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CHARACTERISTICS OF NEW TYPE ROGUE WAVES AND SOLITARY WAVES TO THE EXTENDED (3+1)-DIMENSIONAL JIMBO-MIWA EQUATION

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Abstract On the basis of the binary Bell polynomial scheme, the bilinear form of the extended (3+1)-dimensional Jimbo-Miwa (JM) equation, is constructed. Then, a class of new type rogue waves solutions to the extended (3+1)-dimensional JM equation, is found. It mainly includes the lump solutions, lumpoff solutions and instanton solutions. Their nonlinear evolutionary processes by 3D- and 2D-graphs, are shown. Finally, a direct method which is called the tanh-function method was used to get solitary waves of this considered model. These results can help us better understand interesting physical phenomena and mechanism.

Keywords Extended (3+1)-dimensional JM equation, bilinear form, rogue waves, solitary waves.

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1. Introduction

It is known that the integrability of mathematics physics had been better investigated in recent years. There are different definitions of integrability of nonlinear differential equations(NDEs). Among them, there are existing some indicators, such as the Bäcklund transforms, Lax pairs, infinite conservation laws, N-soliton solutions and infinite symmetry, etc [3, 4, 7–11, 18, 30, 40]. Therefore, as well to study the integrability of NDEs, we need to find these indicators. To the best of our knowledge, the multi-dimensional binary Bell polynomial approach [4, 7, 8] is an effective tool to construct the bilinear equation, Bäcklund transforms and Lax pairs and infinite conservation laws. This method has been developed by Gilson, et al [4], Lambert and Springael [7, 8]. Later, many scholars further developed and promoted this method [3, 9, 10, 18, 30, 40].

Recently, on the basis of the bilinear equation, the lump solutions, lump-type solutions, N-lump and interaction solutions in Refs. [12, 19–25], were presented.

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This field has received a lot of attention from researchers in recent years. For example, Ma, et al [19] discussed the lump solution to the nonlinear partial differential equations in detail, Liu, et al [12] constructed the lump solutions and mixed lump stripe solutions of the (3+1)-dimensional soliton equation, the lump-type solutions, N-lump and interaction solutions for partial differential equations by researcher J. Manafian [20–25], et al, were obtained. The lump solutions and lump-type solutions sometimes were called the rogue waves. It can be used to express meaningful nonlinear wave phenomena in natural science, such as oceanography [26] and nonlinear optics [31]. For example, M, Onorato, et al [29] found the rogue waves generating mechanisms in different physical contexts, Yan [37] found the rogue waves phenomenon appears in financial problems, and others [5, 32]. More recently, Lou [13] constructed a class of new type rogue waves for integrable and non-integrability models.

In this paper, we mainly consider the following nonlinear dynamical model, which can be used to describe some interesting (3+1)-dimensional waves of physics [34]. That is

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{ty} - 3(u_{xz} + u_{yz} + u_{zz}) = 0. \tag{1.1}$$

Its 1-soliton solution, 2-soliton solution and multi-soliton solutions of Eq.(1.1) by utilizing the simplified Hirota bilinear method [34], were obtained. Also, lump solutions have been acquired by Sun and Chen [33]. Some other important results in Refs. [1, 14, 27, 28].

The aim of this paper to obtain a class of new type rogue waves to the extended (3+1)-dimensional JM equation with the help of the bilinear equation. In addition, new solitary waves have been also obtained via the tanh-function method.

2. Multi-dimensional binary Bell polynomials

In this section, we briefly give certain basic knowledge of the Bell polynomials [3, 4, 7–10, 18, 30, 40] which are required for the remaining part of this article.

Making $f = f(x_1, x_2, ..., x_n)$ be a C^∞function with multi-variables, the polynomials of the following form

$$Y_{n_1 x_1, ..., n_l x_l}(f) \equiv Y_{n_1, ..., n_l}(f_{r_1 x_1, ..., r_l x_l}) = e^{-f} \partial_{x_1}^{n_1} ... \partial_{x_l}^{n_l} e^f, \tag{2.1}$$

with

$$f_{r_1 x_1, ..., r_l x_l} = \partial_{x_1}^{r_1} ... \partial_{x_l}^{r_l} f, f_{0x_i} \equiv f, r_1 = 0, ..., n_1; ...; r_l = 0, ..., n_l,$$

are called the multil-dimensional Bell polynomials.

In particular, when $f = f(x)$, then, we have

$$Y_1(f) = f_x, Y_2(f) = f_{2x} + f_x^2, Y_3(f) = f_{3x} + 3f_x f_{2x} + f_x^3, ... \ .$$

When $f = f(x, t)$, then, we get

$$Y_{x,t}(f) = f_{x,t} + f_x f_t, Y_{2x,t}(f) = f_{2x,t} + f_{2x} f_t + 2f_{x,t} f_x + f_x^2 f_t, ... \ . \tag{2.2}$$

According to the definition of the above explained, the multi-dimensional binary

Bell polynomials has the expression of

$$\Xi_{n_1x_1,\ldots,n_lx_l}(v,w) = Y_{n_1,\ldots,n_l}(f)|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & r_1+r_2+\ldots+r_l \text{ is odd},\\ w_{r_1x_1,\ldots,r_lx_l}, & r_1+r_2+\ldots+r_l \text{ is even}.\end{cases}} \tag{2.3}$$

Based on the definition of the above stated, the first few lowest order binary Bell polynomials is given by

$$\begin{aligned} &\Xi_x(v) = v_x,\\ &\Xi_{2x}(v,w) = w_{2x} + v_x^2,\\ &\Xi_{x,t}(v,w) = w_{x,t} + v_xv_t, \ldots \quad . \end{aligned} \tag{2.4}$$

Relations between Ξ-polynomials and the Hirota bilinear equation $D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F\cdot G$ can be given by the following proposition.

Corollary 2.1. *Under the variables*

$$v = \ln(F/G), w = \ln(FG), \tag{2.5}$$

the relations between binary Bell polynomials and Hirota D-operator shall be expressed by

$$\Xi_{n_1x_1,\ldots,n_lx_l}(v,w)|_{v=\ln(F/G),w=\ln(FG)} = (FG)^{-1}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F\cdot G, \tag{2.6}$$

where the Hirota D-operators [2, 6] were defined by

$$D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}FG = (\partial_{x_1} - \partial_{x_1'})^{n_1}\ldots(\partial_{x_l} - \partial_{x_l'})^{n_l}F(x_1,\ldots,x_l)G(x_1',\ldots,x_n')|_{x_1=x_1',\ldots,x_l=x_l'}. \tag{2.7}$$

For $F = G$, the formula (2.6) can be rewritten as

$$\begin{aligned} F^{-2}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F\cdot F &= \Xi_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln(F))\\ &= \begin{cases} 0, & n_1+n_2+\ldots+n_l \text{ is odd},\\ P_{n_1x_1,\ldots,n_lx_l}, & n_1+n_2+\ldots+n_l \text{ is even},\end{cases} \end{aligned} \tag{2.8}$$

where these even ordered Ξ-polynomials are called P-polynomials.

On the basis of the definition of the above stated, the first few lowest order binary Bell polynomials is given by

$$\begin{aligned} &P_{2x} = q_{2x},\\ &P_{x,t}(q) = q_{x,t},\\ &P_{3x,t}(q) = q_{3x,t} + 3q_{x,t}q_{2x}, \ldots \quad . \end{aligned} \tag{2.9}$$

Further, we have the following proposition for the bilinear Bäcklund transformations.

Corollary 2.2. *Given an integrable equation for a field q of the form*

$$E(q) = \sum_i c_iP_{n_1x_1,\ldots,n_lx_l} = 0,$$

one can find a pair of constraint conditions

$$
\begin{cases} \sum_j c_{1j}\Xi_{n_1x_1,\ldots,n_lx_l}(v,w)=0, \\ \sum_j c_{2j}\Xi_{m_1x_1,\ldots,m_lx_l}(v,w)=0, \end{cases} \tag{2.10}
$$

which need to satisfy

$$
\begin{aligned} R(q^{'},q) &= E(q^{'})-E(q) \\ &= E(w+v)-E(w-v)=0, \end{aligned} \tag{2.11}
$$

the system (2.10) *is called the binary Bäcklund transformations.*

We have been known that the binary Bell polynomials $\Xi_{n_1x_1,\ldots,n_lx_l}(v,w)$ can be separated into the generalized Bell polynomials $Y_{n_1x_1,\ldots,n_lx_l}(v)$ and

$$
\begin{aligned} &(FG)^{-1}D_{x_1}^{n_1}...D_{x_l}^{n_l}F\cdot G \\ =&\Xi_{n_1x_1,\ldots,n_lx_l}(v,w)|_{v=\ln(F/G),w=\ln(FG)} \\ =&\Xi_{n_1x_1,\ldots,n_lx_l}(v,v+q)|_{v=\ln(F/G),q=2\ln(G)} \\ =&\sum_{r_1=0}^{n_1}...\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\begin{pmatrix} n_i \\ r_i \end{pmatrix} P_{r_1x_1,\ldots,r_lx_l}(q)Y_{(n_1-r_1)x_1,\ldots,(n_i-r_i)x_i}(v). \end{aligned} \tag{2.12}
$$

Meanwhile, we note that the generalized Bell polynomial can be also linearized by the Cole-Hopf transformation $v=\ln(\psi)$. i.e.

$$
Y_{r_1x_1,\ldots,r_lx_l}(v=\ln(\psi))=\frac{\psi_{n_1x_1,\ldots,n_ix_i}}{\psi}, \tag{2.13}
$$

$$
\begin{aligned} &(FG)^{-1}D_{x_1}^{n_1}...D_{x_l}^{n_l}F\cdot G|_{G=\exp(q/2),F/G=\psi} \\ =&\psi^{-1}\sum_{r_1=0}^{n_1}...\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\begin{pmatrix} n_i \\ r_i \end{pmatrix} P_{r_1x_1,\ldots,r_lx_l}(q)\psi_{(n_1-r_1)x_1,\ldots,(n_i-r_i)x_i}. \end{aligned} \tag{2.14}
$$

Due to the definition of the above stating, we have

$$
\begin{aligned} &\Xi_t(v)=\frac{\psi_t}{\psi}, \\ &\Xi_{2x}(v,w)=q_{2x}+\frac{\psi_{2x}}{\psi}, \\ &\Xi_{2x,y}(v,w)=\frac{q_{2x}\psi_y}{\psi}+\frac{2q_{x,y}\psi_x}{\psi}+\frac{\psi_{2x,y}}{\psi},\ldots\quad . \end{aligned} \tag{2.15}
$$

It helps us construct associated Lax pairs of this considered equation.

3. Bilinear equation to the extended (3+1)-dimensional JM equation

In order to get the existence of linearizing representation, we take the potential field q by considering

$$
u=cq_x, \tag{3.1}
$$

where $c = c(t)$ is a function on time variable t.

Inserting equation (3.1) into equation (1.1), the new equation is

$$E(q) = 3cq_{xy}q_{2x} + 2q_{yt} - 3q_{xz} - 3q_{yz} - 3q_{2z} + q_{xxxy} = 0, \tag{3.2}$$

with the constraint condition

$$c = 1. \tag{3.3}$$

The new equation $E(q)$ can be re-expressed in the form of P-polynomials

$$E(q) = P_{3x,y}(q) + 2P_{y,t}(q) - 3P_{x,z}(q) - 3P_{y,z}(q) - 3P_{2z}(q) = 0. \tag{3.4}$$

Making a change of dependent variable

$$q = 2\ln(F) \Leftrightarrow u = cq_x = 2(\ln(F))_x, \tag{3.5}$$

and owing to the property (2.8), Eq.(3.4) yields the following theorem with the dependent variable (3.5).

Theorem 3.1. *Substituting the following potential field*

$$u = 2(\ln(F))_x, \tag{3.6}$$

into Eq.(1.1)*, the extended (3+1)-dimensional JM equation can be linearized the following bilinear equation*

$$(\mathrm{D}_x^3\mathrm{D}_y + 2\mathrm{D}_y\mathrm{D}_t - 3\mathrm{D}_x\mathrm{D}_z - 3\mathrm{D}_y\mathrm{D}_z - 3\mathrm{D}_z^2)F \cdot F = 0. \tag{3.7}$$

4. New type rogue waves to the extended (3+1)-dimensional JM equation

With the help of the bilinear equation (3.7), we construct the new type rogue waves to the extended (3+1)-dimensional JM equation by setting

$$\begin{aligned} &F = 1 + \mathrm{g}^2 + a\mathrm{e}^{tk_3+xk_1+yk_2+zk_4+k_5} + b\mathrm{e}^{-tk_3-xk_1-yk_2-zk_4-k_5}, \\ &g = ta_3 + xa_1 + ya_2 + za_4 + a_5, \end{aligned} \tag{4.1}$$

where $a_i(i = 1,2,3,4,5), k_j(j = 1,2,3,4,5), a$ and b are arbitrary parameters.

To yield the new type rogue waves of this considered model, we substitute equation (4.1) into Eq.(3.7). As a result, a class of polynomial of the variables x, y, z, t, were obtained. Then, it yields a set of algebraic system in $a_i(i = 1,2,3,4,5), k_j(j = 1,2,3,4,5), a$ and b. Solving this system of equations with the help of symbolic computation Maple, we can obtain the following solutions of parameters:

Case 1

$$\begin{aligned} &a = 0, b = 0, a_1 = -a_4, a_2 = 0, a_3 = a_3, a_4 = a_4, a_5 = a_5, k_1 = k_1, \\ &k_2 = k_2, k_3 = k_3, k_4 = k_4, \end{aligned} \tag{4.2}$$

where $a_i(i = 1,3,5)$ and $k_j(j = 1,2,3,4)$ are arbitrary parameters.

Case 2

$$\begin{aligned} &a = 0, b = 0, a_1 = a_1, a_2 = 0, a_3 = -\frac{3a_1(k_2 + k_4)}{2k_2}, a_4 = -a_1, \\ &a_5 = 0, k_1 = 0, k_2 = k_2, k_3 = \frac{3k_4(k_2 + k_4)}{2k_2}, k_4 = k_4, \end{aligned} \tag{4.3}$$

where a_1, k_2 and k_4 are arbitrary constants.

Case 3

$$a=0, b=0, a_1=-a_4, a_2=0, a_3=\frac{1}{2}\,(3\,{k_1}^2+3)a_4, a_4=a_4,$$
$$a_5=a_5, k_1=-k_4, k_2=k_4, k_3=-\frac{1}{2}{k_1}^3-\frac{3}{2}\,k_1, k_4=k_4, \tag{4.4}$$

where a_1, a_5 and k_1 are free constants.

Case 4

$$a=0, b=b, a_1=-a_4, a_2=0, a_3=a_3, a_4=a_4, a_5=a_5,$$
$$k_1=0, k_2=\frac{3k_3{a_4}^2}{a_3\,(2\,a_3-3\,a_4)}, k_3=k_3, k_4=\frac{k_3a_4}{a_3}, \tag{4.5}$$

where b, a_1, a_3, a_5 and k_3 are arbitrary parameters.

Case 5

$$a=0, b=b, a_1=-a_4, a_2=0, a_3=\frac{3a_4\,(k_2+k_4)}{2k_2}, a_4=a_4,$$
$$a_5=a_5, k_1=0, k_2=k_2, k_3=\frac{3k_4\,(k_2+k_4)}{2k_2}, k_4=k_4, \tag{4.6}$$

where b, a_1, a_5, k_2 and k_4 are arbitrary constants.

Case 6

$$a=a, b=b, a_1=-a_4, a_2=0, a_3=a_3, a_4=a_4, a_5=0,$$
$$k_1=0, k_2=k_2, k_3=\frac{k_2a_3\,(2\,a_3-3\,a_4)}{3{a_4}^2}, k_4=\frac{k_2\,(2\,a_3-3\,a_4)}{3a_4}, \tag{4.7}$$

where a, b, a_1, a_3 and k_2 are arbitrary parameters.

Case 7

$$a=a, b=b, a_1=-a_4, a_2=0, a_3=\frac{3a_4\,(k_2+k_4)}{2k_2}, a_4=a_4,$$
$$a_5=a_5, k_1=0, k_2=k_2, k_3=\frac{3k_4\,(k_2+k_4)}{2k_2}, k_4=k_4, \tag{4.8}$$

where a, b, a_1, a_5, k_2 and k_4 are free constants.

Case 8

$$a_3=\frac{3(a_2{k_1}^3k_2-2\,a_2k_1k_4-a_2{k_4}^2+a_4k_1k_2+a_4{k_2}^2+2\,a_4k_2k_4)}{2{k_2}^2},$$
$$b=b,\ \ a_1=-\frac{a_2k_1}{k_2}, a_2=a_2, a_4=a_4, a_5=0, k_1=k_1, k_2=k_2, \tag{4.9}$$
$$a=\frac{{a_2}^4}{{k_2}^4b}, k_3=-\frac{{k_1}^3k_2-3\,k_1k_4-3\,k_2k_4-3\,{k_4}^2}{2k_2}, k_4=k_4,$$

where b, a_2, a_4, k_1, k_2 and k_4 are arbitrary parameters.

Case 9

$$a=\frac{{a_1}^4}{b{k_1}^4}, b=b, a_1=a_1, a_2=\frac{a_1}{{k_1}^2}, a_3=\frac{3a_1\,(k_4+k_1)\,(k_1k_4-1)}{2k_1}, a_4=-\frac{a_1\,(k_4+k_1)}{k_1},$$
$$a_5=0, k_1=k_1, k_2=-\frac{1}{k_1}, k_3=-\frac{1}{2}{k_1}^3-\frac{3}{2}\,k_4{k_1}^2-\frac{3}{2}\,k_1{k_4}^2+\frac{3}{2}\,k_4, k_4=k_4, \tag{4.10}$$

where b, a_1, k_1 and k_4 are arbitrary constants.

Case 10

$$\begin{aligned}
&a = a, b = b, a_1 = -\frac{a_2}{{k_1}^2 - 2}, a_2 = a_2, a_3 = 0, a_4 = -\frac{a_2\left({k_1}^2 - 3\right)}{{k_1}^2 - 2},\\
&a_5 = a_5, k_1 = k_1, k_4 = k_1, k_2 = \frac{{a_2}^4}{\left({k_1}^2 - 2\right)^3 bak_1{}^3},\\
&k_3 = \frac{(6\,abk_1{}^{10} - 36\,abk_1{}^8 + 72\,abk_1{}^6 - 48\,abk_1{}^4 - {a_2}^4{k_1}^2 + 3\,{a_2}^4)k_1}{2{a_2}^4},
\end{aligned} \tag{4.11}$$

where a, b, a_2, a_5 and k_4 are free constants.

Case 11

$$\begin{aligned}
&a = a, b = b, a_1 = -\frac{a_2}{{k_1}^2 + 2}, a_2 = a_2, a_3 = 0, a_4 = -\frac{a_2\left({k_1}^2 + 1\right)}{{k_1}^2 + 2},\\
&a_5 = a_5, k_1 = k_1, k_2 = \frac{{a_2}^4}{\left({k_1}^2 + 2\right)^3 bak_1{}^3}, k_3 = -\frac{1}{2}{k_1}^3 - \frac{3}{2}\,k_1, k_4 = -k_1,
\end{aligned} \tag{4.12}$$

where a, b, a_2, a_5 and k_4 are arbitrary parameters.

Case 12

$$\begin{aligned}
&a = a, b = b, a_1 = -a_4, a_2 = 0, a_3 = a_3, a_4 = a_4, a_5 = a_5,\\
&k_1 = 0, k_2 = \frac{3k_3{a_4}^2}{a_3\,(2\,a_3 - 3\,a_4)}, k_3 = k_3, k_4 = \frac{k_3 a_4}{a_3},
\end{aligned} \tag{4.13}$$

where a, b, a_1, a_3, a_5 and k_3 are arbitrary constants.

Case 13

$$\begin{aligned}
&a = a, b = b, a_1 = -a_4, a_2 = 0, a_3 = a_3, a_4 = a_4,\\
&a_5 = a_5, k_1 = k_1, k_2 = 0, k_3 = k_3, k_4 = -k_1,
\end{aligned} \tag{4.14}$$

where a, b, a_1, a_3, a_5, k_1 and k_3 are free parameters.

Case 14

$$\begin{aligned}
&a = a, b = b, a_1 = a_1, a_2 = 0, a_3 = \frac{3a_1 k_4}{2k_2}, a_4 = 0, a_5 = a_5,\\
&k_1 = 0, k_2 = k_2, k_3 = \frac{3k_4\,(k_2 + k_4)}{2k_2}, k_4 = k_4,
\end{aligned} \tag{4.15}$$

where a, b, a_1, a_5, k_2 and k_4 are arbitrary constants.

Case 15

$$\begin{aligned}
&a = a, b = b, a_1 = -a_4, a_2 = 0, a_3 = \frac{3a_4\,(k_2 + k_4)}{2k_2}, a_4 = a_4,\\
&a_5 = a_5, k_1 = 0, k_2 = k_2, k_3 = \frac{3k_4\,(k_2 + k_4)}{2k_2}, k_4 = k_4,
\end{aligned} \tag{4.16}$$

where a, b, a_1, a_5, k_2 and k_4 are arbitrary parameters.

From Cases 1–15, the new type rogue waves to the extended (3+1)-dimensional JM equation in detail, were yielded.

(I) Algebraic solitary wave(lump) If we taking $a = b = 0$, then the solutions of Eq.(1.1) have a pure algebraic solitary wave. That is

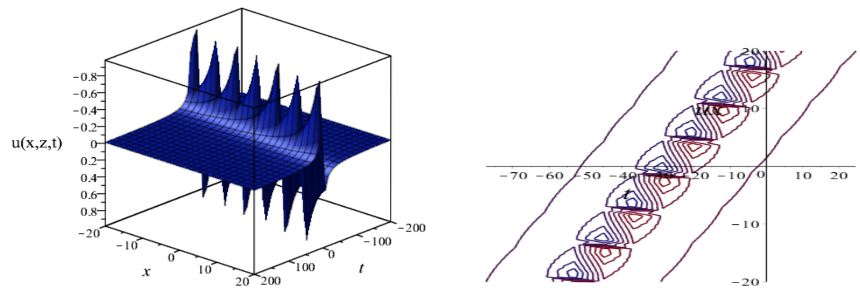
$$u_{cases_{1-3}} = 2(\ln(F_{cases_{1-3}}))_x, \tag{4.17}$$

$$F_{case_1} = 1 + \left(ta_3 - xa_4 + za_4 + a_5\right)^2, \tag{4.18}$$

$$F_{case_2} = 1 + \left(-\frac{3a_1 t\,(k_2 + k_4)}{2k_2} + xa_1 - za_1\right)^2, \tag{4.19}$$

$$F_{case_3} = 1 + \left(-\frac{1}{2}\,t\left(-3\,{k_1}^2 - 3\right)a_4 - xa_4 + za_4 + a_5\right)^2. \tag{4.20}$$

Lump structure of solution u_{case_2} by Figure 1, were plotted.



(a) (b)

Figure 1. Plots of the lump solution u_{case_2} with parameters $a_1 = 1, k_2 = -2, k_4 = 1$. (a) 3D-plot, with $z = 20$. (b) 2D-contour plot.

From Figure 1, we can see that the lump solution exhibits periodic characteristics.

(II) Interaction between lump and exponentially decayed solitary waves (lumpoff). If we letting $a = 0, b \neq 0 (or\ a \neq 0, b = 0)$, then the solutions of Eq.(1.1) is an algebraically decayed soliton and exponentially decayed soliton, which are

$$u_{cases_{4-5}} = 2(\ln(F_{cases_{4-5}}))_x, \tag{4.21}$$

$$F_{case_4} = 1 + \left(ta_3 - xa_4 + za_4 + a_5\right)^2 + b\mathrm{e}^{-tk_3 - \frac{y3k_3{a_4}^2}{a_3(2\,a_3 - 3\,a_4)} - \frac{zk_3a_4}{a_3} - k_5}, \tag{4.22}$$

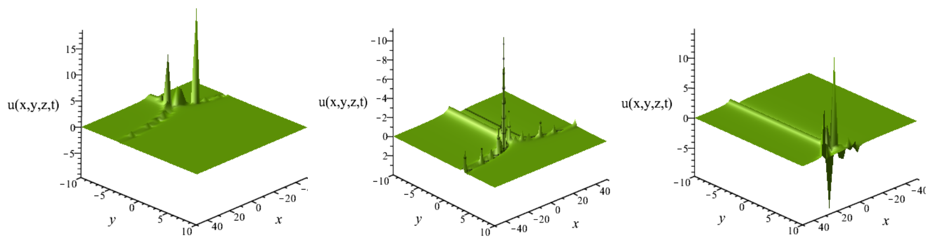
$$F_{case_5} = 1 + \left(\frac{3a_4 t\,(k_2 + k_4)}{2k_2} - xa_4 + za_4 + a_5\right)^2 + b\mathrm{e}^{-\frac{3k_4 t(k_2 + k_4)}{2k_2} - yk_2 - zk_4 - k_5}. \tag{4.23}$$

Lumpoff structures of solution u_{case_5} by Figure 2, were revealed. From Figure 2, we can see that the lumpoff solution is the evolution of the interaction between lump and exponentially decayed solitary waves in XY-plane.

(III) Interaction between lump and exponentially localized twin solitary waves (instanton or rouge wave). If we considering $a \neq 0, b \neq 0$, then the solutions of Eq.(1.1) is an interaction solutions between the lump and the exponentially decayed twin soliton, that are

$$u_{cases_{6-15}} = 2(\ln(F_{cases_{6-15}}))_x, \tag{4.24}$$

$$\begin{aligned} F_{case_6} &= 1 + g^2 + a\mathrm{e}^{\frac{3k_4 t(k_2 + k_4)}{2k_2} + yk_2 + zk_4 + k_5} + b\mathrm{e}^{-\frac{3k_4 t(k_2 + k_4)}{2k_2} - yk_2 - zk_4 - k_5}, \\ g &= \frac{3a_4 t\,(k_2 + k_4)}{2k_2} - xa_4 + za_4 + a_5, \end{aligned} \tag{4.25}$$



(a) (b) (c)

Figure 2. Plots of evolution of the lumpoff structures of solution u_{case_5} with parameters $a_4 = 1, k_2 = -2, k_4 = 1, k_5 = a_5 = 0, b = -10, z = 2$. (a)$t = -20$. (b) $t = 0$. (c) $t = 20$.

$$\begin{aligned} F_{case_7} &= 1 + g^2 + a\mathrm{e}^{\frac{k_2 a_3 t(2a_3-3a_4)}{3{a_4}^2}+yk_2+\frac{zk_2(2a_3-3a_4)}{3a_4}+k_5} \\ &\quad + b\mathrm{e}^{-\frac{tk_2a_3(2a_3-3a_4)}{3{a_4}^2}-yk_2-\frac{zk_2(2a_3-3a_4)}{3a_4}-k_5}, \\ g &= ta_3 - xa_4 + za_4, \end{aligned} \tag{4.26}$$

$$\begin{aligned} F_{case_8} &= 1 + g^2 + \frac{{a_2}^4}{{k_2}^4 b}\mathrm{e}^{-\frac{t\left({k_1}^3k_2-3k_1k_4-3k_2k_4-3{k_4}^2\right)}{2k_2}+xk_1+yk_2+zk_4+k_5} \\ &\quad + b\mathrm{e}^{\frac{t\left({k_1}^3k_2-3k_1k_4-3k_2k_4-3{k_4}^2\right)}{2k_2}-xk_1-yk_2-zk_4-k_5}, \\ g &= \frac{3t\left(a_2{k_1}^3k_2-2a_2k_1k_4-a_2{k_4}^2+a_4k_1k_2+a_4{k_2}^2+2a_4k_2k_4\right)}{2{k_2}^2} \\ &\quad - \frac{xa_2k_1}{k_2} + ya_2 + za_4, \end{aligned} \tag{4.27}$$

$$\begin{aligned} F_{case_9} &= 1 + g^2 + \frac{{a_1}^4}{b{k_1}^4}\mathrm{e}^{t\left(-\frac{1}{2}{k_1}^3-\frac{3}{2}k_4{k_1}^2-\frac{3}{2}k_1{k_4}^2+\frac{3}{2}k_4\right)+xk_1-\frac{y}{k_1}+zk_4+k_5} \\ &\quad + b\mathrm{e}^{-t\left(-\frac{1}{2}{k_1}^3-\frac{3}{2}k_4{k_1}^2-\frac{3}{2}k_1{k_4}^2+\frac{3}{2}k_4\right)-xk_1+\frac{y}{k_1}-zk_4-k_5}, \\ g &= \frac{3a_1t\left(k_4+k_1\right)\left(k_1k_4-1\right)}{2k_1} + xa_1 + \frac{ya_1}{{k_1}^2} - \frac{za_1\left(k_4+k_1\right)}{k_1}, \end{aligned} \tag{4.28}$$

$$\begin{aligned} F_{case_{10}} &= 1 + g^2 + a\mathrm{e}^{\xi} + b\mathrm{e}^{-\xi}, \\ g &= -\frac{xa_2}{{k_1}^2-2} + ya_2 - \frac{za_2\left({k_1}^2-3\right)}{{k_1}^2-2} + a_5, \\ \xi &= \frac{t\left(6abk_1^{10}-36abk_1^{8}+72abk_1^{6}-48abk_1^{4}-{a_2}^4{k_1}^2+3{a_2}^4\right)k_1}{2{a_2}^4} \\ &\quad + xk_1 + \frac{y{a_2}^4}{\left({k_1}^2-2\right)^3ba{k_1}^3} + zk_1 + k_5, \end{aligned} \tag{4.29}$$

$$F_{case_{11}} = 1 + g^2 + a\mathrm{e}^{t\left(-\frac{1}{2}k_1{}^3 - \frac{3}{2}k_1\right) + xk_1 + \frac{ya_2{}^4}{(k_1{}^2+2)^3 bak_1{}^3} - zk_1 + k_5} + b\mathrm{e}^{-t\left(-\frac{1}{2}k_1{}^3 - \frac{3}{2}k_1\right) - xk_1 - \frac{ya_2{}^4}{(k_1{}^2+2)^3 bak_1{}^3} + zk_1 - k_5}, \quad (4.30)$$

$$g = -\frac{xa_2}{k_1{}^2+2} + ya_2 - \frac{za_2\left(k_1{}^2+1\right)}{k_1{}^2+2} + a_5,$$

$$F_{case_{12}} = 1 + g^2 + a\mathrm{e}^{tk_3 + \frac{y3k_3a_4{}^2}{a_3(2a_3-3a_4)} + \frac{zk_3a_4}{a_3} + k_5} + b\mathrm{e}^{-tk_3 - \frac{y3k_3a_4{}^2}{a_3(2a_3-3a_4)} - \frac{zk_3a_4}{a_3} - k_5},$$
$$g = ta_3 - xa_4 + za_4 + a_5, \quad (4.31)$$

$$F_{case_{13}} = 1 + (ta_3 - xa_4 + za_4 + a_5)^2 + a\mathrm{e}^{tk_3 + xk_1 - zk_1 + k_5} + b\mathrm{e}^{-tk_3 - xk_1 + zk_1 - k_5}, \quad (4.32)$$

$$F_{case_{14}} = 1 + g^2 + a\mathrm{e}^{\frac{t3k_4(k_2+k_4)}{2k_2} + yk_2 + zk_4 + k_5} + b\mathrm{e}^{-\frac{t3k_4(k_2+k_4)}{2k_2} - yk_2 - zk_4 - k_5},$$
$$g = \frac{t3a_1k_4}{2k_2} + xa_1 + a_5, \quad (4.33)$$

$$F_{case_{15}} = 1 + g^2 + a\mathrm{e}^{\frac{t3k_4(k_2+k_4)}{2k_2} + yk_2 + zk_4 + k_5} + b\mathrm{e}^{-\frac{t3k_4(k_2+k_4)}{2k_2} - yk_2 - zk_4 - k_5},$$
$$g = \frac{t3a_4\left(k_2+k_4\right)}{2k_2} - xa_4 + za_4 + a_5. \quad (4.34)$$

In particular, when $a = b$, then, we have

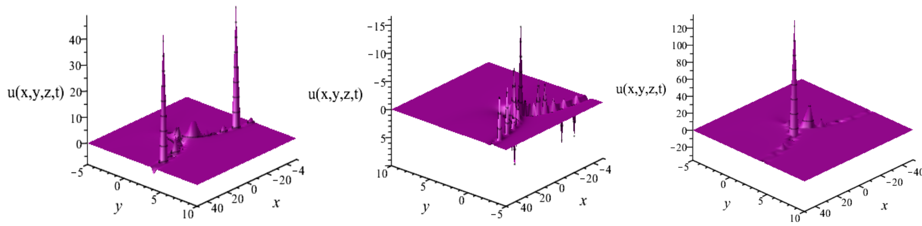
$$\begin{aligned} F_{case_{15-1}} &= 1 + g^2 + 2asinh(\zeta), \\ \zeta &= \frac{3k_4\left(k_2+k_4\right)}{2k_2}t + yk_2 + zk_4 + k_5, \\ g &= \frac{t3a_4\left(k_2+k_4\right)}{2k_2} - xa_4 + za_4 + a_5. \end{aligned} \quad (4.35)$$

When $a = -b$, then, we have

$$\begin{aligned} F_{case_{15-2}} &= 1 + g^2 + 2acosh(\zeta), \\ \zeta &= \frac{3k_4\left(k_2+k_4\right)}{2k_2}t + yk_2 + zk_4 + k_5, \\ g &= \frac{t3a_4\left(k_2+k_4\right)}{2k_2} - xa_4 + za_4 + a_5. \end{aligned} \quad (4.36)$$

Remark 4.1. Other cases ($cases_{6-15}$) are similarly considered.

Instanton structures of solution $u_{case_{15}}$ by Figure 3, were shown.



(a) (b) (c)

Figure 3. Plots of the instanton structures of solution $u_{case_{15}}$ with parameters $a_4 = 1, k_2 = -2, k_4 = 1, k_5 = a_5 = 0, b = -10, z = t = 0$. (a) 3D-plot, $a = 2, b = -2$. (b) 3D-plot, $a = -2, b = 2$.(c) 3D-plot, $a = 2, b = -1$.

From Figure 3, we can see that the instanton solution of the evolution of the interaction between lump and exponentially decayed twins solitary waves.

5. Solitary waves to the extended (3+1)-dimensional JM equation

We know that solitary wave solutions are one form of traveling wave solutions [15–17, 38, 39], which play an important role in understanding physical phenomena and mechanisms. Therefore, in this section, we search for the solitary wave solution to the extended (3+1)-dimensional JM equation by using the tanh-function method [35, 36].

Firstly, plugging the traveling wave transformation

$$U = U(\xi), \xi = k_1 x + k_2 y + k_3 z + k_4 t \tag{5.1}$$

into Eq.(1.1), then, we have

$$\begin{aligned} &k_1^3 k_2 \frac{\partial^4 U(\xi)}{\partial \xi^4} + 6k_1^2 k_2 (\frac{\partial U(\xi)}{\partial \xi} \frac{\partial^2 U(\xi)}{\partial \xi^2}) + 2k_2 k_4 \frac{\partial^2 U(\xi)}{\partial \xi^2} \\ &- 3(k_1 k_3 + k_2 k_3 + k_3^2) \frac{\partial^2 U(\xi)}{\partial \xi^2} = 0. \end{aligned} \tag{5.2}$$

Now, we find the solutions of the following form

$$u = \sum_{i=0}^{N} a_i \left(\tanh\left(tk_4 + xk_1 + yk_2 + zk_3\right)\right)^i, \tag{5.3}$$

where $a_i (i = 0, 1, 2, ..., N)$ and $k_i (i = 1, 2, 3, 4)$are constants.

According to the homogeneous balance principle, we balancing the highest derivative term $\frac{\partial^4 U(\xi)}{\partial \xi^4}$ and highest order nonlinear term $\frac{\partial U(\xi)}{\partial \xi} \frac{\partial^2 U(\xi)}{\partial \xi^2}$, yields

$$N + 4 = 2N + 3 \Rightarrow N = 1. \tag{5.4}$$

Hence equation (5.3) becomes the form

$$u = a_0 + a_1 \tanh\left(tk_4 + xk_1 + yk_2 + zk_3\right). \tag{5.5}$$

Substituting (5.5) into (5.2) and equating coefficients of $tanh(\xi)$ to zero, we can obtain system algebraic equation including $k_i(i=1,2,3,4)$ and $a_j(j=0,1)$

$$\begin{aligned}
&-2\,a_1\left(6\,a_1{k_1}^2k_2-8\,{k_1}^3k_2-3\,k_1k_3-3\,k_2k_3+2\,k_2k_4-3\,{k_3}^2\right)=0,\\
&-2\,a_1\left(12\,a_1{k_1}^2k_2-20\,{k_1}^3k_2-3\,k_1k_3-3\,k_2k_3+2\,k_2k_4-3\,{k_3}^2\right)=0,\\
&-12\,a_1{k_1}^2k_2\,(a_1-2\,k_1)=0.
\end{aligned} \tag{5.6}$$

It can easily solve with the help of Maple. As a result, we have

$$\begin{aligned}
&a_0=a_0, a_1=2k_1, k_1=k_1, k_2=k_2, k_3=k_3,\\
&k_4=-\frac{4\,{k_1}^3k_2-3\,k_1k_3-3\,k_2k_3-3\,{k_3}^2}{2k_2},
\end{aligned} \tag{5.7}$$

where a_0 and $k_i(i=1,2,3)$ are free constants.

Hence we have the following solitary wave solution to Eq.(1.1)

$$u=a_0-2k_1tanh(\frac{(4k_1^3k_2-3k_1k_3-3k_2k_3-3k_3^2)t-2k_1k_2x-2k_2^1y-2k_2k_3z}{2k_2}). \tag{5.8}$$

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References

[1] K. K. Ali, R. I. Nuruddeen and A. R. Hadhoud, *New exact solitary wave solutions for the extended (3+1)-dimensional Jimbo-Miwa equations*, Result. Phys., 2018, 9(9), 12–16.

[2] M. J. Ablowitz, M. A. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, 1991.

[3] E. Fan and K. W. Chow, *Darboux covariant Lax pairs and infinite conservation laws of the (2+1)-dimensional breaking soliton equation*, J. Math. Phys., 2011, 52(2), 023504.

[4] C. Gilson, F. Lambert, J. Nimmo and R. Willox, *On the combinatorics of the Hirota D-operators*, Proc. R. Soc. London, Ser. A., 1996, 452, 223.

[5] J. He, S. Xu and K. Porsezian, *N-order bright and dark rogue waves in a resonant erbium-doped fiber system.* Phys. Rev. E., 2012, 86(6), 066603.

[6] R. Hirota, *The direct method in soliton theory*, Cambridge University Press, 2004.

[7] F. Lambert, I. Loris and J. Springael, *Classical Darboux transformations and the KP hierarchy*, Inverse. Probl., 2001, 17, 1067.

[8] F. Lambert and J. Springael, *Soliton equations and simple combinatorics*, Acta Appl. Math., 2008, 102, 147.

[9] X. Lü, B. Tian and K. Sun, et al, *Bell-polynomial manipulations on the Bäcklund transformations and Lax pairs for some soliton equations with one Tau-function*, J. Math. Phys., 2010, 51(11), 113506.

[10] J. Liu, X. Yang and Y. Feng, *On integrability of the extended (3+1)-dimensional Jimbo-Miwa equation*, Math. Meth. Appl. Sci., 2020, 43(4), 1646–1659.

[11] J. Liu, X. Yang, Y. Feng and L. Geng, *On integrability of the higher-dimensional time fractional KdV-type equation*, J. Geom. Phys., 2021, 160, 104000.

[12] J. Liu and Y. Zhang, *Construction of lump soliton and mixed lump stripe solutions of (3+1)-dimensional soliton equation*, Result. Phys., 2018, 10, 94–98.

[13] S. Lou and J. Lin, *Rogue Waves in Nonintegrable KdV-Type Systems*, Chin. Phys. Lett., 2018, 35(5), 050202.

[14] H. Li and Y. Li, *Meromorphic exact solutions of two extended (3+1)-dimensional JimbošCMiwa equations*, Appl. Math. Comput., 2018, 333, 369–375.

[15] J. Liu, X. Yang and Y. Feng, *Characteristic of the algebraic traveling wave solutions for two extended (2+1)-dimensional Kadomtsev Petviashvili equations*, Moder. Phys. Lett. A., 2020, 35(7), 20500285.

[16] J. Liu, Y. Feng and H. Zhang, *Exploration of the algebraic traveling wave solutions of a higher order model*, Eng. Comput., 2021, 38(2), 618–631.

[17] J. Lenells, *Traveling wave solutions of the Camassa-Holm equation*, J. Diff. Equ., 2005, 217(2), 393–430.

[18] W. Ma, *Bilinear equations and resonant solutions characterized by Bell polynomials*, Rep. Math. Phys., 2013, 72(1), 41–56.

[19] W. Ma and Y. Zhou, *Lump solutions to nonlinear partial differential equations via Hirota bilinear forms*, J. Diff. Equ., 2018, 264(4), 2633–2659.

[20] J. Manafian, B. Mohammadi-Ivatloo and M. Abapour, *Lump-type solutions and interaction phenomenon to the (2+1)-dimensional Breaking Soliton equation*, Appl. Math. Comput., 2019, 356, 13–41.

[21] J. Manafian and M. Lakestani, *Lump-type solutions and interaction phenomenon to the bidirectional Sawada-Kotera equation*, Pramana., 2019, 92(3), 1–13.

[22] J. Manafian and M. Lakestani, *N-lump and interaction solutions of localized waves to the (2+1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation*, J. Geom. Phys., 2020, 150, 103598.

[23] J. Manafian, O. A. Ilhan, L. Avazpour and A. A. Alizadeh, *N-lump and interaction solutions of localized waves to the (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation arise from a model for an incompressible fluid*, Math. Meth. Appl. Sci., 2020, 43(17), 9904–9927.

[24] J. Manafian and M. Lakestani, *Interaction among a lump, periodic waves, and kink solutions to the fractional generalized CBS-BK equation*, Math. Meth. Appl. Sci., 2021, 44(1), 1052–1070.

[25] J. Manafian, S. A. Mohammed, A. A. Alizadeh, H. M. Baskonus and W. Gao, *Investigating lump and its interaction for the third-order evolution equation*

arising propagation of long waves over shallow water, Eur. J. Mech-B/Flui., 2020, 84, 289–301.

[26] P. Müller and C. A. Garrett, *Osborne Rogue waves Oceanography*, 2005, 18, 66–75.

[27] J. Manafian, *Novel solitary wave solutions for the (3+1)-dimensional extended Jimbo-Miwa equations*, Comput. Math. Appl., 2018, 76(5), 1246–1260.

[28] W. Ma and J. H. Lee, *A transformed rational function method and exact solutions to the (3+1)-dimensional Jimbo-Miwa equation*, Chaos, Solitons & Fractals., 2009, 42(3), 1356–1363.

[29] M. Onorato, S. Residori and U. Bortolozzo, et al, *Rogue waves and their generating mechanisms in different physical contexts*, Phys. Rep., 2013, 528(2), 47–89.

[30] M. Singh and R. K. Gupta, *Bäcklund transformations, Lax system, conservation laws and multisoliton solutions for Jimbo-Miwa equation with Bell-polynomials*, Commun. Nonl. Sci. Numer. Simul., 2016, 37, 362–373.

[31] D. R. Solli, C. Ropers, P. Koonath and B. Jalali, *Optical rogue waves*, Nature, 2007, 450, 1054–1057.

[32] D. R. Solli, C. Ropers and P. Koonath, *Active control of rogue waves for stimulated supercontinuum generation*, Phys. Revi. Lett., 2008, 101(23), 233902.

[33] H. Sun and A. Chen, *Lump and lump-kink solutions of the (3+1)-dimensional Jimbo-Miwa and two extended Jimbo-Miwa equations*, Appl. Math. Lett., 2017, 68, 55–61.

[34] A. M. Wazwaz, *Multiple-soliton solutions for extended (3+1)-dimensional Jimbo-Miwa equations*, Appl. Math. Lett., 2016, 64, 21–26.

[35] A. M. Wazwaz, *The tanh method for traveling wave solutions of nonlinear equations*, Appl. Math. Comput., 2004, 154(3), 713–723.

[36] A. M. Wazwaz, *The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations*, Appl. Math. Comput., 2005, 167(2), 1196–1210.

[37] Z. Yan, *Financial rogue waves*, Commun. Theo. Phys., 2010, 54(5), 947.

[38] X. Yang, F. Gao and H. M. Srivastava, *Exact travelling wave solutions for the local fractional two-dimensional burgers-type equations*, Comput. Math. Appl., 2017, 73, 203–210.

[39] X. Yang, J. T. Machado and D. Baleanu, *Exact traveling-wave solution for local fractional boussinesq equation in fractal domain*, Fractals., 2017, 25, 1740006.

[40] Y. Zhang and H. Tam, *Discussion on integrable properties for higher-dimensional variable-coefficient nonlinear partial differential equations*, J. Math. Phys., 2013, 54(1), 013516.