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SMOOTH SOLUTIONS OF THE LANDAU-LIFSHITZ-BLOCH EQUATION*

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Abstract Landau-Lifshitz-Bloch equation is often used to model micromagnetic phenomenon under high temperature. This article proves the existence of smooth solutions of the equation in $\mathbb{R}^2$ and $\mathbb{R}^3$, and a small initial value condition should be added in the latter case. These results can also be generalized to periodical boundary value case.

Keywords Landau-Lifshitz-Bloch equation, a priori estimate, Gagliardo-Nirenberg inequality.

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1. Introduction

In this paper, we consider the following initial value problem, known as Landau-lifshitz-Bloch equation

$$u_t = \Delta u + u \times \Delta u - k(1 + \mu|u|^2)u \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^2 \quad \text{or} \quad \mathbb{R}^3 \tag{1.2}$$

where the constants $k, \mu > 0$.

In [5–7,10,12,13], it is pointed out that the dynamics of the magnetization of ferromagnets is a phase-changing process. When the electronic temperature is higher than θ_c (i.e. the temperature $\theta \geq \theta_c$, where θ_c is the critical Carie temperature), LLB equation has proved to describe the magnetization dynamics. When the temperature $\theta < \theta_c$, it is the normally well-known Landau-Lifshitz equation. For the Landau-Lifshitz equation, many workers has been studied (see [1–4,11,15,16] and the book of Guo-Ding [9]). In [5–7] the following LLB equation has been proposed

$$\frac{\partial u}{\partial t} = \gamma u \times H_{eff} + L_1 \frac{1}{|u|^2}(u \cdot H_{eff}) \cdot u - L_2 \frac{1}{|u|^2} u \times (u \times H_{eff}). \tag{1.3}$$

Here, $|\cdot|$ is Enclideam norm in $\mathbb{R}^3, \gamma > 0$ is the gyromagnetic ration, and L_1 and L_2 are longitudinal and transverse damping parameters, respectively. In [5], Kin

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Ngonle consider (1.3) in the case $L_1 = L_2$ The effective field H_{eff} is given by

$$H_{eff} = \Delta u - \frac{1}{\chi_{11}}(1 + \frac{3}{5}\frac{T}{T - T_c}|u|^2 u)u, \tag{1.4}$$

where χ_{11} is the longitudinal susceptibity and let $L_1 = L_2 = k_1$, then the equation (1.3) can be writed as follows

$$\frac{\partial u}{\partial t} = k_1 \Delta u + \gamma u \times \Delta u - k_2(1 + \mu|u|^2)u, \text{with} \ \ k_2 = \frac{k_1}{\chi_{11}}, \mu = \frac{3T}{5(T - T_c)} \tag{1.5}$$

under the coefficients $k_1, k_2, \gamma, \mu > 0$. He obtained the existence of global weak solution of LLB (1.5), $u \in C^d([0,T], L^{3/2})$, $\sup_{t\in[0,T]} \|u(\cdot .t)\|_{H^1} < \infty$.

In this paper, we study the smooth solution of problem (1.1)-(1.2) and get the following theorem.

Theorem 1.1. *Let dimension $d = 2$ with initial data $u_0 \in H^m (m \geq 2)$ then for any $T > 0$ there ia s unique solution u of problem (1.1),(1.2)satisfying*

$$\partial_t^j \partial_x^\alpha u \in L^\infty([0,T], ; L^2(\mathbb{R}^2)), \tag{1.6}$$
$$\partial_t^k \partial_x^\beta u \in L^2([0,T]; L^2(\mathbb{R}^2)) \tag{1.7}$$

where $2j + |\alpha| \leq m$, and $2k + |\beta| \leq m + 1$.

Theorem 1.2. *Let dimension $d = 3$ with initial data $u_0 \in H^m(m \geq 2)$, and $\|u_0\|_{H^2}$ is sufficiently small. Then for any $T > 0$ there is a unique solution of problem (1.1),(1.2)satisfying*

$$\partial_t^j \partial_x^\alpha u \in L^\infty([0,T], ; L^2(\mathbb{R}^3)), \tag{1.8}$$
$$\partial_t^k \partial_x^\beta u \in L^2([0,T]; L^2(\mathbb{R}^3)) \tag{1.9}$$

where $2j + |\alpha| \leq m$, and $2k + |\beta| \leq m + 1$.

The rest of this paper is divided into three parts. In section 2, we prove the existence of smooth solution in Theorem 1.1; In section 3, we prove the existence of smooth solution in Theorem 1.2; In section 4, we prove the uniqueness of smooth solution in Theorem 1.1 and Theorem 1.2.

2. Proof of existence in Theorem 1.1

From [3, 8, 9, 14, 17] it can be shown that there exist $T > 0$ and a unique smooth solution of problem (1.1) (1.2) in $[0,T]$. Indeed, it is easy to check that $e^{t\Delta}$ is a analytic semigroup generated by Δ in $L^2(\mathbb{R}^d)$, $d = 2$ or 3. Let

$$X = \{u | u \in C([0,T]; H^m(R^d)), t^\alpha u(t) \in C^\alpha([0,T]; H^m(\mathbb{R}^d)), u(0) = u_0\}$$

and

$$Y = \{u | u \in X, \|u\|_{C([0,T];H^m(R^d))} + [t^\alpha u]_{C^\alpha([0,T];H^m(\mathbb{R}^d))} \leq \rho\}$$

with $0 < \alpha < 1, m \geq 2$. Define a nonlinear operator Γ on Y, by $\Gamma(u) = v$, where v is the solution of

$$v_t = \Delta v + u \times \Delta u - k(1 + \mu|u|^2)u, v(0) = u_0 \tag{2.1}$$

by Theorem 4.3.5 of ref [9] (see [9], pages 137–139), for every $u \in Y, \Gamma(u) \in C([0,T]; H^m(\mathbb{R}^d))$ and $t^\alpha \Gamma(u) \in C^\alpha([0,T]; H^m(\mathbb{R}^d))$; then using the same arguments as in the proof of Theorem 8.1.1 of Ref [9] (See [9], pages 290–294), there exists $T > 0, \delta > 0$ such that $\Gamma : Y \to Y$ is constreation, i.e., there exists a unique smooth local solution of problem (1.1), (1.2). In order to prove Theorem 1.1 and Theorem 1.2, it suffices to give a priori estimates for the smooth solution of problem (1.1), (1.2).

Lemma 2.1. *Let dimension $d = 2, 3$ and initial data $u_0 \in H^m (m \geq 2)$ for the smooth solution of problem (1.1), (1.2), we have that*

$$\|u(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\cdot,s)\|_{L^2}^2 ds + 2k\int_0^t (1+\mu|u|^2)u(\cdot,s)ds = \|u_0\|_{L^2}^2, \tag{2.2}$$

$$\|u(\cdot,t)\|_{L^\infty} \leq C\|u_0\|_{H^2},$$

$$\|\nabla u(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta u(\cdot,s)\|_{L^2}^2 ds \leq C(\|u_0\|_{L^2}), \tag{2.3}$$

$$\|u(\cdot,t)\|_{L^\infty} \leq C\|u_0\|_{H^2}, t \geq 0. \tag{2.4}$$

Proof. Taking the scalar product of function u and equation (1.1), and then integrating the resuef over $\mathbb{R}^d$ for the space variable x and over $[0,t]$ for the temporal variable t, we have (2.3).

Now taking the scalar product of $|u|^{p-2}u (p \geq 2)$ with equation (1.1), then integrating the result over $\mathbb{R}^d$ for the space variable x, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u|^{p-2} u u_t dx \\
=&\frac{1}{p}\frac{d}{dt}\|u(\cdot,t)\|_{L^p}^p = \int_{\mathbb{R}^d} |u|^{p-2}u \cdot \Delta u dx - k\int_{\mathbb{R}^d} |u|^{p-2}(1+\mu|u|^2)u^2 dx \\
\leq& -\int_{\mathbb{R}^d} |u|^{p-2}\nabla u \cdot \nabla u dx - (p-2)\int_{\mathbb{R}^d} |u|^{p-4}(u \cdot \nabla u)^2 dx \leq 0.
\end{aligned}$$

This inequality implies that

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{H^2} \quad \forall p \geq 2, t \geq 0 \tag{2.5}$$

where we have used the embedding theorem of Sobolev spaces. Note the constant C is independent of p and let $p \to \infty$, estimate (2.4) is obtained.

Similarly, taking the scalar product of Δu and equation (1.1) and then integrating the result over $\mathbb{R}^d$ for the space variable x and over $[0,t]$ for temporal variable x, we have

$$\|\nabla u(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\Delta u(\cdot,s)\|_{L^2}^2 ds + 2k\int_0^t (1+\mu|u|^2)u\Delta u(\cdot,s)ds = \|\nabla u_0\|_{L^2}^2, \forall t \geq 0. \tag{2.6}$$

$$\begin{aligned}
\left|2k\int_{\mathbb{R}^d} (1+2\mu|u|^2)u \cdot \Delta u ds\right| &\leq 2k\|u\|_{L^\infty}\int_{\mathbb{R}^d} (1+\mu|u|^2)|\Delta u| ds \\
&\leq \int_0^t \|\Delta u(\cdot,s)\|_{L^2}^2 ds + C(\|u_0\|_{H^2})
\end{aligned} \tag{2.7}$$

we can get the estimate (2.2). □

Lemma 2.2. *Let dimension $d = 2$ and initial data $u_0 \in H^m(m \geq 2)$. Then for the smooth solution of problem (1.1), (1.2) one has the following estimates*

$$\|\Delta u(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta\nabla u(\cdot,s)\|_{L^2}^2 ds \leq C(T;\|u_0\|_{H^2}), \forall T > 0, t \in [0,T], \tag{2.8}$$

$$\|u_t(\cdot,t)\|_{L^2} + \int_0^t \|\nabla u_t(\cdot,s)\|_{L^2}^2 ds \leq C(T;\|u_0\|_{H^2}), \forall T > 0, t \in [0,T]. \tag{2.9}$$

Moveover, if $m \geq 3$, then we have

$$\|\Delta\nabla u(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta^2 u(\cdot,s)\|_{L^2}^2 ds \leq C(T;\|u_0\|_{H^3}), \quad \forall T > 0, t \in [0,T], \tag{2.10}$$

$$\|\nabla u_t(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta u_t(\cdot,s)\|_{L^2}^2 ds \leq C(T;\|u_0\|_{H^3}),\ \forall T > 0, t \in [0,T]. \tag{2.11}$$

Proof. By simple calcalation, we get

$$\Delta u_t = \alpha \sum_{j=1}^{2} \partial_{x_j} u \times \Delta\partial_{x_j} u + u \times \Delta^2 u + \Delta^2 u - k\Delta[(1+\mu|u|^2)u]. \tag{2.12}$$

Taking the scalar product of Δu and the equation (2.12), and then integrating the result over $\mathbb{R}^2$ for the space variable x, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \Delta u_t \cdot \Delta u dx = &\int_{\mathbb{R}^2} \Delta^2 u \Delta u dx + 2\sum_{j=1}^{2}\int_{\mathbb{R}^2} (\partial_{x_j} u \times \Delta\partial_j u)\Delta u dx \\ &+ \int_{\mathbb{R}^2} (u \times \Delta^2 u)\Delta u dx - \int_{\mathbb{R}^2} k\Delta[(1+\mu|u|^2)u]\Delta u dx.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta u(x,t)|^2 dx + \int_{\mathbb{R}^2} |\nabla\Delta u(x,t)|^2 dx + k\int_{\mathbb{R}^2} |\Delta u|^2 dx + k\int_{\mathbb{R}^2} \Delta(|u|^2 u)\Delta u dx \\ &= \sum_{j=1}^{2}\int_{\mathbb{R}^2} (\partial_{x_j} u \times \Delta\partial_{x_j} u) \cdot \Delta u dx.\end{aligned} \tag{2.13}$$

By Holder inequality, it follows that

$$\left|\sum_{j=1}^{2}\int_{\mathbb{R}^2} (\partial_{x_j} \times \Delta\partial_{x_j})\Delta u dx\right| \leq 2\|\nabla u\|_{L^4}\|\Delta u\|_{L^4}\|\Delta\nabla u\|_{L^2}.$$

By Gagliardu-Nirenberg's inequality, we have

$$\|\nabla u\|_{L^4} \leq C\|\nabla u\|_{H^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{3}{4}},$$
$$\|\Delta u\|_{L^4} \leq C\|\Delta u\|_{H^1}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}.$$

Hence we get

$$\sum_{j=1}^{2}\int_{\mathbb{R}^2} (\partial_{x_j} u \times \Delta\partial_{x_j} u)\Delta u dx \leq \frac{1}{4}\|\Delta\nabla u\|_{L^2}^2 + C(\|\nabla u_0\|_{L^2})(1 + \|\Delta u\|_{L^2}^2),$$

$$\left|\int_{\mathbb{R}^2}\Delta(|u|^2u)\Delta udx\right| \leq C\|u\|_{L^\infty}^2(\|\nabla u\|_{L^4}^2+\|\Delta u\|_{L^2}^2)$$
$$\leq \frac{1}{4}\|\Delta\nabla u\|_{L^2}^2+C(\|u_0\|_{H^2}).$$

Using (2.13) and Gronwall's inequality, we obtain (2.8).

Next we are going to prove Theorem 1.1 for 2-dimensional case, and it is sufficient to prove the following Theorem.

Theorem 2.1. *Let dimension $d=2$, with the initial data $\nabla u_0 \in H^k (k \geq 2)$. Then any smooth solution of problem (1.3), (1.4) satisfies the following a priori estimate*

$$\sup_{0\leq t\leq T}\|D^{m+1}u(\cdot,t)\|_{L^2}^2+\int_0^t\|D^{m+2}u(\cdot,s)\|_{L^2}^2ds \leq C, 2\leq m\leq k \tag{2.14}$$

where C depends on T and $\|\nabla u_0\|_{H^k}$.

Proof. We will use induction arguments to prove this theorem, but first of all it should be necessary to obtain the boundedness of $\|\nabla u\|_{L^\infty}$. In fact, applying Laplace operator to the both sides of equation (1.3) and taking the scalar product with $\triangle^2 u$, then integrating over $\mathbb{R}^2$, we get

$$-\frac{1}{2}\frac{d}{dt}\|\nabla\triangle u\|_{L^2}^2 = \|\triangle^2u\|_{L^2}^2+2\sum_{j=1}^2\int_{\mathbb{R}^2}\partial_{x_j}u\times\triangle\partial_{x_j}u\cdot\triangle^2udx \tag{2.15}$$
$$-\int_{\mathbb{R}^2}k\triangle[(1+|u|^2)u]\triangle udx. \tag{2.16}$$

By Hölder's inequality, it follows that

$$|\sum_{j=1}^2\int_{\mathbb{R}^2}\partial_{x_j}u\times\triangle\partial_{x_j}u\cdot\triangle^2udx| \leq 2\|\nabla u\|_{L^{\frac{16}{5}}}\|\nabla\triangle u\|_{L^{\frac{16}{3}}}\|\triangle^2u\|_{L^2}.$$

By Gagliardo-Nirenberg's inequality, we have

$$\|\nabla u\|_{L^{\frac{16}{5}}} \leq C\|\nabla u\|_{H^3}^{\frac{1}{8}}\|\nabla u\|_{L^4}^{\frac{7}{8}},$$
$$\|\nabla\triangle u\|_{L^{\frac{16}{3}}} \leq \|\nabla\triangle u\|_{H^1}^{\frac{5}{8}}\|\nabla\triangle u\|_{L^2}^{\frac{3}{8}}.$$

Here note that the boundedness of $\|\nabla u\|_{L^4}$ can be justified by Lemma 2.2, with the same procedures as that in Lemma 2.2, we deduce that

$$\|\nabla\triangle u\|_{L^2}^2 \leq C. \tag{2.17}$$

Then

$$\|\nabla u\|_{L^\infty} \leq C, \tag{2.18}$$

follows from(2.13) and Galiardo-Nirenberg inequality.

Next we utilize induction arguments, the m=1 case has been proved by Lemma 2.2, so it will be supposed that if $m=m$ case holds, then $m=m+1$ also holds. Applying the differential operator D^{m+1} to the both sides of equation (1.4) and taking the scalar product with $D^{m+1}u$, then integrating over $\mathbb{R}^2$, we get

$$-\frac{1}{2}\frac{d}{dt}\|D^{m+1}u\|_{L^2}^2 = \|\nabla D^{m+1}u\|_{L^2}^2 \tag{2.19}$$

$$+2\sum_{j=1}^{2}\int_{\mathbb{R}^2} D^{m+1}(u\times\triangle u)\cdot D^{m+1}udx$$
$$-\int_{\mathbb{R}^2} kD^{m+1}[(1+|u|^2)u]D^{m+1}udx. \tag{2.20}$$

Since

$$\int_{\mathbb{R}^2} D^{m+1}(u\times\triangle u)\cdot D^{m+1}udx = -\int_{\mathbb{R}^2} D^{m+1}(u\times\nabla u)\cdot\nabla D^{m+1}udx, \tag{2.21}$$

and

$$D^{m+1}(u\times\nabla u) = D^{m+1}u\times\nabla u+u\times D^{m+1}\nabla u+\sum_{h=1}^{m} C_h(D^h u\times D^{m+1-h}\nabla u). \tag{2.22}$$

Thus

$$|\int_{\mathbb{R}^2} D^{m+1}(u\times\triangle u)\cdot D^{m+1}udx| \leq |\int_{\mathbb{R}^2} D^{m+1}u\times\nabla u\cdot\nabla D^{m+1}udx| \tag{2.23}$$
$$+|\int_{\mathbb{R}^2}\sum_{h=1}^{m} C_h(D^h u\times D^{m+1-h}\nabla u)\cdot\nabla D^{m+1}udx| \tag{2.24}$$
$$\leq\|\nabla u\|_{L^\infty}\|D^{m+1}u\|_{L^2}+C\|D^m u\|_{L^4}\|D^m\nabla u\|_{L^4}\|\nabla D^{m+1}u\|_{L^2}. \tag{2.25}$$

Consequently,

$$\frac{d}{dt}\|D^{m+1}u\|_{L^2}^2+2\|D^{m+2}u\|_{L^2}^2\leq C\|D^{m+1}u\|_{L^2}^2$$

using Gronwall's inequality, we conclude the theorem. □

3. Proof of existence in Theorem 1.2

The proof of Theorem 1.2 is in line with that of Theorem 1.1, and the difference is that a priori estimates are more difficult to derive as the dimension changes, we propose an additional condition to overcome this situation. Precisely, we have the following Lemma

Lemma 3.1. *Let dimension $d=3$ with initial data $u\in H^m(m\geq 2)$ and $\|u_0\|_{H^2}$ be sufficiently small. Then for the smooth solution of problem (1.1), (1.2) one has the following estimates*

$$\|\Delta u(\cdot,t)\|_{L^2}^2+\int_0^t\|\Delta\nabla u(\cdot.s)\|_{L^2}^2ds\leq C(T,\|u_0\|_{H^2}),\ \forall T>0, t\in[0,T], \tag{3.1}$$
$$\|u_t(\cdot,t)\|_{L^2}^2+\int_0^t\|\nabla u_t(\cdot,s)\|_{L^2}^2ds\leq C(T;\|u_0\|_{H^2}),\quad \forall T>0, t\in[0,T], \tag{3.2}$$
$$\|\Delta\nabla u(\cdot,t)\|_{L^2}^2+\int_0^t\|\Delta^2 u(\cdot,s)\|_{L^2}^2ds\leq C(T,\|u_0\|_{H^3}),\quad \forall T>0, t\in[0,T], \tag{3.3}$$
$$\|\nabla u_t(\cdot,t)\|_{L^2}^2+\int_0^t\|\Delta u_t(\cdot.s)\|_{L^2}^2ds\leq C(T,\|u_0\|_{H^3}),\quad \forall T>0, t\in[0,T]. \tag{3.4}$$

Proof. By using the same arguments as in the proof of Lemma 2.11, we get

$$\Delta u_t = 2\sum_{j=1}^{3}\partial_{x_j} u \times \Delta\partial_{x_j} u + u\times\Delta^2 u + \Delta^2 u - k\Delta(1+\mu|u|^2)u, \tag{3.5}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\|\Delta u(\cdot,t)\|^2dx + \int_{\mathbb{R}^3}|\Delta\nabla u(\cdot,t)|^2dx + k\int_{\mathbb{R}^3}\Delta(1+\mu|u|^2)u\cdot\Delta u\\ =&\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_{x_j})u\times\Delta\partial_{x_j}u\Delta u dx \qquad (3.6)\\ &\le 2\|\nabla u\|_{L^6}\|\Delta u\|_{L^3}\|\Delta\nabla u\|_{L^2} \le C\|u\|_{L^\infty}\|u\|_{H^3}^2 \le \frac{1}{3}\|\Delta\nabla u\|_{L^2}^2\end{aligned}$$

where the estimate (2.4) and hypothesis $\|u_0\|_{H^2}\ll 1$ has been used.

$$\left|\int_{\mathbb{R}^3} k\mu\Delta(|u|^2u)\Delta u dx\right| \le \frac{1}{3}\|\Delta\nabla u\|_{L^2}^2. \tag{3.7}$$

Inserting (3.7) to (3.6), we obtain (3.1).

Now taking the scalar product of $\Delta^2 u$ and the equation (3.5), then integrating the result over $\mathbb{R}^3$ for the space variable x, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\Delta u_t\cdot\Delta^2 u dx = &\int_{\mathbb{R}^3}\Delta^2 u\cdot\Delta^2 u dx + 2\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_{x_j}u\times\partial_{x_j}u)\Delta^2 u\\ &- k\int_{\mathbb{R}^3}\Delta(1+\mu|u|^2)u\Delta^2 u dx.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\Delta\nabla u(x,t)|^2dx + \int_{\mathbb{R}^3}|\Delta^2 u|^2dx + k\int_{\mathbb{R}^3}\Delta(1+\mu|u|^2)u\Delta^2 u dx\\ =&2\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_{x_j}u\times\Delta\partial_{x_j}u)\Delta^2 u dx, \qquad (3.8)\\ &2\left|\sum_{j=1}^{3}\int_{\mathbb{R}^3}(\partial_{x_j}u\times\Delta\partial_{x_j}u)\Delta^2 u dx\right| \le 6\|\nabla u\|_{L^4}\|\Delta\nabla u\|_{L^4}\|\Delta^2 u\|_{L^2}\\ \le& C(T;\|u_0\|_{H^2})\|\Delta\nabla u\|_{L^2}^{\frac{1}{4}}\|\Delta^2 u\|_{H^2}^{\frac{7}{8}} \le \frac{1}{3}\|\Delta^2 u\|_{L^2}^2 + C(\|u_0\|_{H^2})(1+\|\Delta\nabla u\|_{L^2}^2)\end{aligned} \tag{3.9}$$

$$\left|k\mu\int_{\mathbb{R}^3}\Delta(|u|^2u)\Delta^2 u dx\right| \le \frac{1}{3}\|\Delta^2 u\|_{L^2}^2 + C(\|u_0\|_{H^2}). \tag{3.10}$$

Inserting (3.9), (3.10) into (3.8) and using Gronwall's inequality we get (3.3). Using equation (1.1) ,(3.1), (3.3), Holder's inequality and the embedding theorem of Sobolev spaces. we obtain (3.2), (3.4). □

Lemma 3.2. *Let $m\ge 4$. Then under the conditions of Theorem 1.1 and Theorem 1.2 for the smooth solution of problem (1.1), (1.2) one has the following estimates*

$$\|\Delta^2 u(\cdot,t)\|_{L^2}^2 + \int_0^t\|\Delta^2\nabla u(\cdot,s)\|_{L^2}^2ds \le C(T,\|u_0\|_{H^4}),\ \forall T>0, t\in[0,T], \tag{3.11}$$

$$\|\Delta u_t(\cdot,t)\|_{L^2}^2 + \int_0^t \|\Delta\nabla u_t(\cdot,s)\|_{L^2}^2 dt \le C(T;\|u_0\|_{H^4}),\ \ t\in[0,T]. \tag{3.12}$$

By using the induction's method, we can prove this lemma.

Lemma 3.3. *Under the conditions of Theorem 1.1 and Theorem 1.2, for the smooth solution of problem (1.1),(1.2) one has the following estimates*

$$\|\partial_t^j\partial_x^\alpha u(\cdot,t)\|_{L^2}^2 \le C(T;\|u_0\|_{H^m}),\ \ \forall T>0,\ \ T\in[0,T], \tag{3.13}$$

$$\int_0^t \|\partial_t^h\partial_x^\beta u(\cdot,s)\|_{L^2}^2 ds \le C(T;\|u_0\|_{H^m}),\ \ \forall T>0,\ \ t\in[0,T], \tag{3.14}$$

where $2j+|\alpha|\le m, 2k+|\beta|\le m+1$.

Using Lemma 2.1-Lemma 2.2 and Lemma 3.1-Lemma 3.3, the proofs of Theorem 1.1 and Theorem 1.2 are standard and we omitted here.

4. Proof of uniqueness of the solution

In this section we will deal with the uniqueness of the solution in problem (1.1), (1.2), in fact, we have the following generalized result:

Theorem 4.1. *Let* u *and* v *be two smooth solutions of problem (1.1), (1.2), with the same initial data* $u_0=v_0\in H^\infty(\mathbb{R}^d)$*, then for any positive integer* d*,* $u\equiv v$.

Proof. The proof is standard, set $w=u-v$, we'll prove $w\equiv 0$. Since u and v satisfies (1.1) respectively, w satisfies the following equation

$$w_t = \triangle w + u\times\triangle u - v\times\triangle v - kw - k((|u|^2)u-(|v|^2)v), \tag{4.1}$$

the cross product in the above equation can be rewritten as

$$u\times\triangle u - v\times\triangle u + v\times\triangle u - v\times\triangle v = w\times\triangle u + v\times\triangle w.$$

Thus (4.1) becomes

$$w_t = \triangle w + w\times\triangle u + v\times\triangle w - kw - k(|u|^2+|v|^2+u\cdot v)w, \tag{4.2}$$

taking the inner product with w in both sides of (4.2), then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|w|^2dx = &-\int_{\mathbb{R}^d}|\nabla w|^2dx + \int_{\mathbb{R}^d}(v\times\triangle w)\cdot wdx\\ &-k\int_{\mathbb{R}^d}|w|^2dx - k\int_{\mathbb{R}^d}(|u|^2+|v|^2+u\cdot v)|w|^2dx,\end{aligned}$$

where

$$|\int_{\mathbb{R}^d}(v\times\triangle w)\cdot wdx| = |\int_{\mathbb{R}^d}(\nabla v\times\nabla w)\cdot wdx| \le 2\|\nabla v\|_{L^\infty}^2\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2, \tag{4.3}$$

and

$$k|\int_{\mathbb{R}^d}(|u|^2+|v|^2+u\cdot v)|w|^2dx| \le 2k(\|v\|_{L^\infty}^2+\|u\|_{L^\infty}^2)\|w\|_{L^2}^2. \tag{4.4}$$

Since u and v are smooth, the norm $\|\nabla v\|_{L^\infty}^2, \|u\|_{L^\infty}^2$ and $\|v\|_{L^\infty}^2$ can be replaced by a constant C, thus it can be concluded that

$$\frac{d}{dt}\int_{\mathbb{R}^d}|w|^2dx \le C\int_{\mathbb{R}^d}|w|^2dx.$$

Gronwall's inequality and the fact that $w(x,0)\equiv 0$ lead to $w\equiv 0$. □

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