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TRANSMISSION DYNAMICS OF STOCHASTIC SVIR INFLUENZA MODELS WITH MEDIA COVERAGE

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Abstract This paper focuses on the dynamical behaviors of two stochastic SVIR models with media coverage. The first system is based on system perturbation. It is shown that the transmission dynamics can be classified by a critical value R_0^s. If $R_0^s < 1$, the disease will die out. $R_0^s > 1$ implies that the disease will persist. Furthermore, the system has an ergodic stationary distribution if $R_0^s > 1$. The second system is based on transmission parameter perturbation. Sufficient conditions for persistence and extinction are derived. Finally, theoretical results and numerical simulations show the effect of media coverage and environmental white noise.

Keywords SVIR epidemic model, media coverage, stationary distribution, extinction and persistence.

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1. Introduction

Vaccination and media coverage are two important strategies for controlling and preventing diseases [4, 6, 7, 15, 16, 19]. J. M. Tchuenche et.al [21] established the following SIR influenza model with vaccination and media coverage to reflect the transmission dynamics

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = \Lambda + \omega V - (\theta+\mu)S - \left(\beta_1 - \beta_2 \frac{I}{m+I}\right) SI + \sigma R, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \left(\beta_1 - \beta_2 \frac{I}{m+I}\right) SI + \left(\beta_1 - \beta_3 \frac{I}{m+I}\right)(1-\gamma)VI - (\alpha+\mu+\lambda)I, \\ \frac{\mathrm{d}V}{\mathrm{d}t} = \theta S - (\mu+\omega)V - \left(\beta_1 - \beta_3 \frac{I}{m+I}\right)(1-\gamma)VI, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \lambda I - (\mu+\sigma)R. \end{cases} \tag{1.1}$$

Here $S(t)$, $I(t)$, $V(t)$ and $R(t)$ denote the densities of susceptible, infected, vaccinated and recovered individuals, respectively. All the parameters in the model are positive and defined in Table 1. Generally speaking, the immunity acquired by natural infection is more robust and lasts longer than that induced by a vaccine [1], so we assume $\sigma \leq \omega$. Saturated incidence functions $\beta_2 \frac{I}{m+I}$ and $\beta_3 \frac{I}{m+I}$ measure the

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Table 1. Description of the variables and associated parameters

Parameters	Description
Λ	recruitment rate
θ	vaccination rate
μ	natural death rate
β_1	disease transmission rate
ω	vaccine immune decline rate
γ	vaccine efficacy
α	disease-related death rate
λ	recovery rate of infected individuals
σ	natural infection immune decline rate

effect of reduction of the contact rate through reporting by media, respectively. We further assume that $\beta_1 \geq \beta_2$ and $\beta_1 \geq \beta_3$, which ensure that infection rates remain non-negative. From [21] it follows that $R_0 = \frac{\beta_1\Lambda(\mu+\omega)+\beta_1(1-\gamma)\theta\Lambda}{\mu(\alpha+\mu+\lambda)(\theta+\mu+\omega)}$ is a threshold parameter.

In the real world, environmental variations have a critical influence on biomathematical models [2,13,14]. Therefore it is necessary and important to study the transmission dynamics of infectious disease affected by stochastic perturbation. In order to obtain a stochastic system, many scholars use Brownian motion as a stochastic perturbation factor to add to the deterministic model [5, 9, 11, 17, 22]. Generally speaking, all the parameters involved in the epidemic systems exhibit random fluctuation to a greater or lesser extent. In the real situation, the natural death rate μ usually fluctuate around some average values due to the environmental white noise, which should be seen as random variables $\tilde{\mu}$. Therefore, by the well-known central limit theorem, in $[t, t+dt)$,

$$-\tilde{\mu}\mathrm{d}t = -\mu\mathrm{d}t + \delta_i \mathrm{d}B_i(t),$$

here $dB_i(t) = B_i(t+dt) - B_i(t)$ is the increment of a standard Brownian motion, $i = 1, 2, 3, 4$. It is easy to see that in $[t, t+dt)$, $-\tilde{\mu}\mathrm{d}t$ is normally distributed with $\mathbb{E}(-\tilde{\mu}\mathrm{d}t) = -\mu dt$ and variance $Var(-\tilde{\mu}\mathrm{d}t) = \delta_i^2 \mathrm{d}t$. The variances tend to zero as $dt \to 0$. This is a reasonable way of introducing stochastic white noise into population systems. Therefore, replace $-\mu dt$ in (1) with $-\mu\mathrm{d}t + \delta_i \mathrm{d}B_i(t)$ $(i = 1, 2, 3, 4)$, respectively, then model (1) becomes the following stochastic model

$$\begin{cases} \mathrm{d}S = \left(\Lambda + \omega V - (\theta+\mu)S - \left(\beta_1 - \beta_2 \dfrac{I}{m+I}\right) SI + \sigma R\right)\mathrm{d}t + \delta_1 S \mathrm{d}B_1(t), \\ \mathrm{d}I = \left(\left(\beta_1 - \beta_2 \dfrac{I}{m+I}\right) SI + \left(\beta_1 - \beta_3 \dfrac{I}{m+I}\right)(1-\gamma)VI - (\alpha+\mu+\lambda)I\right)\mathrm{d}t \\ \qquad + \delta_2 I \mathrm{d}B_2(t), \\ \mathrm{d}V = \left(\theta S - (\mu+\omega)V - \left(\beta_1 - \beta_3 \dfrac{I}{m+I}\right)(1-\gamma)VI\right)\mathrm{d}t + \delta_3 V \mathrm{d}B_3(t), \\ \mathrm{d}R = (\lambda I - (\mu+\sigma)R)\,\mathrm{d}t + \delta_4 R \mathrm{d}B_4(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, 2, 3, 4)$ are independent Brownian motions and δ_i^2 $(i = 1, 2, 3, 4)$ are their intensities. We will mainly give the threshold for the extinction and persistence of the disease, and also study the stationary distribution of stochastic

model (1.2). On the other hand, the disease transmission coefficient β_1 is a key parameter. If β_1 is perturbed by white noise, that is $\beta_1 \to \beta_1 + \delta\dot{B}(t)$, then system (1.1) becomes

$$\begin{cases} \mathrm{d}S = \left(\Lambda + \omega V - (\theta+\mu)S - \left(\beta_1 - \beta_2\dfrac{I}{m+I}\right)SI + \sigma R\right)\mathrm{d}t - \delta SI\mathrm{d}B(t), \\ \mathrm{d}I = \left[\left(\beta_1 - \beta_2\dfrac{I}{m+I}\right)SI + \left(\beta_1 - \beta_3\dfrac{I}{m+I}\right)(1-\gamma)VI - (\alpha+\mu+\lambda)I\right]\mathrm{d}t \\ \qquad + \delta SI\mathrm{d}B(t) + \delta(1-\gamma)VI\mathrm{d}B(t), \\ \mathrm{d}V = \left(\theta S - (\mu+\omega)V - \left(\beta_1 - \beta_3\dfrac{I}{m+I}\right)(1-\gamma)VI\right)\mathrm{d}t - \delta(1-\gamma)VI\mathrm{d}B(t), \\ \mathrm{d}R = (\lambda I - (\mu+\sigma)R)\,\mathrm{d}t, \end{cases} \tag{1.3}$$

where $B(t)$ is a standard Brownian motion, and δ^2 denotes the intensity of white noise. We aim to obtain conditions for persistence and extinction of infectious disease for model (1.3).

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a σ-field filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We denote $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_+^{4,o} = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_i > 0,\ i = 1,2,3,4\}$. If $f(t)$ is an integrable function on $[0,\infty)$, define $\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)\mathrm{d}s$. If $f(t)$ is a bounded function on $[0,\infty)$, define $\check{f} = \sup_{t\in[0,\infty)} f(t)$. In general, let $x(t)$ be a d-dimensional stochastic process on $t \geq 0$ presented as the following stochastic differential equation

$$dx(t) = f(t)dt + g(t)dB(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R}^d)$, $g \in \mathcal{L}^2(\mathbb{R}_+, \mathbb{R}^{d\times m})$ and B(t)=$\{\left(B_t^1, B_t^2, \cdots, B_t^m\right)^T\}_{t\geq 0}$ is an m-dimentional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$. For any function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$, define the differential operator $\mathcal{L}V(x(t),t)$ as follows:

$$\mathcal{L}V(x(t),t) = V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}trace(g^T(t)V_{xx}(x(t),t)g(t))$$

where

$$V_t = \frac{\partial U}{\partial t},\ V_x = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right),\ V_{xx} = \begin{pmatrix} \frac{\partial^2 U}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_d} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 V}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_d \partial x_d} \end{pmatrix}.$$

The rest of this paper is organized as follows. In section 2, we will analyze the transmission dynamics of stochastic model (1.2) and make numerical simulations to support theoretical results. In section 3, we will study the persistence and extinction of disease of stochastic model (1.3) and employe numerical simulations to support theoretical results.

2. Dynamics of stochastic influenza model (1.2)

2.1. The properties of the global solution

Firstly, we give the result of existence and uniqueness of globally positive solution for model (1.2).

Theorem 2.1. *For any initial value* $(S(0), I(0), V(0), R(0)) \in \mathbb{R}^{4,o}_+$, *system (1.2) has a unique positive solution* $(S(t), I(t), V(t), R(t)) \in \mathbb{R}^{4,o}_+$ *for any* $t \geq 0$ *almost surely.*

This theorem can be proved by using a similar argument as that in the proof of Theorem 3 given by [20]. Here we only define the following C^2-function $\bar{V} : \mathbb{R}^{4,o}_+ \to \mathbb{R}_+$:

$$\bar{V}(S, I, V, R) = (S - a - a\ln(S/a)) + (I - 1 - \ln I) + (V - b - b\ln(V/b)) + (R - 1 - \ln R),$$

where $a = \alpha/\beta_1$, $b = \mu/\beta_1(1-\gamma)$. Applying Itô's formula, we have

$$\begin{aligned}
\mathcal{L}\bar{V} =& \left(1 - \frac{a}{S}\right)\left(\Lambda + \omega V - (\theta + \mu)S - \left(\beta_1 - \beta_2\frac{I}{m+I}\right)SI + \sigma R\right) + \frac{a\delta_1^2}{2} \\
&+ \left(1 - \frac{1}{I}\right)\left[\left(\beta_1 - \beta_2\frac{I}{m+I}\right)SI + \left(\beta_1 - \beta_3\frac{I}{m+I}\right)(1-\gamma)VI\right. \\
&\left. - (\alpha + \mu + \lambda)I\right] + \frac{\delta_2^2}{2} + \left(1 - \frac{b}{V}\right)\left[\theta S - (\mu + \omega)V - \left(\beta_1 - \beta_3\frac{I}{m+I}\right)(1\right. \\
&\left. - \gamma)VI\right] + \frac{b\delta_3^2}{2} + \left(1 - \frac{1}{R}\right)(\lambda I - (\mu + \sigma)R) + \frac{\delta_4^2}{2} \\
\leq& \Lambda - (\alpha + \mu)I + a\beta_1 I + b\beta_1(1-\gamma)I + a\left(\theta + \mu + \frac{\delta_1^2}{2}\right) \\
&+ \left(\alpha + \mu + \lambda + \frac{\delta_2^2}{2}\right) + b\left(\mu + \omega + \frac{\delta_3^2}{2}\right) + \left(\mu + \sigma + \frac{\delta_4^2}{2}\right) \\
=& \Lambda + a\left(\theta + \mu + \frac{\delta_1^2}{2}\right) + \left(\alpha + \mu + \lambda + \frac{\delta_2^2}{2}\right) + b\left(\mu + \omega + \frac{\delta_3^2}{2}\right) + \left(\mu + \sigma + \frac{\delta_4^2}{2}\right) \\
:=& K,
\end{aligned}$$

where K is a positive constant. The remainder of the proof can refer to [20].

According to the results of [23], the solution $(S(t), I(t), V(t), R(t))$ of model (1.2) has the following properties.

Lemma 2.1. *For any initial value* $(S(0), I(0), V(0), R(0)) \in \mathbb{R}^{4,o}_+$, *the solution* $(S(t), I(t), V(t), R(t))$ *of model (1.2) satisfies*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{V(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{R(t)}{t} = 0, \quad a.s. \tag{2.1}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\ln S(t)}{t} \leq 0, \quad \limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq 0, \\
&\limsup_{t\to\infty}\frac{\ln V(t)}{t} \leq 0, \quad \limsup_{t\to\infty}\frac{\ln R(t)}{t} \leq 0, \quad a.s.
\end{aligned} \tag{2.2}$$

Moreover, if $\mu > \max_{k=1,2,3,4} \frac{\delta_k^2}{2}$, *then*

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t S(r)\mathrm{d}B_1(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t I(r)\mathrm{d}B_2(r)}{t} = 0, \\ &\lim_{t\to\infty} \frac{\int_0^t V(r)\mathrm{d}B_3(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t R(r)\mathrm{d}B_4(r)}{t} = 0, \quad a.s. \end{aligned} \tag{2.3}$$

2.2. Threshold for extinction and persistence of the disease

In this subsection, we will study the extinction and persistence of disease $I(t)$ in model (1.2). Denote

$$R_0^s = \frac{\beta_1\Lambda(\mu+\omega) + \beta_1(1-\gamma)\theta\Lambda}{\mu(\alpha+\mu+\lambda+\delta_2^2/2)(\theta+\mu+\omega)}.$$

The following theorem gives the critical value for the extinction and persistence.

Theorem 2.2. *Assume that* $\mu > \max_{k=1,2,3,4} \frac{\delta_k^2}{2}$. *For any initial value* $(S(0), I(0), V(0), R(0)) \in \mathbb{R}_+^{4,o}$, *the following claims hold.*
(i). *If* $R_0^s < 1$, *we have*

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le (\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s - 1) < 0 \quad a.s.,$$

and the disease will die out.
(ii). *If* $R_0^s > 1$, *then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)\mathrm{d}s \ge \frac{(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1)}{(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)} \quad a.s.,$$

where $a_1 = \frac{\beta_1(\mu+\omega)+\beta_1(1-\gamma)\theta}{\mu(\theta+\mu+\omega)}$, $a_2 = \frac{\beta_1\omega+\beta_1(1-\gamma)(\theta+\mu)}{\mu(\theta+\mu+\omega)}$. *This implies the disease will spread.*

Proof. **(i)**. Integrating model (1.2) from 0 to t and and dividing by t, we obtain

$$\begin{aligned} \frac{S(t)-S(0)}{t} =& \lambda + \omega\langle V\rangle_t - (\theta+\mu)\langle S\rangle_t - \frac{1}{t}\int_0^t \left(\beta_1 - \beta_2\frac{I}{m+I}\right) SI\mathrm{d}t + \sigma\langle R\rangle_t \\ &+ \frac{\delta_1}{t}\int_0^t S\mathrm{d}B_1(t), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{I(t)-I(0)}{t} =& \frac{1}{t}\int_0^t \left(\beta_1 - \beta_2\frac{I}{m+I}\right) SI\mathrm{d}t + \frac{1}{t}\int_0^t \left(\beta_1 - \beta_3\frac{I}{m+I}\right)(1-\gamma)VI\mathrm{d}t \\ &- (\alpha+\mu+\gamma)\langle I\rangle_t + \frac{\delta_2}{t}\int_0^t I\mathrm{d}B_2(t), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{V(t)-V(0)}{t} =& \theta\langle S\rangle_t - (\mu+\omega)\langle V\rangle_t - \frac{1}{t}\int_0^t \left(\beta_1 - \beta_3\frac{I}{m+I}\right)(1-\gamma)VI\mathrm{d}t \\ &+ \frac{\delta_3}{t}\int_0^t V\mathrm{d}B_3(t), \end{aligned} \tag{2.6}$$

$$\frac{R(t)-R(0)}{t}=\lambda\langle I\rangle_t-(\mu+\sigma)\langle R\rangle_t+\frac{\delta_4}{t}\int_0^t R\mathrm{d}B_4(t). \tag{2.7}$$

Compute that $(\mu+\delta)(\mu+\omega)\times(2.4)+(\mu+\omega)\sigma\times(2.5)+(\mu+\sigma)\omega\times(2.6)+(\mu+\omega)\sigma\times(2.7)$, we can get

$$\begin{aligned}
&(\mu+\delta)(\mu+\omega)\frac{S(t)-S(0)}{t}+(\mu+\sigma)\omega\frac{V(t)-V(0)}{t}\\
&+(\mu+\omega)\sigma\frac{I(t)-I(0)}{t}+(\mu+\omega)\sigma\frac{R(t)-R(0)}{t}\\
=&(\mu+\delta)(\mu+\omega)\Lambda-\mu(\mu+\delta)(\theta+\mu+\omega)\langle S\rangle_t-(\mu+\omega)(\alpha+\mu)\langle I\rangle_t\\
&-\frac{(\mu+\omega)\mu}{t}\int_0^t\left(\beta_1-\beta_2\frac{I}{m+I}\right)SI\mathrm{d}t-\frac{\mu(\omega-\sigma)}{t}\int_0^t\left(\beta_1-\beta_3\frac{I}{m+I}\right)(1\\
&-\gamma)VI\mathrm{d}t+\frac{(\mu+\sigma)(\mu+\omega)}{t}\int_0^t\delta_1S\mathrm{d}B_1(t)+\frac{(\mu+\sigma)\omega}{t}\int_0^t\delta_2I\mathrm{d}B_2(t)\\
&+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_3V\mathrm{d}B_3(t)+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_4R\mathrm{d}B_4(t)\\
\le&(\mu+\delta)(\mu+\omega)\Lambda-\mu(\mu+\delta)(\theta+\mu+\omega)\langle S\rangle_t-(\mu+\omega)(\alpha+\mu)\langle I\rangle_t\\
&+\frac{(\mu+\sigma)(\mu+\omega)}{t}\int_0^t\delta_1S\mathrm{d}B_1(t)+\frac{(\mu+\sigma)\omega}{t}\int_0^t\delta_2I\mathrm{d}B_2(t)\\
&+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_3V\mathrm{d}B_3(t)+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_4R\mathrm{d}B_4(t).
\end{aligned}$$

Due to $\sigma\le\omega$, we compute that

$$\langle S\rangle_t\le\bar{S}-\frac{(\mu+\omega)(\alpha+\mu)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t+\varphi_1(t), \tag{2.8}$$

where $\bar{S}=\frac{(\mu+\omega)\Lambda}{\mu(\theta+\mu+\omega)}$, and $\varphi_1(t)$ is defined by

$$\begin{aligned}
\varphi_1(t)=&\frac{1}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\left[\frac{(\mu+\sigma)(\mu+\omega)}{t}\int_0^t\delta_1S\mathrm{d}B_1(t)\right.\\
&+\frac{(\mu+\sigma)\omega}{t}\int_0^t\delta_2I\mathrm{d}B_2(t)+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_3V\mathrm{d}B_3(t)\\
&+\frac{(\mu+\omega)\sigma}{t}\int_0^t\delta_4R\mathrm{d}B_4(t)-(\mu+\delta)(\mu+\omega)\frac{S(t)-S(0)}{t}\\
&\left.-(\mu+\sigma)\omega\frac{V(t)-V(0)}{t}-(\mu+\omega)\sigma\frac{I(t)-I(0)}{t}-(\mu+\omega)\sigma\frac{R(t)-R(0)}{t}\right].
\end{aligned}$$

Note Lemma 2.1, so

$$\lim_{t\to\infty}\varphi_1(t)=0,\quad a.s.$$

On the other hand, from the third equation of model (1.2), we get

$$\frac{V(t)-V(0)}{t}\le\theta\langle S\rangle_t-(\mu+\omega)\langle V\rangle_t+\frac{1}{t}\int_0^t\delta_3V\mathrm{d}B_3(t).$$

This, combining (2.8), leads to

$$\begin{aligned}\langle V\rangle_t \leq & \frac{\theta}{\mu+\omega}\bar{S}-\frac{\theta(\alpha+\mu)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t+\varphi_2(t)\\ =&\bar{V}-\frac{\theta(\alpha+\mu)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t+\varphi_2(t),\end{aligned}\tag{2.9}$$

where $\bar{V}=\frac{\Lambda\theta}{\mu(\theta+\mu+\omega)}$, $\varphi_2(t)=\frac{\theta}{\mu+\omega}\varphi_1(t)-\frac{1}{\mu+\omega}\frac{V(t)-V(0)}{t}+\frac{1}{(\mu+\omega)t}\int_0^t\delta_3 V\mathrm{d}B_3(t)$, and $\varphi_2(t)\to\ 0$ as $t\to\infty$.

Applying Itô's formula, one has

$$\begin{aligned}\mathrm{d}\ln I(t)=&\left[\left(\beta_1-\beta_2\frac{I}{m+I}\right)S+\left(\beta_1-\beta_3\frac{I}{m+I}\right)(1-\gamma)V-(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})\right]\mathrm{d}t\\&+\delta_2\mathrm{d}B_2(t).\end{aligned}$$

Integrating from 0 to t, combining (2.8) and (2.9), we deduce

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t}\leq&\beta_1\langle S\rangle_t+\beta_1(1-\gamma)\langle V\rangle_t-(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})+\frac{\delta_2B_2(t)}{t}\\ \leq&\beta_1\bar{S}+\beta_1(1-\gamma)\bar{V}-(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})\\&-\frac{(\alpha+\mu)(\beta_1(\mu+\omega)+\beta_1(1-\gamma)\theta)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t\\&+\beta_1\varphi_1(t)+\beta_1(1-\gamma)\varphi_2(t)+\frac{\delta_2B_2(t)}{t}\\ =&(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1)-\frac{(\alpha+\mu)(\beta_1(\mu+\omega)+\beta_1(1-\gamma)\theta)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t\\&+\beta_1\varphi_1(t)+\beta_1(1-\gamma)\varphi_2(t)+\frac{\delta_2B_2(t)}{t}.\end{aligned}\tag{2.10}$$

Using strong law of large numbers [12] it follows

$$\lim_{t\to\infty}\frac{\delta_2B_2(t)}{t}=0,\quad a.s.\tag{2.11}$$

Thus, (2.10) implies that

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1),\quad a.s.$$

Therefore, we obtain that if $R_0^s<1$, then

$$\lim_{t\to\infty}I(t)=0,\quad a.s.$$

which means the disease dies out with probability one.

(ii). Applying Itô's formula again, we have

$$\begin{aligned}\mathcal{L}(\ln I+\frac{I}{m})=&\beta_1S-\frac{\beta_2}{m}SI+\frac{\beta_1}{m}SI+\beta_1(1-\gamma)V-\frac{\beta_3(1-\gamma)}{m}VI+\frac{\beta_1(1-\gamma)}{m}VI\\&-(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})-\frac{\alpha+\mu+\lambda}{m}I\\ \geq&\beta_1S+\beta_1(1-\gamma)V-(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})-\frac{\alpha+\mu+\lambda}{m}I,\end{aligned}$$

where we used $\beta_1 \geq \beta_2$ and $\beta_1 \geq \beta_3$. Then we define a C^2-function

$$V_1(S, I, V) = \ln I + \frac{I}{m} + a_1(S + I) + a_2(V + I),$$

where a_1 and a_2 are positive constants which will be defined later. So

$$\begin{aligned}\mathcal{L}V_1 \geq & \beta_1 S + \beta_1(1-\gamma)V - (\alpha+\mu+\lambda+\frac{\delta_2^2}{2}) - \frac{\alpha+\mu+\lambda}{m}I + a_1\Lambda \\ & - (a_1(\theta+\mu) - a_2\theta)S - (a_2(\mu+\omega) - a_1\omega)V - (a_1+a_2)(\alpha+\mu+\lambda)I.\end{aligned}$$

Let

$$a_1(\theta+\mu) - a_2\theta = \beta_1, \quad a_2(\mu+\omega) - a_1\omega = \beta_1(1-\gamma),$$

by calculation,

$$a_1 = \frac{\beta_1(\mu+\omega) + \beta_1(1-\gamma)\theta}{\mu(\theta+\mu+\omega)}, \quad a_2 = \frac{\beta_1\omega + \beta_1(1-\gamma)(\theta+\mu)}{\mu(\theta+\mu+\omega)}.$$

Then

$$\begin{aligned}\mathcal{L}V_1 \geq & a_1\Lambda - (\alpha+\mu+\lambda+\frac{\delta_2^2}{2}) - (\frac{1}{m}+a_1+a_2)I \\ = & (\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s - 1) - (\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)I.\end{aligned} \tag{2.12}$$

Therefore, we obtain

$$\begin{aligned}& \frac{\ln I(t) + \frac{I(t)}{m} + a_1(S(t)+I(t)) + a_2(V(t)+I(t))}{t} \\ & - \frac{\ln I(0) + \frac{I(0)}{m} + a_1(S(0)+I(0)) + a_2(V(0)+I(0))}{t} \\ \geq & (\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s - 1) - (\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t + \frac{\delta_2 B_2(t)}{t} + M(t),\end{aligned}$$

where

$$\begin{aligned}M(t) = & \frac{1}{mt}\int_0^t \delta_2 I \mathrm{d}B_2(t) + \frac{a_1}{t}\left(\int_0^t \delta_1 S \mathrm{d}B_1(t) + \int_0^t \delta_2 I \mathrm{d}B_2(t)\right) \\ & + \frac{a_2}{t}\left(\int_0^t \delta_3 V \mathrm{d}B_1(t) + \int_0^t \delta_2 I \mathrm{d}B_2(t)\right).\end{aligned}$$

From (2.2) and (2.3) it follows that

$$\begin{aligned}& \liminf_{t\to\infty}(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t \\ \geq & \liminf_{t\to\infty}\frac{\ln I(t)}{t} + \liminf_{t\to\infty}(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t \\ \geq & (\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s - 1).\end{aligned}$$

Therefore,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s \geq \frac{(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1)}{(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)},$$

which means that the disease will spread if $R_0^s > 1$. The proof is complete. □

Remark 2.1. R_0^s can be seen as the basic reproduction number of stochastic model (1.2), which is irrelevant to media coverage. On the other hand, R_0^s is smaller than R_0, which means environmental noise can suppress the outbreak of the epidemic.

2.3. Existence of ergodic stationary distribution of system (1.2)

In this subsection, we will give the conditions for the existence of an ergodic stationary distribution which is corresponding to the stability of the endemic equilibrium of deterministic version. The method we adopt is the theory of Khasminskii [10] (see Appendix).

Theorem 2.3. *Assume that $R_0^s > 1$. Then model (1.2) has a unique stationary distribution, and the solution $(S(t), I(t), V(t), R(t))$ is ergodic.*

Proof. From the theory of Khasminskii, in order to prove Theorem 2.3, we should verify Assumptions (A1) and (A2) in Appendix. Firstly, we will construct a series of Lyapunov functions and a bounded set to verify $(B2)$. Applying Itô's formula, we have

$$\begin{aligned}
\mathcal{L}(-\ln S) &\le -\frac{\Lambda}{S} + \beta_1 I + (\theta + \mu + \frac{\delta_1^2}{2}),\\
\mathcal{L}(-\ln V) &\le -\frac{\theta S}{V} + \beta_1(1-\gamma) I + (\mu + \omega + \frac{\delta_3^2}{2}),\\
\mathcal{L}(-\ln R) &\le -\frac{\lambda I}{R} + (\mu + \delta + \frac{\delta_4^2}{2}).
\end{aligned}$$

Taking $V_1 = \ln I + \frac{I}{m} + a_1(S+I) + a_2(V+I)$, from the proof of Theorem 2.2, we obtain

$$\mathcal{L}(-V_1) \le -(\alpha + \mu + \lambda + \frac{\delta_2^2}{2})(R_0^s - 1) + (\frac{1}{m} + a_1 + a_2) I.$$

We further denote

$$V_2 = \frac{1}{\theta+1}(S+I+V+R)^{\theta+1},$$

where θ is a sufficiently small number satisfying $\rho := \mu - \frac{\theta}{2}(\max_{k=1,2,3,4} \delta_k^2) > 0$. Then

$$\begin{aligned}
\mathcal{L}V_2 =& (S+I+V+R)^{\theta} \left(\Lambda - \mu(S+I+V+R) - \alpha I\right)\\
&+ \frac{\theta}{2}(S+I+V+R)^{\theta-1}\left(\delta_1^2 S^2 + \delta_2^2 I^2 + \delta_3^2 V^2 + \delta_4^2 R^2\right)\\
\le& \Lambda(S+I+V+R)^{\theta} - \mu(S+I+V+R)^{\theta+1}\\
&+ \frac{\theta}{2}(\max_{k=1,2,3,4} \delta_k^2)(S+I+V+R)^{\theta+1}\\
\le& \Lambda(S+I+V+R)^{\theta} - \left(\mu - \frac{\theta}{2}(\max_{k=1,2,3,4} \delta_k^2)\right)(S+I+V+R)^{\theta+1}\\
\le& A - \frac{\rho}{2}(S+I+V+R)^{\theta+1}\\
\le& A - \frac{\rho}{2}\left(S^{\theta+1} + V^{\theta+1} + I^{\theta+1} + R^{\theta+1}\right),
\end{aligned} \tag{2.13}$$

with $A = \sup_{(S,I,V,R)\in\mathbb{R}_+^{4,o}} \left\{\mu(S+I+V+R)^{\theta} - \frac{\rho}{2}(S+I+V+R)^{\theta+1}\right\}$.

To this end, we define a C^2-function in the following form

$$V_0(S,I,V,R) = M(-V_1) + V_2 - \ln S - \ln V - \ln R,$$

where M is sufficiently large such that

$$-M(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1)+B \le -2,$$

here $B = A + (\theta+\mu+\frac{\delta_1^2}{2}) + (\mu+\omega+\frac{\delta_3^2}{2}) + (\mu+\sigma+\frac{\delta_4^2}{2})$. It is easy to check that

$$\liminf_{\epsilon\to\infty,(S,I,V,R)\in\mathbb{R}_+^{4,o}\setminus U_\epsilon} V_0(S,I,V,R) = +\infty,$$

where $U_\epsilon = (1/\epsilon,\epsilon)\times(1/\epsilon,\epsilon)\times(1/\epsilon,\epsilon)\times(1/\epsilon,\epsilon)$. The monotonicity and continuity of V_0 implies that $V_0(S,I,V,R)$ has the minimum value point $(\hat{S},\hat{I},\hat{V},\hat{R})$.

Then define the nonnegative C^2-function as follows

$$\hat{V} = M(-V_1) + V_2 - \ln S - \ln V - \ln R - V_0(\hat{S},\hat{I},\hat{V},\hat{R}).$$

Therefore, from above analysis, we obtain

$$\begin{aligned}
\mathcal{L}\hat{V}(S,I,V,R) \le & -M(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1) \\
& + \left(M(\frac{1}{m}+a_1+a_2)+\beta_1+\beta_1(1-\gamma)\right)I + B \\
& - \frac{\Lambda}{S} - \frac{\theta S}{V} - \frac{\lambda I}{R} - \frac{\rho}{2}S^{\theta+1} - \frac{\rho}{2}I^{\theta+1} - \frac{\rho}{2}V^{\theta+1} - \frac{\rho}{2}R^{\theta+1} \\
= & -M(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1) + f(I) \\
& - \frac{\Lambda}{S} - \frac{\theta S}{V} - \frac{\lambda I}{R} - \frac{\rho}{2}S^{\theta+1} - \frac{\rho}{2}V^{\theta+1} - \frac{\rho}{2}R^{\theta+1},
\end{aligned}$$

where $f(I) = -\frac{\rho}{2}I^{\theta+1} + \left(M(\frac{1}{m}+a_1+a_2)+\beta_1+\beta_1(1-\gamma)\right)I + B$.

Up to now, we have construct the suitable Lyapunov function $\hat{V}(S,I,V,R)$. Now, we are in the position to construct a compact set $D \subset \mathbb{R}_+^{4,o}$ such that $\mathcal{L}\hat{V}(S,I,V,R) \le -1$ for $(S,I,V,R) \in \mathbb{R}_+^{4,o}\setminus D$. Define

$$D = \left\{\varepsilon \le S \le 1/\varepsilon,\ \varepsilon \le I \le 1/\varepsilon,\ \varepsilon^2 \le V \le 1/\varepsilon^2,\ \varepsilon^2 \le R \le 1/\varepsilon^2\right\},$$

where ε is a sufficiently small positive constant such that the following conditions hold

$$-\frac{\Lambda}{\varepsilon}+\check{f} \le -1,\quad -\frac{\theta}{\varepsilon}+\check{f} \le -1,\quad -\frac{\lambda}{\varepsilon}+\check{f} \le -1, \tag{2.14}$$

$$-M(\alpha+\mu+\lambda+\frac{\delta_2^2}{2})(R_0^s-1)+\left(M(\frac{1}{m}+a_1+a_2)+\beta_1+\beta_1(1-\gamma)\right)\varepsilon+B \le -1, \tag{2.15}$$

$$\check{f}-\frac{\rho}{2}\frac{1}{\varepsilon^{\theta+1}} \le -1, \tag{2.16}$$

$$\check{f}-\frac{\rho}{2}\frac{1}{\varepsilon^{(2\theta+2)}} \le -1. \tag{2.17}$$

Then

$$\mathbb{R}_+^{4,o} \setminus D = D_1^c \cup D_2^c \cup D_3^c \cup D_4^c \cup D_5^c \cup D_6^c \cup D_7^c \cup D_8^c,$$

with

$$\begin{aligned}
&D_1^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ 0 < S < \varepsilon\right\}, \quad D_2^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ 0 < I < \varepsilon\right\},\\
&D_3^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ S > \varepsilon,\ 0 < V < \varepsilon^2\right\},\\
&D_4^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ I > \varepsilon,\ 0 < R < \varepsilon^2\right\},\\
&D_5^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ S > \frac{1}{\varepsilon}\right\}, \quad D_6^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ I > \frac{1}{\varepsilon}\right\},\\
&D_7^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ V > \frac{1}{\varepsilon^2}\right\}, \quad D_8^c = \left\{(S,I,V,R) \in \mathbb{R}_+^{4,o} |\ R > \frac{1}{\varepsilon^2}\right\}.
\end{aligned}$$

From inequalities (2.14), (2.15), (2.16) and (2.17), we can easily deduce that

$$\mathcal{L}\hat{V}(S,I,V,R) \le -1, \quad (S,I,V,R) \in D_i^c, \quad i = 1,2,3,4,5,6,7,8.$$

Therefore, we obtain

$$\mathcal{L}\hat{V}(S,I,V,R) \le -1, \quad (S,I,V,R) \in \mathbb{R}_+^{4,o} \setminus D,$$

which implies that Assumption (B2) is satisfied.

On the other hand, the diffusion matrix of system (1.2) is

$$\bar{A}(X) = \begin{pmatrix} \delta_1^2 S^2 & 0 & 0 & 0 \\ 0 & \delta_2^2 I^2 & 0 & 0 \\ 0 & 0 & \delta_3^2 V^2 & 0 \\ 0 & 0 & 0 & \delta_4^2 R^2 \end{pmatrix}.$$

There is a $c = \min\{\delta_1^2 S^2, \delta_2^2 I^2, \delta_3^2 V^2, \delta_4^2 R^2\} > 0$ such that

$$\sum_{i,j=1}^{4} \bar{a}_{ij}(X)\xi_i\xi_j = \delta_1^2 S^2 \xi_1^2 + \delta_2^2 I^2 \xi_2^2 + \delta_3^2 V^2 \xi_3^2 + \delta_4^2 R^2 \xi_4^2 \ge c|\xi|^2$$

for $(S,I,V,R) \in \bar{D}$ and $\xi \in \mathbb{R}_+^{4,o}$. That is to say, Assumption (B1) holds. Consequently, system (1.2) has an ergodic stationary distribution. □

2.4. Numerical simulations

In this subsection, in order to show the transmission dynamics of stochastic SVIR model with media coverage, we present some numerical simulations. We use the Milstein's high-order method [8] to simulate the stochastic model (1.2). The nu-

merical scheme for stochastic model (1.2) is given by:

$$\begin{cases} S^{k+1} =& S^k + [\Lambda + \omega V^k - (\theta+\mu)S^k - (\beta_1 - \beta_2\dfrac{I^k}{m+I^k})S^kI^k + \sigma R^k]\Delta t \\ & + \delta_1 S^k\sqrt{\Delta t}\xi_k + \dfrac{\delta_1^2}{2}S^k(\xi_k^2-1)\Delta t, \\ I^{k+1} =& I^k + [(\beta_1 - \beta_2\dfrac{I^k}{m+I^k})S^kI^k + (\beta_1 - \beta_3\dfrac{I^k}{m+I^k})(1-\gamma)V^kI^k \\ & - (\alpha+\mu+\lambda)I^k]\Delta t + \delta_2 I^k\sqrt{\Delta t}\zeta_k + \dfrac{\delta_2^2}{2}I^k(\zeta_k^2-1)\Delta t, \\ V^{k+1} =& V^k + [\theta S^k - (\mu+\omega)V^k - (\beta_1 - \beta_3\dfrac{I^k}{m+I^k})(1-\gamma)V^kI^k]\Delta t \\ & + \delta_3 V^k\sqrt{\Delta t}\eta_k + \dfrac{\delta_3^2}{2}V^k(\eta_k^2-1)\Delta t, \\ R^{k+1} =& R^k + [\lambda I^k - (\mu+\sigma)R^k]\Delta t + \delta_4 R^k\sqrt{\Delta t}\varsigma_k + \dfrac{\delta_4^2}{2}V^k(\varsigma_k^2-1)\Delta t, \end{cases}$$

where the time increment $\Delta t > 0$, ξ_k, η_k, ζ_k and ς_k are independent Gaussian random variables which follow the distribution $N(0,1)$ for $k = 1, 2, ..., n$.

We use the data of Florida form 2018 to 2019 [3] to estimate the parameters Λ and μ by the following equations

$$\begin{cases} \mathcal{S}'(t) = \Lambda - \mu\mathcal{S}(t), \\ \mathcal{D}'(t) = \mu\mathcal{S}(t), \\ \mathcal{S}(0) = 21,299,325, \quad \mathcal{S}(365) = 21,477,737, \quad \mathcal{D}(0) = 0, \quad \mathcal{D}(365) = 207,002, \end{cases}$$

where $\mathcal{S}(t)$ and $\mathcal{D}(t)$ denote the number of susceptible people and the number of deaths at time t, respectively. Then we obtain $\Lambda = 1056$ person day^{-1} and $\mu = 0.01393$ person^{-1} day^{-1}.

Table 2. List of Parameters

Parameters	Description	Value	Reference
Λ	Recruitment rate	1056 person day^{-1}	Estimated
ω	Rate at which vaccine wanes	0.15 day^{-1}	[21]
θ	Vaccine uptake rate	0.7 day^{-1}	[18]
μ	Natural death rate	0.01393 day^{-1}	Estimated
σ	Loss of immunity	0.01 day^{-1}	[19]
γ	Vaccine efficacy	0.8	[19]
α	Infection death rate	0.1 day^{-1}	[19]
λ	Recovery rate of Infectives	0.68 day^{-1}	Assumed
m	Reaction due to media coverage	500 person	Assumed

Therefore, we take parameter values which are listed in Table 2, $\beta_1 = 3\times10^{-5}$ and $\beta_2 = \beta_3 = 6\times10^{-6}$. With the parameters in Table 2, we can obtain that

$$R_0 = \frac{\beta_1\Lambda(\mu+\omega) + \beta_1(1-\gamma)\theta\Lambda}{\mu(\alpha+\mu+\lambda)(\theta+\mu+\omega)} = 1.0077 > 0.$$

Figure 1 shows that the disease in deterministic model (1.1) is persistent. Now we analyze the effect of white noise.

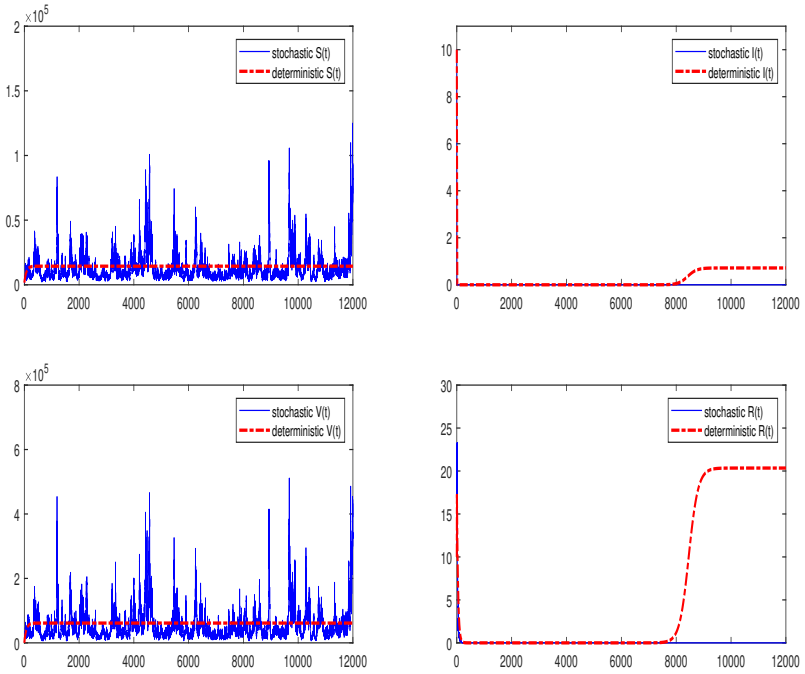

Figure 1. Simulations of solution $(S(t), I(t), V(t), R(t))$ for deterministic model (1.1) and stochastic model (1.2) with white noise $\delta_k = 0.16$, $(k = 1, 2, 3, 4)$.

Case 1. Let the environmental white noise intensities be fixed at $\delta_k = 0.16$, $(k = 1, 2, 3, 4)$. Then $\mu = 0.01393 > \delta_k^2/2 = 0.0128$ and

$$R_0^s = \frac{\beta_1\Lambda(\mu+\omega) + \beta_1(1-\gamma)\theta\Lambda}{\mu(\alpha+\mu+\lambda+\delta_2^2/2)(\theta+\mu+\omega)} = 0.9917 < 1.$$

From Theorem 2.2 (i) it follows that the influenza in model (1.2) can not spread and Fig. 1 confirms this. This also means that larger white noise can suppress the outbreak of the influenza.

Case 2. Let the environmental white noise intensities be fixed at $\delta_k = 0.001$, $(k = 1, 2, 3, 4)$. Then

$$R_0^s = \frac{\beta_1\Lambda(\mu+\omega) + \beta_1(1-\gamma)\theta\Lambda}{\mu(\alpha+\mu+\lambda+\delta_2^2/2)(\theta+\mu+\omega)} = 1.0077 > 1.$$

According to the results of Theorem 2.2 (ii) and Theorem 2.3, the influenza will be persistent in the long time and Fig. 2 confirms this.

On the other hand, since the terms $\beta_2\frac{I}{m+I}$ and $\beta_3\frac{I}{m+I}$ measure the effect of reduction of the contact rate when infectious and vaccinated individuals are reported in the media, so if β_2 and β_3 are more larger, the number of infected population is more smaller. We fix white noise intensities at $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0.05$, and choose three different values of $\beta_2 = \beta_3$ as 0.04 (see green line), 0.02 (see red line)

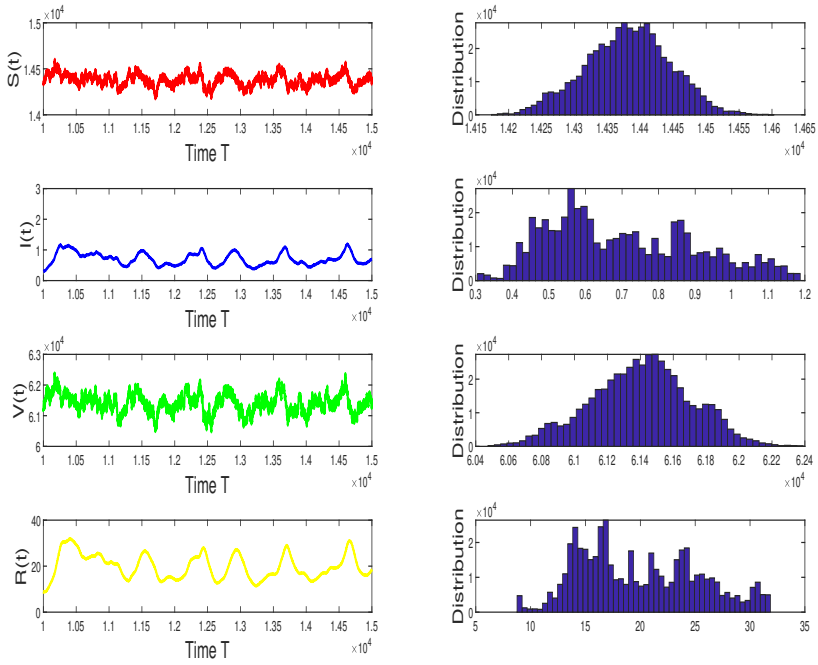

Figure 2. Left panels: Simulations of solution $(S(t), I(t), V(t), R(t))$ for stochastic model (1.2) . Right panels: Histogram stochastic system (1.2) with white noise $\delta_k = 0.001$, $(k = 1, 2, 3, 4)$.

and 0.00 (see blue line), then Fig. 4 support the analytical results. Theoretical results and numerical simulations show that environmental white noise and media coverage can reduce the transmission of the disease.

3. Dynamics of stochastic influenza model (1.3)

3.1. The properties of the global solution

By constructing the same Lyapunov function as that in Theorem 2.1, we can show that model (1.3) has a unique global positive solution. Here we only give the result.

Theorem 3.1. *For any initial value* $(S(0), I(0), V(0), R(0)) \in \mathbb{R}^{4,o}_+$, *there is a unique positive solution* $(S(t), I(t), V(t), R(t)) \in \mathbb{R}^{4,o}_+$ *of model (1.3) on* $t \geq 0$, *and the solution will remain in* $\mathbb{R}^{4,o}_+$ *with probability 1.*

Theorem 3.1 shows that model (1.3) has a unique global positive solution. Then from the expression of model (1.3) it leads to

$$\begin{aligned}\frac{\mathrm{d}(S+I+V+R)}{\mathrm{d}t} =& \Lambda - \mu(S+I+V+R) - \alpha I \\ \leq& \Lambda - \mu(S+I+V+R), \quad a.s.\end{aligned}$$

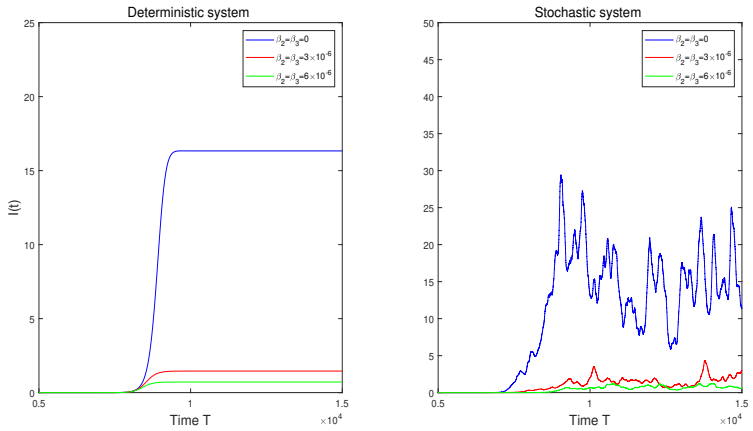

Figure 3. Simulations of $I(t)$ of deterministic model (1.1) and stochastic model (1.2) with different β_2 and β_3 values.

Therefore, we deduce that if the initial value $S(0)+I(0)+V(0)+R(0) \leq \frac{\Lambda}{\mu}$, then $S(t)+I(t)+V(t)+R(t) \leq \frac{\Lambda}{\mu}$ a.s., that is to say

$$\Gamma = \{(S,I,V,R) : S>0,\ I>0,\ V>0,\ R>0,\ S+I+V+R \leq \frac{\Lambda}{\mu}\}$$

is a positive invariant set of model (1.3). Hence we have

Lemma 3.1. *Let $(S(t),I(t),V(t),R(t))$ be a positive solution of model (1.3), the following properties hold*

$$\lim_{t\to\infty}\frac{S(t)}{t}=0,\quad \lim_{t\to\infty}\frac{I(t)}{t}=0,\quad \lim_{t\to\infty}\frac{V(t)}{t}=0,\quad \lim_{t\to\infty}\frac{R(t)}{t}=0,\quad a.s. \tag{3.1}$$

$$\begin{aligned} &\limsup_{t\to\infty}\frac{\ln S(t)}{t}\leq 0,\quad \limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq 0,\\ &\limsup_{t\to\infty}\frac{\ln V(t)}{t}\leq 0,,\quad \limsup_{t\to\infty}\frac{\ln R(t)}{t}\leq 0,\quad a.s. \end{aligned} \tag{3.2}$$

On the other hand, denote $M(t)=\int_0^t S(r)\mathrm{d}B(r)$, then $M(t)$ is a locally continuous martingale with $M(0)=0$, and

$$\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t}\leq\frac{\Lambda^2}{\mu^2}<\infty,\quad a.s.$$

Strong law of large numbers leads to

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t S(r)\mathrm{d}B(r)=0,\quad a.s.$$

Similarly, we have the following lemma.

Lemma 3.2. *Let $(S(t), I(t), V(t), R(t))$ be a positive solution of model (1.3). Then*

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t S(r)\mathrm{d}B(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t V(r)\mathrm{d}B(r)}{t} = 0, \\ &\lim_{t\to\infty} \frac{\int_0^t S(r)I(r)\mathrm{d}B(r)}{t} = 0, \quad \lim_{t\to\infty} \frac{\int_0^t V(r)I(r)\mathrm{d}B(r)}{t} = 0, \quad a.s. \end{aligned} \tag{3.3}$$

3.2. Persistence and extinction of disease

In this subsection, we will find the conditions which determine the persistence and extinction of disease in model (1.3).

Theorem 3.2. *Let $(S(t), I(t), V(t), R(t))$ be a positive solution of system (1.3).*
(i). If

$$\delta^2 > \frac{\beta_1^2}{2(\alpha+\mu+\lambda)} \tag{3.4}$$

then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le -(\alpha+\mu+\lambda) + \frac{\beta_1^2}{2\delta^2} < 0 \quad a.s.;$$

if

$$R_{01}^s = R_0 - \frac{\delta^2}{2(\alpha+\mu+\lambda)}(\bar{S} + (1-\gamma)\bar{V})^2 < 1 \quad \textit{and} \quad \delta^2 \le \frac{\beta_1}{\bar{S} + (1-\gamma)\bar{V}}, \tag{3.5}$$

then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le (\alpha+\mu+\lambda)(R_{01}^s - 1) < 0 \quad a.s.$$

These mean the disease dies out with probability one.
(ii). If $R_{02}^s = R_0 - \frac{\delta^2}{2(\alpha+\mu+\lambda)}(\frac{\Lambda}{\mu} + (1-\gamma)\frac{\Lambda}{\mu})^2 > 1$, then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)\mathrm{d}s \ge \frac{(R_{02}^s - 1)}{\frac{1}{m} + a_1 + a_2} \quad a.s.,$$

where a_1 and a_2 are same as those in Theorem 2.2. This implies the disease will persist in a long term.

Proof. On the one hand, using the similar analysis as that in the proof of Theorem 2.2, we have

$$\langle S\rangle_t \le \bar{S} - \frac{(\mu+\omega)(\alpha+\mu)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t + \varphi_3(t), \tag{3.6}$$

where $\bar{S} = \frac{(\mu+\omega)\Lambda}{\mu(\theta+\mu+\omega)}$, and $\varphi_3(t)$ is defined by

$$\begin{aligned} \varphi_3(t) =& \frac{1}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\left[-\frac{(\mu+\omega)\mu}{t}\int_0^t \delta SI\mathrm{d}B(t) - \frac{\mu(\omega-\sigma)}{t}\int_0^t \delta(1-\gamma)VI\mathrm{d}B(t)\right. \\ &-(\mu+\delta)(\mu+\omega)\frac{S(t)-S(0)}{t} - (\mu+\sigma)\omega\frac{V(t)-V(0)}{t} - (\mu+\omega)\sigma\frac{I(t)-I(0)}{t} \\ &\left. -\ (\mu+\omega)\sigma\frac{R(t)-R(0)}{t}\right]. \end{aligned}$$

From (3.1) and (3.3), it follows

$$\lim_{t\to\infty}\varphi_3(t)=0,\quad a.s.$$

We also obtain

$$\frac{V(t)-V(0)}{t}\le\theta\langle S\rangle_t-(\mu+\omega)\langle V\rangle_t-\frac{1}{t}\int_0^t\delta(1-\gamma)VI\mathrm{d}B(t). \tag{3.7}$$

Substituting (3.6) into (3.7), one has

$$\langle V\rangle_t\le\bar{V}-\frac{\theta(\alpha+\mu)}{\mu(\mu+\sigma)(\theta+\mu+\omega)}\langle I\rangle_t+\varphi_4(t), \tag{3.8}$$

where $\bar{V}=\frac{\Lambda\theta}{\mu(\theta+\mu+\omega)}$, $\varphi_4(t)=\frac{\theta}{\mu+\omega}\varphi_3(t)-\frac{1}{\mu+\omega}\frac{V(t)-V(0)}{t}-\frac{1}{(\mu+\omega)t}\int_0^t\delta_(1-\gamma)VI\mathrm{d}B(t)$, and $\varphi_4(t)\to\ 0$ as $t\to\infty$. Therefore,

$$\begin{aligned}\langle S\rangle_t+(1-\gamma)\langle V\rangle_t\le&\bar{S}+(1-\gamma)\bar{V}-\frac{\alpha+\mu}{\mu(\mu+\sigma)}\langle I\rangle_t+\varphi_3(t)+\varphi_4(t)\\=&\frac{(\mu+\omega)\Lambda+(1-\gamma)\Lambda\theta}{\mu(\theta+\mu+\omega)}-\frac{\alpha+\mu}{\mu(\mu+\sigma)}\langle I\rangle_t+\varphi_3(t)+\varphi_4(t).\end{aligned} \tag{3.9}$$

On the other hand, we can also obtain

$$\begin{aligned}&a_1\frac{S(t)-S(0)+I(t)-I(0)}{t}+a_2\frac{V(t)-V(0)+I(t)-I(0)}{t}\\\ge&a_1\Lambda-\beta_1\langle S\rangle_t-\beta_1(1-\gamma)\langle V\rangle_t-(a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t\\&+\frac{a_1}{t}\int_0^t(1-\gamma)\delta VI\mathrm{d}B(t)+\frac{a_2}{t}\int_0^t\delta SI\mathrm{d}B(t),\end{aligned}$$

where a_1 and a_2 are the same as those in Theorem 2.2. This leads to

$$\beta_1\langle S\rangle_t+\beta_1(1-\gamma)\langle V\rangle_t\ge a_1\Lambda-(a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t+\varphi_5(t), \tag{3.10}$$

where $\varphi_5(t)=\frac{a_1}{t}\int_0^t(1-\gamma)\delta VI\mathrm{d}B(t)+\frac{a_2}{t}\int_0^t\delta SI\mathrm{d}B(t)-a_1\frac{S(t)-S(0)+I(t)-I(0)}{t}+a_2\frac{V(t)-V(0)+I(t)-I(0)}{t}$ tends to zero as $t\to\infty$.

(i). By Itô's formula, we deduce that

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t}\le&\beta_1\langle S+(1-\gamma)V\rangle_t-(\alpha+\mu+\lambda)-\frac{\delta^2}{2}\langle(S+(1-\gamma)V)^2\rangle_t+\frac{M_2(t)}{t}\\\le&\beta_1\langle S+(1-\gamma)V\rangle_t-(\alpha+\mu+\lambda)-\frac{\delta^2}{2}\langle S+(1-\gamma)V\rangle_t^2+\frac{M_2(t)}{t}\\\le&-\frac{\delta^2}{2}\left(\langle S+(1-\gamma)V\rangle_t-\frac{\beta_1}{\delta^2}\right)^2-(\alpha+\mu+\lambda)+\frac{\beta_1^2}{2\delta^2}+\frac{M_2(t)}{t},\end{aligned} \tag{3.11}$$

where $M_2(t)=\delta\int_0^t(S+(1-\gamma)V)\mathrm{d}B(t)$, and $\lim_{t\to\infty}\frac{M_2(t)}{t}=0$ due to Lemma 3.2. On the one hand, in view of (3.11), if condition (3.4) holds, taking super limit on both sides of (3.11), we obtain

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le-(\alpha+\mu+\lambda)+\frac{\beta_1^2}{2\delta^2}<0,\quad a.s.$$

On the other hand, if condition (3.5) holds, combining (3.9), then

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln I(t)}{t}\leq&-\frac{\delta^2}{2}\left(\bar{S}+(1-\gamma)\bar{V}-\frac{\beta_1}{\delta^2}\right)^2-(\alpha+\mu+\lambda)+\frac{\beta_1^2}{2\delta^2}\\=&\beta_1(\bar{S}+(1-\gamma)\bar{V})-\frac{\delta^2}{2}(\bar{S}+(1-\gamma)\bar{V})^2-(\alpha+\mu+\lambda)\\=&(\alpha+\mu+\lambda)\left(R_0-1-\frac{\delta^2}{2(\alpha+\mu+\lambda)}(\bar{S}+(1-\gamma)\bar{V})^2\right)\\=&(\alpha+\mu+\lambda)(R_{01}^s-1)<0\quad a.s.\end{aligned}$$

(ii). From the proof of Theorem 2.2, in view of (3.10), we also have

$$\begin{aligned}&\frac{\ln I(t)-\ln I(0)}{t}+\frac{1}{m}\frac{I(t)-I(0)}{t}\\\geq&\beta_1\langle S+(1-\gamma)V\rangle_t-(\alpha+\mu+\lambda)-\frac{\alpha+\mu+\lambda}{m}\langle I\rangle_t-\frac{\delta^2}{2}\langle(S+(1-\gamma)V)^2\rangle_t\\&+\frac{M_2(t)}{t}+\frac{M_3(t)}{t}\\\geq&a_1\Lambda-(\alpha+\mu+\lambda)-(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t-\frac{\delta^2}{2}\left(\frac{\Lambda}{\mu}+(1-\gamma)\frac{\Lambda}{\mu}\right)^2\\&+\varphi_5(t)+\frac{M_2(t)}{t}+\frac{M_3(t)}{t}\\=&(\alpha+\mu+\lambda)(R_{02}^s-1)-(\frac{1}{m}+a_1+a_2)(\alpha+\mu+\lambda)\langle I\rangle_t+\varphi_5(t)+\frac{M_2(t)}{t}+\frac{M_3(t)}{t},\end{aligned}$$

where $M_3(t)=\frac{\delta}{m}\int_0^t(S+(1-\gamma)V)I\mathrm{d}B(t)$. Therefore, from Lemmas 3.1 and 3.2, we obtain that if $R_{02}^s>1$, then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s\geq\frac{(R_{02}^s-1)}{\frac{1}{m}+a_1+a_2}\quad a.s.$$

This completes the proof. □

Remark 3.1. Theorem 3.2 gives sufficient conditions for persistence and extinction of the disease. Conditions (3.5) show that if $R_{01}^s<1$ provided the white noise is small, the disease will extinct exponentially. But the disease will also go extinct if the environmental white noise is large enough. In the mean while, we obtain that the disease will persist if $R_{02}^s>1$ when the white noise is small.

3.3. Numerical simulations

In this subsection, we also use the Milstein's high-order method to simulate the stochastic model (1.3). The numerical scheme for stochastic model (1.3) is given

by:

$$\begin{cases} S^{k+1} = S^k + [\Lambda + \omega V^k - (\theta+\mu)S^k - (\beta_1 - \beta_2\dfrac{I^k}{m+I^k})S^kI^k + \sigma R^k]\Delta t \\ \qquad - \delta S^kI^k\sqrt{\Delta t}\xi_k - \dfrac{\delta^2}{2}S^kI^k(\xi_k^2-1)\Delta t, \\ I^{k+1} = I^k + [(\beta_1 - \beta_2\dfrac{I^k}{m+I^k})S^kI^k + (\beta_1 - \beta_3\dfrac{I^k}{m+I^k})(1-\gamma)V^kI^k \\ \qquad - (\alpha+\mu+\lambda)I^k]\Delta t + \delta(S^kI^k + (1-\gamma)V^kI^k)\sqrt{\Delta t}\xi_k \\ \qquad + \dfrac{\delta^2}{2}(S^kI^k + (1-\gamma)V^kI^k)(\xi_k^2-1)\Delta t, \\ V^{k+1} = V^k + [\theta S^k - (\mu+\omega)V^k - (\beta_1 - \beta_3\dfrac{I^k}{m+I^k})(1-\gamma)V^kI^k]\Delta t \\ \qquad - \delta(1-\gamma)V^kI^k\sqrt{\Delta t}\xi_k - \dfrac{\delta^2}{2}(1-\gamma)V^kI^k(\xi_k^2-1)\Delta t, \\ R^{k+1} = R^k + [\lambda I^k - (\mu+\sigma)R^k]\Delta t, \end{cases}$$

where the time increment $\Delta t > 0$, ξ_k is Gaussian random variables which follow the distribution $N(0,1)$ for $k = 1, 2, ..., n$.

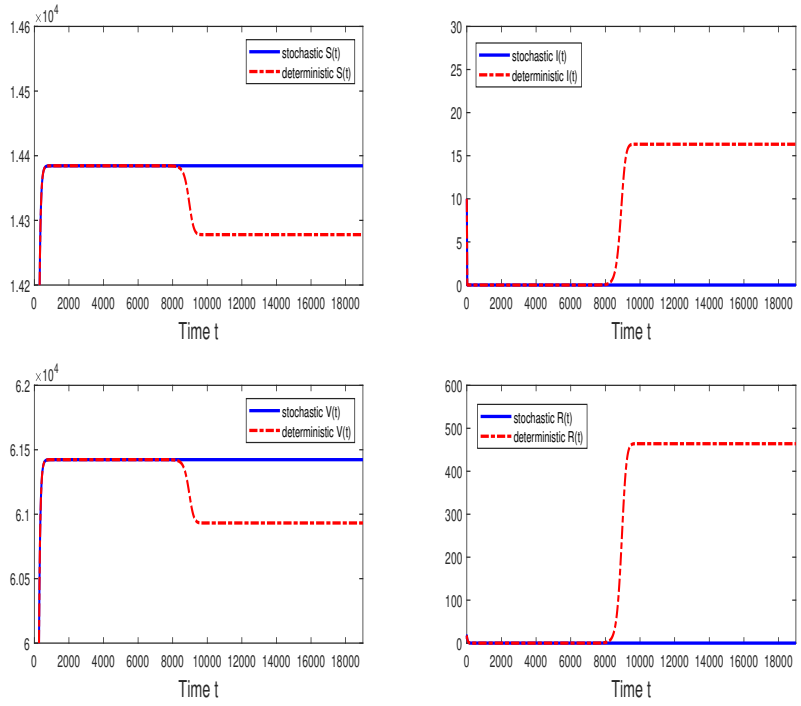

Figure 4. Simulations of solution $(S(t), I(t), V(t), R(t))$ for deterministic model (1.1) and stochastic model (1.3) with white noise $\delta = 0.05$.

We also take parameter values which are similar with Section 2.4. Choose white

noise intensities $\delta = 6 \times 10^{-6}$, then we have $\delta^2 = 3.6 \times 10^{-11} \leq \beta_1/[\bar{S} + (1-\gamma)\bar{V}] = 1.1249 \times 10^{-9}$ and

$$R_{01}^s = R_0 - \frac{\delta^2}{2(\alpha+\mu+\lambda)}(\bar{S} + (1-\gamma)\bar{V})^2 = 0.9916 < 1.$$

Theorem 3.2 (i) shows that the influenza will be extinct in the long time, and Fig. 4 confirms this. If we choose white noise $\delta = 3 \times 10^{-7}$, then

$$R_{02}^s = R_0 - \frac{\delta^2}{2(\alpha+\mu+\lambda)}(\frac{\Lambda}{\mu} + (1-\gamma)\frac{\Lambda}{\mu})^2 = 1.0073 > 1.$$

From (ii) in Theorem 3.2, it follows the influenza will spread in a long run and Fig. 5 confirms this. The impact of media coverage can be simulated by Fig. 6. Theoretical results and numerical simulations of model (1.3) also show that environmental white noise and media coverage can reduce the spread of disease.

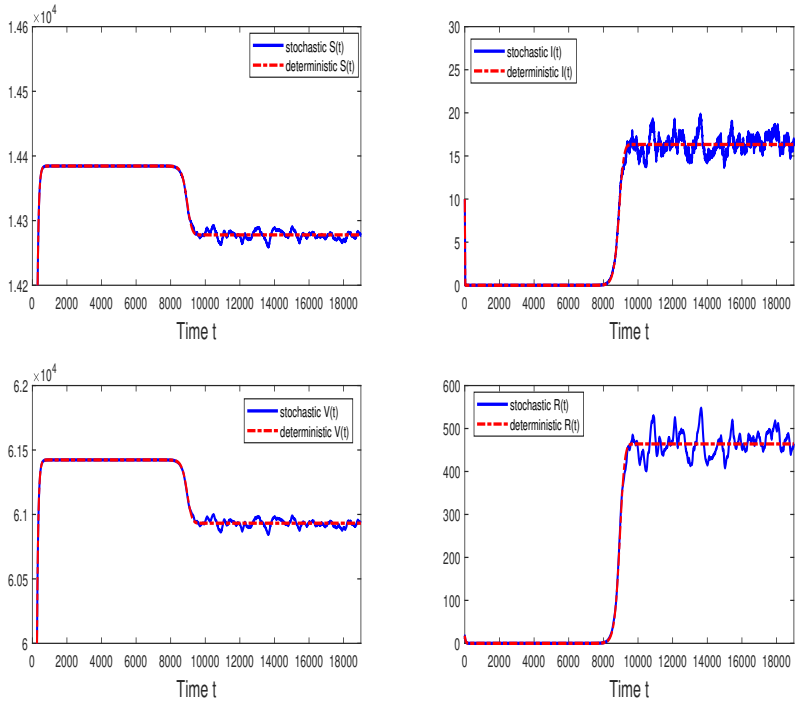

Figure 5. Simulations of solution $(S(t), I(t), V(t), R(t))$ for deterministic model (1.1) and stochastic model (1.3) with white noise $\delta = 0.008$.

Appendix

Since the proof of our result is based on the theory of Khasminskii, we introduce some definitions and results concerning stationary distribution and periodic Markov processes (see [10]).

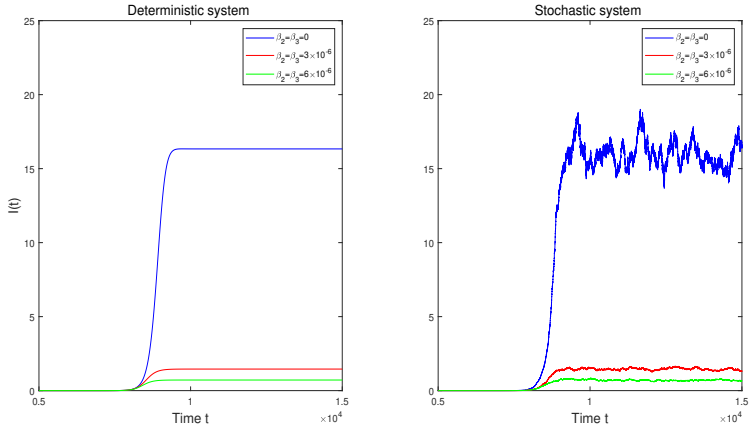

Figure 6. Simulations of $I(t)$ of deterministic model (1.1) and stochastic model (1.3) with different β_2 and β_3 values.

Let $X(t)$ be a homogeneous Markov process in E^l (E^l denotes euclidean l-space) satisfying the stochastic equation

$$dX(t) = h(X)dt + \sum_{m=1}^{k} g_m(X)dB_m(t).$$

The diffusion matrix is

$$\bar{A}(x) = (\bar{a}_{ij}(x)), \quad \bar{a}_{ij}(x) = \sum_{m=1}^{k} g_m^{(i)}(x)g_m^{(j)}(x).$$

Assumption. There is a bounded domain $U \subset E^l$ with regular boundary Γ, which has the properties that

(A1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $\bar{A}(x)$ is bounded away from zero.

(A2) If $x \in E^l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in\mathbb{K}} \mathbb{E}_x\tau < +\infty$ for every compact subset $\mathbb{K} \in E^l$.

Lemma 3.3. *If Assumption holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$\mathbb{P}\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \int_{E^l} f(x)\mu(dx)\right\} = 1.$$

In order to verify $(B1)$, we only need to show that F is uniformly elliptical in U, where $F(u) = h(x)u_x + 0.5\text{trace}(\bar{A}(x)u_{xx})$, that is to say, there is $M > 0$ such that

$$\sum_{i,j=1}^{k} \bar{a}_{ij}(x)\xi_i\xi_j > M|\xi|^2, \quad x \in U, \quad \xi \in \mathbb{R}^k. \tag{3.12}$$

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