Journal of Applied Analysis and Computation
Volume 11, Number 6, December 2021, 2736–2761
Website:http://www.jaac-online.com
DOI:10.11948/20200405

FAST COMPACT DIFFERENCE SCHEME FOR THE FOURTH-ORDER TIME MULTI-TERM FRACTIONAL SUB-DIFFUSION EQUATIONS WITH THE FIRST DIRICHLET BOUNDARY*

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Abstract In this paper, a fast compact difference scheme is proposed for the initial-boundary value problem of fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions. Using the method of order reduction, the original problem can be converted to an equivalent lower-order system. Then at some super-convergence points, the multi-term Caputo derivatives are fast evaluated based on the sum-of-exponentials (SOE) approximation for the kernel functions appeared in Caputo fractional derivatives. The difficulty caused by the first Dirichlet boundary conditions is carefully handled. The energy method is used to illustrate the unconditional stability and convergence of the proposed fast compact scheme. The convergence accuracy is second-order in time and fourth-order in space if the solution has enough regularity. Compared with the direct scheme without the acceleration in time direction, the CPU time of the current fast scheme is largely reduced, which is shown by numerical examples.

Keywords Fast evaluation, sum-of-exponentials approximation, multi-term fractional derivatives, the first Dirichlet boundary, stability, convergence.

MSC(2010) 65M06, 65M12, 65M15.

1. Introduction

In the modern era, many researchers have placed more and more attentions on fractional differential equations (FDEs). Numerous processes in physics [19], astronomy [32], medicine [10], finance [35], etc. can be modelled by FDEs. There are many different definitions of fractional derivatives, such as Riemann-Liouville derivative, Riesz derivative, Caputo derivative, etc. Generally speaking, the time-fractional derivatives are described in the Caputo sense, while the space-fractional derivatives are often defined in the Riemann-Liouville or Riesz sense. In most cases, the analytical solution to fractional partial differential equations (FPDEs) is difficult to obtain, thus it is quite necessary to find some effective numerical methods for solving FPDEs.

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*The authors were supported by Natural Science Foundation of Jiangsu Province of China (No. BK20191375) and the project NUPTSF (No. NY220037).

For the fourth-order time fractional diffusion-wave equations, there have been extensive results. In [14], Jafari et al. used the Adomian decomposition method to obtain solutions of fourth-order fractional diffusion-wave equations defined in a bounded space domain. Hu and Zhang [12] explored a compact difference scheme for the fourth-order fractional diffusion-wave equation by the method of order reduction. The stability and convergence of the scheme were proved. Combining the average operator for the spatial derivatives, Ji, Sun and Hao [15] presented a novel technique to deal with the first Dirichlet boundary conditions, where an average of functions at four points near the boundary was introduced. In [33], Vong and Wang studied a high-order compact difference scheme for the fourth-order fractional subdiffusion system with the first Dirichlet boundary, where an average operator and a discrete fourth-order difference quotient operator were defined to build the scheme. In [38], Yao and Wang also considered the fourth-order fractional sub-diffusion equations, but the boundary conditions under consideration were Neumann ones. For one- and two-dimensional time fractional fourth-order sub-diffusion equations with the first Dirichlet boundary conditions, Cui [4] constructed the compact finite difference schemes. The boundary conditions were firstly transformed into the homogeneous ones by the Hermite interpolating polynomial, and the spatial derivatives were discretized by the Stephenson scheme. Whereas, the theoretical analysis on the resultant scheme was a little complicated. The Ref. [3] was devoted to the spline approach to numerical solution of a class of fourth-order time FPDEs. The backward Euler method was used for temporal discretization whereas the spatial discretization was achieved by non-polynomial quintic spline method. In [5], Fei and Huang considered the Galerkin-Legendre spectral method for solving the two-dimensional distributed-order time fractional fourth-order partial differential equation with the first Dirichlet boundary, where the composite Simpson formula was used to discretize the distributed-order integral and the $L2-1_\sigma$ formula was applied to approximate the involved multi-term Caputo fractional derivatives. The proposed spectral scheme was shown to be unconditionally stable and convergent with fourth-order accuracy in distributed order, second-order accuracy in time and spectral accuracy in space. By utilizing the $L2-1_\sigma$ formula for temporal dimension, some difference schemes were constructed for a class of fourth-order fractional equations with time delay or variable coefficients in [23,24], where the second Dirichlet boundary value problem was considered. In [36], Xu et al. proposed and analyzed a compact finite difference scheme for the fourth-order time-fractional integro-differential equation with a weakly singular kernel. The stability and convergence were proved by the discrete energy method, the Cholesky decomposition and the reduced-order method.

In practice, it is often not enough to describe some phenomena by the single-term FDEs. Many processes should be described by the multi-term time FDEs, such as the underlying processes with loss [21], viscoelastic damping [27], oxygen delivery through a capillary to tissues [29], the anomalous diffusion in highly heterogeneous aquifers and complex viscoelastic materials [17]. In particular, the multi-term time-fractional diffusion-wave equations can successfully describe the power-law frequency dependence in a continuous time random walk model [20].

For the multi-term time-fractional diffusion-wave equations, considerable research achievements can be found. In [17], the authors developed a simple numerical scheme based on the Galerkin finite element method for a multi-term time fractional diffusion equation which involves multiple Caputo fractional derivatives.

Gao et al. [6] constructed a numerical formula for the multi-term Caputo fractional derivatives at the super-convergence point, where the proposed formula can achieve at least second-order accuracy. Gao and Liu [7] developed a spatial compact difference scheme for a class of fourth-order temporal multi-term fractional wave equations with the second Dirichlet boundary. Combined with the method of order reduction, the original problem was reduced to a lower order system and the $L1$ formula was used to approximate the corresponding time fractional derivatives. In [1], Abdel-Rehim et al. gave the simulations of the approximation solutions of time-fractional wave, forced wave (shear wave) and damped wave equations, respectively. The Caputo time-fractional derivatives were discretized by the G-L formula and the Von-Neumann stability conditions were discussed for these proposed models. Wei [34] constructed a fully discrete local discontinuous Galerkin method for solving a class of multi-term time fractional diffusion equations and this method was proved to be unconditionally stable and convergent. Two temporal second-order accurate difference schemes at the super-convergence point were presented by the order reduction technique for time multi-term fractional diffusion-wave equation in [31]. The two difference schemes were proved to be uniquely solvable. In [39], Zhang et al. developed a multi-term time-fractional Burgers' fluid model and obtained the analytical solution by the method of separation of variables. Then they proposed a unified numerical method for this model by weighted and shifted Grünwald difference operators in time and Legendre spectral method in space. Also they considered a modified scheme to improve the convergence accuracy. Huang and Stynes [13] considered a multi-term time-fractional initial-boundary value problem, where the spatial derivative was discretized by the standard finite element method, while each fractional derivative was approximated by the $L1$ formula on a graded temporal mesh in view of the weak singularity which appeared in typical solutions to this class of problems.

As we know, in order to get the value of fractional derivative at the current time, the values at all previous levels need be in storage, which results in huge computational cost. Hence, some efficient and fast evaluations of fractional derivatives will be quite essential. Jiang et al. [16] proposed a fast evaluation for Caputo fractional derivative which employed the SOE approximation to the kernel function $t^{-\alpha}$. The fast algorithm achieves the accuracy $O(\tau^{2-\alpha}+\epsilon)$ based on the $L1$ formula, with α the order of the fractional derivative. Yan et al. [37] made the further study on the fast evaluation of the Caputo fractional derivative based on the work [16]. They considered the fast evaluation of the time-fractional derivative on the basis of the $L2-1_\sigma$ scheme proposed in [2]. The obtained $\mathcal{F}L2-1_\sigma$ approximation can achieve high-order accuracy and can reduce the storage and computational cost greatly. In [9], a linear combination of Caputo fractional derivatives was fast evaluated based on the SOE approximation for the kernels in Caputo fractional derivatives. Numerical examples showed that the CPU time was largely reduced while the accuracy was kept with the large value of temporal levels. In addition, Sun and Sun [30] established a fast temporal second-order compact ADI difference scheme for the 2D multi-term fractional wave equation. The fast scheme can be solved by the recursion which reduces the storage and computational cost significantly. Liao et al. [18] proposed a fast two-level linearized scheme with nonuniform time-steps for an initial-boundary-value problem of semilinear subdiffusion equations. The two-level fast $L1$ formula of the Caputo derivative was derived based on the SOE technique. Gu et al. [11] developed two fast implicit difference schemes for solving a class of variable coefficient time-space fractional diffusion equations with integral fractional Laplacian (IFL). The $L1$ formula for the Caputo fractional derivative and a special finite difference discretization for IFL were applied to derive the schemes. Moreover, the fast SOE approximation and Toeplitz matrix algorithms were used to reduce the computational cost of time and space fractional derivatives, respectively. In [22], Lyu et al. considered a fast and linearized finite difference method to solve a nonlinear multi-term time-fractional wave problem. In order to construct the fast numerical scheme, based on the $L2-1_\sigma$ formula and the SOE approximation to the kernel function in the Caputo derivative, they developed the $\mathcal{F}L2-1_\sigma$ formula, which was proposed in [37] for the single time fractional derivative.

It's noted that both Refs. [15] and [33] handle the problem with the single-term time fractional derivative, which was approximated by the $L1$ formula, and the convergence order of the resultant schemes in time was less than two. Different from the previous works, for the the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary, in the present work, we are devoted to find the higher-order numerical solutions to the problem by the novel handling of the boundary and higher-order approximation for the multi-term time fractional derivatives. In addition, inspired by [9], the time multi-term Caputo fractional derivatives will be fast evaluated based on the SOE approximation for the kernel functions appeared in Caputo fractional derivatives. Finally, the unconditional stability and convergence of the derived fast scheme will be proved. Several numerical examples are calculated to illustrate the efficiency of the fast scheme. The main advantages of the current work cover:

- In view of the first Dirichlet boundary value problem, a novel and simple average operator is defined so that the global fourth-order accuracy of the proposed difference scheme in space can be easily achieved;
- By introducing an intermediate function, the original spatial fourth-order problem is converted to a lower-order second-order system, for which the numerical algorithm is developed. Whereas, in the practical calculation, the intermediate function can be cancelled and a numerical scheme involving only the original unknowns need be calculated on each time level.
- To speed up the evaluation of time multi-term fractional derivatives, based on the $L2-1_\sigma$ approximation, the SOE approximation for the kernel function appeared in fractional derivatives is applied.
- The strict prior estimate on the proposed fast compact scheme is carried out by the energy method and some novel techniques.

This paper is arranged as follows. In section 2, we introduce some useful notations and lemmas to prepare for the construction of the difference scheme. A fast compact difference scheme is proposed for the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary in section 3. In section 4, the unconditional stability and convergence of the difference scheme are discussed by the energy method. In section 5, some numerical examples are provided to further validate our theoretical results and verify the efficiency of the proposed scheme. It ends with a brief conclusion.

2. Preliminaries

Consider the following initial-boundary value problem of fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary conditions:

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} + q u(x,t) = f(x,t), \quad x \in (0,L),\ t \in (0,T], \tag{2.1}$$

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad t \in (0,T], \tag{2.2}$$

$$u_x(0,t) = \gamma_1(t), \quad u_x(L,t) = \gamma_2(t), \quad t \in (0,T], \tag{2.3}$$

$$u(x,0) = \phi(x), \quad x \in [0,L], \tag{2.4}$$

where $g_1(0) = \phi(0), g_2(0) = \phi(L), \phi'(0) = \gamma_1(0), \phi'(L) = \gamma_2(0)$, $q, \lambda_0, \lambda_1, \ldots, \lambda_m$ are some positive constants, $0 \leqslant \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \leqslant 1$ and at least one of α_i's belongs to $(0,1)$, ${}_0^C D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative defined by

$${}_0^C D_t^{\alpha} u(x,t) = \begin{cases} u(x,t) - u(x,0), & \alpha = 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{u_s(x,s)}{(t-s)^{\alpha}} \mathrm{d}s, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1. \end{cases}$$

As described in [28], the considered equations occur in many applications in real-life problems such as modelling of plates and thin beams, strain gradient elasticity and phase separation in binary mixtures, which are basic elements in engineering structures and are of great practical significance to civil, mechanical and aerospace engineering.

Gao et al. [6] found some particular points where the linear combination of Caputo fractional derivatives is approximated. We will recall this result here simply. Let τ be the step size, $t_n = n\tau, n = 0,1,2,\ldots,N$ with $N\tau = T$, $t_{n-\frac{1}{2}} = (n-\frac{1}{2})\tau, n = 1,2,\ldots,N$. The constant σ is the root of nonlinear equation

$$F(x) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(3-\alpha_r)} x^{1-\alpha_r} \big[x - (1 - \frac{\alpha_r}{2})\big] \tau^{2-\alpha_r} = 0, \quad x \geqslant 0,$$

and $t_{n-1+\sigma} = (n-1+\sigma)\tau$. It can be known from [6] that $\sigma \in (\frac{1}{2}, 1)$.

For any $\alpha \in [0,1]$, when $n = 1$, denote

$$c_0^{(n,\alpha)} = \sigma^{1-\alpha};$$

When $n \geqslant 2$, denote

$$\begin{aligned} c_0^{(n,\alpha)} &= \frac{(1+\sigma)^{2-\alpha} - \sigma^{2-\alpha}}{2-\alpha} - \frac{(1+\sigma)^{1-\alpha} - \sigma^{1-\alpha}}{2}, \\ c_k^{(n,\alpha)} &= \frac{1}{2-\alpha}\Big[(k+1+\sigma)^{2-\alpha} - 2(k+\sigma)^{2-\alpha} + (k-1+\sigma)^{2-\alpha}\Big] \\ &\quad -\frac{1}{2}\Big[(k+1+\sigma)^{1-\alpha} - 2(k+\sigma)^{1-\alpha} + (k-1+\sigma)^{1-\alpha}\Big], \quad 1 \le k \le n-2, \\ c_{n-1}^{(n,\alpha)} &= \frac{1}{2}\Big[3(n-1+\sigma)^{1-\alpha} - (n-2+\sigma)^{1-\alpha}\Big] \\ &\quad -\frac{1}{2-\alpha}\Big[(n-1+\sigma)^{2-\alpha} - (n-2+\sigma)^{2-\alpha}\Big]. \end{aligned}$$

Lemma 2.1 ([6]). *Suppose $u(\cdot,t) \in C^3[0,T]$. Then it holds*

$$\begin{aligned}&\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i, t_{n-1+\sigma})\\ =&\sum_{r=0}^{m} \lambda_r \frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)} \sum_{k=0}^{n-1} c_k^{(n,\alpha_r)} [u(x_i, t_{n-k}) - u(x_i, t_{n-k-1})] + O(\tau^{3-\alpha_0}),\end{aligned}$$

where $0 \leq \alpha_m < \alpha_{m-1} < \cdots < \alpha_0 \leq 1$ and the constant σ satisfies $F(\sigma) = 0$.

Next the SOE approximation of the kernel function in the Caputo fractional derivative will be introduced.

Lemma 2.2 ([9, 16, 37]). *For any given $\alpha \in (0,1)$, tolerance error ε, cut-off time step size τ_0 and final time T, there is one positive integer $N_{exp}^{(\alpha)}$, some positive numbers $s_i^{(\alpha)} (i = 1, 2, \ldots, N_{exp}^{(\alpha)})$ and corresponding positive weights $w_i^{(\alpha)} (i = 1, 2, \ldots, N_{exp}^{(\alpha)})$ satisfing*

$$\left| t^{-\alpha} - \sum_{i=1}^{N_{exp}^{(\alpha)}} \omega_i^{(\alpha)} \mathrm{e}^{-s_i^{(\alpha)} t} \right| \leq \epsilon, \quad \forall\, t \in [\tau_0, T],$$

and the number of exponentials needed, $N_{exp}^{(\alpha)}$, is of the order

$$O\Bigg(\log \frac{1}{\epsilon} \Big(\log\log \frac{1}{\epsilon} + \log \frac{T}{\tau_0} \Big) + \log \frac{1}{\tau_0} \Big(\log\log \frac{1}{\epsilon} + \log \frac{1}{\tau_0} \Big) \Bigg).$$

In [9], the authors presented a fast numerical differentiation formula to approximate the multi-term Caputo derivative $\sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} f(t)$ at the point $t = t_{n-1+\sigma}$. By approximating the kernel function $(t_{n-1+\sigma} - s)^{-\alpha_r} (s \in (0, t_{n-1}))$ by SOE approximation and $f'(s) (s \in (t_{n-1}, t_{n-1+\sigma}))$ by linear interpolation, it can be obtained that

$$\sum_{r=0}^{m} \lambda_r {}_0^C D_t^{\alpha_r} f(t_{n-1+\sigma}) \approx \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \sum_{l=1}^{N_{exp}^{(\alpha_r)}} \omega_l^{(\alpha_r)} F_l^{(n-1,\alpha_r)} + c_0 [f(t_n) - f(t_{n-1})]$$

with

$$\begin{aligned}c_0 &= \frac{1}{\tau} \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \int_{t_{n-1}}^{t_{n-1+\sigma}} (t_{n-1+\sigma} - s)^{-\alpha_r} \, \mathrm{d}s = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(2-\alpha_r)} \sigma^{1-\alpha_r} \tau^{-\alpha_r},\\ F_l^{(n-1,\alpha_r)} &= \int_0^{t_{n-1}} f'(s) \mathrm{e}^{-s_l^{(\alpha_r)} (t_{n-1+\sigma} - s)} \, \mathrm{d}s, \quad l = 1, 2, \ldots, N_{exp}^{(\alpha_r)}.\end{aligned}$$

The term $F_l^{(n-1,\alpha_r)}$ can be evaluated by a recursive relation, that is,

$$F_l^{(n-1,\alpha_r)} = \mathrm{e}^{-s_l^{(\alpha_r)} \tau} F_l^{(n-2,\alpha_r)} + \int_{t_{n-2}}^{t_{n-1}} f'(s) \mathrm{e}^{-s_l^{(\alpha_r)} (t_{n-1+\sigma} - s)} \, \mathrm{d}s, \quad n \geqslant 2.$$

The function $f(s)$ in the second term of the above equality can be approximated by a quadratic interpolation as follows:

$$\begin{aligned}&\int_{t_{n-2}}^{t_{n-1}} f'(s)\mathrm{e}^{-s_l^{(\alpha_r)}(t_{n-1+\sigma}-s)}\mathrm{d}s\\ \approx\ & A_l^{(\alpha_r)}[f(t_{n-1})-f(t_{n-2})]+B_l^{(\alpha_r)}[f(t_n)-f(t_{n-1})],\end{aligned}$$

with

$$\begin{aligned}A_l^{(\alpha_r)}&=\int_0^1(\frac{3}{2}-s)\mathrm{e}^{-s_l^{(\alpha_r)}\tau(\sigma+1-s)}\mathrm{d}s>0,\\ B_l^{(\alpha_r)}&=\int_0^1(s-\frac{1}{2})\mathrm{e}^{-s_l^{(\alpha_r)}\tau(\sigma+1-s)}\mathrm{d}s>0.\end{aligned}$$

Thus the fast numerical differentiation formula ${}^{\mathcal{F}}Df(t_{n-1+\sigma})$ for multi-term Caputo derivative $\sum_{r=0}^{m}\lambda_r {}_0^C D_t^{\alpha_r} f(t)\mid_{t=t_{n-1+\sigma}}$ can be obtained as

$${}^{\mathcal{F}}Df(t_{n-1+\sigma})=\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\sum_{l=1}^{N_{exp}^{(\alpha_r)}}\omega_l^{(\alpha_r)}\hat{F}_l^{(n-1,\alpha_r)}+c_0[f(t_n)-f(t_{n-1})], \quad (2.5)$$

where $\hat{F}_l^{(n-1,\alpha_r)}$ can be evaluated by the following recursive relation

$$\begin{aligned}\hat{F}_l^{(n-1,\alpha_r)}=&\mathrm{e}^{-s_l^{(\alpha_r)}\tau}\hat{F}_l^{(n-2,\alpha_r)}+A_l^{(\alpha_r)}[f(t_{n-1})-f(t_{n-2})]\\&+B_l^{(\alpha_r)}[f(t_n)-f(t_{n-1})],\end{aligned}$$

with $\hat{F}_l^{(0,\alpha_r)}=0, \quad l=1,2,\ldots,N_{exp}^{(\alpha_r)}$.

Equality (2.5) can be equivalently rewritten as

$${}^{\mathcal{F}}Df(t_{n-1+\sigma})=\sum_{k=0}^{n-1}{}^{\mathcal{F}}c_k^{(n)}[f(t_{n-k})-f(t_{n-k-1})], \quad (2.6)$$

where ${}^{\mathcal{F}}c_0^{(1)}=c_0$ and for $n=2,3,\ldots,N$,

$${}^{\mathcal{F}}c_k^{(n)}=\begin{cases}\displaystyle\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}}\omega_l^{(\alpha_r)}B_l^{(\alpha_r)}+\frac{\tau^{-\alpha_r}}{1-\alpha_r}\sigma^{1-\alpha_r}\Big], & k=0,\\ \displaystyle\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\sum_{l=1}^{N_{exp}^{(\alpha_r)}}\omega_l^{(\alpha_r)}\Big[\mathrm{e}^{-(k-1)s_l^{(\alpha_r)}\tau}A_l^{(\alpha_r)}+\mathrm{e}^{-ks_l^{(\alpha_r)}\tau}B_l^{(\alpha_r)}\Big], & 1\leqslant k\leqslant n-2,\\ \displaystyle\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\sum_{l=1}^{N_{exp}^{(\alpha_r)}}\omega_l^{(\alpha_r)}\mathrm{e}^{-(n-2)s_l^{(\alpha_r)}\tau}A_l^{(\alpha_r)}, & k=n-1.\end{cases} \quad (2.7)$$

Lemma 2.3 ([9]). *For any function $f\in C^3[0,t_N]$, a sufficiently small ϵ and the constant σ satisfying $F(\sigma)=0$, it holds*

$$\Big|\sum_{r=0}^{m}\lambda_r\,{}_0^C D_t^{\alpha_r}f(t)|_{t=t_{n-1+\sigma}}-{}^{\mathcal{F}}Df(t_{n-1+\sigma})\Big|=O(\tau^{3-\alpha_0}+\epsilon), \quad n=1,2,\ldots,N.$$

For the finite difference approximation, the spatial mesh partition is essential. Let $h = L/M$ be the spatial step size with M a positive integer. Denote $x_i = ih(0 \le i \le M)$. Define $\Omega_h = \{x_i \mid 0 \le i \le M\}, \Omega_\tau = \{t_n \mid 0 \le n \le N\}$. For any mesh function $u = \{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ defined on $\Omega_h \times \Omega_\tau$, introduce the following operators:

$$\delta_x u_{i-\frac{1}{2}}^n = \frac{1}{h}(u_i^n - u_{i-1}^n), \quad \delta_x^2 u_i^n = \frac{1}{h}\big(\delta_x u_{i+\frac{1}{2}}^n - \delta_x u_{i-\frac{1}{2}}^n\big)$$

and the average operator

$$(\mathcal{H}u)_i^n = \begin{cases} \frac{2}{3}u_0^n + \frac{1}{3}u_1^n, \quad i = 0, \\ \frac{1}{12}(u_{i-1}^n + 10u_i^n + u_{i+1}^n), \quad 1 \le i \le M-1, \\ \frac{1}{3}u_{M-1}^n + \frac{2}{3}u_M^n, \quad i = M. \end{cases}$$

Lemma 2.4 ([8]). *Denote $\theta(s) = (1-s)^3[10 - 3(1-s)^2]$ and $\xi(s) = (1-s)^3[5 - 3(1-s)^2]$.*
(1) If function $g \in C^6[x_0, x_1]$, then it holds

$$\begin{aligned} &\Big[\frac{2}{3}g''(x_0) + \frac{1}{3}g''(x_1)\Big] - \frac{2}{h}\Big[\frac{g(x_1) - g(x_0)}{h} - g'(x_0)\Big] \\ =&\frac{h^2}{12}g^{(4)}(x_0) + \frac{7h^3}{180}g^{(5)}(x_0) + \frac{h^4}{180}\int_0^1 \theta(s)g^{(6)}(x_0 + sh)\mathrm{d}s. \end{aligned} \tag{2.8}$$

(2) If function $g \in C^6[x_{M-1}, x_M]$, then it holds

$$\begin{aligned} &\Big[\frac{1}{3}g''(x_{M-1}) + \frac{2}{3}g''(x_M)\Big] - \frac{2}{h}\Big[g'(x_M) - \frac{g(x_M) - g(x_{M-1})}{h}\Big] \\ =&\frac{h^2}{12}g^{(4)}(x_M) - \frac{7h^3}{180}g^{(5)}(x_M) + \frac{h^4}{180}\int_0^1 \theta(s)g^{(6)}(x_M - sh)\mathrm{d}s. \end{aligned} \tag{2.9}$$

(3) If function $g \in C^6[x_{i-1}, x_{i+1}]$, then it holds

$$\begin{aligned} &\frac{1}{12}[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})] \\ =\ &\frac{1}{h^2}[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})] \\ &+\frac{h^4}{360}\int_0^1 \xi(s)[g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh)]\mathrm{d}s, \quad 1 \le i \le M-1. \end{aligned} \tag{2.10}$$

Remark 2.1. These equalities can be easily obtained by the formula of Taylor expansion with integral remainder.

Lemma 2.5 ([38]). *If function $u \in C^3[t_{n-1}, t_n]$, σ is a constant and $0 < \sigma < 1$, it holds*

$$u(t_{n-1+\sigma}) = \sigma u(t_n) + (1-\sigma)u(t_{n-1}) + O(\tau^2).$$

For simplicity, for any mesh function $u = \{u^0, u^1, \cdots, u^N\}$ defined on Ω_τ, denote

$$u^{\sigma_n} = \sigma u(t_n) + (1-\sigma)u(t_{n-1}), \quad 1 \le n \le N$$

with the constant σ satisfying $F(\sigma) = 0$.

3. The derivation of the fast compact finite difference scheme

In this section, we construct a fast difference scheme for the problem (2.1)-(2.4) by the fast super-convergent approximation and the method of order reduction.

Let $v(x,t) = u_{xx}(x,t)$. Then Eqs. (2.1)–(2.4) can be transformed into

$$\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x,t) + v_{xx}(x,t) + qu(x,t) = f(x,t), \quad x \in (0,L), \ t \in (0,T], \tag{3.1}$$

$$v(x,t) = u_{xx}(x,t), \quad x \in (0,L), \quad t \in [0,T], \tag{3.2}$$

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad t \in (0,T], \tag{3.3}$$

$$u_x(0,t) = \gamma_1(t), \quad u_x(L,t) = \gamma_2(t), \quad t \in (0,T], \tag{3.4}$$

$$u(x,0) = \phi(x), \quad x \in [0,L]. \tag{3.5}$$

Suppose the exact solution $u \in C^{(8,3)}([0,L] \times [0,T])$. Define the grid functions

$$U_i^n = u(x_i,t_n), \quad V_i^n = v(x_i,t_n), \quad 0 \le i \le M, 0 \le n \le N;$$
$$f_i^{n-1+\sigma} = f(x_i,t_{n-1+\sigma}), \quad 0 \le i \le M, 1 \le n \le N.$$

Considering Eq. (3.1) at the point $(x_i, t_{n-1+\sigma})$, we have

$$\begin{aligned}&\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i,t_{n-1+\sigma}) + v_{xx}(x_i,t_{n-1+\sigma}) + qu(x_i,t_{n-1+\sigma}) \\ &= f(x_i,t_{n-1+\sigma}), \quad 0 \le i \le M, \ 1 \le n \le N.\end{aligned} \tag{3.6}$$

Acting the average operator $\mathcal{H}$ on both hand sides of (3.6), we get

$$\begin{aligned}&\mathcal{H}\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i,t_{n-1+\sigma}) + \mathcal{H}v_{xx}(x_i,t_{n-1+\sigma}) + q\mathcal{H}u(x_i,t_{n-1+\sigma}) \\ &= \mathcal{H}f(x_i,t_{n-1+\sigma}), \quad 1 \le i \le M-1, \ 1 \le n \le N.\end{aligned} \tag{3.7}$$

Applying the fast superconvergent approximation to the multi-term time-fractional derivatives in the above equation, by Lemma 2.3, it yields

$$\left\{\begin{aligned}&\sum_{r=0}^{m} \lambda_r \, {}_0^C D_t^{\alpha_r} u(x_i,t_{n-1+\sigma}) = \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} \omega_l^{(\alpha_r)} \hat{u}_{l,i}^{(n-1,\alpha_r)} \Big] \\ &+c_0\big(U_i^n - U_i^{n-1}\big) + O(\tau^{3-\alpha_0} + \epsilon), \quad 1 \leqslant i \leqslant M-1, \ 1 \leqslant n \leqslant N, &(3.8)\\ &\hat{u}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)}, \ 1 \leqslant i \leqslant M-1, &(3.9)\\ &\hat{u}_{l,i}^{(n-1,\alpha_r)} = \mathrm{e}^{-s_l^{(\alpha_r)}\tau} \hat{u}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)}(U_i^{n-1} - U_i^{n-2}) + B_l^{(\alpha_r)}(U_i^n - U_i^{n-1}), \\ &\qquad 1 \le l \le N_{exp}^{(\alpha_r)}, \ 1 \le i \le M-1, \ 2 \le n \le N. &(3.10)\end{aligned}\right.$$

Applying (3.8)–(3.10) into (3.7), by Lemma 2.4 and Lemma 2.5, we obtain

$$\begin{cases} \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} \omega_l^{(\alpha_r)} \mathcal{H} \hat{u}_{l,i}^{(n-1,\alpha_r)} \Big] + c_0 \mathcal{H}\big(U_i^n - U_i^{n-1}\big) + \delta_x^2 V_i^{\sigma_n} \\ +q\mathcal{H}U_i^{\sigma_n} = \mathcal{H} f_i^{n-1+\sigma} + R_i^{\sigma_n}, \quad 1 \leqslant i \leqslant M-1,\ 1 \leqslant n \leqslant N, & (3.11) \\ \hat{u}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \leqslant i \leqslant M-1, & (3.12) \\ \hat{u}_{l,i}^{(n-1,\alpha_r)} = \mathrm{e}^{-s_l^{(\alpha_r)}\tau} \hat{u}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)} (U_i^{n-1} - U_i^{n-2}) + B_l^{(\alpha_r)} (U_i^n - U_i^{n-1}), \\ \qquad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \le i \le M-1,\ 2 \le n \le N, & (3.13) \end{cases}$$

where there is a positive constant c_1, independent of h and τ, such that

$$|R_i^{\sigma_n}| \le c_1(\tau^2 + h^4 + \varepsilon), \quad 1 \le i \le M-1,\ 1 \le n \le N. \tag{3.14}$$

Next, considering Eq. (3.2) at the grid point (x_i, t_n) and performing the average operator $\mathcal{H}$ on both hand sides of the resultant equation, we obtain

$$\mathcal{H} v(x_i, t_n) = \mathcal{H} u_{xx}(x_i, t_n), \quad 0 \le i \le M,\ 0 \le n \le N. \tag{3.15}$$

According to Lemma 2.4, we get

$$\mathcal{H} V_i^n = \delta_x^2 U_i^n + S_i^n, \quad 1 \le i \le M-1,\ 0 \le n \le N \tag{3.16}$$

and there exists a positive constant c_2 such that

$$|S_i^n| \le c_2 h^4, \quad 1 \le i \le M-1,\ 0 \le n \le N. \tag{3.17}$$

When $i = 0$, Eq. (3.15) reads

$$\mathcal{H} v(0, t_n) = \mathcal{H} u_{xx}(0, t_n), \quad 0 \le n \le N. \tag{3.18}$$

Letting $x \to 0^+$ in Eq. (2.1), using the boundary condition (2.2), one can obtain

$$\frac{\partial^4 u(0,t)}{\partial x^4} = f(0,t) - \sum_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} g_1(t) - q g_1(t). \tag{3.19}$$

Meanwhile, differentiating the both hand sides of Eq. (2.1) with respect to x once and letting $x \to 0^+$, using the boundary condition (2.3), we obtain

$$\frac{\partial^5 u(0,t)}{\partial x^5} = f_x(0,t) - \sum_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} \gamma_1(t) - q \gamma_1(t). \tag{3.20}$$

Noticing Lemma 2.4 and substituting (3.19), (3.20) into (3.18), we get

$$\begin{aligned} \mathcal{H} V_0^n =& \frac{2}{h}\Big[\delta_x U_{\frac{1}{2}}^n - \gamma_1(t_n)\Big] + \frac{h^2}{12}\Big[f(0,t_n) - \sum_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} g_1(t_n) - q g_1(t_n)\Big] \\ &+ \frac{7h^3}{180}\Big[f_x(0,t_n) - \sum_{r=0}^{m} \lambda_r\, {}_0^C D_t^{\alpha_r} \gamma_1(t_n) - q\gamma_1(t_n)\Big] + S_0^n \\ :=& \frac{2}{h}\delta_x U_{\frac{1}{2}}^n + p(t_n) + S_0^n, \quad 0 \le n \le N, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} p(t) = & -\frac{2}{h}\gamma_1(t) + \frac{h^2}{12}\Big[f(0,t) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r} g_1(t) - q g_1(t)\Big] \\ & + \frac{7h^3}{180}\Big[f_x(0,t) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r}\gamma_1(t) - q\gamma_1(t)\Big] \end{aligned}$$

and there exists a positive constant c_3 such that

$$|S_0^n| \le c_3 h^4, \quad 0 \le n \le N. \tag{3.22}$$

We note that the similar result can be obtained on the right boundary as

$$\begin{aligned} \mathcal{H}V_M^n = & \frac{2}{h}\Big[\gamma_2(t_n) - \delta_x U_{M-\frac{1}{2}}^n\Big] + \frac{h^2}{12}\Big[f(x_M,t_n) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r} g_2(t_n) - q g_2(t_n)\Big] \\ & - \frac{7h^3}{180}\Big[f_x(x_M,t_n) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r}\gamma_2(t_n) - q\gamma_2(t_n)\Big] + S_M^n \\ := & -\frac{2}{h}\delta_x U_{M-\frac{1}{2}}^n + q(t_n) + S_M^n, \quad 0 \le n \le N, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} q(t) = & \frac{2}{h}\gamma_2(t) + \frac{h^2}{12}\Big[f(x_M,t) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r} g_2(t) - q g_2(t)\Big] \\ & - \frac{7h^3}{180}\Big[f_x(x_M,t) - \sum_{r=0}^{m}\lambda_r\, {}_0^C D_t^{\alpha_r}\gamma_2(t) - q\gamma_2(t)\Big] \end{aligned}$$

and there exists a positive constant c_4 such that

$$|S_M^n| \le c_4 h^4, \quad 0 \le n \le N. \tag{3.24}$$

Noticing the initial-boundary value conditions (3.3) and (3.5), we have

$$U_0^n = g_1(t_n), \quad U_M^n = g_2(t_n), \quad 1 \le n \le N, \tag{3.25}$$

$$U_i^0 = \phi(x_i), \quad 0 \le i \le M. \tag{3.26}$$

Omitting the small terms $R_i^{\sigma_n}, S_i^n, S_0^n, S_M^n$ in Eqs. (3.11), (3.16), (3.21), (3.23), respectively, and replacing the exact solution $\{U_i^n, V_i^n | 0 \le i \le M, 0 \le n \le N\}$ with its numerical one $\{u_i^n, v_i^n | 0 \le i \le M, 0 \le n \le N\}$, we construct a fast compact difference scheme for the problem (3.1)–(3.5) as follows:

$$\begin{aligned} & \sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,i}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(u_i^n - u_i^{n-1}) \\ & +\delta_x^2 v_i^{\sigma_n} + q\mathcal{H}u_i^{\sigma_n} = \mathcal{H}f_i^{n-1+\sigma}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \end{aligned} \tag{3.27}$$

$$\hat{u}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \leqslant i \leqslant M-1, \tag{3.28}$$

$$\hat{u}_{l,i}^{(n-1,\alpha_r)} = \mathrm{e}^{-s_l^{(\alpha_r)}\tau}\hat{u}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)}(u_i^{n-1} - u_i^{n-2}) + B_l^{(\alpha_r)}(u_i^n - u_i^{n-1}),$$

$$1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \le i \le M-1,\ 2 \le n \le N, \tag{3.29}$$

$$\mathcal{H}v_i^n = \delta_x^2 u_i^n, \quad 1 \le i \le M-1,\ 0 \le n \le N, \tag{3.30}$$

$$\mathcal{H}v_0^n = \frac{2}{h}\delta_x u_{\frac{1}{2}}^n + p(t_n), \quad 0 \le n \le N, \tag{3.31}$$

$$\mathcal{H}v_M^n = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^n + q(t_n), \quad 0 \le n \le N, \tag{3.32}$$

$$u_0^n = g_1(t_n), \quad u_M^n = g_2(t_n), \quad 1 \le n \le N, \tag{3.33}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M. \tag{3.34}$$

Theorem 3.1. *The difference scheme* (3.27)–(3.34) *is equivalent to*

$$\begin{aligned}
&\frac{19}{36}\Big\{\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,1}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(u_1^n - u_1^{n-1}) + q\mathcal{H}u_1^{\sigma_n}\Big\}\\
&+\frac{1}{18}\Big\{\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,2}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(u_2^n - u_2^{n-1}) + q\mathcal{H}u_2^{\sigma_n}\Big\}\\
&+\frac{2}{h^3}\delta_x u_{\frac{1}{2}}^{\sigma_n} - \frac{5}{3h^2}\delta_x^2 u_1^{\sigma_n} + \frac{2}{3h^2}\delta_x^2 u_2^{\sigma_n}\\
=\ &\frac{19}{36}\mathcal{H}f_1^{n-1+\sigma} + \frac{1}{18}\mathcal{H}f_2^{n-1+\sigma} - \frac{1}{h^2}p^{n-1+\sigma}, \quad 1 \le n \le N,
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}^2\hat{u}_{l,i}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}^2(u_i^n - u_i^{n-1}) + \delta_x^4 u_i^{\sigma_n} + q\mathcal{H}^2 u_i^{\sigma_n}\\
=\ &\mathcal{H}^2 f_i^{n-1+\sigma}, \quad 2 \le i \le M-2,\ 1 \le n \le N,
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
&\frac{19}{36}\Big\{\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,M-1}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(u_{M-1}^n - u_{M-1}^{n-1}) + q\mathcal{H}u_{M-1}^{\sigma_n}\Big\}\\
&+\frac{1}{18}\Big\{\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,M-1}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(u_{M-1}^n - u_{M-1}^{n-1}) + q\mathcal{H}u_{M-2}^{\sigma_n}\Big\}\\
&-\frac{2}{h^3}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} - \frac{5}{3h^2}\delta_x^2 u_{M-1}^{\sigma_n} + \frac{2}{3h^2}\delta_x^2 u_{M-2}^{\sigma_n}\\
=\ &\frac{19}{36}\mathcal{H}f_{M-1}^{n-1+\sigma} + \frac{1}{18}\mathcal{H}f_{M-2}^{n-1+\sigma} - \frac{1}{h^2}q^{n-1+\sigma}, \quad 1 \le n \le N,
\end{aligned} \tag{3.37}$$

$$\hat{u}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \leqslant i \leqslant M-1, \tag{3.38}$$

$$\begin{aligned}
\hat{u}_{l,i}^{(n-1,\alpha_r)} &= \mathrm{e}^{-s_l^{(\alpha_r)}\tau}\hat{u}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)}(u_i^{n-1} - u_i^{n-2}) + B_l^{(\alpha_r)}(u_i^n - u_i^{n-1}),\\
&1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \le i \le M-1,\ 2 \le n \le N,
\end{aligned} \tag{3.39}$$

$$u_0^n = g_1(t_n), \quad u_M^n = g_2(t_n), \quad 1 \le n \le N, \tag{3.40}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M \tag{3.41}$$

and

$$\mathcal{H}v_i^n = \delta_x^2 u_i^n, \quad 1 \le i \le M-1, \quad 0 \le n \le N, \tag{3.42}$$

$$\mathcal{H}v_0^n = \frac{2}{h}\delta_x u_{\frac{1}{2}}^n + p(t_n), \quad 0 \le n \le N, \tag{3.43}$$

$$\mathcal{H}v_M^n = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^n + q(t_n), \quad 0 \le n \le N, \tag{3.44}$$

where $p^{n-1+\sigma} = \sigma p(t_n) + (1-\sigma)p(t_{n-1})$ *and* $q^{n-1+\sigma}$ *is similarly defined.*

Proof. By (3.42), we can get

$$\mathcal{H}v_i^{\sigma_n} = \delta_x^2 u_i^{\sigma_n}, \quad 1 \le i \le M-1, \quad 1 \le n \le N. \tag{3.45}$$

In a similar way, we have

$$\mathcal{H}v_0^{\sigma_n} = \frac{2}{h}\delta_x u_{\frac{1}{2}}^{\sigma_n} + p^{n-1+\sigma}, \quad 1 \le n \le N, \tag{3.46}$$

$$\mathcal{H}v_M^{\sigma_n} = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} + q^{n-1+\sigma}, \quad 1 \le n \le N. \tag{3.47}$$

Performing the average operator $\mathcal{H}$ and the operator δ_x^2 on both hand sides of (3.27) and (3.45), respectively, we have

$$\begin{aligned}&\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}^2\hat{u}_{l,i}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}^2(u_i^n - u_i^{n-1}) + \mathcal{H}\delta_x^2 v_i^{\sigma_n}\\ &+q\mathcal{H}^2 u_i^{\sigma_n} = \mathcal{H}^2 f_i^{n-1+\sigma}, \quad 2 \le i \le M-2,\ 1 \le n \le N, \end{aligned}\tag{3.48}$$

$$\mathcal{H}\delta_x^2 v_i^{\sigma_n} = \delta_x^4 u_i^{\sigma_n}, \quad 2 \le i \le M-2,\ 1 \le n \le N. \tag{3.49}$$

Substituting (3.49) into (3.48), we obtain

$$\begin{aligned}&\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}^2\hat{u}_{l,i}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}^2(u_i^n - u_i^{n-1}) + \delta_x^4 u_i^{\sigma_n}\\ &+q\mathcal{H}^2 u_i^{\sigma_n} = \mathcal{H}^2 f_i^{n-1+\sigma}, \quad 2 \le i \le M-2,\ 1 \le n \le N,\end{aligned}\tag{3.50}$$

which is exactly (3.36).

For $i = 0$, rewriting the left hand side of (3.46) gives

$$\mathcal{H}v_0^{\sigma_n} = h^2(b_1\delta_x^2 v_1^{\sigma_n} + b_2\delta_x^2 v_2^{\sigma_n}) + b_3\mathcal{H}v_1^{\sigma_n} + b_4\mathcal{H}v_2^{\sigma_n}, \quad 1 \le n \le N. \tag{3.51}$$

Comparing the coefficients on both hand sides, we get the following system of linear equations in the unknown $\{b_1, b_2, b_3, b_4\}$:

$$\begin{cases} b_1 + \dfrac{b_3}{12} = \dfrac{2}{3},\\ -2b_1 + b_2 + \dfrac{5}{6}b_3 + \dfrac{1}{12}b_4 = \dfrac{1}{3},\\ b_1 - 2b_2 + \dfrac{b_3}{12} + \dfrac{5}{6}b_4 = 0,\\ b_2 + \dfrac{b_4}{12} = 0,\end{cases}$$

which implies $b_1 = \frac{19}{36}, b_2 = \frac{1}{18}, b_3 = \frac{5}{3}$ and $b_4 = -\frac{2}{3}$. Applying the results of (3.27) and (3.45) with $i = 1, 2$ into (3.51), noticing (3.46), it yields

$$\frac{2}{h}\delta_x u_{\frac{1}{2}}^{\sigma_n} + p^{n-1+\sigma} = h^2\Bigg\{\frac{19}{36}\Big[-\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big(\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{u}_{l,1}^{(n-1,\alpha_r)}\Big)$$

$$
\begin{aligned}
&- c_0\mathcal{H}(u_1^n - u_1^{n-1}) - q\mathcal{H}u_1^{\sigma_n} + \mathcal{H}f_1^{n-1+\sigma}\Big] \\
&+ \frac{1}{18}\Big[-\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big(\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)} \mathcal{H}\hat{u}_{l,2}^{(n-1,\alpha_r)} \Big) \\
&- c_0\mathcal{H}(u_2^n - u_2^{n-1}) - q\mathcal{H}u_2^{\sigma_n} + \mathcal{H}f_2^{n-1+\sigma}\Big]\Big\} \\
&+ \frac{5}{3}\delta_x^2 u_1^{\sigma_n} - \frac{2}{3}\delta_x^2 u_2^{\sigma_n}, \quad 1 \le n \le N. \qquad (3.52)
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
&- \frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} + q^{n-1+\sigma} \\
=&h^2\Big\{ \frac{19}{36}\Big[-\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big(\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)} \mathcal{H}\hat{u}_{l,M-1}^{(n-1,\alpha_r)} \Big) - c_0\mathcal{H}(u_{M-1}^n - u_{M-1}^{n-1}) \\
&- q\mathcal{H}u_{M-1}^{\sigma_n} + \mathcal{H}f_{M-1}^{n-1+\sigma}\Big] + \frac{1}{18}\Big[-\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big(\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)} \mathcal{H}\hat{u}_{l,M-2}^{(n-1,\alpha_r)} \Big) \\
&- c_0\mathcal{H}(u_{M-2}^n - u_{M-2}^{n-1}) - q\mathcal{H}u_{M-2}^{\sigma_n} + \mathcal{H}f_{M-2}^{n-1+\sigma}\Big]\Big\} + \frac{5}{3}\delta_x^2 u_{M-1}^{\sigma_n} - \frac{2}{3}\delta_x^2 u_{M-2}^{\sigma_n}, \\
&1 \le n \le N. \qquad (3.53)
\end{aligned}
$$

Rearranging the above two equalities, we can acquire (3.35) and (3.37) easily. The proof ends. $\square$

Based on Theorem 3.1, we can calculate the numerical solution $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ directly from the difference scheme (3.35)–(3.41) for the problem (2.1)-(2.4), whereas, the theoretical analysis which follows will still start from the difference scheme (3.27)-(3.34), which is more convenient for the analysis.

4. Stability and convergence analysis of the fast difference scheme

In this section, the stability and convergence of the fast compact difference scheme (3.27)–(3.34) will be studied. To this end, we introduce some lemmas, which will play a vital role in the subsequent analysis.

Lemma 4.1 ([8]). *Let v be a grid function defined on Ω_h, then it holds*

$$
h\sum_{i=1}^{M-1} (\mathcal{H}v_i)^2 \ge \frac{5}{12}h\sum_{i=1}^{M-1}(v_i)^2 - \frac{5}{72}h(v_0^2 + v_M^2). \qquad (4.1)
$$

Lemma 4.2 ([8]). *For any grid functions u and v defined on Ω_h, if $u_0 = u_M = 0$, we have*

$$
h\sum_{i=1}^{M-1} \delta_x^2 v_i \cdot \mathcal{H}u_i - h\sum_{i=1}^{M-1} \delta_x^2 u_i \cdot \mathcal{H}v_i = (\delta_x u_{\frac{1}{2}})v_0 - (\delta_x u_{M-\frac{1}{2}})v_M. \qquad (4.2)
$$

Lemma 4.3 ([9]). *For a sufficiently small ϵ, the coefficient $\{{}^{\mathcal{F}}c_k^{(n)} \mid 0 \le k \le n-1\}$ defined by (2.7) satisfies*

$$ {}^{\mathcal{F}}c_1^{(n)} > {}^{\mathcal{F}}c_2^{(n)} > \cdots > {}^{\mathcal{F}}c_{n-1}^{(n)} \ge C^{\mathcal{F}} > 0, \tag{4.3} $$

$$ (2\sigma - 1)\ {}^{\mathcal{F}}c_0^{(n)} - \sigma\ {}^{\mathcal{F}}c_1^{(n)} > 0, \tag{4.4} $$

$$ {}^{\mathcal{F}}c_0^{(n)} > {}^{\mathcal{F}}c_1^{(n)}, \tag{4.5} $$

where the constant $C^{\mathcal{F}}$ can be taken as

$$ \min\Big\{ \sum_{r=0}^{m} \frac{\lambda_r \sigma}{\Gamma(2-\alpha_r)} t_\sigma^{-\alpha_r}, \quad \sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} \omega_l^{(\alpha_r)} e^{-Ts_l^{(\alpha_r)}} A_l^{(\alpha_r)} \Big] \Big\}. $$

Lemma 4.4 ([2,6]). *Define $\mathring{\mathcal{U}}_h = \{u | u = (u_0, u_1, \dots, u_M), u_0 = u_M = 0\}$. Let $(\cdot,\cdot)$ be an inner product defined in $\mathring{\mathcal{U}}_h$ with the induced norm $\|\cdot\|$. Suppose $\{{}^{\mathcal{F}}c_k^{(n)} | 0 \le k \le n-1, n \ge 1\}$ satisfies the conditions (4.3)-(4.5). For any grid functions $u^0, u^1, \dots, u^n, \dots \in \mathring{\mathcal{U}}_h$, it holds that*

$$ \begin{aligned} &\sum_{k=0}^{n-1} {}^{\mathcal{F}}c_k^{(n)} (u^{n-k} - u^{n-k-1}, u^{\sigma_n}) \\ \ge &\frac{1}{2} \sum_{k=0}^{n-1} {}^{\mathcal{F}}c_k^{(n)} (\|u^{n-k}\|^2 - \|u^{n-k-1}\|^2), \quad n = 1, 2, \dots. \end{aligned} $$

The following theorem presents a priori estimation on the difference scheme (3.27)–(3.34).

Theorem 4.1 (A priori estimate). *Suppose that $\{u_i^n, v_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ is the solution of the following difference scheme*

$$ \begin{aligned} &\sum_{r=0}^{m} \frac{\lambda_r}{\Gamma(1-\alpha_r)} \Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)} \mathcal{H} \hat{u}_{l,i}^{(n-1,\alpha_r)} \Big] + c_0 \mathcal{H}(u_i^n - u_i^{n-1}) + \delta_x^2 v_i^{\sigma_n} + q \mathcal{H} u_i^{\sigma_n} \\ &= P_i^{n-1+\sigma}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \end{aligned} \tag{4.6} $$

$$ \hat{u}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \leqslant i \leqslant M-1, \tag{4.7} $$

$$ \begin{aligned} \hat{u}_{l,i}^{(n-1,\alpha_r)} &= \mathrm{e}^{-s_l^{(\alpha_r)}\tau} \hat{u}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)}(u_i^{n-1} - u_i^{n-2}) + B_l^{(\alpha_r)}(u_i^n - u_i^{n-1}), \\ &\qquad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \le i \le M-1,\ 2 \le n \le N, \end{aligned} \tag{4.8} $$

$$ \mathcal{H} v_i^n = \delta_x^2 u_i^n + Q_i^n, \quad 1 \le i \le M-1,\ 0 \le n \le N, \tag{4.9} $$

$$ \mathcal{H} v_0^n = \frac{2}{h} \delta_x u_{\frac{1}{2}}^n + Q_0^n, \quad 0 \le n \le N, \tag{4.10} $$

$$ \mathcal{H} v_M^n = -\frac{2}{h} \delta_x u_{M-\frac{1}{2}}^n + Q_M^n, \quad 0 \le n \le N, \tag{4.11} $$

$$ u_i^0 = \omega_i, \quad 0 \le i \le M, \tag{4.12} $$

$$ u_0^n = 0, \quad u_M^n = 0, \quad 1 \le n \le N. \tag{4.13} $$

Then we have

$$ h \sum_{i=1}^{M-1} (u_i^n)^2 \le \frac{12}{5} \Big\{ h \sum_{i=1}^{M-1} (\mathcal{H} u_i^0)^2 + \frac{1}{\mathcal{C}^{\mathcal{F}}} \max_{1 \le l \le n} \Big[\frac{1}{2q} h \sum_{i=1}^{M-1} (P_i^{l-1+\sigma})^2 $$

$$+ 2h\sum_{i=1}^{M-1}(Q_i^{l-1+\sigma})^2 + h(Q_0^{l-1+\sigma})^2 + h(Q_M^{l-1+\sigma})^2\Big]\Big\}, \quad 1 \le n \le N, \tag{4.14}$$

where the constant $\mathcal{C}^{\mathcal{F}}$ is defined in Lemma 4.3.

Proof. By (4.9)–(4.11), one can know that

$$\mathcal{H}v_i^{\sigma_n} = \delta_x^2 u_i^{\sigma_n} + Q_i^{n-1+\sigma}, \quad 1 \le i \le M-1, \quad 1 \le n \le N, \tag{4.15}$$

$$\mathcal{H}v_0^{\sigma_n} = \frac{2}{h}\delta_x u_{\frac{1}{2}}^{\sigma_n} + Q_0^{n-1+\sigma}, \quad 1 \le n \le N, \tag{4.16}$$

$$\mathcal{H}v_M^{\sigma_n} = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^{\sigma_n} + Q_M^{n-1+\sigma}, \quad 1 \le n \le N. \tag{4.17}$$

Substituting (4.7) and (4.8) into (4.6), we can achieve

$$\begin{aligned}&\sum_{k=0}^{n-1} {}^{\mathcal{F}}c_k^{(n)}\mathcal{H}(u_i^{n-k} - u_i^{n-k-1}) + \delta_x^2 v_i^{\sigma_n} + q\mathcal{H}u_i^{\sigma_n} = P_i^{n-1+\sigma},\\ &1 \le i \le M-1, \quad 1 \le n \le N.\end{aligned} \tag{4.18}$$

Multiplying the both hand sides of (4.18) and (4.15) by $h\mathcal{H}u_i^{\sigma_n}, h\mathcal{H}v_i^{\sigma_n}$, respectively, and summing up for i from 1 to $M-1$, and multiplying the both hand sides of (4.16) and (4.17) by $\frac{h}{2}v_0^{\sigma_n}$, $\frac{h}{2}v_M^{\sigma_n}$, respectively, adding the results, we get

$$\begin{aligned}&h\sum_{i=1}^{M-1}\sum_{k=0}^{n-1} {}^{\mathcal{F}}c_k^{(n)}(\mathcal{H}u_i^{n-k} - \mathcal{H}u_i^{n-k-1})\mathcal{H}u_i^{\sigma_n} + h\sum_{i=1}^{M-1}\delta_x^2 v_i^{\sigma_n}\cdot\mathcal{H}u_i^{\sigma_n}\\ &+ qh\sum_{i=1}^{M-1}(\mathcal{H}u_i^{\sigma_n})^2 + h\sum_{i=1}^{M-1}(\mathcal{H}v_i^{\sigma_n})^2 + \frac{h}{2}\mathcal{H}v_0^{\sigma_n}\cdot v_0^{\sigma_n} + \frac{h}{2}\mathcal{H}v_M^{\sigma_n}\cdot v_M^{\sigma_n}\\ =&\, h\sum_{i=1}^{M-1}P_i^{n-1+\sigma}\cdot\mathcal{H}u_i^{\sigma_n} + h\sum_{i=1}^{M-1}\delta_x^2 u_i^{\sigma_n}\cdot\mathcal{H}v_i^{\sigma_n} + h\sum_{i=1}^{M-1}Q_i^{n-1+\sigma}\cdot\mathcal{H}v_i^{\sigma_n}\\ &+ \delta_x u_{\frac{1}{2}}^{\sigma_n}\cdot v_0^{\sigma_n} + \frac{h}{2}Q_0^{n-1+\sigma}\cdot v_0^{\sigma_n} - \delta_x u_{M-\frac{1}{2}}^{\sigma_n}\cdot v_M^{\sigma_n} + \frac{h}{2}Q_M^{n-1+\sigma}\cdot v_M^{\sigma_n}, \quad 1 \le n \le N.\end{aligned} \tag{4.19}$$

Using the Cauchy-Schwarz inequality, we obtain

$$h\sum_{i=1}^{M-1}P_i^{n-1+\sigma}\cdot\mathcal{H}u_i^{\sigma_n} \le qh\sum_{i=1}^{M-1}(\mathcal{H}u_i^{\sigma_n})^2 + \frac{1}{4q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2,\ 1 \le n \le N, \tag{4.20}$$

$$h\sum_{i=1}^{M-1}Q_i^{n-1+\sigma}\cdot\mathcal{H}v_i^{\sigma_n} \le \frac{h}{4}\sum_{i=1}^{M-1}(\mathcal{H}v_i^{\sigma_n})^2 + h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2,\ 1 \le n \le N. \tag{4.21}$$

Noticing Lemmas 4.1-4.3 and substituting (4.20), (4.21) into (4.19), we get

$$\frac{1}{2}\sum_{k=0}^{n-1} {}^{\mathcal{F}}c_k^{(n)}\Big[h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k})^2 - h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k-1})^2\Big] + \frac{h}{2}\mathcal{H}v_0^{\sigma_n}\cdot v_0^{\sigma_n}$$

$$
\begin{aligned}
&+\frac{h}{2}\mathcal{H}v_M^{\sigma_n}\cdot v_M^{\sigma_n}+\frac{5}{16}h\sum_{i=1}^{M-1}(v_i^{\sigma_n})^2\\
&\le \frac{1}{4q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2+h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2+\frac{h}{2}Q_0^{n-1+\sigma}\cdot v_0^{\sigma_n}\\
&+\frac{h}{2}Q_M^{n-1+\sigma}\cdot v_M^{\sigma_n}+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2,\quad 1\le n\le N. \qquad (4.22)
\end{aligned}
$$

For some terms related with the boundaries, one has

$$
\begin{aligned}
&-\frac{h}{2}(\mathcal{H}v_0^{\sigma_n}-Q_0^{n-1+\sigma})v_0^{\sigma_n}-\frac{h}{2}(\mathcal{H}v_M^{\sigma_n}-Q_M^{n-1+\sigma})v_M^{\sigma_n}+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
&=-\frac{h}{2}\Big(\frac{2}{3}v_0^{\sigma_n}+\frac{1}{3}v_1^{\sigma_n}-Q_0^{n-1+\sigma}\Big)v_0^{\sigma_n}-\frac{h}{2}\Big(\frac{2}{3}v_M^{\sigma_n}+\frac{1}{3}v_{M-1}^{\sigma_n}-Q_M^{n-1+\sigma}\Big)v_M^{\sigma_n}\\
&\quad+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
&\le -\frac{h}{3}(v_0^{\sigma_n})^2+\frac{h}{6}\Big[\frac{1}{4}(v_0^{\sigma_n})^2+(v_1^{\sigma_n})^2\Big]+\frac{h}{2}\Big[\frac{1}{4}(v_0^{\sigma_n})^2+(Q_0^{n-1+\sigma})^2\Big]-\frac{h}{3}(v_M^{\sigma_n})^2\\
&\quad+\frac{h}{6}\Big[\frac{1}{4}(v_M^{\sigma_n})^2+(v_{M-1}^{\sigma_n})^2\Big]+\frac{h}{2}\Big[\frac{1}{4}(v_M^{\sigma_n})^2+(Q_M^{n-1+\sigma})^2\Big]+\frac{5h}{96}(v_0^{\sigma_n})^2+\frac{5h}{96}(v_M^{\sigma_n})^2\\
&=-\frac{11h}{96}(v_0^{\sigma_n})^2+\frac{h}{6}(v_1^{\sigma_n})^2-\frac{11h}{96}(v_M^{\sigma_n})^2+\frac{h}{6}(v_{M-1}^{\sigma_n})^2\\
&\quad+\frac{h}{2}(Q_0^{n-1+\sigma})^2+\frac{h}{2}(Q_M^{n-1+\sigma})^2\\
&\le \frac{h}{6}[(v_1^{\sigma_n})^2+(v_{M-1}^{\sigma_n})^2]+\frac{h}{2}[(Q_0^{n-1+\sigma})^2+(Q_M^{n-1+\sigma})^2],\quad 1\le n\le N. \qquad (4.23)
\end{aligned}
$$

Noticing

$$
\frac{h}{6}[(v_1^{\sigma_n})^2+(v_{M-1}^{\sigma_n})^2]\le\frac{5}{16}h\sum_{i=1}^{M-1}(v_i^{\sigma_n})^2,
$$

the substitution of (4.23) into (4.22) produces

$$
\begin{aligned}
&\sum_{k=0}^{n-1}{}^{\mathcal{F}}c_k^{(n)}\Big[h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k})^2-h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k-1})^2\Big]\\
&\le\frac{1}{2q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2+2h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2+h[(Q_0^{n-1+\sigma})^2+(Q_M^{n-1+\sigma})^2],
\end{aligned}
$$

that is

$$
\begin{aligned}
&{}^{\mathcal{F}}c_0^{(n)}h\sum_{i=1}^{M-1}(\mathcal{H}u_i^n)^2\le\sum_{k=1}^{n-1}({}^{\mathcal{F}}c_{k-1}^{(n)}-{}^{\mathcal{F}}c_k^{(n)})h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k})^2\\
&+{}^{\mathcal{F}}c_{n-1}^{(n)}h\sum_{i=1}^{M-1}(\mathcal{H}u_i^0)^2+\frac{1}{2q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2+2h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2\\
&+h[(Q_0^{n-1+\sigma})^2+(Q_M^{n-1+\sigma})^2],\quad 1\le n\le N.
\end{aligned}
$$

Noticing [9] ${}^{\mathcal{F}}c_{n-1}^{(n)}\ge C^{\mathcal{F}}>0$, further one can get

$$
{}^{\mathcal{F}}c_0^{(n)}h\sum_{i=1}^{M-1}(\mathcal{H}u_i^n)^2
$$

$$\begin{aligned}
&\le \sum_{k=1}^{n-1}({}^{\mathcal{F}}c_{k-1}^{(n)} - {}^{\mathcal{F}}c_k^{(n)})h\sum_{i=1}^{M-1}(\mathcal{H}u_i^{n-k})^2 \\
&\quad + {}^{\mathcal{F}}c_{n-1}^{(n)}\Bigg\{h\sum_{i=1}^{M-1}(\mathcal{H}u_i^0)^2 + \frac{1}{C^{\mathcal{F}}}\Bigg[\frac{1}{2q}h\sum_{i=1}^{M-1}(P_i^{n-1+\sigma})^2 + 2h\sum_{i=1}^{M-1}(Q_i^{n-1+\sigma})^2 \\
&\quad + h(Q_0^{n-1+\sigma})^2 + h(Q_M^{n-1+\sigma})^2\Bigg]\Bigg\}, \quad 1 \le n \le N.
\end{aligned}$$

The induction method applied to the above inequality will lead to

$$\begin{aligned}
h\sum_{i=1}^{M-1}(\mathcal{H}u_i^n)^2 \le & h\sum_{i=1}^{M-1}(\mathcal{H}u_i^0)^2 + \frac{1}{C^{\mathcal{F}}}\max_{1\le l\le n}\Bigg[\frac{1}{2q}h\sum_{i=1}^{M-1}(P_i^{l-1+\sigma})^2 \\
& + 2h\sum_{i=1}^{M-1}(Q_i^{l-1+\sigma})^2 + h(Q_0^{l-1+\sigma})^2 + h(Q_M^{l-1+\sigma})^2\Bigg], \quad 1 \le n \le N.
\end{aligned}$$

Noticing (4.13) and Lemma 4.1, further one can reach the desired inequality. The proof ends. □

Theorem 3.1 and Theorem 4.1 reveal the stability of the difference scheme (3.35)–(3.41) with respect to the initial value and the right hand term $f(x,t)$.

Theorem 4.2 (Stability)**.** *The difference scheme* (3.35)–(3.41) *is unconditionally stable with respect to the right hand term f and the initial value.*

In what follows, the convergence of the difference scheme (3.27)–(3.34) will be concerned.

Theorem 4.3 (Convergence)**.** *Suppose that* $\{U_i^n, V_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ *and* $\{u_i^n, v_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ *are the solution of the problem* (3.1)–(3.5) *and the difference scheme* (3.27)–(3.34)*, respectively. Let*

$$e_i^n = U_i^n - u_i^n, \quad \epsilon_i^n = V_i^n - v_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N,$$

then there exists a positive constant C *independent of* h *and* τ*, such that*

$$\sqrt{h\sum_{i=1}^{M-1}(e_i^n)^2} \le C(\tau^2 + h^4 + \epsilon), \quad 1 \le n \le N,$$

where

$$C^2 = \frac{12L}{5\mathcal{C}^{\mathcal{F}}}\left(\frac{1}{2q}c_1^2 + 2c_2^2 + c_3^2 + c_4^2\right).$$

Proof. Subtracting Eqs. (3.27)–(3.34) from (3.11)–(3.13), (3.16), (3.21), (3.23) and (3.25)–(3.26), respectively, we have the system of error equations

$$\begin{aligned}
&\sum_{r=0}^{m}\frac{\lambda_r}{\Gamma(1-\alpha_r)}\Big[\sum_{l=1}^{N_{exp}^{(\alpha_r)}} w_l^{(\alpha_r)}\mathcal{H}\hat{e}_{l,i}^{(n-1,\alpha_r)}\Big] + c_0\mathcal{H}(e_i^n - e_i^{n-1}) \\
&+\delta_x^2\epsilon_i^{\sigma_n} + q\mathcal{H}e_i^{\sigma_n} = R_i^{\sigma_n}, \quad 1 \le i \le M-1, \quad 1 \le n \le N, \qquad (4.24) \\
&\hat{e}_{l,i}^{(0,\alpha_r)} = 0, \quad 1 \le l \le N_{exp}^{(\alpha_r)},\ 1 \leqslant i \leqslant M-1, \qquad (4.25)
\end{aligned}$$

$$\hat{e}_{l,i}^{(n-1,\alpha_r)} = \mathrm{e}^{-s_l^{(\alpha_r)}\tau}\hat{e}_{l,i}^{(n-2,\alpha_r)} + A_l^{(\alpha_r)}(e_i^{n-1} - e_i^{n-2}) + B_l^{(\alpha_r)}(e_i^n - e_i^{n-1}),$$
$$1 \le l \le N_{exp}^{(\alpha_r)}, 1 \le i \le M-1, 2 \le n \le N, \tag{4.26}$$
$$\mathcal{H}\epsilon_i^n = \delta_x^2 e_i^n + S_i^n, \quad 1 \le i \le M-1, \quad 0 \le n \le N, \tag{4.27}$$
$$\mathcal{H}\epsilon_0^n = \frac{2}{h}\delta_x e_{\frac{1}{2}}^n + S_0^n, \quad 0 \le n \le N, \tag{4.28}$$
$$\mathcal{H}\epsilon_M^n = -\frac{2}{h}\delta_x e_{M-\frac{1}{2}}^n + S_M^n, \quad 0 \le n \le N, \tag{4.29}$$
$$e_i^0 = 0, \quad 0 \le i \le M, \tag{4.30}$$
$$e_0^n = 0, \quad e_M^n = 0, \quad 1 \le n \le N. \tag{4.31}$$

The application of Theorem 4.1 into (4.24)–(4.31) produces

$$h\sum_{i=1}^{M-1}(e_i^n)^2 \le \frac{12}{5}\Bigg\{h\sum_{i=1}^{M-1}(\mathcal{H}e_i^0)^2 + \frac{1}{\mathcal{C}^{\mathcal{F}}}\max_{1\le l\le n}\Bigg[\frac{1}{2q}h\sum_{i=1}^{M-1}(R_i^{\sigma_l})^2 + 2h\sum_{i=1}^{M-1}(S_i^{\sigma_l})^2 + h(S_0^{\sigma_l})^2 + h(S_M^{\sigma_l})^2\Bigg]\Bigg\}, \quad 1 \le n \le N.$$

Noticing (3.14), (3.17), (3.22) and (3.24), together with (4.30)–(4.31), further it follows

$$\begin{aligned} h\sum_{i=1}^{M-1}(e_i^n)^2 &\le \frac{12}{5\mathcal{C}^{\mathcal{F}}}\left[\frac{L}{2q}c_1^2(\tau^2+h^4+\epsilon)^2 + (2Lc_2^2 + hc_3^2 + hc_4^2)(h^4)^2\right] \\ &\le \frac{12L}{5\mathcal{C}^{\mathcal{F}}}\left(\frac{1}{2q}c_1^2 + 2c_2^2 + c_3^2 + c_4^2\right)(\tau^2+h^4+\epsilon)^2, \quad 1 \le n \le N. \end{aligned}$$

The proof ends. □

5. Numerical examples

In this section, two numerical examples will be implemented to illustrate the correctness of theoretical analysis and the validity of the fast difference scheme (FDS) proposed in current work. In addition, we aim to compare the direct difference scheme (DDS) without the acceleration in time direction with the current fast difference scheme (FDS) , which shows that the FDS can reduce the CPU time greatly.

Applying Lemma 2.1 to handle the time-fractional derivatives and the same techniques to treat the space derivatives, similar to the derivation of FDS, the DDS scheme for the problem (2.1)–(2.4) has been obtained as follows [8]:

$$\sum_{r=0}^{m}\lambda_r\frac{\tau^{-\alpha_r}}{\Gamma(2-\alpha_r)}\sum_{k=0}^{n-1}c_k^{(n,\alpha_r)}\mathcal{H}(u_i^{n-k} - u_i^{n-k-1}) + \delta_x^2 v_i^{\sigma_n} + q\mathcal{H}u_i^{\sigma_n} = \mathcal{H}f_i^{n-1+\sigma}, \quad 1 \le i \le M-1,\ 1 \le n \le N, \tag{5.1}$$
$$\mathcal{H}v_i^n = \delta_x^2 u_i^n, \quad 1 \le i \le M-1,\ 0 \le n \le N, \tag{5.2}$$
$$\mathcal{H}v_0^n = \frac{2}{h}\delta_x u_{\frac{1}{2}}^n + p(t_n), \quad 0 \le n \le N, \tag{5.3}$$
$$\mathcal{H}v_M^n = -\frac{2}{h}\delta_x u_{M-\frac{1}{2}}^n + q(t_n), \quad 0 \le n \le N, \tag{5.4}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{5.5}$$
$$u_0^n = g_1(t_n), \quad u_M^n = g_2(t_n), \quad 1 \le n \le N, \tag{5.6}$$

with the global convergence order $O(\tau^2 + h^4)$.

Denote

$$\mathrm{err}(h,\tau) = \max_{\substack{0\leqslant i\leqslant M\\0\leqslant n\leqslant N}} |u(x_i,t_n) - u_i^n|, \quad \mathrm{order}_t = \log_2 \frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h,\tau/2)},$$
$$\mathrm{order}_x = \log_2 \frac{\mathrm{err}(h,\tau)}{\mathrm{err}(h/2,\tau)}.$$

Example 5.1. In (2.1)–(2.4), take $L = 1$, $T = 1$, $m = 2$, $q = 1$, $f(x,t) = \left[24 \sum_{r=0}^{m} \lambda_r \frac{t^{4-\alpha_r}}{\Gamma(5-\alpha_r)} + \pi^4 t^4 + t^4\right] \sin(\pi x)$, $\phi(x) = 0$, $g_1(t) = 0$, $g_2(t) = 0$, $\gamma_1(t) = \pi t^4$, $\gamma_2(t) = -\pi t^4$.

The exact solution of this example is $u(x,t) = t^4 \sin(\pi x)$.

Taking the parameter $(\lambda_0, \lambda_1, \lambda_2) = (1,2,3)$ and varying values of $(\alpha_0, \alpha_1, \alpha_2)$, the numerical examples are calculated using the FDS (3.35)–(3.41) with the tolerance error $\epsilon = 1e-12$. We present the corresponding results using the DDS (5.1)–(5.6) for comparison.

On the one hand, we examine the numerical accuracy in time. Fixing the spatial step size $h = \frac{1}{100}$, the maximum errors and convergence orders are shown in Table 1, from which, one can read off the second-order convergence of both schemes in time.

On the other hand, the numerical accuracies of the difference scheme (3.35)–(3.41) and the scheme (5.1)–(5.6) in space are tested. Fix the temporal step size $\tau = \frac{1}{10000}$. Table 2 presents the maximum errors and convergence orders with the different spatial step sizes, from which, the fourth-order convergence of both schemes in space is verified.

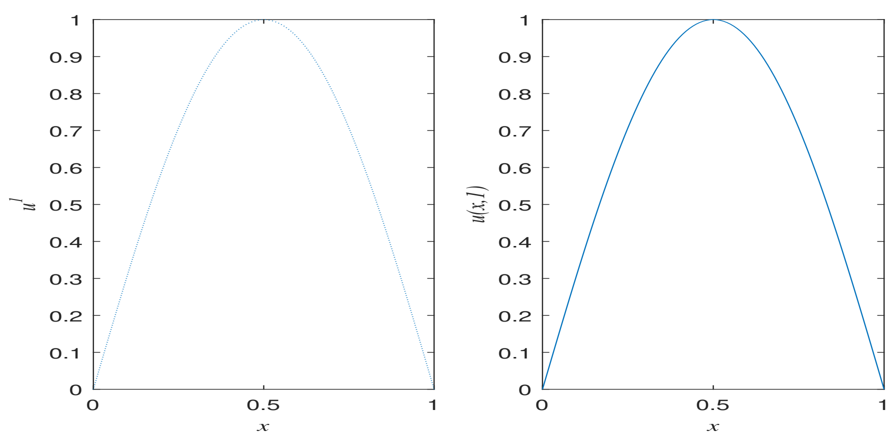

Figure 1. Numerical solution and exact solution plots for Example 5.1 with $t = 1$, $M = 100$, $N = 160$, $(\alpha_0, \alpha_1, \alpha_2) = (1/3, 1/4, 1/5)$, $(\lambda_0, \lambda_1, \lambda_2) = (1,2,3)$ (*Left*: Numerical solution; *Right*: Exact solution)

Table 1. Numerical errors and convergence orders of the DDS (5.1)–(5.6) and the FDS (3.35)–(3.41) in time for solving Example 5.1 ($M = 100$).

		DDS (5.1) − (5.6)		FDS (3.35) − (3.41)	
$(\alpha_0, \alpha_1, \alpha_2)$	τ	err(h, τ)	order$_t$	err(h, τ)	order$_t$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/10	1.324210e − 3	1.96	1.324210e − 3	1.96
	1/20	3.412309e − 4	1.97	3.412308e − 4	1.97
	1/40	8.682708e − 5	1.98	8.682677e − 5	1.98
	1/80	2.196241e − 5	1.99	2.196265e − 5	1.99
	1/160	5.539102e − 6	—	5.539134e − 6	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/10	2.335417e − 3	1.95	2.335417e − 3	1.95
	1/20	6.027641e − 4	1.97	6, 027640e − 4	1.97
	1/40	1.540140e − 4	1.98	1.540138e − 4	1.98
	1/80	3.915502e − 5	1.98	3.915514e − 5	1.98
	1/160	9.925806e − 6	—	9.925680e − 6	—
$(1, \frac{1}{2}, \frac{1}{4})$	1/10	3.015353e − 3	1.95	3.015353e − 3	1.95
	1/20	7.797358e − 4	1.98	7.797359e − 4	1.98
	1/40	1.981352e − 4	1.99	1.981352e − 4	1.99
	1/80	4.990732e − 5	2.00	4.990741e − 5	2.00
	1/160	1.251736e − 5	—	1.251730e − 5	—

Table 2. Numerical errors and convergence orders of the DDS (5.1)–(5.6) and the FDS (3.35)–(3.41) in space for solving Example 5.1 ($N = 10000$).

		DDS (5.1) − (5.6)		FDS (3.35) − (3.41)	
$(\alpha_0, \alpha_1, \alpha_2)$	h	err(h, τ)	order$_x$	err(h, τ)	order$_x$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/4	6.067967e − 4	3.90	6.067967e − 4	3.90
	1/8	4.063221e − 5	3.97	4.063221e − 5	3.97
	1/16	2.588908e − 6	4.01	2.588908e − 6	4.01
	1/32	1.612457e − 7	—	1.612436e − 7	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/4	6.031255e − 4	3.90	6.031255e − 4	3.90
	1/8	4.038819e − 5	3.97	4.038819e − 5	3.97
	1/16	2.572191e − 6	4.02	2.572190e − 6	4.02
	1/32	1.590621e − 7	—	1.590627e − 7	—
$(1, \frac{1}{2}, \frac{1}{4})$	1/4	6.020751e − 4	3.90	6.020951e − 4	3.90
	1/8	4.031953e − 5	3.97	4.031953e − 5	3.97
	1/16	2.567315e − 6	4.02	2.567315e − 6	4.02
	1/32	1.582629e − 7	—	1.582661e − 7	—

Example 5.2. In (2.1)–(2.4), take $L = 1$, $T = 1$, $m = 2$, $q = 1$, $f(x,t) = [24\sum_{r=0}^{m}\lambda_r\frac{t^{4-\alpha_r}}{\Gamma(5-\alpha_r)} + \pi^4t^4 + t^4]\sin(\pi x) + 2\cos x$, $\phi(x) = \cos x$, $g_1(t) = 1$, $g_2(t) = \cos 1$, $\gamma_1(t) = \pi t^4$, $\gamma_2(t) = -\pi t^4 - \sin 1$.

The exact solution for this example is $u(x,t) = t^4\sin(\pi x) + \cos x$.

Table 3 lists the maximum errors and the temporal convergence orders when the spatial step size $h = 1/100$ and τ is taken as 1/10, 1/20, 1/40, 1/80 and 1/160, respectively, where the parameter $(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$. From this table, we can see that, as what is expected, the convergence order in time of the fast difference scheme (3.35)–(3.41) and the direct scheme (5.1)–(5.6) is both two. Besides, in order to examine the convergence order in space, we choose a sufficiently small

Table 3. Numerical errors and convergence orders of the DDS (5.1)–(5.6) and the FDS (3.35)–(3.41) in time for solving Example 5.2 ($M = 100$).

		DDS (5.1) − (5.6)		FDS (3.35) − (3.41)	
$(\alpha_0, \alpha_1, \alpha_2)$	τ	err(h, τ)	order$_t$	err(h, τ)	order$_t$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/10	1.324210e − 3	1.96	1.324210e − 3	1.96
	1/20	3.412310e − 4	1.97	3.412310e − 4	1.97
	1/40	8.682729e − 5	1.98	8.682667e − 5	1.98
	1/80	2.196228e − 5	1.99	2.196276e − 5	1.99
	1/160	5.539019e − 6	—	5.539074e − 6	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/10	2.335417e − 3	1.95	2.335417e − 3	1.95
	1/20	6.027641e − 4	1.97	6.027639e − 4	1.97
	1/40	1.540141e − 4	1.98	1.540136e − 4	1.98
	1/80	3.915488e − 5	1.98	3.915119e − 5	1.98
	1/160	9.925790e − 6	—	9.925521e − 6	—
$(1, \frac{1}{2}, \frac{1}{4})$	1/10	3.015353e − 3	1.95	3.015353e − 3	1.95
	1/20	7.797358e − 4	1.98	7.797360e − 4	1.98
	1/40	1.981352e − 4	1.99	1.981353e − 4	1.99
	1/80	2.286938e − 5	2.00	2.286940e − 5	2.00
	1/160	1.251734e − 5	—	1.251723e − 5	—

Table 4. Numerical errors and convergence orders of the DDS (5.1)–(5.6) and the FDS (3.35)–(3.41) in space for solving Example 5.2 ($N = 10000$).

		DDS (5.1) − (5.6)		FDS (3.35) − (3.41)	
$(\alpha_0, \alpha_1, \alpha_2)$	h	err(h, τ)	order$_x$	err(h, τ)	order$_x$
$(\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$	1/8	4.061643e − 5	3.97	4.061643e − 5	3.97
	1/16	2.588559e − 6	4.00	2.588559e − 6	4.00
	1/32	1.612435e − 7	—	1.612394e − 7	—
$(\frac{2}{3}, \frac{1}{2}, \frac{1}{3})$	1/8	4.037236e − 5	3.97	4.037236e − 5	3.97
	1/16	2.571840e − 6	4.02	2.571840e − 6	4.02
	1/32	1.590599e − 7	—	1.590612e − 7	—
$(1, \frac{1}{2}, \frac{1}{4})$	1/8	4.030370e − 5	3.97	4.030370e − 5	3.97
	1/16	2.566965e − 6	4.02	2.566965e − 6	4.02
	1/32	1.582586e − 7	—	1.582649e − 7	—

temporal step size $\tau = 1/10000$ and h is taken as 1/8, 1/16 and 1/32, respectively. Table 4 presents the computational results which are in accord with the theoretical results we proved in last section. From Table 4, we can find that the fourth-order convergence of both schemes in space can be numerically achieved.

Table 5 lists the maximum errors and the CPU time of the difference schemes with $\tau = \frac{1}{200000}$, $(\alpha_0, \alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$ and $(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$. The CPU time for both schemes is shown in Table 5 which verifies the efficiency of the FDS (3.35)–(3.41). From Table 5, we find that the computational cost of the fast difference scheme (3.35)–(3.41) can be reduced significantly. The larger N is, more obvious the advantage of the FDS (3.35)–(3.41) over the DDS (5.1)–(5.6) is.

For visualization, Figure 1 and Figure 2 illustrate the numerical and exact solution plots when $t = 1$, $M = 100$, $N = 160$, $(\alpha_0, \alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{4}, \frac{1}{5})$, $(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$ indicating an excellent closeness between the numerical and exact ones.

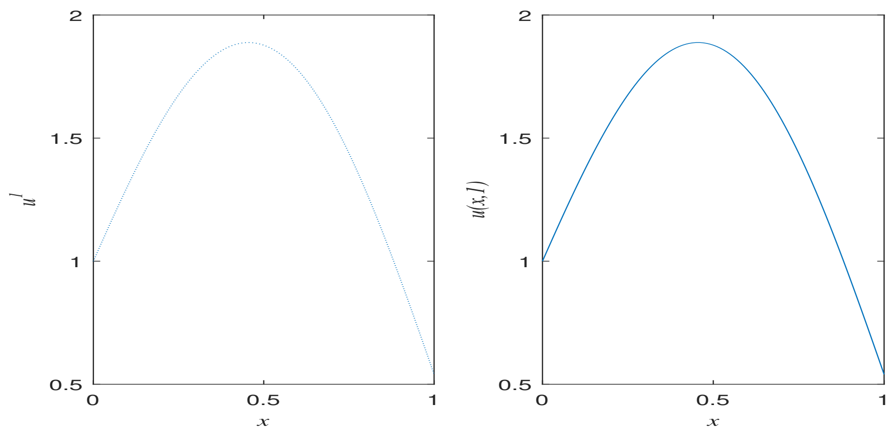

Figure 2. Numerical solution and exact solution plots for Example 5.2 with $t = 1$, $M = 100$, $N = 160$, $(\alpha_0, \alpha_1, \alpha_2) = (1/3, 1/4, 1/5)$, $(\lambda_0, \lambda_1, \lambda_2) = (1, 2, 3)$ (*Left*: Numerical solution; *Right*: Exact solution)

Table 5. The comparsion of the DDS (5.1)–(5.6) and the FDS (3.35)–(3.41) when $N = 200000$

		DDS (5.1) − (5.6)		FDS (3.35) − (3.41)	
	h	err(h,τ)	CPU time	err(h,τ)	CPU time
Example 5.1	1/8	4.063368e − 5	9.46h	4.032271e − 5	34.80s
	1/16	2.590384e − 6	11.19h	2.570491e − 6	37.65s
	1/32	1.627221e − 7	15.42h	1.614391e − 7	45.42s
Example 5.2	1/8	4.037505e − 5	9.90h	4.061789e − 5	34.95s
	1/16	2.574522e − 6	12.12h	2.590021e − 6	39.31s
	1/32	1.617457e − 7	16.67h	1.627071e − 7	49.63s

Therefore, we conclude that the proposed scheme is efficient for the considered problem.

Remark 5.1. Both the numerical results and theoretical analyses show that the expected accuracy of the proposed fast difference scheme (3.35)–(3.41) can be attained if the solution has enough regularity. If no so, the scheme is still valid but with a possible polluted accuracy.

6. Conclusion

In this paper, a fast compact difference scheme for the fourth-order time multi-term fractional sub-diffusion equations with the first Dirichlet boundary is derived by the SOE approximation and the method of order reduction, then some basic properties of this difference scheme are investigated. The unconditional convergence and stability of the scheme are proved by the discrete energy method and the convergence accuracy is second-order in time and fourth-order in space. In order to show the efficiency and advantages of the proposed fast scheme in current work, we compare it with the direct scheme without acceleration in time. The numerical examples reveal that the CPU time of the fast difference scheme is markedly reduced compared with the direct difference scheme, especially for the case with the large value of temporal levels.

Acknowledgements

The authors would like to express their gratitude to the anonymous referees for so many valuable comments and suggestions to greatly improve this work.

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