1.2) has at least a solution in X provided that

$$(p+q)\varphi < 1 \tag{3.10}$$

where $\varphi = \max\{\|\psi(t)\|_\infty, \|\varphi_0(t)\|_\infty, \|\varphi_1(t)\|_\infty, \|\varphi_2(t)\|_\infty\}$.

Proof. Let

$$\Omega_1 = \big\{u \in \mathrm{dom}L \setminus \mathrm{Ker}L : \ Lu = \lambda Nu, \lambda \in (0,1)\big\}.$$

For $Lu = \lambda Nu \in \mathrm{Im}L = \mathrm{Ker}Q$, we have $QN(u) \neq 0$. According to (H_2), there exists $t_0 \in (0,1)$ such that $|D_{0+}^{\alpha+\beta+\gamma-1}u(t_0)| \leq A$. By $Lu = \lambda Nu, \ u \in \mathrm{dom}L \setminus \mathrm{Ker}L$, that is $\big(D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0+}^{\alpha}u))\big)(t) = \lambda Nu$, we have

$$D_{0+}^{\alpha+\beta+\gamma-1}u(0) = D_{0+}^{\alpha+\beta+\gamma-1}u(t_0) - \int_0^{t_0} D_{0+}^{\alpha+\beta+\gamma}u(s)ds.$$

This gives

$$\begin{aligned} \big|D_{0+}^{\alpha+\beta+\gamma-1}u(0)\big| &= \big|D_{0+}^{\alpha+\beta+\gamma-1}u(t_0)\big| + \left|\int_0^{t_0} D_{0+}^{\alpha+\beta+\gamma}u(s)ds\right| \\ &\leq A + \int_0^{t_0} \big|D_{0+}^{\alpha+\beta+\gamma}u(s)\big|ds \\ &= A + \int_0^{t_0} \big|Lu\big|ds \\ &\leq A + \|Nu\|_\infty. \end{aligned} \tag{3.11}$$

By (3.5), (3.9), (3.11) and (H_1), we have

$$\begin{aligned} \|u\|_X &= \|Pu + (I-P)u\|_X \leq \|Pu\|_Y + \|(I-P)u\|_X \\ &\leq q|D_{0+}^{\alpha+\beta+\gamma-1}u(0)| + p\|Nu\|_\infty \\ &\leq qA + q\|Nu\|_\infty + p\|Nu\|_\infty \\ &= qA + (p+q)\big|f(s,u,u',u'')\big| \\ &\leq qA + (p+q)\big(\psi(t) + \varphi_0(t)|u_1| + \varphi_1(t)|u_2| + \varphi_2(t)|t^{3-\alpha-\beta-\gamma}u_3|\big) \\ &\leq qA + (p+q)\big(\|\psi\|_\infty + \|\varphi_0\|_\infty\|u\|_\infty + \|\varphi_1\|_\infty\|u'\|_\infty \\ &\quad + \|\varphi_2\|_\infty\|t^{3-\alpha-\beta-\gamma}u''\|_\infty\big) \\ &\leq qA + (p+q)\big(\varphi + \varphi\|u\|_\infty + \varphi\|u'\|_\infty + \varphi\|t^{3-\alpha-\beta-\gamma}u''\|_\infty\big) \end{aligned}$$

$$\begin{aligned}&\leq qA+(p+q)\varphi\big(1+\|u\|_X\big)\\&=qA+(p+q)\varphi+(p+q)\varphi\|u\|_X.\end{aligned}$$

According to 3.10, we can derive

$$\|u\|_X \leq \frac{qA+(p+q)\varphi}{1-(p+q)\varphi} := M.$$

Thus, we have Ω_1 is bounded.

Let

$$\Omega_2=\big\{u\in \mathrm{Ker}L : Nu\in \mathrm{Im}L\big\}.$$

For $u\in\Omega_2$, then $u(t)=ct^{\alpha+\beta+\gamma-1}$, $c\in\mathbb{R}$. In view of $Nu\in \mathrm{Im}L=\mathrm{Ker}Q$, then $QN(u)=0$. From (H_2), we have $|c|\leq k$. Thus,

$$\begin{aligned}\|u\|_X &= \|u\|_\infty+\|u'\|_\infty+\|t^{3-\alpha-\beta-\gamma}u''(t)\|_\infty\\&=\|ct^{\alpha+\beta+\gamma-1}\|_\infty+\|(ct^{\alpha+\beta+\gamma-1})'\|_\infty+\|t^{3-\alpha-\beta-\gamma}(ct^{\alpha+\beta+\gamma-1})''\|_\infty\\&=\|ct^{\alpha+\beta+\gamma-1}\|_\infty+\|(\alpha+\beta+\gamma-1)ct^{\alpha+\beta+\gamma-2}\|_\infty\\&\quad+\|t^{3-\alpha-\beta-\gamma}(\alpha+\beta+\gamma-1)(\alpha+\beta+\gamma-2)ct^{\alpha+\beta+\gamma-3}\|_\infty\\&\leq k+2k+2k\\&=5k,\end{aligned}$$

which implies Ω_2 is bounded.

Let

$$\Omega_3=\big\{u\in \mathrm{Ker}L : \lambda u+(1-\lambda)QNu=0, \lambda\in[0,1]\big\}.$$

Without loss of generality, we suppose that the first part of (H_3) holds. For any $u\in\Omega_3$, then $u(t)=ct^{\alpha+\beta+\gamma-1}$. By the definition of the set Ω_3, we have

$$\lambda ct^{\alpha+\beta+\gamma-1}+(1-\lambda)QN(ct^{\alpha+\beta+\gamma-1})=0. \tag{3.12}$$

Here, the following cases arises:

Case (1) If $\lambda=1$, then $c=0$. So, $\|u\|_X=\|u\|_\infty+\|u'\|_\infty+\|t^{3-\alpha-\beta-\gamma}u''(t)\|_\infty=0$.
Case (2) If $\lambda=0$, similar to the proof of the boundness of Ω_2, we get $\|u\|_X\leq 5k$.
Case (3) If $\lambda\in(0,1)$, we also have $|c|\leq k$. Otherwise, if $|c|>k$, we obtain

$$\lambda c^2t^{\alpha+\beta+\gamma-1}+(1-\lambda)c\cdot QN(ct^{\alpha+\beta+\gamma-1})>0,$$

which contradicts (3.12). It follows $|c|\leq k$ that $\|u\|_X\leq 5k$.

Thus, Ω_3 is bounded.

If the second part of (H_3) holds, we can prove the set

$$\Omega_3'=\big\{u\in \mathrm{Ker}L : -\lambda u+(1-\lambda)QNu=0, \lambda\in[0,1]\big\}$$

is bounded.

Finally, let Ω to be a bounded open set of Y, such that $\cup_{i=1}^3\overline{\Omega}_i\subset\Omega$. By Lemma 3.3, N is L-compact on Ω. Then by the above arguments, we get
(1) $Lu\neq\lambda Nu$, for every $u\in[(\mathrm{dom}L\setminus KerL)\cap\partial\Omega]\times(0,1)$;
(2) $Nu\notin \mathrm{Im}L$ for every $u\in \mathrm{Ker}L\cap\partial\Omega$;
(3) Let $H(u,\lambda)=\pm\lambda Iu+(1-\lambda)JQNu$, where I is the identical operator. Via the homotopy property of degree, we obtain that

$$\deg\big(JQN|_{\mathrm{Ker}L},\Omega\cap\mathrm{Ker}L,0\big)=\deg\big(H(\cdot,0),\Omega\cap\mathrm{Ker}L,0\big)$$

$$
\begin{aligned}
&= \deg\left(H(\cdot,1), \Omega \cap \mathrm{Ker} L, 0\right) \\
&= \deg\left(I, \Omega \cap \mathrm{Ker} L, 0\right) \\
&= 1 \neq 0.
\end{aligned}
$$

Applying Theorem 3.1, we conclude that $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. □

4. Example

Consider the following initial-boundary value problems of fractional jerk equation:

$$
\begin{cases}
\big(D_{0+}^{0.6}(D_{0+}^{0.7}(D_{0+}^{0.9}u))\big)(t) = f\big(t, u(t), u'(t), u''(t)\big), \\
u(0) = D_{0+}^{0.6}u(0) = 0, D_{0+}^{1.6}u(1) = \displaystyle\sum_{i=1}^{\infty} \lambda_i D_{0+}^{1.6}u(\xi_i),
\end{cases} \tag{4.1}
$$

where

$$
f(t, x_1, x_2, x_3) = \frac{t}{15} + \frac{1 + \sin x_1}{16} - \frac{\operatorname{arccot} x_2}{20\pi} + \frac{\arctan x_3}{10\pi}
$$

and $\xi_i = \frac{1}{2^i}, \lambda = (\frac{1}{2})^{1.4i}$, $i = 1, 2, \cdots, \infty$.

Comparing with the problem (1.2), it can be easily seen that $\alpha = 0.9$, $\beta = 0.7$, $\gamma = 0.6$, $\alpha + \beta + \gamma = 2.2$, $\sum_{i=1}^{\infty} \lambda_i \xi_i^{\gamma-1} = 1$ and

$$
\begin{aligned}
p &= \frac{1 + (\alpha+\beta+\gamma)^2}{\Gamma(1+\alpha+\beta+\gamma)} = \frac{5.84}{\Gamma(3.3)} \approx 2.68, \\
q &= \frac{1}{\Gamma(\alpha+\beta+\gamma)}[(\alpha+\beta+\gamma)^2 - 2(\alpha+\beta+\gamma) + 2] = \frac{2.44}{\Gamma(2.2)} \approx 2.22.
\end{aligned}
$$

By a simple calculation, we have

$$
\begin{aligned}
|f(t, x_1, x_2, x_3)| &= \left| \frac{t}{20} + \frac{3 + \sin x_1}{50} - \frac{\operatorname{arccot} x_2}{40\pi} + \frac{\arctan x_3}{30\pi} \right| \\
&\leq \frac{t}{20} + \frac{4}{50} + \frac{1}{40\pi}\frac{\pi}{2} + \frac{1}{30\pi}\frac{\pi}{2} \\
&\leq \frac{3}{20} + \frac{t}{20}.
\end{aligned}
$$

We can choose $\psi(t) = \frac{3}{20} + \frac{t}{20}$, $\varphi_1(t) = \varphi_2(t) = \varphi_3(t) = 0$, then we have (H_1) is satisfied. In addition, we can get f is a positive function. By choosing $A = k = 1$, then (H_2) and the first inequality of (H_3) are satisfied.
By a direct calculation, we have

$$
(p+q)\varphi \approx \frac{4.9}{5} < 1
$$

which implies (3.10) holds. Therefore, it follows by Theorem 3.1, BVP (4.1) has at least one solution.

Acknowledgments

The authors are grateful to the anonymous referees for their useful suggestions which improve the contents of this article.

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