Journal of Applied Analysis and Computation Website:http://www.jaac-online.com
Volume 11, Number 6, December 2021, 2815–2839 DOI:10.11948/20200485

ORTHOGONAL ARRAYS OBTAINED BY ARRAY SUBTRACTION

Chun Luo[1], Min Yang[1,†], Xuanhang Ma[1], Yingshan Zhang[2] and Sihui He[2]

Abstract In this paper, by using the orthogonal decompositions of projection matrices, a new general approach is proposed to construct asymmetrical OAs, namely array subtraction, the operation of which is not the usual subtraction but it is interesting since many so called atoms of asymmetrical OAs can be obtained by the array subtraction. It is important to find these atoms from some known asymmetrical OAs since they can make up of many new asymmetrical OAs. As an application of the method, some old and new mixed-level OAs of run sizes 72 and 100 are constructed.

Keywords Mixed-level OA, Kronecker product, projection matrix, permutation matrix.

MSC(2010) 62K15, 05B15.

1. Introduction

An $n\times m$ matrix A, having k_i columns with p_i levels, $i=1,\ldots,t, m=\sum_{i=1}^{t}k_i, p_i\neq p_j$, for $i\neq j$, is called an orthogonal array(OA) of strength d and size n if each $n\times d$ submatrix of A contains all possible $1\times d$ row vectors with the same frequency. Unless stated otherwise, an OA of strength 2 is considered, using the notation $L_n(p_1^{k_1}\cdots p_t^{k_t})$ for such an array. An OA is said to have mixed level (or asymmetrical) if $t\geq 2$. Asymmetrical OAs was formally introduced by Rao [27, 28], although examples of such arrays appeared in earlier publications, for example Addelman [1,2] Addelman and Kempthome [3, 4].

Clearly n must be a multiple of $p_ip_j, i\neq j$. If $k_i\geq 2, n$ must be a multiple of p_i^2. Therefore, without loss of generality, we assume that $n=prq$ for mixed level OA's. The proceeding definition also includes the case that $t=1$, but the array is usually called a symmetrical OA, denoted by $L_n(p^m)$. These arrays were introduced by Rao [27], although the adjective "orthogonal" seems to have been added by Bush [6] and Bose etc [7]. For simplicity, it will be no longer demanded that $t\geq 2$ and that $p_i\neq p_j$, for $i\neq j$. The symmetrical and asymmetrical will only be used when needed.

Orthogonal arrays are not only beautiful but also useful. Furthermore they are essential in Statistics and they play important roles in coding theory, computer

†The corresponding author. Email: YangMin20201226@163.com(M. Yang)
[1]School of Sciences, Shanghai Institute of Technology, Shanghai, 201418, China
[2]School of Statistics, Faculty of Econ omics and Management, East China Normal University, Shanghai 200241, China

science and cryptography. More details are illustrated by Heydayet etc [14].

In Statistics they are primarily used in designing experiments, which simply means that they are immensely important in all areas of human investigation: for example in medicine, agriculture and manufacturing.

A new theory or procedure of constructing asymmetrical OAs by using the orthogonal decompositions of projection matrices has been given by Zhang etc [37]. Suen [31], Suen etc [32] and Luo etc [19] have obtained some OAs by this procedure. Similarly, Leng etc [20] have constructed some fusion frames by investigating decompositions of positive matrices as weighted sums of orthogonal projections. The idea of the orthogonal decompositions of projection matrices for constructing designs comes from the theory of multilateral matrices in Zhang [40]– a mathematical technique to solve the problems of system with complexity. In general, the procedure of constructing asymmetrical OAs in our theory has been partitioned mainly into five parties: orthogonal-array addition, subtraction, multiplication, division and replacement.

The technique, namely generalized Kronecker product which belongs to the class of orthogonal-array multiplications, has also been proposed for the construction of asymmetrical OAs by Zhang [40] in the theory of multilateral matrices. Pang et al. [26] have discussed the generalized concept of matrices orthogonal-array multiplications. Zhang [41] has discussed the special technique of Kronecker sum from generalized Hadamard product and Zhang etc [42] have proposed a particular generalized Kronecker product about generalized difference matrices. Luo etc [21] continue to develop the technique of generalized Kronecker product.

The technique, namely generalized Hadamard product which belongs to the class of orthogonal-array additions, has also been proposed for the construction of asymmetrical OAs by Zhang [40] in the theory of multilateral matrices. Zhang etc [43] construct a lot of new asymmetrical OAs by the generalized Hadamard matrices $D(r^m(r+1), r^m(r+1); p)$. Zhang etc [44] construct a lot of new asymmetrical OAs by a generalized Hadamard product.

Furthermore, Luo [22] has discussed the relationship between generalized difference matrices and mixed OAs. The relationship is similar to a general "expansive replacement method" for constructing mixed-level OAs of an arbitrary strength established by Jiang etc [17], a construction and decomposition of OAs with non-prime-power numbers of symbols on the complement of a Baer subplane demonstrated by Yamada etc [36], and the existence of mixed OAs with four and five factors of strength two investigated by Chen etc [8] .

The current emphasis on quality control and product improvement has rejuvenated research in the area of asymmetrical factorial design, or namely asymmetrical OAs. Practical considerations have spurred research in various new or newly emphasized directions. Among these is that of the use and construction of asymmetrical OAs, examplified by research of Taguchi [33], Cheng [11], Agrawal etc [5], Dey etc [13], Wu etc [34] and Hedayet etc [16] .

In the past three years, scholars have made outstanding contributions to the construction of orthogonal arrays. For the first time, a new general iterative construction method for asymmetric OAs of high strength was proposed by Pang etc [23]. Pang etc [24] have put forward a new method for constructing OAs, which is a general method to construct symmetric and asymmetric OAs of strength t by orthogonal partition. Pang etc [25] have given new construction methods for symmetric and asymmetric orthogonal arrays (OAs) with high strength are proposed by using lower strength orthogonal partitions of spaces and OAs. Moreover, Charles etc [12] claims a way that completely solves the problem of constructing basic OAs with $n = 2$, assuming the truth of the Hadamard matrix conjecture. Literature [15] mainly studies the system construction of COAs (component orthogonal arrays as fractional designs of all possible permutations on experimental factors). As a consequence, the proposed COAs not only possess reasonable run sizes, but also have high efficiencies under the PWO (pair-wise ordering) model. A new method for identifying and evaluating irregular design complex alias structures of all symmetric and asymmetric orthogonal, regular and irregular asymmetric arrays is proposed by X. P. Xue etc [9].X. P. Xue etc [10] also constructed a new orthogonal-array based on composite minimax loss designs, which are more robust in terms of D-efficiency and the generalized standard deviation for a missing design point.

The mathematical theory is extremely wonderful: OAs are related to combinatorics, finite fields, geometry and error-correcting codes. The definition of an OA is simple and natural and many elegant constructions can be known-yet there are at least as many unsolved problems. In fact, the construction of orthogonal arrays as fractional factorial designs is so sufficiently documented that it requires no further explanation. Many new construction methods on the asymmetrical OAs have been proposed, for example, a grouping scheme by Wu [35], a method from the theory of group or domain [34], and Baer subplane by Ryoh [29]. Most of these methods on the asymmetrical OAs are only or mainly from the theory of group or domain. The theory of the construction of asymmetrical OAs has not received yet a considerable amount of attention in the literature since the problem is too difficulty to solve. But the work must be done because of the requirement of the quality control and product improvement.

The technique, namely array subtraction (Definition 2.3), has been first proposed for the construction of asymmetrical OAs by Zhang [40] in the theory of multilateral matrices. In this paper, the technique will further be explained and extend to construct some new asymmetrical (or mixed-level) OAs by using the orthogonal decompositions of projection matrix in Zhang etc [37].

Section 2 contains the basic concepts and main theorems. In Section 3 the method of construction is described. Some old and new mixed level OAs with run sizes 72 and 100 are constructed in Section 4.

2. Basic Concepts and Main Theorems

In our procedure, an important idea is to find the relationship among OAs, permutation matrices and projection matrices. The following notations had to be used.

0 Let $x \in E^n = V \bigoplus W$.Then x can be uniquely decomposed into

$$x = x_1 + x_2 (\text{where} \quad x_1 \in V \quad \text{and} \quad x_2 \in W).$$

The transformation that maps x into x_1 is called the projection matrix (or simply projector) onto V along W and is denoted as ϕ.

Let 1_r be the $r \times 1$ vector of 1's, 0_r the $r \times 1$ vector of 0's, I_r the identity matrix of order r and $J_{r,s}$ the $r \times s$ matrix of 1's, also $J_r =: J_{r,r}$. Of course, the two matrices $P_r = (1/r)1_r 1_r^T = (1/r)J_r$ and $\tau_r = I_r - P_r$ are projection matrices.

Define

$$(r) = (0, \ldots, r-1)^T_{r\times 1}, (r)^{\circ 2} = (r) + (r), \mod r, e_i(r) = (0 \cdots 0 \overset{i}{1} 0 \cdots 0)^T_{r\times 1},$$

where $*^T$ means the transpose of matrix $*$ and $e_i(r)$ is the base vector of R^r (r-dim vector space) for any i. Two $r \times r$ and $pq \times pq$ permutation matrices can be constructed by the $r \times 1$,$p \times 1$and $q \times 1$ base vectors $e_i(r)$,$e_i(p)$and $e_j(q)$respectively as follows:

$$N_r = e_1(r)e_2^T(r) + \cdots + e_{r-1}(r)e_r^T(r) + e_r(r)e_1^T(r)$$

and

$$K(p,q) = \sum_{i=1}^{p}\sum_{j=1}^{q} e_i(p)e_j^T(q) \otimes e_j(q)e_i^T(p), \tag{2.1}$$

where $\otimes$ is the usual Kronecker product in the theory of matrices. The permutation matrices N_r and $K(p,q)$ have the following properties:

$$N_r \cdot (r) = 1_r + (r), \text{ mod } r, \text{ and } K(p, \lambda p) \cdot ((\lambda p) \oplus (p)) = (p) \oplus (\lambda p),$$

where $\oplus$ is the usual Kroneker sum in the theory of matrices (Shrikhande [30]).

Definition 2.1. Let A be an OA of strength 1, i.e.,

$$A = (a_1, \ldots, a_m) = (S_1(0_{r_1} \oplus (p_1)), \ldots, S_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, S_i$ is a permutation matrix for any $i = 1, \ldots, m$. The following projection matrix,

$$A_j = S_j(P_{r_j} \otimes \tau_{p_j})S_j^T, \tag{2.2}$$

is called the **matrix image** (MI) of the jth column a_j of A, denoted by $m(a_j) = A_j$ for $j = 1, \ldots, m$. In general,the MI of a subarray of A is defined as the sum of the MI's of all its columns. In particular, denoted the MI of A by $m(A)$.

If a design is an OA, then the MI's of its columns has some interesting properties which can be used to construct OAs. For example, by the definition, there is

$$m(0_r) = P_r \text{ and } m((r)) = \tau_r.$$

Theorem 2.1. *For any permutation matrix S and any array L,*

$$m(S(L \oplus 0_r)) = S(m(L) \otimes P_r)S^T \quad \textit{and} \quad m(S(0_r \oplus L)) = S(P_r \otimes m(L))S^T.$$

Theorem 2.2. *Let the array A be an OA of strength 1, i.e.,*

$$A = (a_1, \ldots, a_m) = (S_1(0_{r_1} \oplus (p_1)), \ldots, S_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n, S_i$ is a permutation matrix, for $i = 1, \ldots, m$.

The following statements are equivalent.

(1). A is an OA of strength 2.

(2). The MI of A is a projection matrix.

(3). The MI's of any two columns of A are orthogonal, i.e $m(a_i)m(a_j) = 0 (i \neq j)$.

(4). The projection matrix τ_n can be decomposed as

$$\tau_n = m(a_1) + \ldots + m(a_m) + \triangle,$$

where $rk(\triangle) = n - 1 - \sum_{j=1}^{m}(p_j - 1)$ is the rank of the matrix $\triangle$.

Definition 2.2. An OA $L_n = L_n(p_1 \cdots p_m)$ of run size n is said to be saturated if $\sum_{j=1}^{m}(p_j - 1) = n - 1$ (or,equivalently, $m(L_n) = \tau_n$).

Corollary 2.1. *Let (L, H) and K be OAs of run size n. Then (K, H) is an orthogonal array if $m(K) \le m(L)$, where $m(K) \le m(L)$ means that the difference $m(K) - m(L)$ is nonnegative definite.*

Corollary 2.2. *Suppose L and H are OAs. Then $K = (L, H)$ is also an OA if $m(L)$ and $m(H)$ are orthogonal, i.e., $m(L)m(H) = 0$. In this case $m(K) = m(L)+m(H)$.*

These theorems and corollaries can be found in Zhang [38–40].

The following theorem is elementary for the procedure of so called array subtraction although it is simple. The concept of array subtraction can be introduced from it.

Theorem 2.3. *Let $L_n = (K, H)$ be an OA of run size n. If there exists an $n \times n$ projection matrix Θ such that $m(K) \ge \Theta$, then the sub-array H of L_n is also an orthogonal whose MI is less than or equal to $\tau_n - \Theta$, i.e.,*

$$m(H) \le \tau_n - \Theta. \tag{2.3}$$

Proof. From Definition 2.1 and Corollary 2.2, we have

$$m(L_n) = m(K) + m(H),$$

i.e.,$m(H) = m(L_n) - m(K)$. By Theorem 2.2, the inequality $m(L_n) \le \tau_n$ holds for any orthogonal arrays of run size n. Thus by the condition $m(K) \ge \Theta$, the projection matrix $m(H) \le \tau_n - \Theta$ can be obtained. Of course H is an OA since it is a subarray of OA, completing the proof. □

Definition 2.3. Let K be an OA of run size n. If there exists an OA H such that $m(H) \le \tau_n - m(K)$, then the OA H is called an **atom** of asymmetrical OAs of run size n corresponding to the OA K.

In general, let both K and L_n be two OAs of run size n, if there exists an OA H such that $m(H) \le m(L_n) - m(K)$, then the OA H is called an **atom** of the OA L_n corresponding to the OA K (or a **difference** of L_n and K).When $K = 0_n$, the OA H is called simply an **atom** of L_n.

For given asymmetrical OAs K and L_n, the operation of finding all atoms H of L_n corresponding K from known asymmetrical OAs is called **array subtraction.**

In our procedure, it is a key to constructing asymmetrical OAs that for a given projection matrix A an OA H can be found from known OAs such that $m(H) \le A$. The problem is often solved by the array subtraction. Therefore the operation of array subtraction is important.

The following theorems are very useful in the operation of array subtraction.

Theorem 2.4. *Let H be an atom of L_n corresponding to K and let T be an $n \times n$ permutation matrix.Then the OA TH is also an* **atom** *of TL_n corresponding to TK.*

Proof. Let H be an **atom** of L_n corresponding to K, i.e.

$$m(H) \le m(L_n) - m(K). \tag{2.4}$$

Let T be an $n \times n$ permutation matrix, i.e.

$$m(TH) = Tm(H)T^T. \tag{2.5}$$

By 2.4 and 2.5,it is gotten that

$$m(TH) = Tm(H)T^T \leq Tm(L_n)T^T - Tm(K)T^T = m(TL_n) - m(TK).$$

By Definition 2.3,it is gotten that the OA TH is an **atom** of TL_n corresponding to TK. □

Theorem 2.5. *Let H be an atom of L_n corresponding to K and let H^1 be an atom of the OA H. Then the OA H^1 is also an atom of L_n corresponding to K.*

Proof. Let H be an **atom** of L_n corresponding to K, i.e.

$$m(H) \leq m(L_n) - m(K). \tag{2.6}$$

Let H^1 be an **atom** of the OA H, i.e.

$$m(H^1) \leq m(H). \tag{2.7}$$

By 2.6 and 2.7, it is gotten that

$$m(H^1) \leq m(L_n) - m(K).$$

By Definition 2.3, it is gotten that the OA H^1 is an **atom** of L_n corresponding to K. □

Theorem 2.6. *Let H be an atom of L_n corresponding to K and let K^1 be an atom of the OA K. Then the OA H is also an atom of L_n corresponding to K^1.*

Proof. Let H be an **atom** of L_n corresponding to K,i.e.

$$m(H) \leq m(L_n) - m(K). \tag{2.8}$$

Let K^1 be an **atom** of the OA K,i.e.

$$m(K^1) \leq m(K). \tag{2.9}$$

By 2.8 and 2.9, it is gotten that

$$m(H) \leq m(L_n) - m(K) \leq m(L_n) - m(K^1).$$

By Definition 2.3, it is gotten that the OA H is an **atom** of L_n corresponding to K^1. □

3. General Methods for Constructing OA's by Array Subtraction

Our procedure of constructing mixed-level OAs by using the array subtraction based on the orthogonal decomposition of the projection matrix τ_n consists of the following three steps:

Step 1. Orthogonally decompose the projection matrix τ_n :

$$\begin{aligned}\tau_n =&T_1[P_{r_1}\otimes(\tau_{p_1}-\Theta_1)]T_1^T+\ldots+T_{k_1}[P_{r_{k_1}}\otimes(\tau_{p_{k_1}}-\Theta_{k_1})]T_{k_1}^T\\&+C_1+\ldots+C_{k_2}+\triangle,\end{aligned}$$

where $r_ip_i=n$, $\Theta_i\leq\tau_{p_i}$, $\Theta_i, C_s, \triangle$ are projection matrices and T_t is a permutation matrix for any i,s,t.

Step 2. Suppose that there exists an OA L^i such that $m(L_i)\geq\Theta_i$ for any $i=1,2,\ldots,k_1$. Find all (or many) atoms H^i of L_{p_i} (an OA of run size p_i) corresponding to L^i and find all (or many) orthogonal H_s from some known OAs such that

$$m(H^i)\leq\tau_{p_i}-\Theta_i \text{ and } m(H_s)\leq C_s,$$

for any $i=1,2,\ldots,k_1, s=1,2,\ldots,k_2$.

Step 3. Lay out the new OA L by Theorem 2.1,Corollaries 2.1 and 2.2 :

$$L=(T_1(0_{r_1}\oplus H^1),\ldots,T_{k_1}(0_{r_{k_1}}\oplus H^{k_1}),H_1,\ldots,H_{k_2}).$$

Based on Step 1, the following orthogonal decomposition of τ_n is very useful,

$$\begin{aligned}&\tau_{pq}=I_p\otimes\tau_q+\tau_p\otimes P_q=\tau_p\otimes P_q+P_p\otimes\tau_q+\tau_p\otimes\tau_q=\tau_p\otimes I_q+P_p\otimes\tau_q,\\&\tau_{prq}=\tau_p\otimes I_r\otimes P_q+P_p\otimes\tau_{rq}+\tau_p\otimes I_r\otimes\tau_q.\end{aligned}\tag{3.1}$$

These equations are easy to verify from $\tau_p=I_p-P_p, P_{pq}=P_p\otimes P_q$ and $I_{pq}=I_p\otimes I_q$.

Definition 3.1. Let L_n be an OA. The L_n is called having a clear structure if there exist two OAs A, B and m permutation matrices $T_j=f_j(N_r,K(p,q)), j=1,\ldots,m$, such that

$$L_n=(T_1(A\oplus 0_\lambda),\ldots,T_m(A\oplus 0_\lambda),0_\mu\oplus B),$$

where $T_j=f_j(N_r,K(p,q))$ means that T_i can be written into a matrix function of N_r and $K(p,q)$.

Lemma 3.1. *There exist OAs $L_4(2^3), L_9(3^4)$ and $L_{18}(3^66^1)$ having the following clear structures:*

$$\begin{aligned}&L_4(2^3)=[(2)\oplus 0_2, Q_1((2)\oplus 0_2), Q_2((2)\oplus 0_2)],\\&L_9(3^4)=[(3)\oplus 0_3, S_1((3)\oplus 0_3), S_2((3)\oplus 0_3), 0_3\oplus(3)],\end{aligned}$$

and

$$\begin{aligned}&L_{18}(3^66^1)=[(3)\oplus D^0(3,2;3)\oplus 0_2,\\&M_1((3)\oplus D^0(3,2;3)\oplus 0_2), M_2((3)\oplus D^0(3,2;3)\oplus 0_2), 0_3\oplus(6)],\end{aligned}$$

where $Q_1=K(2,2), Q_2=K(2,2)diag(I_2,N_2)K(2,2)^T$;

$$S_1=K(3,3)diag(I_3,N_3,N_3^2)K(3,3)^T,\ S_2=K(3,3)diag(I_3,N_3^2,N_3)K(3,3)^T;$$

and

$$\begin{aligned}&M_1=K(3,6)diag(N_3,N_3^2,Q_1\otimes I_3)K(3,6)^T,\\&M_2=K(3,6)diag(N_3^2,N_3,Q_2\otimes I_3)K(3,6)^T;\end{aligned}$$

and where $D^0(3,2;3) = ((3),(3)^{\circ 2}) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$ *(an atom of difference matrix* $D(3,3;3)$*).*

A key to construct asymmetrical OAs by using array subtraction is to find a clear structure for known orthgonal arrays.

In applying Step 2, the following theorems play very useful roles in the procedure:

Theorem 3.1. *There exist three atoms:*

$$L_{12}^{(-)}(2^9), L_{12}^{(-)}(2^2 3^1), L_{12}^{(-)}(6^1), \tag{3.2}$$

of L_{12} *corresponding to* $[0_6 \oplus (2), 0_3 \oplus (2) \oplus (2)]$*, i.e., there exist three OAs in (3.2) such that their matrix images are less than or equal to the following projection matrix:*

$$\tau_{12} - P_6 \otimes \tau_2 - P_3 \otimes \tau_2 \otimes \tau_2 = \tau_3 \otimes I_4 + P_3 \otimes \tau_2 \otimes P_2.$$

Proof. Consider the following OA $L_{12}(2^{11})$(in (3.3)).

$$L_{12}(2^{11}) = (b_1,\ldots,b_{11}) = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&1&1&0&1\\
0&1&0&1&0&1&1&1&0&0&1\\
1&0&1&1&0&1&0&1&0&1&0\\
0&0&0&1&1&0&0&1&1&1&1\\
1&1&0&0&1&1&0&0&0&1&1\\
0&1&1&0&1&0&1&1&0&1&0\\
1&0&1&1&1&0&1&0&0&0&1\\
0&0&1&0&0&1&1&0&1&1&1\\
1&1&0&1&0&0&1&0&1&1&0\\
0&1&1&1&1&1&0&0&1&0&0\\
1&0&0&0&1&1&1&1&1&0&0
\end{pmatrix}, \tag{3.3}$$

which is obtained by a computer searches from a Hadamard matrix $D(12,12;2)$ in Zhang etc [43].

Define

$$L_{12}^{(-)}(2^9) = (b_3,\ldots,b_{11}),$$
$$L_{12}^{(-)}(2^2 3^1) = (b_3, b_4, (3) \oplus 0_4),$$

and

$$L_{12}^{(-)}(6^1) = (6) \oplus 0_2.$$

They are really the desired atoms (or OAs) by using array subtraction. □

Corollary 3.1. *There exist two atoms:* $L_{12}^{(-)}(2^7)$ *and* $L_{12}^{(-)}(3^1)$ *of* $L_{12}^{(-)}(2^9)$ *(in (3.2)) corresponding to* (b_3, b_4) *(in (3.3)).*

Above Theorem 3.1 and Corollary 3.1 have been used to construct some new asymmetrical OAs of run sizes 36 such as $L_{36}(2^{27}3^1)$, $L_{36}(2^{20}3^2)$, $L_{36}(2^{18}3^16^1)$ by using the orthogonal decomposition of projection matrices in Zhang etc [37].

$$\begin{aligned} L_{36}(2^{27}3^1) =&[(S_1 \otimes Q_1)(1_3 \otimes L_{12}(2^9)), (S_2 \otimes Q_2)(1_3 \otimes L_{12}(2^9)),\\ &(S_3 \otimes Q_3)(1_3 \otimes L_{12}(2^9), (S_4 \otimes I_4)(1_3 \otimes (3) \otimes I_4)],\\ L_{36}(2^{20}3^2) =&[(S_1 \otimes Q_1)(1_3 \otimes L_{12}(3^1), b_3b_4), (S_2 \otimes Q_2)(1_3 \otimes L_{12}(3^1), b_3b_4),\\ &(S_3 \otimes Q_3)(1_3 \otimes L_{12}(3^1), b_3b_4), (S_4 \otimes I_4)(1_3 \otimes (3) \otimes I_4)],\\ L_{36}(2^{18}3^16^1) =&[(S_1 \otimes Q_1)(1_3 \otimes L_{12}(6^1)), (S_2 \otimes Q_2)(1_3 \otimes L_{12}(6^1)),\\ &(S_3 \otimes Q_3)(1_3 \otimes L_{12}(6^1)), (S_4 \otimes I_4)(1_3 \otimes (3) \otimes I_4)]. \end{aligned}$$

In the above formulae,

$$S_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix}$$

Theorem 3.2. *There exists an atom $L_{18}^{(-)}(3^4)$ of L_{18} corresponding to*

$$[0_3 \oplus (6), (3) \oplus (3) \oplus 0_2, (3) \oplus (3)^{\circ 2} \oplus 0_2],$$

where $(3)^{\circ 2} = (021)^T = (3) + (3) mod 3$.

On the other words, there exists an OA $L_{18}^{(-)}(3^4)$ such that its matrix image is equal to the following projection matrix:

$$\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2 = \tau_3 \otimes I_3 \otimes \tau_2 + \tau_3 \otimes P_6,$$

where $m([(3) \oplus (3), (3) \oplus (3)^2]) = \tau_3 \otimes \tau_3$.

proof. Consider the following OA $L_{18}(3^6 6^1)$(in (3.4)).

$$L_{18}(3^6 6^1) = (c_1, \ldots, c_6, f) = \begin{pmatrix} 0\,0\,1\,1\,2\,2\,0 \\ 0\,0\,2\,2\,1\,1\,1 \\ 1\,2\,1\,2\,1\,2\,2 \\ 1\,2\,2\,1\,2\,1\,3 \\ 2\,1\,1\,2\,2\,1\,4 \\ 2\,1\,2\,1\,1\,2\,5 \\ 1\,1\,2\,2\,0\,0\,0 \\ 1\,1\,0\,0\,2\,2\,1 \\ 2\,0\,2\,0\,2\,0\,2 \\ 2\,0\,0\,2\,0\,2\,3 \\ 0\,2\,2\,0\,0\,2\,4 \\ 0\,2\,0\,2\,2\,0\,5 \\ 2\,2\,0\,0\,1\,1\,0 \\ 2\,2\,1\,1\,0\,0\,1 \\ 0\,1\,0\,1\,0\,1\,2 \\ 0\,1\,1\,0\,1\,0\,3 \\ 1\,0\,0\,1\,1\,0\,4 \\ 1\,0\,1\,0\,0\,1\,5 \end{pmatrix}, \tag{3.4}$$

which is really the structure in Lemma 3.1 and obtained by the construction of generalized Hadamard matrices $D(r^m(r+1), r^m(r+1); p)$ (Zhang etc [43]). □

Define

$$L_{18}^{(-)}(3^4) = (c_3, \ldots, c_6).$$

The OA is really the desired atom by using array subtraction.

Corollary 3.2. *There exists one atom* $L_{18}^{(-)}(3^4 6^1)$ *of* L_{18} *corresponding to* $[(3) \oplus (3) \oplus 0_2, (3) \otimes (3)^{\circ 2} \oplus 0_2]$ *whose matrix image is*

$$\tau_{18} - \tau_3 \otimes \tau_3 \otimes P_2 = I_9 \otimes \tau_2 + (\tau_3 \otimes P_3 + P_3 \otimes \tau_3) \otimes P_2,$$

where $\tau_3 \otimes P_3 + P_3 \otimes \tau_3 = m((3) \oplus 0_3, 0_3 \oplus (3)) = m(M[(3) \oplus (3), (3) \oplus (3)^2]) = M\tau_3 \otimes \tau_3 M^T$ *in which* $M = diag(I_3, N_3^2, N_3)K(3,3)diag(I_3, N_3^2, N_3)K(3,3)^T$.

Proof. By (3.4), let $L_{18}^{(-)}(3^4 6^1) = (c_3, \ldots, c_6, f)$. The orthogonal is really the desired atom by using subtraction.

Above Theorem 3.2 and Corollary 3.2 have been used to construct some asymmetrical OAs of run sizes 36 and 72 such as $L_{36}(2^1 3^8 6^2)$, $L_{36}(2^{10} 3^8 6^1)$, $L_{36}(2^9 3^4 6^2)$, $L_{72}(2^{11} 3^{20} 6^1 12^1)$ by using the methods of so called orthogonal decomposition of projection matrices (Zhang etc [37]) and so called generalized Hadamard product (Zhang etc [44]).

$$\begin{aligned}
L_{36}(2^1 3^8 6^2) =& [(T_1 \otimes Q_1)(L_{18}(6 \cdot 3^4) \otimes 1_2), (T_2 \otimes Q_2)(L_{18}(6 \cdot 3^4) \otimes 1_2), \\
& (I_9 \otimes Q_3)(1_9 \otimes (2) \otimes 1_2)], \\
L_{36}(2^{10} 3^8 6^1) =& [1_3 \otimes L_{12}(2^9), (S_1 \otimes Q_1)(L_{18}(3^4) \otimes 1_2), 1_{18} \otimes (2), \\
& (S_2 \otimes Q_2)((3) \otimes 1_{12})\square(1_9 \otimes (2) \otimes 1_2), L_{18}(3^4) \otimes 1_2], \\
L_{36}(2^9 3^4 6^2) =& [1_3 \otimes L_{12}(2^9), (S_1 \otimes Q_1)((3) \otimes 1_{12} \square 1_9 \otimes (2) \otimes 1_2), \\
& (S_2 \otimes Q_2)((3) \otimes 1_{12} \square 1_9 \otimes (2) \otimes 1_2), L_{18}(3^4) \otimes 1_2].
\end{aligned}$$

In the above formulae,

$$T_1 = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0 \end{pmatrix}.$$

$\square$

Theorem 3.3. *There exist three atoms:*

$$L_{20}^{(-)}(2^{17}), L_{20}^{(-)}(2^6 5^1), L_{20}^{(-)}(10^1), \tag{3.5}$$

of L_{20} corresponding to $[0_{10} \oplus (2), 0_5 \oplus (2) \oplus (2)]$, i.e., there exist three OAs in (8) such that their matrix images are less than or equal to the following projection matrix:

$$\tau_{20} - P_{10} \otimes \tau_2 - P_5 \otimes \tau_2 \otimes \tau_2 = \tau_5 \otimes I_4 + P_5 \otimes \tau_2 \otimes P_2.$$

Proof. Consider the following OA $L_{20}(2^{19})$(in (9)).

Define

$$L_{20}^{(-)}(2^{17}) = (b_3, \ldots, b_{19}),$$
$$L_{20}^{(-)}(2^6 5^1) = (b_3, \ldots, b_8, (5) \oplus 0_4),$$

and

$$L_{20}^{(-)}(10^1) = (10) \oplus 0_2.$$

They are the desired atoms (or OAs) by using array subtraction.

Corollary 3.3. *There exist two atoms:* $L_{20}^{(-)}(2^{11})$ *and* $L_{20}^{(-)}(5^1)$ *of* $L_{20}^{(-)}(2^{17})$ *(in (8)) corresponding to* $(b_3, b_4, b_5, b_6, b_7, b_8)$ *(in (3.6)).*

A particular form of OAs having above properties is

$$L_{20}(2^{19}) = (b_1, \ldots, b_{19}) = \begin{pmatrix}
0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0 \\
1\,1\,1\,0\,1\,1\,1\,0 & 0\,0\,1\,1\,1\,0\,1 & 0\,0\,0\,0 \\
0\,1\,0\,1\,1\,1\,0\,1 & 0\,0\,1\,1\,0\,1\,0 & 0\,1\,1\,0 \\
1\,0\,1\,1\,0\,0\,1\,1 & 0\,0\,1\,1\,0\,1\,0 & 1\,0\,0\,1 \\
0\,0\,0\,0\,1\,0\,1\,1 & 0\,1\,0\,1\,1\,1\,1 & 0\,1\,0\,1 \\
1\,1\,0\,1\,0\,1\,1\,1 & 0\,1\,0\,0\,0\,0\,1 & 1\,1\,0\,0 \\
0\,1\,1\,1\,0\,0\,0\,0 & 0\,1\,0\,1\,1\,1\,1 & 1\,0\,1\,0 \\
1\,0\,1\,0\,1\,1\,0\,0 & 0\,0\,0\,0\,0\,1\,1 & 1\,1\,1\,1 \\
0\,0\,0\,0\,0\,1\,1\,1 & 1\,0\,1\,0\,1\,1\,1 & 1\,0\,1\,0 \\
1\,1\,0\,1\,0\,0\,0\,0 & 1\,0\,1\,0\,1\,1\,1 & 0\,1\,0\,1 \\
0\,1\,1\,0\,1\,0\,1\,0 & 1\,1\,1\,0\,0\,1\,0 & 1\,1\,0\,0 \\
1\,0\,1\,1\,1\,1\,0\,1 & 1\,1\,0\,0\,1\,1\,0 & 0\,0\,0\,0 \\
0\,0\,0\,1\,1\,1\,0\,0 & 1\,1\,1\,1\,0\,0\,1 & 1\,0\,0\,1 \\
1\,1\,0\,0\,0\,1\,1\,0 & 1\,1\,0\,1\,0\,1\,0 & 0\,0\,1\,1 \\
0\,1\,1\,1\,1\,0\,1\,1 & 1\,0\,0\,0\,0\,0\,1 & 0\,0\,1\,1 \\
1\,0\,1\,0\,0\,0\,0\,1 & 1\,1\,1\,1\,0\,0\,1 & 0\,1\,1\,0 \\
0\,0\,1\,1\,0\,1\,1\,0 & 0\,1\,1\,0\,1\,0\,0 & 0\,1\,1\,1 \\
1\,1\,0\,0\,1\,0\,0\,1 & 0\,1\,1\,0\,1\,0\,0 & 1\,0\,1\,1 \\
0\,1\,1\,0\,0\,1\,0\,1 & 1\,0\,0\,1\,1\,0\,0 & 1\,1\,0\,1 \\
1\,0\,0\,1\,1\,0\,1\,0 & 1\,0\,0\,1\,1\,0\,0 & 1\,1\,1\,0
\end{pmatrix}, \qquad (3.6)$$

which is obtained by a computer searches from a Hadamard matrix $D(20, 20; 2)$ in Zhang etc [43].

Above Theorem 3.3 and Corollary 3.3 have been used to construct some new asymmetrical OAs of run sizes 60 such as $L_{60}(2^{28}3^1)$ and $L_{60}(2^{26}6^1)$ by using a method of so called generalized difference matrices (Zhang etc [42]).

$$L_{60}(2^{28}3^1) = [D_5(3, 14; 2, 19) \overset{k}{\otimes} (c_1, \ldots, c_{19}), 1_3 \otimes (c_6, \ldots, c_{19}), (3) \otimes 1_{20}]$$

$$L_{60}(2^{26}6^1) = [D^5(3, 13; 2, 18) \overset{k}{\otimes} (c_2, \ldots, c_{19}), 1_3 \otimes (c_7, \ldots, c_{19}), ((3)) \otimes 1_{20} \diamond (1_3 \otimes c_1)]$$

$$D^t(3, 3t; 2, 4t) = [D^1(3, 3; 2, 4) \overset{k}{\otimes} (a_1, a_{t+1}, a_{t+2}, a_{t+3}), \overset{t}{\ldots}, D^1(3, 3; 2, 4) \overset{k}{\otimes} (a_t, a_{4t-2}, a_{4t-1}, a_{4t})];$$

$$D^{t+1}(3,3t+2;2,4t+3) = [D^1(3,3;2,4) \overset{k}{\otimes} (a_1,a_{t+2},a_{t+3},a_{t+4}), \overset{t}{\ldots}, D^1(3,3;2,4) \overset{k}{\otimes} (a_t,a_{4t-1},a_{4t},a_{4t+1})];$$

$$D^{t+2}(3,3t+4;2,4t+6) = [D^1(3,3;2,4) \overset{k}{\otimes} (a_1,a_{t+3},a_{t+4},a_{t+5}), \overset{t}{\ldots}, D^1(3,3;2,4) \overset{k}{\otimes} (a_t,a_{4t},a_{4t+1},a_{4t+2}), D^1(3,2;2,3) \overset{k}{\otimes} (a_{t+1},a_{4t+3},a_{4t+4}), D^1(3,2;2,3) \overset{k}{\otimes} (a_{t+2},a_{4t+5},a_{4t+6})];$$

$$D^1(3,2;2,3) = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_1 \\ -a_2 & -a_3 \end{pmatrix}.$$

Theorem 3.4. *There exist four atoms $L_{24}^{(-)}(2^8 12^1)$, $L_{24}^{(-)}(2^{16}4^1)$, $L_{24}^{(-)}(2^7 4^1 6^1)$, $L_{24}^{(-)}(2^9 3^1 4^1)$ of L_{24}(in (3.7)) corresponding to $[b^1,b^2,b^3,b^4]$ (in (3.7)), where the four columns (b^1,b^2,b^3,b^4) in (10) satisfy the following equation*

$$\sum_{i=1}^{2}(M_i \otimes Q_i)\cdot(P_9 \otimes I_2 \otimes \tau_2 \otimes P_2)\cdot(M_i \otimes Q_i)^{\cdot}T = P_3 \otimes m(b^1,b^2,b^3,b^4),$$

where M_1, M_2, Q_1, Q_2 are defined in Lemma 3.1.

On the other words, the forms of four columns b^1, b^2, b^3, b^4 are

$$0_3 \oplus (b^1,b^2) = (M_1 \otimes Q_1)(0_9 \oplus [0_2,(2)] \oplus (2) \oplus 0_2) = 0_3 \oplus [0_6,(010011)^T] \oplus 0_2 \oplus (2),$$
$$0_3 \oplus (b^3,b^4) = (M_2 \otimes Q_2)(0_9 \oplus [0_2,(2)] \oplus (2) \oplus 0_2) = 0_3 \oplus [0_6, 0_3 \oplus (2)] \oplus (2) \oplus (2).$$

Proof. Consider the following array $(b_1,\ldots,b_{23},c,d,f,l)$(in (10)).

Define

$$L_{24}^{(-)}(2^8 12^1) = (b_{16},\ldots,b_{23},l),$$
$$L_{24}^{(-)}(2^{16}4^1) = (b_2,\ldots,b_{17},b_{20},\ldots,b_{23},d),$$
$$L_{24}^{(-)}(2^9 3^1 4^1) = (b_2,b_3,b_4,b_{12},\ldots,b_{17},b_{20},\ldots,b_{23},c,d),$$
$$L_{24}^{(-)}(2^7 4^1 6^1) = (b_2,b_{12},\ldots,b_{17},b_{20},\ldots,b_{23},d,f).$$

They are really the desired atoms (or OAs) by using array subtraction because $(b_1,\ldots,b_{23})$ is an OA $L_{24}(2^{23})$ and $b^j = b_{11+j}, j = 1,\ldots,4$. □

Corollary 3.4. *There exists an atom $L_{24}^{(-)}(2^8) = (b_{16},\ldots,b_{23})$ of L_{24}(in (3.7)) corresponding to $[b^1,b^2,b^3,b^4,(12)\oplus 0_2]$ (in (10)), where b^j is the same as that in Theorem 3.4 for any j.*

A particular form of OAs having the property is

$$(b_1, \ldots, b_{23}, c, d, f, l) = \begin{pmatrix}
0&0&0&1&0&1&1&1&1&0&1&0&0&0&0&0&1&1&1&1&1&0&1&0&3&0&0\\
0&0&0&1&0&1&1&1&1&0&1&1&1&1&1&1&0&0&0&0&0&1&0&0&0&0&0\\
0&1&1&1&1&1&0&1&0&0&0&0&0&1&1&1&0&1&1&0&1&0&0&0&3&1&1\\
0&1&1&1&1&1&0&1&0&0&0&1&1&0&0&0&1&0&0&1&0&1&1&0&0&1&1\\
1&1&1&0&0&1&1&0&1&0&0&0&1&0&1&1&1&1&0&1&0&0&0&0&2&0&2\\
1&1&1&0&0&1&1&0&1&0&0&1&0&1&0&0&0&0&1&0&1&1&1&0&1&0&2\\
1&0&0&0&1&1&0&1&1&1&0&0&1&1&0&0&0&1&0&1&1&1&0&0&2&1&3\\
1&0&0&0&1&1&0&1&1&1&0&1&0&0&1&1&1&0&1&0&0&0&1&0&1&1&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&4\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&3&2&4\\
1&1&0&1&0&0&1&1&0&1&0&0&0&1&1&0&1&0&1&1&0&1&0&1&1&2&5\\
1&1&0&1&0&0&1&1&0&1&0&1&1&0&0&1&0&1&0&0&1&0&1&1&2&2&5\\
0&1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&0&0&0&1&1&1&1&1&0&3&6\\
0&1&1&0&0&0&0&1&1&1&1&1&1&1&0&0&1&1&1&0&0&0&0&1&3&3&6\\
1&0&1&1&0&1&0&0&0&1&1&0&0&1&0&1&1&1&0&0&0&1&1&1&2&3&7\\
1&0&1&1&0&1&0&0&0&1&1&1&1&0&1&0&0&0&1&1&1&0&0&1&1&3&7\\
1&1&0&1&1&0&0&0&1&0&1&0&1&0&0&1&1&0&1&0&1&1&0&2&1&4&8\\
1&1&0&1&1&0&0&0&1&0&1&1&0&1&1&0&0&1&0&1&0&0&1&2&2&4&8\\
0&0&1&1&1&0&1&0&1&1&0&0&1&1&1&0&1&0&0&0&1&0&1&2&0&4&9\\
0&0&1&1&1&0&1&0&1&1&0&1&0&0&0&1&0&1&1&1&0&1&0&2&3&4&9\\
0&1&0&0&1&1&1&0&0&1&1&0&1&0&1&0&0&1&1&0&0&1&1&2&3&5&10\\
0&1&0&0&1&1&1&0&0&1&1&1&0&1&0&1&1&0&0&1&1&0&0&2&0&5&10\\
1&0&1&0&1&0&1&1&0&0&1&0&1&1&0&1&0&0&1&1&0&0&1&2&1&5&11\\
1&0&1&0&1&0&1&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0&2&2&5&11
\end{pmatrix}, \tag{3.7}$$

which is obtained by a computer searches from an OA $L_{24}(2^{12}12^1)$.

Above Theorem 3.4 and Corollary 3.4 have been used to construct some new asymmetrical OAs of run sizes 72 such as $L_{72}(2^{10}3^{16}6^2 12^1)$ and $L_{72}(2^9 3^{12}6^3 12^1)$ by using a method of so called generalized Hadamard product (Zhang etc [44]). In this paper, the result will be also used to construct an asymmetrical orthogonal array $L_{72}(2^{18}3^4 6^5)$.

Theorem 3.5. *There exist three atoms:*

$$L_{36}^{(-)}(2^2 3^8), L_{36}^{(-)}(2^1 3^4 6^1), L_{36}^{(-)}(6^2), \tag{3.8}$$

of L_{36} *corresponding to* $[(L_{18}^{(-)}(3^4 6^1)) \oplus 0_3, 0_2 \oplus (L_{12}^{(-)}(2^9))]$, *where the OA* $L_{18}^{(-)}(3^4 6^1)$ *is the atom in Corollary 3.2 of* L_{18} *corresponding to* $[(3) \oplus (3), (3) \oplus (3)^{\circ 2}] \oplus 0_2$ *and* *similarly the OA* $L_{12}^{(-)}(2^9)$ *is the atom in Theorem 3.1 of* L_{12} *corresponding to* $0_3 \oplus [0_2 \oplus (2), (2) \oplus (2)]$.

On the other words, there exist three OAs in (11) such that their matrix images are less than the following projection matrix:

$$\begin{aligned} &\tau_{36} - [(\tau_{18} - \tau_3 \otimes \tau_3 \otimes P_2)] \otimes P_2 - P_3 \otimes [(\tau_{12} - P_3 \otimes I_2 \otimes \tau_2)] \\ =&\tau_3 \otimes I_6 \otimes \tau_2 + \tau_3 \otimes \tau_3 \otimes P_2 + P_2 \otimes I_2 \otimes \tau_2. \end{aligned}$$

Proof. In general,let $\Omega_1 = \{0, 1, \dots, p-1\}$,$\Omega_2 = \{0, 1, \dots, q-1\}$,$V = \{0, 1, \dots, pq-1\}$ and $h(i, j) = iq + j$.In this case,the generalized Hadamard product $\overset{h}{\circ}$ also called a jointing or repeating operation and denoted by $\Box$,can be used for the construction of asymmetrical orthogonal arrays.

By the generalized Hadamard product $\overset{h}{\circ} = \diamond$, $h(i, j) = i3 + j$, in Zhang etc [44], the following orthogonal OAs can be obtained

$$\begin{aligned} &L_{36}^{(-)}(2^2 3^8) = [(S_1 \otimes Q_1)(L_{18}^{(-)}(3^4) \oplus 0_2), (S_2 \otimes Q_2)(L_{18}^{(-)}(3^4) \oplus 0_2), 0_9 \oplus (0_2, (2)) \oplus (2)], \\ &L_{36}^{(-)}(2^1 3^4 6^1) = [(S_1 \otimes Q_1)(L_{18}^{(-)}(3^4) \oplus 0_2), [(3) \oplus (3)^{\circ 2} \oplus 0_4] \diamond [0_9 \oplus (2) \oplus (2)], 0_{18} \oplus (2)], \\ &L_{36}^{(-)}(6^2) = [[(3) \oplus (3) \oplus 0_4] \diamond (0_{18} \oplus (2)], [(3) \oplus (3)^{\circ 2} \oplus 0_4] \diamond [0_9 \oplus (2) \oplus (2)]], \end{aligned}$$

in which $L_{18}^{(-)}(3^4)$ is the atom in Theorem 3.2 of L_{18} corresponding to $[(3) \oplus (3) \oplus 0_2, (3) \oplus (3)^2 \oplus 0_2, 0_3 \oplus (6)]$. This completes the proof. $\Box$

Above Theorem 3.5 has been used to construct some asymmetrical OAs of run sizes 72 such as $L_{72}(2^{19}3^{20}4^1 6^1)$ and $L_{72}(2^{18}3^{16}4^1 6^2)$ by using a method of so called generalized Hadamard product in Zhang etc [44]. In this paper, the result will be also used to construct some asymmetrical orthogonal arrays of run sizes 72 such as $L_{72}(2^{18}3^4 6^5)$.

$$\begin{aligned} L_{72}(2^{19}3^{20}4^1 6^1) =& L_{72}(12^1 6^2 3^{16} 2^1 0) \\ =& [1_3 \otimes L_{24}(12^1 \cdot 2^8), L_{36}(3^8) \otimes 1_2, (M_1 \otimes Q_1)(L_{36}(3^8) \otimes 1_2), \\ & (M_1 \otimes Q_1)(L_{36}(3^8) \otimes 1_2), (M_1 \otimes Q_1) 1_9 \otimes [Q_1((2) \otimes 1_2), \\ & Q_2((2) \otimes 1_2)] \otimes 1_2, \\ & (M_2 \otimes Q_2) \cdot [S_1((3) \otimes 1_3)] \otimes 1_8 \Box 1_9 \otimes [Q_1((2) \otimes 1_2)] \otimes 1_2, \\ & (M_2 \otimes Q_2) \cdot [S_2((3) 1_3)] \otimes 1_8 \Box 1_9 \otimes [Q_2((2) \otimes 1_2)] \otimes 1_2]. \\ L_{72}(2^{18}3^{16}4^1 6^2) =& [1_3 \otimes (L_{24}^{(-)}(4^1 \cdot 2^{16})), L_{36}(3^8) \otimes 1_2, \\ & (M_1 \otimes Q_1) \cdot (L_{36}(3^8) \otimes 1_2), \\ & (M_1 \otimes Q_1) \cdot 1_9 \otimes [Q_1((2) \otimes 1_2), Q_2((2) \otimes 1_2)] \otimes 1_2, \\ & (M_2 \otimes Q_2) \cdot [S_1((3) \otimes 1_3)] \otimes 1_8 \Box 1_9 \otimes [Q_1((2) \otimes 1_2)] \otimes 1_2, \\ & (M_2 \otimes Q_2) \cdot [S_2((3) \otimes 1_3)] \otimes 1_8 \Box 1_9 \otimes [Q_2((2) \otimes 1_2)] \otimes 1_2]. \end{aligned}$$

$$S_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix},$$

$$M_1 = \begin{pmatrix}
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}, \; M_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}.$$

Theorem 3.6. *There exist eleven atoms of 36-run OAs corresponding to* $L_{18}^{(-)}(3^4) \oplus 0_2$ *which is the same as that in Theorem 3.5 as follows:*

$$L_{36}^{(-)}(2^{11}3^8), L_{36}^{(-)}(2^{10}3^4 6^1), L_{36}^{(-)}(2^9 6^2), L_{36}^{(-)}(2^4 3^9),$$

$$L_{36}^{(-)}(2^3 3^5 6^1), L_{36}^{(-)}(2^2 3^8 6^1), L_{36}^{(-)}(2^2 3^1 6^2), L_{36}^{(-)}(2^1 3^4 6^2),$$
$$L_{36}^{(-)}(3^9 4^1), L_{36}^{(-)}(3^8 12^1), L_{36}^{(-)}(6^3), \tag{3.9}$$

whose MI's are less than or equal to

$$\begin{aligned}\tau_{36} - (\tau_{18} - \tau_3 \otimes \tau_3 \otimes P_2 - P_3 \otimes \tau_6) \otimes P_2 &= I_{18} \otimes \tau_2 + \tau_3 \otimes \tau_3 \otimes P_2 + P_3 \otimes \tau_6 \otimes P_2 \\ &= P_3 \otimes \tau_{12} + \tau_3 \otimes I_6 \otimes \tau_2 + \tau_3 \otimes \tau_3 \otimes P_4,\end{aligned}$$

where $\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2 = \tau_3 \otimes I_3 \otimes \tau_2 + \tau_3 \otimes P_6$.

Proof. By Theorem 3.2, there exists an atom $L_{18}^{(-)}(3^4)$ of L_{18} corresponding to $[0_3 \oplus (6), (3) \oplus (3) \oplus 0_2, (3) \oplus (3)^2 \oplus 0_2]$, whose matrix image is equal to

$$\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2 = \tau_3 \otimes I_3 \otimes \tau_2 + \tau_3 \otimes P_6.$$

By the array subtraction (Theorem 2.3), in order to find OAs such that whose MI's are less than or equal to

$$\tau_{36} - m(L_{18}^{(-)}(3^4) \otimes P_2) = \tau_{36} - (\tau_{18} - \tau_6 - P_2 \otimes \tau_3 \otimes \tau_3) \oplus P_2,$$

it is needed to construct the 36-run OAs whose forms are as follows:

$$L_{36}(2^x 3^y 6^z) = (L_{18}^{(-)}(3^4) \oplus 0_2, L_{36}^{(-)}(2^x 3^{y-4} 6^z)),$$

where $y \geq 4$ and $2^0 = 3^0 = 6^0 = 1^1$.

In fact, from the constructions in Zhang etc [44], the following seven OAs can be found, denoted by L_{36},

$$\begin{aligned}&L_{36}(2^{11} 3^{12}), L_{36}(2^{10} 3^8 6^1), L_{36}(2^9 3^4 6^2), L_{36}(2^4 3^9),\\ &L_{36}(2^3 3^9 6^1), L_{36}(2^2 3^{12} 6^1), L_{36}(2^2 3^5 6^2), L_{36}(2^1 3^8 6^2),\\ &L_{36}(3^{13} 4^1), L_{36}(3^{12} 12^1), L_{36}(3^4 6^3),\end{aligned}$$

each of which contains the OA $L_{18}^{(-)}(3^4) \oplus 0_2$.

By the array subtraction (Theorem 2.3), deleting the OA $L_{18}^{(-)}(3^4) \oplus 0_2$ from L_{36}, the desired OAs can be obtained. This completes the proof. □

The result in Theorem 3.6 will be used to construct the new asymmetrical orthogonal arrays of run sizes 72 such as $L_{72}(2^{18} 3^4 6^5)$. A clear structure of each of atoms in Theorem 3.5 and 3.6 is need. In the following equation (3.10), there are

$$\begin{aligned}&L_{36}^{(-)}(2^2 3^8) = (b_1, b_2, c_5, \ldots, c_{12}), \quad L_{36}^{(-)}(2^1 3^4 6^1) = (b_1, c_5, \ldots, c_8, f_3),\\ &L_{36}^{(-)}(6^2) = (f_2, f_3), \quad L_{36}^{(-)}(2^{11} 3^8) = (b_1, \ldots, b_{11}, c_5, \ldots, c_{12}),\\ &L_{36}^{(-)}(2^{10} 3^4 6^1) = (b_1, b_3, \ldots, b_{11}, c_5, \ldots, c_8, f_3), \quad L_{36}^{(-)}(2^9 6^2) = (b_3, \ldots, b_{11}, f_2, f_3),\\ &L_{36}^{(-)}(2^4 3^9) = (b_1, \ldots, b_4, c_5, \ldots, c_{12}, c), \quad L_{36}^{(-)}(2^3 3^5 6^1) = (b_1, b_3, b_4, c_5, \ldots, c_8, c, f_3),\\ &L_{36}^{(-)}(2^2 3^8 6^1) = (b_1, b_2, c_5, \ldots, c_{12}, f), \quad L_{36}^{(-)}(2^2 3^1 6^2) = (b_3, b_4, c, f_2, f_3),\\ &L_{36}^{(-)}(2^1 3^4 6^2) = (b_1, c_5, \ldots, c_8, f, f_3), \quad L_{36}^{(-)}(3^9 4^1) = (c_5, \ldots, c_{12}, c, 0_9 \oplus (4)),\\ &L_{36}^{(-)}(3^8 12^1) = (c_5, \ldots, c_{12}, l), \quad L_{36}^{(-)}(6^3) = (f, f_2, f_3).\end{aligned}$$

A particular form of 36-run arrays having above properties is

$$
\begin{aligned}
&(L_{36}(2^{11}3^{12}), c, f, f_1 - f_3, l)\\
=&(b_1, \ldots, b_{11}, c_1, \ldots, c_{12}, c, f, f_1 - f_3, l)\\
=&\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&1&1&2&2&1&1&2&2&1&1&2&2&0&0&1&0&0&0\\
1&1&1&0&0&0&1&0&1&0&1&1&1&2&2&2&2&1&1&2&2&1&1&0&0&4&1&1&1\\
0&1&0&1&1&1&1&0&0&0&1&2&2&1&1&1&1&2&2&2&2&1&1&0&1&2&0&1&2\\
1&0&1&1&1&0&1&0&0&1&0&2&2&1&1&2&2&1&1&1&1&2&2&0&1&3&1&0&3\\
0&0&0&1&0&0&1&1&1&1&1&1&2&1&2&2&0&2&0&0&1&0&1&1&2&1&2&4&4\\
1&1&0&0&1&0&0&1&0&1&1&1&2&1&2&0&2&0&2&1&0&1&0&1&2&2&3&5&5\\
0&1&1&0&0&1&1&1&0&1&0&2&1&2&1&2&0&2&0&1&0&1&0&1&3&4&2&5&6\\
1&0&1&1&0&1&0&1&0&0&1&2&1&2&1&0&2&0&2&0&1&0&1&1&3&3&3&4&7\\
0&0&1&0&1&1&0&0&1&1&1&1&2&2&1&0&1&1&0&2&0&0&2&2&4&3&4&2&8\\
1&1&0&1&0&1&0&0&1&1&0&1&2&2&1&1&0&0&1&0&2&2&0&2&4&2&5&3&9\\
0&1&1&1&1&0&0&1&1&0&0&2&1&1&2&0&1&1&0&0&2&2&0&2&5&4&4&3&10\\
1&0&0&0&1&1&1&1&1&0&0&2&1&1&2&1&0&0&1&2&0&0&2&2&5&1&5&2&11\\
0&0&0&0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&0&2&2&0&0&0&0&2&2&2&0\\
1&1&1&0&0&0&1&0&1&0&1&2&2&0&0&0&0&2&2&0&0&2&2&0&0&5&3&3&1\\
0&1&0&1&1&1&1&0&0&0&1&0&0&2&2&2&2&0&0&0&0&2&2&0&1&0&2&3&2\\
1&0&1&1&1&0&1&0&0&1&0&0&0&2&2&0&0&2&2&2&2&0&0&0&1&4&3&2&3\\
0&0&0&1&0&0&1&1&1&1&1&2&0&2&0&0&1&0&1&1&2&1&2&1&2&2&4&0&4\\
1&1&0&0&1&0&0&1&0&1&1&2&0&2&0&1&0&1&0&2&1&2&1&1&2&0&5&1&5\\
0&1&1&0&0&1&1&1&0&1&0&0&2&0&2&0&1&0&1&2&1&2&1&1&3&5&4&1&6\\
1&0&1&1&0&1&0&1&0&0&1&0&2&0&2&1&0&1&0&1&2&1&2&1&3&4&5&0&7\\
0&0&1&0&1&1&0&0&1&1&1&2&0&0&2&1&2&2&1&0&1&1&0&2&4&4&0&4&8\\
1&1&0&1&0&1&0&0&1&1&0&2&0&0&2&2&1&1&2&1&0&0&1&2&4&0&1&5&9\\
0&1&1&1&1&0&0&1&1&0&0&0&2&2&0&1&2&2&1&1&0&0&1&2&5&5&0&5&10\\
1&0&0&0&1&1&1&1&1&0&0&0&2&2&0&2&1&1&2&0&1&1&0&2&5&2&1&4&11\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&0&4&4&0\\
1&1&1&0&0&0&1&0&1&0&1&0&0&1&1&1&1&0&0&1&1&0&0&0&0&3&5&5&1\\
0&1&0&1&1&1&1&0&0&0&1&1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&4&5&2\\
1&0&1&1&1&0&1&0&0&1&0&1&1&0&0&1&1&0&0&0&0&1&1&0&1&5&5&4&3\\
0&0&0&1&0&0&1&1&1&1&1&0&1&0&1&1&2&1&2&2&0&2&0&1&2&0&0&2&4\\
1&1&0&0&1&0&0&1&0&1&1&0&1&0&1&2&1&2&1&0&2&0&2&1&2&1&1&3&5\\
0&1&1&0&0&1&1&1&0&1&0&1&0&1&0&1&2&1&2&0&2&0&2&1&3&3&0&3&6\\
1&0&1&1&0&1&0&1&0&0&1&1&0&1&0&2&1&2&1&2&0&2&0&1&3&5&1&2&7\\
0&0&1&0&1&1&0&0&1&1&1&0&1&1&0&2&0&0&2&1&2&2&1&2&4&5&2&0&8\\
1&1&0&1&0&1&0&0&1&1&0&0&1&1&0&0&2&2&0&2&1&1&2&2&4&1&3&1&9\\
0&1&1&1&1&0&0&1&1&0&0&1&0&0&1&2&0&0&2&2&1&1&2&2&5&3&2&1&10\\
1&0&0&0&1&1&1&1&1&0&0&1&0&0&1&0&2&2&0&1&2&2&1&2&5&0&3&0&11
\end{pmatrix},
\end{aligned}
\tag{3.10}
$$

which is obtained by using a method of so called generalized Hadamard product (Zhang etc [44]).

4. Constructions of OA's of Run Sizes 72 and 100

4.1. Construction of OA $L_{72}(2^{19}3^86^4)$

Step 1. Orthogonally decompose the projection matrix τ_{72}. From (3.1), there is

$$\tau_{72} = P_{24} \otimes \tau_{24} + \tau_3 \otimes I_6 \otimes P_4 + \tau_3 \otimes I_6 \otimes \tau_4. \tag{4.1}$$

By Lemma 3.1 and array subtraction, there is

$$\tau_4 = \sum_{i=0}^{2} Q_i \cdot (\tau_2 \otimes P_2) \cdot Q_i^T, \tag{4.2}$$

and

$$\tau_3 \otimes I_6 = \sum_{i=0}^{2} M_i \cdot (\tau_3 \otimes \tau_3 \otimes P_2) \cdot M_i^T, \tag{4.3}$$

where $Q_0 = I_4, M_0 = I_{18}$ and Q_1, Q_2, M_1, M_2 are defined in Lemma 3.1.

By (4.1),(4.2) and (4.3), an orthogonal decomposition of projection matrix τ_{72} can be obtained as follows:

$$\tau_{72} = P_3 \otimes \tau_{24} + \sum_{i=0}^{2} (M_i \otimes Q_i) \cdot (\tau_3 \otimes \tau_3 \otimes P_8 + \tau_3 \otimes I_6 \otimes \tau_2 \otimes P_2) \cdot (M_i \otimes Q_i)^T. \tag{4.4}$$

Now orthogonally decompose the sum of the first two items of (4.4):

$$A =: P_3 \otimes \tau_{24} + \tau_3 \otimes \tau_3 \otimes P_8 + \tau_3 \otimes I_6 \otimes \tau_2 \otimes P_2.$$

Since $\tau_{24} = I_{12} \otimes \tau_2 + \tau_{12} \otimes P_2$ and

$$P_3 \otimes \tau_{12} + \tau_3 \otimes \tau_3 \otimes P_4 + \tau_3 \otimes I_6 \otimes \tau_2 = \tau_{36} - (\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2) \otimes P_2,$$

the following projection matrix can be obtained

$$A = P_3 \otimes I_{12} \otimes \tau_2 + (\tau_{36} - (\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2)) \otimes P_2.$$

On the other hand, by Theorem 3.4 and Corollary 3.4, an orthogonal decomposition of projection matrix $P_3 \otimes I_{12} \otimes \tau_2$ can be obtained as follows:

$$P_3 \otimes I_{12} \otimes \tau_2 = \sum_{i=1}^{2} (M_i \otimes Q_i) \cdot (P_9 \otimes I_2 \otimes \tau_2 \otimes P_2) \cdot (M_i \otimes Q_i)^T + P_3 \otimes m(L_{24}^{(-)}(2^8)),$$

where the OA $L_{24}^{(-)}(2^8)$ is the atom of L_{24} in Corollary 3.4. Thus from (4.4) the following projection matrix decomposition can be obtained

$$\begin{aligned}\tau_{72} =& P_3 \otimes m(L_{24}^{(-)}(2^8)) + (\tau_{36} - (\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2) \otimes P_2) \otimes P_2 \\ &+ \sum_{i=1}^{2} (M_i \otimes Q_i) \cdot (P_9 \otimes I_2 \otimes \tau_2 \otimes P_2 + \tau_3 \otimes \tau_3 \otimes P_8 + \tau_3 \otimes I_6 \otimes \tau_2 \otimes P_2)\end{aligned}$$

$$\cdot (M_i \otimes Q_i)^T. \quad (4.5)$$

The above decompositions are orthogonal because of the orthogonality in each step.

Step 2. Now the following OAs can be found $L^1_{72}(2^8)$, $L^2_{72}(\cdots)$, and $L^3_{72}(\cdots)$ such that

$$m(L^1_{72}(2^8)) \leq P_3 \otimes m(L^{(-)}_{24}(2^8)),$$
$$m(L^2_{72}(\cdots)) \leq (\tau_{36} - (\tau_{18} - P_3 \otimes \tau_6 - \tau_3 \otimes \tau_3 \otimes P_2) \otimes P_2) \otimes P_2,$$

and

$$m(L^3_{72}(\cdots)) \leq (P_9 \otimes I_2 \otimes \tau_2 + \tau_3 \otimes \tau_3 \otimes P_8 + \tau_3 \otimes I_6 \otimes \tau_2) \otimes P_2.$$

By Theorems 2.1 and 3.4, only need to take

$$L^1_{72}(2^8) = 0_3 \oplus L^{(-)}_{24}(2^8),$$

where $L^{(-)}_{24}(2^8)$ has been given in Corollary 3.4.

Similarly, by Theorems 2.1 and 3.6, only need to take

$$L^2_{72}(\cdots) = L^{(-)}_{36}(\cdots) \oplus 0_2,$$

where $L^{(-)}_{36}(\cdots)'s$ have been given in Theorem 3.6(there exist eleven OAs $L^{(-)}_{36}(\cdots)$).

Also similarly, by Theorems 2.1 and 3.5, only need to take

$$L^3_{72}(\cdots) = L^{(-)}_{36}(\cdots) \oplus 0_2,$$

where $L^{(-)}_{36}(\cdots)'s$ have been given in Theorem 3.5(there exist three OAs $L^{(-)}_{36}(\cdots)$).

Step 3. By Theorems 3.5 and 3.6, the new OA is lay out.

$$\begin{aligned} L_{72}(2^{18}3^4 6^5) =& [0_3 \oplus L^{(-)}_{24}(2^8), L^{(-)}_{36}(2^{10}3^4 6^1) \oplus 0_2, \\ & (M_1 \otimes Q_1)(L^{(-)}_{36}(6^2) \oplus 0_2), (M_2 \otimes Q_2)(L^{(-)}_{36}(6^2) \oplus 0_2)], \end{aligned} \quad (4.6)$$

where both $L^{(-)}_{36}(6^2)$ and $L^{(-)}_{36}(2^{10}3^4 6^1)$ are given in Theorems 3.5 and 3.6, the permutation matrices Q_1, Q_2, M_1, M_2 are the same as those in Step 1 or those in Lemma 3.1. The OA $L_{72}(2^{18}3^4 6^5)$ is not new which has been included in Hedayat etc [14] or Kuhfeld [18] yet.

Furthermore, in (4.6) replacing $L^{(-)}_{36}(6^2)$ and $L^{(-)}_{36}(2^{10}3^4 6^1)$ by the three atoms in Theorem 3.5 and the eleven atoms in Theorem 3.6 respectively, the $11 \times 3 \times 3 = 99$ OAs can be obtained, in which many arrays (Table 1) are new which are not included in Hedayat etc [14] or Kuhfeld [18] yet.

4.2. Construction of OA $L_{100}(2^{51}5^3)$

By the definition of OA, without loss of generality, assume that

$$L_4(2^3) = (Q_1((2) \oplus 0_2), \ldots, Q_3((2) \oplus 0_2)),$$

and

$$L_{25}(5^6) = (T_1(0_5 \oplus (5)), \ldots, T_6(0_5 \oplus (5))),$$

where $Q_i (i = 1, 2)$ are defined in Lemma 3.1 and $T_1 = I_{25}$, $T_2 = \text{diag}(I_5, N_5, \ldots, N_5^4)$, $T_3 := T_2 \cdot T_2 = T_2^2$, $T_4 = T_2^3$, $T_5 = T_2^4$, $T_6 = K(5, 5)$.

Since $L_{25}(5^6)$ and $L_4(2^3)$ are saturated OAs, from the definition of matrix images and by Theorem 2.1 and 2.2, there are

$$\tau_{25} = \sum_{i=1}^{6} T_i(P_5 \otimes \tau_5)T_i^T,$$

and

$$\tau_4 = \sum_{i=1}^{3} Q_i(\tau_2 \otimes P_2)Q_i^T.$$

From (3.1), there is

$$\tau_{100} = \tau_{25} \otimes I_4 + P_{25} \otimes \tau_4 = [\sum_{i=1}^{6} T_i(P_5 \otimes \tau_5)T_i^T] \otimes I_4 + P_{25} \otimes [\sum_{i=1}^{3} Q_i(\tau_2 \otimes P_2)Q_i^T].$$

Using the matrix properties $I_4 = Q_i I_4 Q_i^T$,$P_{25} = T_j P_{25} T_j^T$,$(ABC) \otimes (DEF) = (A \otimes D)(B \otimes E)(C \otimes F)$ and $I_4 = P_4 + \tau_5$, there is

$$\begin{aligned}\tau_{100} =& \sum_{i=1}^{3}(T_i \otimes Q_i)(P_5 \otimes (\tau_5 \otimes I_4 + P_5 \otimes \tau_2 \otimes P_2)(T_i \otimes Q_i)^T \\ &+ \sum_{i=4}^{6}(T_i \otimes I_4)(P_5 \otimes \tau_5 \otimes P_4)(T_i \otimes I_4)^T \\ &+ \sum_{i=4}^{6}(T_i \otimes I_4)(P_5 \otimes \tau_5 \otimes \tau_4)(T_i \otimes I_4)^T. \qquad (4.7)\end{aligned}$$

The above decompositions are orthogonal because of the orthogonality in each step.

Now it is wanted to find an OA whose matrix image is less than or equal to $\tau_5 \otimes I_4 + P_5 \otimes \tau_2 \otimes P_2$. From (3.1) it is seen that

$$\tau_5 \otimes I_4 + P_5 \otimes \tau_2 \otimes P_2 = \tau_{20} - P_5 \otimes I_2 \otimes \tau_2.$$

By Theorem 3.3, there exist three atoms in (3.5) of L_{20} corresponding to $0_5 \oplus [0_2 \oplus (2), (2) \oplus (2)]$. On the other words, the matrix images of the three atoms $L_{20}^{(-)}(2^{17})$,$L_{20}^{(-)}(2^6 5^1)$ and $L_{20}^{(-)}(10^1)$ are less than or equal to $\tau_5 \otimes I_4 + P_5 \otimes \tau_2 \otimes P_2$.

By (4.7) and Theorem 2.1, 2.2 and Corollary 2.1, an OA $L_{100}(2^{51}5^3)$ can be obtained as follows:

$$\begin{aligned}L_{100}(2^{51}5^3) =& [(T_1 \otimes Q_1)(0_5 \oplus L_{20}^{(-)}(2^{17})), (T_2 \otimes Q_2)(0_5 \oplus L_{20}^{(-)}(2^{17})), \\ & (T_3 \otimes Q_3)(0_5 \oplus L_{20}^{(-)}(2^{17})), (T_4 \otimes I_4)(0_5 \oplus (5) \oplus 0_4), \\ & (T_5 \otimes I_4)(0_5 \oplus (5) \oplus 0_4), (T_6 \otimes I_4)(0_5 \oplus (5) \oplus 0_4)]. \qquad (4.8)\end{aligned}$$

Furthermore, replacing the atom $L_{20}^{(-)}(2^{17})$ in (4.8) by the atoms $L_{20}^{(-)}(2^6 5^1)$ and $L_{20}^{(-)}(10^1)$ in Theorem 3.3, the $3 \times 3 \times 3 = 27$ asymmetrical OAs can be constructed in which many arrays (Table 1) are new which are not included in Hedayat etc [14] or Kuhfeld [18] yet.

Table 1. Orthogonal arrays Obtained in Section 4

$L_{100}(2^{51}5^3)$(new)	$L_{72}(2^{23}3^{24})$	$L_{72}(2^{12}3^9 6^4)$
$L_{100}(2^{40}5^4)$(new)	$L_{72}(2^{22}3^{20}6^1)$(new)	$L_{72}(2^{11}3^{21}4^1 6^1)$
$L_{100}(2^{34}5^3 10^1)$(new)	$L_{72}(2^{21}3^{16}6^2)$(new)	$L_{72}(2^{11}3^{20}6^1 12^1)$(new)
$L_{100}(2^{29}5^5)$(new)	$L_{72}(2^{20}3^{12}6^3)$(new)	$L_{72}(2^{11}3^{12}6^4)$(new)
$L_{100}(2^{23}5^4 10^1)$(new)	$L_{72}(2^{19}3^8 6^4)$(new)	$L_{72}(2^{11}3^5 6^5)$
$L_{100}(2^{18}5^6)$(new)	$L_{72}(2^{18}3^4 6^5)$	$L_{72}(2^{10}3^{17}4^1 6^2)$(new)
$L_{100}(2^{17}5^3 10^2)$(new)	$L_{72}(2^{17}6^6)$	$L_{72}(2^{10}3^{16}6^2 12^1)$(new)
$L_{100}(2^{12}5^5 10^1)$(new)	$L_{72}(2^{16}3^{25})$	$L_{72}(2^{10}3^8 6^5)$(new)
$L_{100}(2^6 5^4 10^2)$(new)	$L_{72}(2^{15}3^{21}6^1)$(new)	$L_{72}(2^{10}3^1 6^6)$
$L_{100}(5^3 10^3)$	$L_{72}(2^{14}3^{24}6^1)$	$L_{72}(2^9 3^{13}4^1 6^3)$(new)
	$L_{72}(2^{14}3^{17}6^2)$(new)	$L_{72}(2^9 3^{12}6^3 12^1)$(new)
	$L_{72}(2^{13}3^{20}6^2)$(new)	$L_{72}(2^9 3^4 6^6)$
	$L_{72}(2^{13}3^{13}6^3)$(new)	$L_{72}(2^8 3^9 4^1 6^4)$
	$L_{72}(2^{12}3^{25}4^1)$	$L_{72}(2^8 3^8 6^4 12^1)$
	$L_{72}(2^{12}3^{24}12^1)$	$L_{72}(2^8 6^7)$
	$L_{72}(2^{12}3^{16}6^3)$(new)	

References

[1] S. Addelman, *Orthogonal main effect plans for asymmetrical experiments*, Technometrics, 1962, 4, 21–46.

[2] S. Addelman, *Symmetrical and asymmetrical fractional factorial plans*, Technometrics, 1962, 4, 47–58.

[3] S. Addelman and O. Kempthome, *Orthogonal main-effect plans*, Aeronautical Research Laboratory Technical Report, 1961, 79.

[4] S. Addelman and O. Kempthome, *Some main-effect plans and OAs of strength*

two, Ann. Math. Statist., 1961, 32, 1167–1176.

[5] V. Agrawal and A. Dey, *A note on orthogonal main effect plans for asymmetrical factorials*, Sankhya. Ser. B., 1982, 44, 278–282.

[6] K. A. Bush, *Orthogonal arrays*, Ph. D. dessertation. Univ. North Carolina., Chapel Hill, 1950.

[7] R. C. Bose and K. A. Bush, *Orthogonal arrays of strength two and three*, Ann. Math. Statist., 1952, 23, 508–524.

[8] G. Chen, L. Ji and J. Lei, *The existence of mixed OAs with four and five factors of strength two*, Journal of Combinatorial Designs, 2014, 22(8), 323–342.

[9] X. Chen, J. Lin, X. Chen and X. Wang, *Matrix Image Method for Ranking Nonregular Fractional Factorial Designs*, Acta Mathematicae Applicatae Sinica, 2018, 34(1), 742–751.

[10] X. Chen, B. Guo, M. Liu and X. Wang, *Robustness of orthogonal-array based composite designs to missing data*, Journal of Statistical Planning and Inference, 2018, 194, 15–24.

[11] C. Cheng, *Some orthogonal main-effect plans for asymmetrical factorials*, Technometrics, 1989, 31, 475–477.

[12] C. J. Colbourn, D. R. Stinson and S. Veitch, *Constructions of optimal orthogonal arrays with repeated rows*, Discrete Mathematics, 2019, 342(9), 2455–2466.

[13] A. Dey and G. U. S. Ramakrishma, *A note on orthogonal main effect plans*, Technometics, 1977, 19, 511–512.

[14] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer-Verlag, New York, 1999.

[15] H. Huang, *Construction of component orthogonal arrays with any number of components*, Journal of Statistical Planning and Inference, 2021, 213, 72–79.

[16] A. S. Hedayat, K. Pu and J. Stufken, *On the construction of asymmetrical OAs*, Ann. Statist., 1992, 20, 2142–2152.

[17] L. Jiang and J. Yin, *An approach of constructing mixed-level orthogonal arrays of strength* ≥ 3, Science China Mathematics, 2013, 56(6), 1109–1115.

[18] W. F. Kuhfeld, *Orthogonal arrays*, http://support.sas.com/techsup/technote/ts723.html, 2015.

[19] C. Luo, Y. Zhang and S. He, *Asymmetrical OAs with run size 100*, Communications in Statistics Theory and Methods, 2015, 44(6), 1222–1240.

[20] J. Leng and D. Han, *Orthogonal projection decomposition of matrices and construction of fusion frames*, Advances in Computational Mathematics. 2013, 38(2), 369–381.

[21] C. Luo, Y. Zhang and X. Chen, *Orthogonal arrays obtained by generalized Kronecker product*, Journal of Applied Analysis and Computation, 2017, 7(2), 728–744.

[22] C. Luo, *Theory of generalized difference matrices and construction of OAs*, Chinese Science Press, 2015.

[23] S. Pang, X. Lin and J. Wang, *Construction of Asymmetric Orthogonal Arrays of Strength t from Orthogonal Partition of Small Orthogonal Arrays*, IEICE

Transactions on Fundamentals of Electronics Communications and Computer Sciences, 2018, E101A(8), 1267–1272.

[24] S. Pang, W. Xu, G. Chen, et al, *Construction of Symmetric and Asymmetric Orthogonal Arrays of Strength t from Orthogonal Partition*, Indian Journal of Pure and Applied Mathematics, 2018, 49(4), 663–669.

[25] S. Pang, J. Wang, D. Lin, and M. Liu, *Construction of Mixed Orthogonal Arrays with High Strength*, accepted to the Annals of Statistics, https://imstat.org/journals-and-publications/annals-of-statistics/annals-of-statistics-future-papers/, 2021.

[26] S. Pang and Y. Zhang, *Multiplication of OAs*, Acta Mathematica Scientia, 2007, 27A(3), 568–576.

[27] C. Rao, *Factorial experiments derivable from combinational arrangements of arrays*, Journal of the Royal Statal Society, 1947, 9(1), 128–239.

[28] C. Rao, *Some combinatorial problem of arrays and applications to design of experiments*, In a Survey of Combinatorial Theory.(J. N. Srivastava et al ed.), North-Holland. Amsterdam, 1973, 349–359.

[29] F. H. Ryoh, *Orthogonal array from Baer Subplanes*, Utilitas Mathematica, 1993, 43, 65–70.

[30] S. Shrikhande, *Generalized Hadamard matrices and OAs strength two*, Canadian Journal of mathematics, 1964, 16, 736–740.

[31] C. Y. Suen, *Some mixed orthogonal arrays obtained by orthogonal projection matrices*, Journal of Statistical Planning and Inference, 2007, 137(5), 1704–1710.

[32] C. Y. Suen and W. F. Kuhfeld, *On the construction of mixed orthogonal arrays of strength two*, Journal of Statistical Planning and Inference, 2005, 133, 555–560.

[33] G. Taguchi, *System of experimental design*, White Plains: UNIPUB, 1987, 1(2).

[34] C. Wu, R. Zhang and R. Wang, *Construction of asymmetrical orthogonal array of the type* $OA(s^k, s^m(s_1^r)^{n_1} \cdots (s_t^r)^{n_t})$., Statistica Sinica, 1992, 1, 203–219.

[35] C. Wu, *Construction of* $2^m 4^n$ *design via group scheme*, Ann. Statist., 1989, 17, 1880–1885.

[36] K. Yamada and N. Miyamoto, *A construction and decomposition of orthogonal arrays with non-prime-power numbers of symbols on the complement of a Baer subplane*, Designs, Codes and Cryptography, 2015, 14, 1–12.

[37] Y. Zhang, Y. Lu and S. Pang, *Orthogonal arrays obtained by orthogonal decomposition of projection matrices*, Statistica Sinica, 1999, 9, 595–604.

[38] Y. Zhang, *Asymmetrical orthogonal design by multi-matrix methods*, Journal of the Chinese Statistical Association, 1991, 29, 197–218.

[39] Y. Zhang, *Orthogonal array and matrices*, Journal of Mathematical Research And Exposition, 1992, 3, 438–440.

[40] Y. Zhang, *Theory of multilateral matrix*, Chinese Statistic Press, 1993.

[41] Y. Zhang, *Orthogonal arrays obtained by repeating-column difference matrices*, Discrete Mathematics, 2007, 307(4), 246–261.

[42] Y. Zhang, W. Li, S. Mao and Z. Zheng, *Orthogonal arrays obtained by generalized difference matrices with g levels*, SCIENCE CHINA Mathematics (Science in China Series A: Mathematics), 2011, 54(1), 133–143.

[43] Y. Zhang, L. Duan, Y. Lu and Z. Zheng, *Construction of Generalized Hadamard Matrix* $D(r^m(r+1), r^m(r+1); p)$., Journal of Statistical Planning and Inference, 2002, 104, 239–258.

[44] Y. Zhang, S. Pang and Y. Wang, *Orthogonal arrays obtained by generalized Hadamard produc*, Discrete Mathematics, 2001, 238, 153–170.